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Conditional value at risk and related linear programming models for portfolio optimization

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Abstract Many risk measures have been recently introduced which (for discrete random variables) result in Linear Programs (LP). While some LP computable risk measures may be viewed as approximations to the variance (e.g., the mean absolute deviation or the Gini's mean absolute difference), shortfall or quantile risk measures are recently gaining more popularity in various financial applications. In this paper we study LP solvable portfolio optimization models based on extensions of the Conditional Value at Risk (CVaR) measure. The models use multiple CVaR measures thus allowing for more detailed risk aversion modeling. We study both the theoretical properties of the models and their performance on real-life data.

Keywords Portfolio optimization · Mean-risk models · Linear programming · Stochastic dominance · Conditional Value at Risk · Gini's mean difference

Following the seminal work by Markowitz (1952), the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the expected return is maximized and some (scalar) risk measure is minimized. In the original Markowitz model the risk is measured by the standard deviation or variance. Several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz-type) models. While the original Markowitz model forms a quadratic programming problem, following Sharpe (1971a), many attempts have been made to linearize the portfolio optimization procedure (c.f.,

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Speranza, 1993 and references therein). The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints (including the minimum transaction lots (Mansini and Speranza, 1999; Mansini and Speranza, 2005), transaction costs (Kellerer, Mansini and Speranza, 2000; Konno and Wijayanayake, 2001) and mutual funds characteristics (Chiodi, Mansini, and Speranza, 2003). The introduction of these features leads to mixed integer LP problems. In order to guarantee that the portfolio takes advantage of diversification, no risk measure can be a linear function of the portfolio weights. Nevertheless, a risk measure can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios.

The simplest LP computable risk measures are dispersion measures similar to the variance. The mean absolute deviation was very early considered in portfolio analysis (Sharpe, 1971b and references therein) while Konno and Yamazaki (1991) presented and analyzed the complete portfolio optimization model (the so-called MAD model). Yitzhaki (1982) introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. Both the mean absolute deviation and the Gini's mean difference turn out to be special aggregation techniques of the multiple criteria LP model (Ogryczak, 2000) based on the pointwise comparison of the absolute Lorenz curves. The latter leads the quantile shortfall risk measures which are more commonly used and accepted. Recently, the second order quantile risk measures have been introduced in different ways by many authors (Artzner et al., 1999; Ogryczak, 1999; Rockafellar and Uryasev, 2000). The measure, now commonly called the Conditional Value at Risk (CVaR) (after Rockafellar and Uryasev (2000)) or Tail VaR, represents the mean shortfall at a specified confidence level. It leads to LP solvable portfolio optimization models in the case of discrete random variables represented by their realizations under specified scenarios. The CVaR has been shown by Pflug (2000) to satisfy the requirements of the so-called coherent risk measures (Artzner et al., 1999) and is consistent with the second degree stochastic dominance as shown by Ogryczak and Ruszczyński, 2002a). Several empirical analyses (Andersson et al., (2001; Rockafellar and Uryasev, 2002; Mansini, Ogryczak and Speranza, 2003b; Topaloglou, Vladimirou and Zenios, 2002) confirm its applicability to various financial optimization problems. Thus, the CVaR models seem to overstep the measure of Value-at-Risk (VaR) defined as the maximum loss at a specified confidence level which is commonly used in banking (c.f., Jorion, 2001 and references therein).

This paper deals with portfolio optimization models based on the use of multiple CVaR risk measures. Such an extension allows for more detailed risk aversion modeling while preserving the simplicity of the original CVaR model. Both the theoretical properties of the models and their performance on real data are analyzed. The paper is organized as follows. In the next section we introduce basics of the mean-risk portfolio optimization, the CVaR risk measures and the concepts necessary to make the paper self-contained. Section 3 is devoted to the extended multiple CVaR model. Our analysis has been focused on the weighted CVaR (WCVaR) measures defined as simple combinations of a few CVaR measures. The general model is presented and its two specific weight-setting schemes relating the WCVaR measure to the Gini's mean difference and its tail version, respectively, are analyzed in detail. Moreover, a CVaR-related LP technique to directly enforce portfolio diversification is introduced. Section 4 presents the experimental analysis on real data from the Milan Stock Exchange. Extensive in-sample and out-of-sample computational results are provided and commented. Finally, some concluding remarks are given.

1 Basic models

1.1 Mean-safety portfolio optimization

At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. Let $J = \{1, 2, \dots, n\}$ denote a set of securities considered for investment. For each security $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Further, let $\mathbf{x} = (x_j)_{j=1,2,\dots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. To represent a portfolio, the weights must satisfy a set of constraints that form a feasible set $\mathcal{P}$. The simplest way of defining a feasible set is by a requirement that the weights must sum to one and short sales are not allowed, i.e. $\sum_{j=1}^n x_j = 1$ and $x_j \geq 0$ for $j = 1, \dots, n$. Hereafter, it is assumed that $\mathcal{P}$ is a general LP feasible set defined by linear equations and/or inequalities. This allows one to include upper bounds on single shares as well as several more complex portfolio structure restrictions which may be faced by a real-life investor.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^n R_j x_j$ that represents the portfolio rate of return. We consider T scenarios with probabilities p_t (where $t = 1, \dots, T$). We assume that for each random variable R_j its realization r_{jt} under the scenario t is known. Typically, the realizations are derived from historical data treating T historical periods as equally probable scenarios ($p_t = 1/T$). The realizations of the portfolio return $R_{\mathbf{x}}$ are given as $y_t = \sum_{j=1}^n r_{jt} x_j$ and the expected value can be computed as $\mu(\mathbf{x}) = \sum_{t=1}^T y_t p_t = \sum_{t=1}^T [\sum_{j=1}^n r_{jt} x_j] p_t$. Similarly, several risk measures can be LP computable with respect to the realizations y_t.

Following Markowitz (1952), the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the mean $\mu(\mathbf{x})$ is maximized and the risk measure $\varrho(\mathbf{x})$ is minimized. In the original Markowitz model, the standard deviation $\sigma(\mathbf{x}) = [\mathbb{E}\{(R_{\mathbf{x}} - \mu(\mathbf{x}))^2\}]^{1/2}$ was used as the risk measure. Several other risk measures have been later considered thus creating the entire family of mean-risk models (see Mansini, Ogryczak and Speranza, 2003a, 2003b). These risk measures, similar to the standard deviation, are not affected by any shift of the outcome scale and are equal to 0 in the case of a risk-free portfolio while taking positive values for any risky portfolio. Unfortunately, such risk measures are not consistent with the stochastic dominance order (Whitmore and Findlay, 1978) or other axiomatic models of risk-averse preferences (Rothschild and Stiglitz, 1969) and risk measurement (Artzner et al., 1999).

In stochastic dominance, uncertain returns (modeled as random variables) are compared by pointwise comparison of some performance functions constructed from their distribution functions. The first performance function $F_{\mathbf{x}}^{(1)}$ is defined as the right-continuous cumulative distribution function: $F_{\mathbf{x}}^{(1)}(\eta) = F_{\mathbf{x}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$ and it defines the first degree stochastic dominance (FSD). The second function is derived from the first as $F_{\mathbf{x}}^{(2)}(\eta) = \int_{-\infty}^{\eta} F_{\mathbf{x}}(\xi)\, d\xi$ and it defines the (weak) relation of *second degree stochastic dominance* (SSD): $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ if $F_{\mathbf{x}'}^{(2)}(\eta) \leq F_{\mathbf{x}''}^{(2)}(\eta)$ for all η. We say that portfolio $\mathbf{x}'$ *dominates* $\mathbf{x}''$ *under the SSD* ($R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''}$), if $F_{\mathbf{x}'}^{(2)}(\eta) \leq F_{\mathbf{x}''}^{(2)}(\eta)$ for all η, with at least one strict inequality. A feasible portfolio $\mathbf{x}^0 \in \mathcal{P}$ is called *SSD efficient* if there is no $\mathbf{x} \in \mathcal{P}$ such that $R_{\mathbf{x}} \succ_{SSD} R_{\mathbf{x}^0}$.

Several other portfolio performance measures were introduced as safety measures to be maximized, like the worst realization, analyzed by Young (1998), and the CVaR risk measures we consider further. Contrary to risk measures, the safety measures may be consistent with formal models of risk-averse preferences (Rothschild and Stiglitz, 1969) and risk measurement (Artzner et al., 1999). It has been shown by Mansini, Ogryczak and Speranza (2003a, 2003b) that for any risk measure $\varrho(\mathbf{x})$ a corresponding *safety* measure $\mu_\varrho(\mathbf{x}) = \mu(\mathbf{x}) - \varrho(\mathbf{x})$ can be defined and viceversa. Note that while risk measure $\varrho(\mathbf{x})$ is a convex function of $\mathbf{x}$, the corresponding safety measure $\mu_\varrho(\mathbf{x})$ is concave. A safety measure is considered risk relevant if for any risky portfolio its value is less than the value for the risk-free portfolio with the same expected returns. We say that the safety measure $\mu_\varrho(\mathbf{x})$ is *SSD consistent* (or that the risk measure $\varrho(\mathbf{x})$ is *SSD safety consistent*) if $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ implies $\mu_\varrho(\mathbf{x}') \geq \mu_\varrho(\mathbf{x}'')$. If the safety measure is SSD consistent, then except for portfolios with identical values of $\mu(\mathbf{x})$ and $\mu_\varrho(\mathbf{x})$ (and thereby $\varrho(\mathbf{x})$), every efficient solution of the bicriteria problem

$$\max\{[\mu(\mathbf{x}), \mu_\varrho(\mathbf{x})] : \mathbf{x} \in \mathcal{P}\} \tag{1}$$

is an SSD efficient portfolio (Ogryczak and Ruszczyński, 1999). Therefore, we will focus on the mean-safety bicriteria optimization (1) rather than on the classical mean-risk model.

The commonly accepted approach to implement the Markowitz-type mean-risk models is based on the use of a specified lower bound μ_0 on expected return while minimizing the risk criterion. In our analysis we use the bounding approach applied to the maximization of the safety measures, i.e.

$$\max\{\mu_\varrho(\mathbf{x}) : \mathbf{x} \in \mathcal{P},\ \mu(\mathbf{x}) \geq \mu_0\}. \tag{2}$$

For small values of the bound μ_0, the constraint $\mu(\mathbf{x}) \geq \mu_0$ does not influence the optimization (2). In this case, the portfolio obtained is the so called Maximum Safety Portfolio (MSP), whose return is referred to as μ(MSP). The MSP is the solution of $\max_{\mathbf{x}\in\mathcal{P}} \mu_\varrho(\mathbf{x})$. When $\mu_0 \geq \mu$(MSP), then the optimal solution of the corresponding problem represents a mean-safety efficient solution. In our computational analysis we will examine the MSPs for the different models. We will obtain the MSPs by solving (2), with μ_0 set to zero.

1.2 Absolute Lorenz curve and related measures

Stochastic dominance relates the notion of risk to a possible failure of achieving some targets. Note that function $F^{(2)}_{\mathbf{x}}$, used to define the SSD relation, can also be presented as follows (Ogryczak and Ruszczyński, 1999, 2001): $F^{(2)}_{\mathbf{x}}(\eta) = \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}$ and its values are LP computable for returns represented by their realizations y_t as:

$$F^{(2)}_{\mathbf{x}}(\eta) = \min \sum_{t=1}^{T} d_t^- p_t \quad \text{subject to} \quad d_t^- \geq \eta - y_t, \quad d_t^- \geq 0 \quad \text{for } t = 1, \ldots, T. \tag{3}$$

In this paper we focus on quantile shortfall risk measures related to the so-called *Absolute Lorenz Curves* (ALC) (Levy and Kroll (1978), Shorrocks (1983), Shalit and Yitzhaki (1994), Ogryczak (1999), Ogryczak and Ruszczyński (2002a)) which represent the second quantile functions defined as

$$F^{(-2)}_{\mathbf{x}}(p) = \int_0^p F^{(-1)}_{\mathbf{x}}(\alpha)d\alpha \quad \text{for } 0 < p \leq 1 \quad \text{and} \quad F^{(-2)}_{\mathbf{x}}(0) = 0, \tag{4}$$

where $F^{(-1)}_{\mathbf{x}}(p) = \inf\{\eta : F_{\mathbf{x}}(\eta) \geq p\}$ is the left-continuous inverse of the cumulative distribution function $F_{\mathbf{x}}$. Actually, the pointwise comparison of ALCs provides an alternative characterization of the SSD relation (Ogryczak and Ruszczyński, 2002a) in the sense that

$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ if and only if $F_{\mathbf{x}'}^{(-2)}(\beta) \geq F_{\mathbf{x}''}^{(-2)}(\beta)$ for all $0 < \beta \leq 1$. The duality (conjugency) relation between $F^{(-2)}$ and $F^{(2)}$ (Ogryczak, 1999; Ogryczak and Ruszczyński, 2002a) leads to the following formula:

$$F_{\mathbf{x}}^{(-2)}(\beta) = \max_{\eta \in R} \left[\beta\eta - F_{\mathbf{x}}^{(2)}(\eta)\right] = \max_{\eta \in R} \left[\beta\eta - \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}\right] \tag{5}$$

where η is a real variable taking the value of β-quantile $Q_\beta(\mathbf{x})$ at the optimum.

For any real tolerance level $0 < \beta \leq 1$, the normalized value of the ALC defined as $M_\beta(\mathbf{x}) = F_{\mathbf{x}}^{(-2)}(\beta)/\beta$ is now commonly called the *Conditional Value-at-Risk (CVaR)*. This name was introduced by Rockafellar and Uryasev (2000) who considered (similar to the Expected Shortfall by Embrechts, Klüppelberg and Mikosch (1997)) the measure CVaR defined as $\mathbb{E}\{R_{\mathbf{x}} | R_{\mathbf{x}} \leq F_{\mathbf{x}}^{(-1)}(\beta)\}$ for continuous distributions showing that it could then be expressed by a formula analogous to (5) and thus be potentially LP computable. The approach has been further expanded to general distributions (Rockafellar and Uryasev, 2002). For additional discussion of relations between various definitions of the measures we refer to (Ogryczak and Ruszczyński, 2002b).

The CVaR measure is a safety measure according to our classification (Mansini, Ogryczak and Speranza, 2003a). The corresponding risk measure $\Delta_\beta(\mathbf{x}) = \mu(\mathbf{x}) - M_\beta(\mathbf{x})$ (Ogryczak and Ruszczyński, 2002b) is called hereafter the (worst) *conditional semideviation*. Note that, for any $0 < \beta < 1$, the CVaR measures defined by $F^{(-2)}(\beta)$, opposite to below-target mean deviations $F^{(2)}(\eta)$, are risk relevant. They are also coherent (Pflug, 2000) and SSD consistent (Ogryczak and Ruszczyński, 2002a). For a discrete random variable represented by its realizations y_t, due to (3), problem (5) becomes an LP. Thus

$$M_\beta(\mathbf{x}) = \max\left[\eta - \frac{1}{\beta}\sum_{t=1}^{T} d_t^- p_t\right] \quad \text{s.t.} \quad d_t^- \geq \eta - y_t, \quad d_t^- \geq 0 \quad \text{for } t = 1, \ldots, T. \tag{6}$$

The CVaR measure is an increasing function of the tolerance level β, with $M_1(\mathbf{x}) = \mu(\mathbf{x})$. For β approaching 0, the CVaR measure tends to the Minimax safety measure (Young, 1998)

$$M(\mathbf{x}) = \min_{t=1,\ldots,T} y_t \tag{7}$$

whose corresponding risk measure is $\Delta(\mathbf{x}) = \mu(\mathbf{x}) - M(\mathbf{x})$. One may also notice that $\Delta_{0.5}(\mathbf{x})$ represents the mean absolute deviation from the median (Mansini, Ogryczak, and Speranza, 2003a), the risk measure suggested by Sharpe (1971b) as the right MAD model.

Yitzhaki (1982) introduced the GMD model using Gini's mean (absolute) difference as the risk measure. For a discrete random variable represented by its realizations y_t, the *Gini's mean difference* $\Gamma(\mathbf{x}) = \frac{1}{2}\sum_{t'=1}^{T}\sum_{t''=1}^{T} |y_{t'} - y_{t''}| p_{t'} p_{t''}$ is LP computable (when minimized). Actually, Yitzhaki (1982) suggested to use the corresponding safety measure

$$\mu_\Gamma(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}} \wedge R_{\mathbf{x}}\} \tag{8}$$

to take advantages of its SSD consistency. The measure is LP computable as:

$$\begin{aligned} \mu_\Gamma(\mathbf{x}) = \max & \sum_{t=1}^{T} p_t^2 y_t + 2\sum_{t'=1}^{T-1}\sum_{t''=t'+1}^{T} p_{t'} p_{t''} u_{t't''} \\ \text{s.t.} \quad & u_{t't''} \leq y_{t'}, \quad u_{t't''} \leq y_{t''} \quad \text{for} \quad t' = 1, \ldots, T-1; \quad t'' = t'+1, \ldots, T. \end{aligned} \tag{9}$$

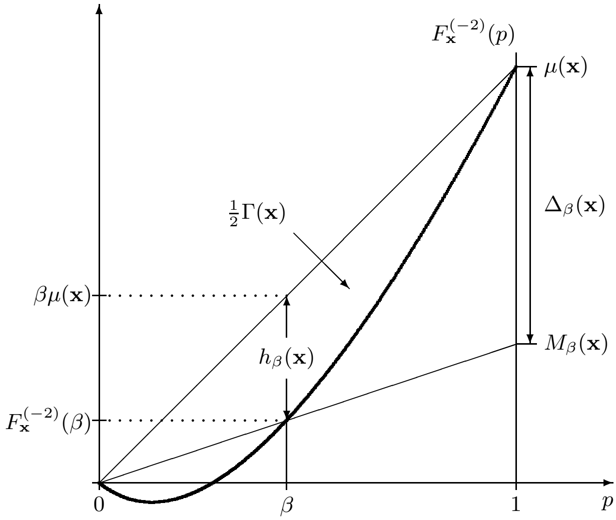

Fig. 1 The absolute Lorenz curve and risk measures

Both the Gini's mean difference and the CVaR measures are related to the absolute Lorenz curve (4). The graph of $F_{\mathbf{x}}^{(-2)}$ is a continuous convex curve connecting points (0, 0) and (1, $\mu(\mathbf{x})$), whereas a deterministic outcome with the same expected value $\mu(\mathbf{x})$, yields the chord (straight line) connecting the same points. Hence, the space between the curve $(p, F_{\mathbf{x}}^{(-2)}(p))$, $0 \leq p \leq 1$, and its chord represents the dispersion (and thereby the riskiness) of $R_{\mathbf{x}}$ in comparison to the deterministic outcome of $\mu(\mathbf{x})$. It is called the *Lorenz dispersion space*. One may notice that $\Delta_\beta(\mathbf{x}) = \frac{1}{\beta}h_\beta(\mathbf{x})$ where $h_\beta(\mathbf{x})$ denotes the vertical diameter of the Lorenz dispersion space at point $p = \beta$ (Fig. 1). Actually, all the classical LP computable risk measures are well defined size characteristics of the Lorenz dispersion space (Ogryczak, 2000, Ogryczak and Ruszczyński, 2002a). The Gini's mean difference may be expressed as $\Gamma(\mathbf{x}) = 2\int_0^1(\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha))d\alpha = 2\int_0^1 h_\alpha(\mathbf{x})d\alpha$ thus representing the doubled area of the Lorenz dispersion space. Therefore, any CVaR measure (the conditional semideviation) is a rather rough (diameter) measure of the Lorenz dispersion space when comparing to the Gini's mean difference.

The GMD safety measure summarizes all the CVaR measures as $\mu_\Gamma(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = 2\int_0^1 F_{\mathbf{x}}^{(-2)}(\alpha)d\alpha = 2\int_0^1 \alpha M_\alpha(\mathbf{x})d\alpha$. Therefore, the stronger SSD consistency results have been recently shown for the GMD model by Ogryczak and Ruszczyński (2002a), i.e., $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''}$ implies $\mu_\Gamma(\mathbf{x}') > \mu_\Gamma(\mathbf{x}'')$ which guarantees that every efficient solution of the bicriteria problem (1) is an SSD efficient portfolio. On the other hand, its computational LP model (9) requires T^2 variables which makes it much more complicated than the CVaR model (6) using only T variables. In the next sections we will demonstrate that models based on a few CVaR criteria offer a very good compromise between the computationally complex GMD model and simplified CVaR.

2 Enhanced CVaR measures

2.1 The multiple CVaR model

Although any CVaR measure (for $0 < \beta < 1$) is risk relevant, it represents only the mean within a part (tail) of the distribution of returns. Therefore, such a single criterion is in some manner crude for modeling various risk aversion preferences. In order to enrich the modeling capabilities, one needs to treat differently some more or less extreme events (Haimes, 1993). For this purpose one may consider multiple CVaR measures as risk (safety) criteria. In particular, one may consider several, say m, tolerance levels $0 < \beta_1 < \beta_2 < \cdots < \beta_m \leq 1$ and use the corresponding CVaR measures $M_{\beta_k}(\mathbf{x})$ to build a multiple criteria portfolio selection model:

$$\max\{[M_{\beta_1}(\mathbf{x}), M_{\beta_2}(\mathbf{x}), \ldots, M_{\beta_m}(\mathbf{x})]: \quad \mathbf{x} \in \mathcal{P}\}. \tag{10}$$

The model may contain the original mean value as the last criterion $M_1(\mathbf{x}) = \mu(\mathbf{x})$, if $\beta_m = 1$. One may notice that, for any portfolio $\mathbf{x}$, one gets $[M_{\beta_1}(\mathbf{x}), M_{\beta_2}(\mathbf{x}), \ldots, M_{\beta_m}(\mathbf{x})] \leq [\mu(\mathbf{x}), \mu(\mathbf{x}), \ldots, \mu(\mathbf{x})]$ with at least one inequality strict. Hence, the multiple criteria model (10) is risk relevant in the sense that for any risky portfolio its outcome vector is dominated by that for the risk-free portfolio with the same expected return. Actually, the model (10) is SSD consistent in the sense that $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ implies $[M_{\beta_1}(\mathbf{x}'), M_{\beta_2}(\mathbf{x}'), \ldots, M_{\beta_m}(\mathbf{x}')] \geq [M_{\beta_1}(\mathbf{x}''), M_{\beta_2}(\mathbf{x}''), \ldots, M_{\beta_m}(\mathbf{x}'')]$. Actually, the following assertion is valid.

Theorem 1. *For any set of levels $0 < \beta_1 < \beta_2 < \ldots < \beta_m \leq 1$, except for portfolios with identical values of all the corresponding CVaR measures $M_{\beta_k}(\mathbf{x})$, every efficient solution of the multiple criteria problem (10) is an SSD efficient portfolio.*

Proof: Let $\mathbf{x}^0 \in \mathcal{P}$ be an efficient solution of (10). Suppose that there exists $\mathbf{x}' \in \mathcal{P}$ such that $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}^0}$. Then, due to SSD consistency of the CVaR measures, $M_{\beta_k}(\mathbf{x}') \geq M_{\beta_k}(\mathbf{x}^0)$ for all $k = 1, \ldots, m$. The latter together with the fact that $\mathbf{x}^0$ is efficient, implies that $M_{\beta_k}(\mathbf{x}') = M_{\beta_k}(\mathbf{x}^0)$ for $k = 1, \ldots, m$, which completes the proof. $\square$

The weighted sum is the simplest aggregation technique in multiple criteria optimization. It can also be used to combine the CVaR criteria in (10). The weighted CVaR objective was first introduced by Ogryczak, 2000) (not using the name CVaR introduced later by Rockafellar and Uryasev, 2000); the portfolio optimization model based on historical data and its LP computability was then proven. Later it was extended and considered in various forms for portfolio optimization (Ogryczak and Ruszczyński, 2002b; Acerbi and Simonetti, 2002), general decisions under risk (Ogryczak, 2002), as well as for regression analysis (Rockafellar, Uryasev and Zabarankin, 2002).

In order to distinguish clearly the $\mu(\mathbf{x})$ criterion, further we will consider it separately from the m tolerance levels $0 < \beta_1 < \beta_2 < \cdots < \beta_m < 1$ (thus excluding $\beta = 1$). However, to simplify some formulas, we will use the expanded notation: $\beta_0 = 0$ and $\beta_{m+1} = 1$. Combining $\mu(\mathbf{x})$ and the CVaR values with positive (and normalized) weights we introduce the Weighted multiple CVaR (WCVaR) measure as

$$M_{\mathbf{w}}^{(m)}(\mathbf{x}) = w_0\mu(\mathbf{x}) + \sum_{k=1}^{m} w_k M_{\beta_k}(\mathbf{x}),$$

$$\sum_{k=0}^{m} w_k = 1, \quad w_0 \geq 0, \quad w_k > 0 \quad \text{for} \quad k = 1, \ldots, m. \tag{11}$$

The WCVaR measure is a safety measure and it is risk relevant. The corresponding risk measure turns out to be the weighted sum of the $\Delta_{\beta_k}(\mathbf{x})$ measures thus forming the weighted conditional semideviation:

$$\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) = \mu(\mathbf{x}) - M_{\mathbf{w}}^{(m)}(\mathbf{x}) = \sum_{k=1}^{m} w_k \Delta_{\beta_k}(\mathbf{x}), \ \sum_{k=1}^{m} w_k \leq 1, \quad w_k > 0 \quad \text{for } k = 1, \ldots, m. \tag{12}$$

The latter is not affected by any shift of the outcome scale and it is equal to 0 in the case of a risk-free portfolio while taking positive value for any risky portfolio, thus representing a translation invariant and risk relevant dispersion parameter. Therefore, we can consider the corresponding Markowitz-type model and its mean-safety formalization (1):

$$\max\left\{\left[\mu(\mathbf{x}), M_{\mathbf{w}}^{(m)}(\mathbf{x})\right]: \mathbf{x} \in \mathcal{P}\right\} = \max\left\{\left[\mu(\mathbf{x}), \mu(\mathbf{x}) - \Delta_{\mathbf{w}}^{(m)}(\mathbf{x})\right]: \mathbf{x} \in \mathcal{P}\right\}. \tag{13}$$

Since the CVaR measures are coherent (Pflug, 2000) and SSD consistent (Ogryczak and Ruszczyński, 2002a), the same applies to the WCVaR measure. In particular, $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ implies $M_{\mathbf{w}}^{(m)}(\mathbf{x}') \geq M_{\mathbf{w}}^{(m)}(\mathbf{x}'')$ (Ogryczak and Ruszczyński, 2002b). Actually, the SSD consistency relation for the WCVaR measure is stronger since it takes into account all the individual CVaR measures as shown in the following assertion.

Theorem 2. *For any set of levels* $0 < \beta_1 < \beta_2 < \cdots < \beta_m \leq 1$*, except for portfolios with identical values of* $\mu(\mathbf{x})$ *and all conditional semideviations* $\Delta_{\beta_k}(\mathbf{x})$*, respectively, every efficient solution of the bicriteria problem (13) is an SSD efficient portfolio.*

Proof: Let $\mathbf{x}^0 \in \mathcal{P}$ be an efficient solution of (13). Suppose that there exists $\mathbf{x}' \in \mathcal{P}$ such that $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}^0}$. Then, due to SSD consistency of the CVaR measures, $\mu(\mathbf{x}') \geq \mu(\mathbf{x}^0)$ and $M_{\beta_k}(\mathbf{x}') \geq M_{\beta_k}(\mathbf{x}^0)$ for all $k = 1, \ldots, m$. The latter, together with the fact that $\mathbf{x}^0$ is efficient, implies that $\mu(\mathbf{x}') = \mu(\mathbf{x}^0)$ and $\sum_{k=1}^{m} w_k M_{\beta_k}(\mathbf{x}') = \sum_{k=1}^{m} w_k M_{\beta_k}(\mathbf{x}^0)$. Hence, $M_{\beta_k}(\mathbf{x}') = M_{\beta_k}(\mathbf{x}^0)$ for $k = 1, \ldots, m$, and therefore, $\Delta_{\beta_k}(\mathbf{x}') = \Delta_{\beta_k}(\mathbf{x}^0)$ for all $k = 1, \ldots, m$, which completes the proof. □

For returns represented by their realizations we get an LP model. The model contains the following core LP constraints to define a feasible portfolio, portfolio realizations, and portfolio expected return:

$$\mathbf{x} \in \mathcal{P}, \quad z \geq \mu_0, \quad \sum_{j=1}^{n} \mu_j x_j = z \quad \text{and} \quad \sum_{j=1}^{n} r_{jt} x_j = y_t \quad \text{for } t = 1, \ldots, T \tag{14}$$

where z is an unbounded variable representing the mean return of the portfolio $\mathbf{x}$ and y_t $(t = 1, \ldots, T)$ are unbounded variables to represent the realizations of the portfolio return under the scenario t. The general WCVaR model (13) leads us to the following LP problem:

$$\begin{aligned} &\text{maximize } w_0 z + \sum_{k=1}^{m} w_k q_k - \sum_{k=1}^{m} \frac{w_k}{\beta_k} \sum_{t=1}^{T} p_t d_{tk} \\ &\text{subject to (14) and } d_{tk} - q_k + y_t \geq 0, \ d_{tk} \geq 0 \quad \text{for } t = 1, \ldots, T; \quad k = 1, \ldots, m \end{aligned} \tag{15}$$

where q_k (for $k = 1, \ldots, m$) are unbounded variables taking the values of the corresponding β_k-quantiles (in the optimal solution). Except from the core constraints (14), model (15) contains T nonnegative variables d_{tk} and T corresponding linear inequalities for each k. Thus, its dimensionality is proportional to the number of scenarios T and to the number of tolerance levels m. Note that model (15) with $m = 1$ and $w_0 = 0$ covers the standard CVaR model, while $m > 1$ and various settings of positive weights w_k allow us to model a wide gamut of risk averse preferences. The model does not require any specific relation between the number of scenarios T and the number of securities n or the number of tolerance levels m. Similar to the Markowitz model, a very low number of scenarios may result in much less diversified portfolios. Increasing the number of tolerance levels m, generally, enables a larger diversification. However, such diversification is not guaranteed since, as demonstrated later, it also depends on a specific weight-setting.

Recall that the absolute Lorenz curve, and thereby the CVaR measures, represent a dual characterization of the SSD relation (Ogryczak and Ruszczyński, 2002a). Hence, the weighted combination of the CVaR measures may be interpreted as the dual utility criterion within the theory developed by Yaari (1987) which was recently reintroduced into the finance literature in a simplified form of the spectral risk measures (Acerbi, 2002). Indeed, according to (11),

$$M_{\mathbf{w}}^{(m)}(\mathbf{x}) = w_0 \int_0^1 F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha + \sum_{k=1}^{m} \frac{w_k}{\beta_k} \int_0^{\beta_k} F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha = \int_0^1 \phi(\alpha) F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha$$

where

$$\phi(\alpha) = \begin{cases} w_0 + \sum_{k=i}^{m} \frac{w_k}{\beta_k}, & \beta_{i-1} < \alpha \le \beta_i \\ w_0, & \beta_m < \alpha \le 1 \end{cases} \tag{16}$$

is a decreasing risk aversion function (note the sign change for our safety measures to be maximized).

As pointed out by Acerbi (2002), the subjective risk aversion of an investor can be encoded in a function $\phi(\alpha)$ defined for all possible confidence levels $\alpha \in (0, 1]$ and from a financial point of view one cannot see any natural choice of function $\phi(\alpha)$. The use of a wide class of risk aversion functions in portfolio optimization (Acerbi and Simonetti, 2002) seems to be rather far from the simplicity necessary to make possible an effective implementation of the portfolio optimization procedure. In the following we will focus on the WCVaR measures defined as simple combinations of a very few CVaR measures (thus stepwise risk aversion functions ϕ with a very few steps). On the other hand, we introduce two specific types of weight-settings which relate the WCVaR measure to the Gini's mean difference and its tail version. This allows us to use a few tolerance levels β_k as the only parameters specifying the entire WCVaR measures (modeling risk aversion function) while the corresponding weights are automatically predefined by the requirements of the corresponding Gini's measures. In other words, the investor's preferences are modeled by a selection of a few tolerance levels. It turns out that we have managed to identify a class of simple WCVaR measures performing better in a real-life portfolio optimization environment than typical CVaR measures and the GMD model.

2.2 Wide WCVaR measures

In the case of equally probable T scenarios with $p_t = 1/T$ (historical data for T periods), the weighted CVaR measure $M_{\mathbf{w}}^{(T-1)}(\mathbf{x})$ defined with $m = T - 1$ tolerance levels $\beta_k = k/T$

for $k = 1, 2, \dots, T-1$ represents the standard weighting approach in the multiple criteria LP portfolio optimization model with criteria $F^{(-2)}(k/T)$ (Ogryczak, 2000). The use of weights $w_k = (2k)/T^2$ for $k = 1, 2, \dots, T-1$ and $w_0 = 1/T$ results then in $\Delta_{\mathbf{w}}^{(T-1)}(\mathbf{x}) = \frac{2}{T}\sum_{k=1}^{T-1} h_{\frac{k}{T}}(\mathbf{x}) = \Gamma(\mathbf{x})$ (c.f. Fig. 1). Hence, the WCVaR model is then equivalent to the GMD model and it cannot provide us with any new modeling capabilities.

In the general case of T scenarios with arbitrary probabilities p_t, one may use an approximation to $\Gamma(\mathbf{x})$ with $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x})$ based on some reasonably chosen grid of tolerance levels β_k, $k = 1, \dots, m$ and weights w_k expressing the corresponding trapezoidal approximation to the integral formula $\Gamma(\mathbf{x}) = 2\int_0^1 (\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha))d\alpha$. Such an approximation is a very attractive risk measure itself as it allows us to dramatically reduce the computational burden caused by T^2 dimensionality of the LP implementation of the GMD model (9) while introducing new modeling capabilities connected to the grid selection. Exactly, for any grid of m tolerance levels $0 < \beta_1 < \cdots < \beta_k < \cdots < \beta_m < 1$ one gets the trapezoidal approximation:

$$\Gamma(\mathbf{x}) \cong \sum_{k=1}^{m} (\beta_{k+1} - \beta_{k-1}) h_{\beta_k}(\mathbf{x}) = \sum_{k=1}^{m} (\beta_{k+1} - \beta_{k-1}) \beta_k \Delta_{\beta_k}(\mathbf{x}).$$

Note that $\sum_{k=1}^{m} (\beta_{k+1} - \beta_{k-1})\beta_k = \beta_m < 1$. This leads us to the WCVaR measure with weights:

$$w_k = (\beta_{k+1} - \beta_{k-1})\beta_k, \quad \text{for } k = 1, \dots, m, \quad \text{and} \quad w_0 = 1 - \beta_m. \tag{17}$$

Precisely, when using the weights given by (17), the corresponding WCVaR measure defined by (11) is an approximation to the GMD safety measure (8) (i.e., $M_{\mathbf{w}}^{(m)}(\mathbf{x}) \cong \mu_{\Gamma}(\mathbf{x})$), and the corresponding weighted conditional semideviation (12) is an approximation to the Gini's mean difference $\Gamma(\mathbf{x})$. This can be also illustrated in terms of the spectral measures (Acerbi, 2002) as integrating by parts one gets

$$\mu_{\Gamma}(\mathbf{x}) = 2\int_0^1 F_{\mathbf{x}}^{(-2)}(\alpha)d\alpha = 2F_{\mathbf{x}}^{(-2)}(1) - 2\int_0^1 \alpha F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha = \int_0^1 2(1-\alpha)F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha$$

which allows us to express the GMD safety measure by the risk aversion function $\phi(\alpha) = 2(1-\alpha)$ while formula (16) with the weights (17) defines a stepwise approximation to this function.

Again, the WCVaR measures may be considered the exact GMD measure applied to $(m+1)$-point distributions approximating the original distribution of returns $R_{\mathbf{x}}$, thus providing a trapezoidal approximation to the original Lorenz dispersion space. In particular, for the $(m+1)$-point distribution $R_{\mathbf{x}^{(\beta_1,\dots,\beta_m)}}$

$$\mathbb{P}\{R_{\mathbf{x}^{(\beta_1,\dots,\beta_m)}} = \xi\} = \begin{cases} \beta_k - \beta_{k-1}, & \xi = a_k \quad \text{for } k = 1, \dots, m+1 \\ 0, & \text{otherwise} \end{cases}$$

such that $a_1 \le a_2 \le \cdots \le a_{m+1}$, the weighted conditional semideviation $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}^{(\beta_1,\dots,\beta_m)})$ with weights (17) is equal to $\Gamma(\mathbf{x}^{(\beta_1,\dots,\beta_m)})$. In general, $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x})$ is a lower approximation to $\Gamma(\mathbf{x})$.

It must be emphasized that despite being only an approximation to the Gini's mean difference, any WCVaR measure with weights defined by (17) is a well defined LP computable risk measure with guaranteed SSD consistency in the sense of Theorem 2. In other words, we are interested in the GMD approximation properties only for a reasonable weights definition. We will refer to the WCVaR measures with weights defined by (17) as the Wide WCVaR as covering (spanning) a wide area of the quantile scale. The Wide WCVaR measures need not to employ a very dense grid to provide a proper modeling of risk averse preferences. This allows us to build relatively small LP models with mT variables. In our computational analysis we have considered $m = 3$ while testing three different patterns of the tolerance levels (see Table 1) corresponding to three types of preferences defined by the tolerance levels location.

2.3 Tail WCVaR measures

The Wide WCVaR measures, based on the approximation to the Gini's measure, contain the risk neutral term $M_1(\mathbf{x}) = \mu(\mathbf{x})$ with the weight $w_0 = 1 - \beta_m$. This may cause the measure to pay too much attention to very low probable but very large returns. Actually, the measure can be more sensitive to large returns than the Gini's mean difference. We encountered such a situation in our computational analysis where in a few cases all the models based on the WCVaR approximation to GMD selected a single security portfolio with very high expectation caused by a very few but extremely high return realizations.

In order to overcome this flaw one may use the Tail WCVaR measures, built with an approximation to the tail GMD measures instead of the GMD itself. The tail GMD (Ogryczak and Ruszczyński (2002a, 2002b)) is defined for any $\beta \in (0, 1]$ by averaging the vertical diameters $h_p(\mathbf{x})$ within the tail interval $p \leq \beta$ as:

$$\Gamma_\beta(\mathbf{x}) = \frac{2}{\beta^2} \int_0^\beta (\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha))d\alpha. \tag{18}$$

A simple analysis of the absolute Lorenz curve (Ogryczak and Ruszczyński, 2002a) shows that, for any $0 < \beta \leq 1$, the tail Gini's measure $\Gamma_\beta(\mathbf{x})$ is SSD safety consistent. One may notice that the corresponding safety measure $\mu_{\Gamma_\beta}(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma_\beta(\mathbf{x})$ can be expressed as

$$\mu_{\Gamma_\beta}(\mathbf{x}) = \mu(\mathbf{x}) - \frac{2}{\beta^2} \int_0^\beta (\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha))d\alpha = \frac{2}{\beta^2} \int_0^\beta F_{\mathbf{x}}^{(-2)}(\alpha)d\alpha$$

which allows us to consider it as a second degree CVaR measure.

In the simplest case of equally probable T scenarios with $p_t = 1/T$, the tail Gini's measure for $\beta = K/T$ may be expressed as the weighted conditional semideviation $\Delta_{\mathbf{w}}^{(K)}(\mathbf{x})$ with tolerance levels $\beta_k = k/T$ for $k = 1, 2, \ldots, K$ and properly defined weights (Ogryczak and Ruszczyński, 2002a). In a general case, we may resort to an approximation with the weighted CVaR measure based on some reasonably chosen grid β_k, $k = 1, \ldots, m$ and weights w_k expressing the corresponding trapezoidal approximation of the integral in the formula (18). Exactly, for any $0 < \beta \leq 1$, while using the grid of m tolerance levels $0 < \beta_1 < \cdots < \beta_k < \cdots < \beta_m = \beta$ one may define the weights:

$$w_k = \frac{(\beta_{k+1} - \beta_{k-1})\beta_k}{\beta^2}, \quad \text{for } k = 1, \ldots, m-1, \quad \text{and} \quad w_m = \frac{(\beta_m - \beta_{m-1})\beta_m}{\beta^2} \tag{19}$$

where $\beta_0 = 0$. This results in the weighted sum $\sum_{k=1}^m w_k \Delta_{\beta_k}(\mathbf{x})$ expressing the trapezoidal approximation to the tail Gini's measure (18). Note that $\sum_{k=1}^m w_k = \beta_m^2/\beta^2 = 1$ and thus we get a regular weighted conditional semideviation (12) $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) \cong \Gamma_\beta(\mathbf{x})$. Further, weights (19) together with $w_0 = 0$ generate a WCVaR measure (11) such that $M_{\mathbf{w}}^{(m)}(\mathbf{x}) \cong \mu_{\Gamma_\beta}(\mathbf{x})$. This can also be illustrated in terms of the spectral measures (Acerbi, 2002) as integrating by parts one gets

$$\mu_{\Gamma_\beta}(\mathbf{x}) = \frac{2}{\beta^2}\int_0^\beta F_{\mathbf{x}}^{(-2)}(\alpha)d\alpha = \frac{2}{\beta}F_{\mathbf{x}}^{(-2)}(\beta) - \frac{2}{\beta^2}\int_0^\beta \alpha F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha = \int_0^1 \frac{2(\beta-\alpha)^+}{\beta^2}F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha$$

allowing us to express the tail GMD safety measure by the risk aversion function $\phi(\alpha) = 2(\beta-\alpha)^+/\beta^2$ where $(.)^+$ denotes the nonnegative part of a number. Formula (16) with the weights (19) defines a stepwise approximation to this function.

Again, we emphasize that despite being only an approximation to (18), any Tail WCVaR measure (e.g., a WCVaR measure with weights defined according to (19)) is a well defined LP computable measure with guaranteed SSD consistency in the sense of Theorem 2. They need not be built on a very dense grid to provide proper modeling of risk averse preferences. Actually, we are interested in a direct preference modeling with simple Tail WCVaR measures rather than strict approximation to the Tail GMD measure. In our computational analysis we have tested two Tail WCVaR models with $m = 2$ and $m = 3$ (see Table 1). Obviously, all the Tail WCVaR model measures are implemented as LP problems (15) but with $w_0 = 0$. Again, for a small value of m we get rather small LP models with mT variables.

2.4 Direct diversification enforcement

Since the seminal work of Markowitz (1952), the notion of investing in diversified portfolios is considered one of the most fundamental concepts of portfolio management. Diversification should be enforced by the mean/risk preference model. Indeed, in the original Markowitz model it was usually guaranteed by the standard deviation (variance) minimization. In general, it may happen that a single security or a low diversified portfolio is SSD dominating over all other (more diversified) portfolios, and the SSD consistent Markowitz-type models will select such an undiversified solution. Especially, the SSD consistent models based on the LP computable risk measures may fail to generate sufficiently diversified portfolios, although this also happens for the original Markowitz model (Mansini, Ogryczak and Speranza, 2003b). Therefore, additional restrictions may be posed on the feasible portfolios to guarantee the required diversification. The simplest way to enforce portfolio diversification is to limit the maximum share. This, however, allows us to form a portfolio with a few shares at the maximum level. A better modeling alternative would be to allow for a relatively large maximum share provided that the other shares are smaller. Such a rich diversification scheme may be introduced with the CVaR constructs applied to the right tail of the distribution of shares.

A natural generalization of the maximum share is the (right-tail) conditional mean defined as the mean within the specified tolerance level (amount) of the worst shares. One may simply define the conditional mean as the mean of the k largest shares. This can be formalized as follows. First, we introduce the ordering map $\Theta : R^n \to R^n$ such that $\Theta(\mathbf{x}) = (\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \ldots, \theta_n(\mathbf{x}))$, where $\theta_1(\mathbf{x}) \geq \theta_2(\mathbf{x}) \geq \cdots \geq \theta_n(\mathbf{x})$ and there exists a permutation τ of set J such that $\theta_j(\mathbf{x}) = x_{\tau(j)}$ for $j = 1, \ldots, n$. The use of ordered outcome vectors $\Theta(\mathbf{x})$ allows us to focus on distributions of shares impartially. Next, the linear cumulative map is applied to ordered vectors to get $\bar{\theta}_k(\mathbf{x}) = \sum_{j=1}^k \theta_j(\mathbf{x})$ for $k = 1, \ldots, n$. The coefficients of vector $\bar{\Theta}(\mathbf{x}) = (\bar{\theta}_1(\mathbf{x}), \bar{\theta}_2(\mathbf{x}), \ldots, \bar{\theta}_n(\mathbf{x}))$ express, respectively: the largest share, the total of the two largest shares, the total of the three largest shares, etc. Hence, the (worst) $\frac{k}{n}$*–conditional mean* share is given as $\frac{1}{k}\bar{\theta}_k(\mathbf{x})$, for $k = 1, \ldots, n$.

Similar to the CVaR formulas, for a given vector $\mathbf{x}$, the value of $\bar{\theta}_k(\mathbf{x})$ may be found by solving the linear program (Ogryczak and Tamir, 2003):

$$\bar{\theta}_k(\mathbf{x}) = \min \left\{ ks_k + \sum_{j=1}^{n} d_{kj}^s \ : \ d_{kj}^s \geq x_j - s_k,\ d_{kj}^s \geq 0 \quad \text{for } j = 1, \ldots, n \right\},$$

where s_k is an unbounded variable (representing the k-th largest share at the optimum) and d_{kj}^s are additional nonnegative (deviational) variables. Hence, any model under consideration can easily be extended with direct diversification constraints specified as $\bar{\theta}_k(\mathbf{x}) \leq c_k$ upper bounding total of the k largest shares and implemented with linear inequalities:

$$ks_k + \sum_{j=1}^{n} d_{kj}^s \leq c_k \quad \text{and} \quad d_{kj}^s \geq x_j - s_k, \quad d_{kj}^s \geq 0 \quad \text{for } j = 1, \ldots, n. \tag{20}$$

3 Experimental analysis

3.1 Testing environment

The present section is devoted to the experimental analysis in a real framework of all the described LP models based on extensions of the CVaR measure. Models have been tested on a PC with a 500 MHz Pentium processor by using CPLEX 6.5 package. First we present the test problems. Then the results of the in-sample analysis, both on the original models and on their modifications to enforce diversification, are described. Next, the out-of-sample analysis including the results obtained through the simulation of a "multiperiod-type" portfolio investment is presented. Finally, portfolio performances in a separated period characterized by a strong drawdown trend are discussed.

Historical data are represented by weekly rates of return from Milan Stock Exchange. The rates are computed as relative stock price variations. Dividends are not included. The data set consists of 157 securities quoted with continuity from 1994 to 1999. In the first years of this historical period the Italian Stock Exchange has shown alternate short periods of up and down trends while entering a positive growing trend at the end. This is shown in Fig. 2, where the performance of the Milan Stock Exchange index MIB30 is depicted in the period (1994–2002).

A set of 13 instances has been created, each of which takes into account the complete set of securities over a different time period. For this reason, from now on, we will indifferently refer to them as instances or periods. In particular, each instance is based on two years realizations (about 104 weekly observations) as in-sample period and one year as out-of-sample. The choice of weekly periodicity is consistent with the objective of reducing estimation errors through an adequate number of observations (Simaan, 1997). Two consecutive instances differ from each other for a three months period, e.g. the first instance covers the two years 1994–1995 as in-sample period, while the second instance does not include the first three months of 1994 and does include the first three months of 1996. For each instance the Maximum Safety Portfolio (MSP) has been obtained through the use of the various tested

Fig. 2 The Milan Stock Exchange Index MIB30: weekly quotations in the years 1994–2002 (source: DATASTREAM)

models. In this section we only summarize and comment the main figures out of the huge amount of computational results we obtained.

The model introduced by Young (1998), with safety measure the maximization of the worst realization (7), is identified as Minimax. The model based on the safety measure corresponding to the Gini's mean difference (9), i.e. the mean worse return, is referred simply as GMD. The CVaR model associated to a given tolerance level β is identified as CVaR(β). We have tested the CVaR model for five different values of β, i.e. CVaR(0.05), CVaR(0.1), CVaR(0.25), CVaR(0.5) and CVaR(0.75). All the CVaR and the weighted CVaR models have been formulated according to (15). Among the weighted models we have tested three Wide WCVaR models (with $m = 3$ tolerance levels) and two Tail WCVaR models (with $m = 2$ and $m = 3$, respectively). The corresponding tolerance levels and weights are summarized in Table 1.

Since SSD consistent models based on the LP computable risk measures may fail to generate diversified enough portfolios we have added the following additional restrictions to guarantee sufficient diversification: any stock share cannot exceed 0.20, while any three shares cannot exceed 0.50 in total and any six shares cannot globally exceed 0.75 of the

Table 1 Weighted CVaR models

Wide models	Tolerance levels	Weights			
		w_0	w_1	w_2	w_3
WCVaR(AGD)	Downside Approximation				
	$\beta_1 = 0.1, \beta_2 = 0.25, \beta_3 = 0.5$	0.5	0.025	0.1	0.375
WCVaR(AGS)	Symmetric Approximation				
	$\beta_1 = 0.25, \beta_2 = 0.5, \beta_3 = 0.75$	0.25	0.125	0.25	0.375
WCVaR(AGT)	Tails Approximation				
	$\beta_1 = 0.1, \beta_2 = 0.5, \beta_3 = 0.9$	0.1	0.05	0.4	0.45
Tail models		–	w_1	w_2	w_3
WCVaR(TG2)	Two-Point Tail				
	$\beta_1 = 0.1, \beta_2 = 0.25$	–	0.4	0.6	–
WCVaR(TG3)	Three-Point Tail				
	$\beta_1 = 0.1, \beta_2 = 0.25, \beta_3 = 0.5$	–	0.1	0.4	0.5

portfolio investment. This requires the following side constraints:

$$\begin{aligned}
&x_j \le 0.2 && \text{for } j = 1, \dots, n \\
&3s_3 + \textstyle\sum_{j=1}^n d_{3j}^s \le 0.5 \quad \text{and} \quad d_{3j}^s \ge x_j - s_3, \quad d_{3j}^s \ge 0 && \text{for } j = 1, \dots, n \\
&6s_6 + \textstyle\sum_{j=1}^n d_{6j}^s \le 0.75 \quad \text{and} \quad d_{6j}^s \ge x_j - s_6, \ d_{6j}^s \ge 0 && \text{for } j = 1, \dots, n
\end{aligned} \tag{21}$$

For each model we have tested the corresponding version obtained by adding constraints (21).

3.2 In-sample analysis

In the following we present and comment the characteristics of the MSPs selected by the different models with and without the introduction of diversification enforcement. In Table 2, the complete computational results for model CVaR(0.1) are presented as an example of the type of information obtained by solving a single model over all the 13 periods. The table consists of a first part corresponding to the results for the model without diversification enforcement and a second one for the model with the diversification enforcement constraints (21). Each of the two parts of the table has five columns: the objective function value (Obj.), the portfolio per cent mean return (z), the portfolio diversification (Div.) represented by the number of selected securities, the minimum and the maximum share within the portfolio, respectively. The average return is given on a weekly basis (a good yearly approximation can be obtained by multiplying the figures by 52). Notice that the introduction of diversification constraints may result in an unmodified optimal portfolio. This is the case for the portfolios selected in the instances 7 and 8.

To simplify results presentation, we have decided to focus our attention only on a subset of instances (periods). In Tables 3 and 4 we show the results obtained by the different models in the first and the twelfth instance without and with diversification constraints. The tables have the same structure of Table 2. Tables 3 shows the large values obtained by the Wide WCVaR: the fact can be explained by the relevance given to high returns by these models which are much more sensitive to large returns than the Gini's mean difference. This also explains why,

Table 2 CVaR(0.1) model without and with diversification constraints: Optimal portfolio characteristics in each of the 13 periods

	Without diversification					With diversification				
				Shares					Shares	
Periods	Obj. 10^{-2}	z %	Div. #	Min	Max	Obj. 10^{-2}	z %	Div. #	Min	Max
1	−1.567	0.21	18	$1.77\ 10^{-3}$	0.293	−1.638	0.17	20	$1.93\ 10^{-3}$	0.200
2	−1.543	0.24	18	$2.09\ 10^{-3}$	0.281	−1.604	0.21	20	$4.45\ 10^{-4}$	0.200
3	−1.129	0.61	18	$3.33\ 10^{-3}$	0.204	−1.137	0.52	20	$1.10\ 10^{-3}$	0.192
4	−0.999	0.19	23	$1.68\ 10^{-3}$	0.144	−0.999	0.19	23	$1.68\ 10^{-3}$	0.144
5	−0.955	0.10	24	$6.24\ 10^{-3}$	0.138	−0.955	0.10	24	$8.28\ 10^{-3}$	0.138
6	−0.736	0.43	26	$1.18\ 10^{-4}$	0.165	−0.736	0.43	26	$1.18\ 10^{-4}$	0.132
7	−0.721	0.41	27	$1.48\ 10^{-4}$	0.112	−0.721	0.41	27	$1.48\ 10^{-4}$	0.112
8	−0.649	0.54	27	$1.53\ 10^{-3}$	0.138	−0.649	0.54	27	$1.53\ 10^{-3}$	0.138
9	−0.560	0.61	29	$7.64\ 10^{-4}$	0.128	−0.560	0.61	29	$7.64\ 10^{-4}$	0.128
10	−0.549	1.01	24	$2.07\ 10^{-3}$	0.121	−0.549	1.01	24	$2.07\ 10^{-3}$	0.121
11	−1.401	0.99	19	$7.81\ 10^{-5}$	0.245	−1.412	0.98	20	$1.05\ 10^{-3}$	0.200
12	−2.188	0.91	18	$1.11\ 10^{-4}$	0.210	−2.188	0.91	18	$1.11\ 10^{-4}$	0.210
13	−2.476	0.90	17	$4.49\ 10^{-3}$	0.223	−2.478	0.87	18	$5.30\ 10^{-4}$	0.200

Table 3 Period 1—Maximum safety portfolios: Optimal portfolio characteristics

	Without diversification					With diversification				
				Shares					Shares	
Models	Obj. 10^{-2}	z %	Div. #	Min	Max	Obj. 10^{-2}	z %	Div. #	Min	Max
Minimax	−1.812	0.06	14	$1.40\ 10^{-3}$	0.193	−1.823	0.12	16	$3.98\ 10^{-4}$	0.194
CVaR(0.05)	−1.767	0.15	14	$5.14\ 10^{-3}$	0.214	−1.789	0.13	16	$3.06\ 10^{-3}$	0.187
CVaR(0.1)	−1.567	0.21	18	$1.77\ 10^{-3}$	0.293	−1.638	0.17	20	$1.93\ 10^{-3}$	0.200
CVaR(0.25)	−1.155	0.29	11	$1.57\ 10^{-3}$	0.316	−1.199	0.29	18	$5.13\ 10^{-3}$	0.200
CVaR(0.5)	−0.597	0.39	17	$3.31\ 10^{-5}$	0.288	−0.611	0.40	19	$4.87\ 10^{-5}$	0.200
CVaR(0.75)	−0.055	0.62	17	$1.12\ 10^{-4}$	0.179	−0.056	0.61	18	$1.18\ 10^{-4}$	0.172
GMD	−0.313	0.60	17	$2.60\ 10^{-4}$	0.246	−0.318	0.61	18	$7.90\ 10^{-4}$	0.200
WCVaR(AGD)	40.577	94.26	1	1	1	8.422	19.52	12	$1.49\ 10^{-2}$	0.200
WCVaR(AGS)	16.147	94.26	1	1	1	3.532	19.48	13	$1.19\ 10^{-2}$	0.200
WCVaR(AGT)	1.603	94.26	1	1	1	0.632	19.47	13	$2.74\ 10^{-3}$	0.200
WCVaR(TG2)	−1.393	0.21	16	$1.34\ 10^{-3}$	0.343	−1.455	0.16	18	$2.63\ 10^{-3}$	0.200
WCVaR(TG3)	−0.986	0.31	15	$8.99\ 10^{-5}$	0.304	−1.025	0.36	18	$2.60\ 10^{-5}$	0.200

in some instances, the Wide WCVaR models select only one security portfolio characterized by a large expected return (about 94% per week) generated by very few realizations with dramatically high return. Moreover, it is worth noticing that in the Wide WCVaR models the mean return z value is larger than that of all the other models. This is also true in the twelfth period where the returns are dramatically lower than in other periods, thus reflecting the downwards trend of the whole market. From Tables 3, 4 and the analogous results obtained for the other periods (Mansini, Ogryczak and Speranza, 2003c) we observed that the MSPs mean return tends to increase over the years by reaching a pick in the first quarter of the year 1998 and then decreasing. In general, the basic CVaR models are the most diversified.

Table 4 Period 12—Maximum safety portfolios: Optimal portfolio characteristics

	Without diversification					With diversification				
				Shares					Shares	
Models	Obj. 10^{-2}	z %	Div. #	Min	Max	Obj. 10^{-2}	z %	Div. #	Min	Max
Minimax	−2.848	0.99	13	$3.77\ 10^{-3}$	0.289	−2.927	0.99	14	$1.41\ 10^{-3}$	0.200
CVaR(0.05)	−2.486	0.88	14	$1.09\ 10^{-3}$	0.198	−2.488	0.88	15	$1.51\ 10^{-3}$	0.200
CVaR(0.1)	−2.188	0.91	18	$1.11\ 10^{-4}$	0.210	−2.188	0.90	18	$9.13\ 10^{-4}$	0.200
CVaR(0.25)	−1.529	0.94	20	$4.65\ 10^{-3}$	0.134	−1.529	0.94	20	$4.65\ 10^{-3}$	0.134
CVaR(0.5)	−0.608	1.05	20	$7.87\ 10^{-4}$	0.129	−0.608	1.05	20	$7.87\ 10^{-4}$	0.129
CVaR(0.75)	0.204	1.35	14	$4.30\ 10^{-4}$	0.222	0.203	1.35	14	$3.99\ 10^{-3}$	0.200
GMD	−0.186	1.35	20	$1.01\ 10^{-2}$	0.129	−0.186	1.35	20	$1.01\ 10^{-2}$	0.129
WCVaR(AGD)	0.182	1.51	17	$2.71\ 10^{-3}$	0.179	0.182	1.51	17	$2.71\ 10^{-3}$	0.179
WCVaR(AGS)	−0.028	1.39	18	$6.10\ 10^{-3}$	0.158	−0.028	1.39	18	$6.10\ 10^{-3}$	0.158
WCVaR(AGT)	0.029	1.41	20	$3.36\ 10^{-3}$	0.135	0.029	1.41	20	$3.36\ 10^{-3}$	0.135
WCVaR(TG2)	−1.882	0.99	18	$1.25\ 10^{-3}$	0.187	−1.882	0.99	18	$1.25\ 10^{-3}$	0.187
WCVaR(TG3)	−1.270	0.99	23	$1.47\ 10^{-4}$	0.131	−1.270	0.99	23	$1.47\ 10^{-4}$	0.131

Table 5 Diversification of the optimal portfolios (MSPs)

Models	MSP without Diversification enforcement	MSP with Diversification enforcement
Minimax	6–29	12–29
CVaR(0.05)	14–29	15–29
CVaR(0.1)	17–29	18–29
CVaR(0.25)	11–30	18–30
CVaR(0.5)	16–29	18–27
CVaR(0.75)	12–23	13–22
GMD	12–26	16–26
WCVaR(AGD)	1–21	11–21
WCVaR(AGS)	1–23	13–21
WCVaR(AGT)	1–25	11–25
WCVaR(TG2)	15–30	17–30
WCVaR(TG3)	15–29	16–29

Table 5 shows, for all the models over all the periods, the diversification of the optimal portfolios (MSPs). For instance, the number of selected securities for the Minimax model varies, out of the 13 solved instances, from 6 to 29 securities, while with the introduction of diversification constraints the corresponding range becomes 12–29. Similar considerations can be made by analyzing portfolios composition in terms of minimum and maximum portfolio shares. On average, the Wide WCVaR provide the ranges with the lowest upper limits and result in extremely low diversified portfolios (or rather undiversified portfolios as the lower limit can be equal to 1). These models seem to require the use of an additional technique to guarantee enough diversification. Also the Minimax model may generate some low diversified portfolios (6 securities). The other models have always selected more than 10 securities. Nevertheless, in many cases they generate portfolios with some very large shares (exceeding 30%). Thus, for all the LP computable models under consideration we may recommend a support of some direct technique for diversification enforcement. One may notice that the application of the CVaR based diversification enforcement constraints (21) has resulted in portfolios always containing at least 10 securities.

3.3 Out-of-sample analysis

In this section the behavior of all the MSPs is examined in the twelve months following the date of each portfolio selection. To describe out-of-sample results we have used the following nine ex-post parameters: the minimum, the average, the maximum and the median portfolio return ($r_{\min}$, r_{av}, $r_{\max}$ and r_{med}, respectively); the standard deviation (std) and the semi-standard deviation (s-std); the mean absolute deviation (MAD) and the mean downside semideviation (s-MAD); the maximum downside deviation (D-DEV). Such performance criteria have been computed for all the models over all the periods and can be used to compare the out-of-sample behavior of the maximum safety portfolios selected by the different models. The minimum, average, maximum and median ex-post portfolio returns are expressed on a yearly basis. All the dispersion measures (std, s-std, MAD, s-MAD and D-DEV) have been computed with respect to the target return μ_0 (which is zero for the MSP) to make them directly comparable in the different models.

In Table 6 we present the average value of each criterion, over the thirteen periods, for the various models in the cases without and with diversification enforcement, respectively. One may notice extremely high average returns of the Wide WCVaR models (without diver-

Table 6 Out-of-sample statistics for MSPs: Average values over the 13 periods

Models	$r_{\min}$	r_{av}	r_{med}	$r_{\max}$	std	s-std	MAD	s-MAD	D-DEV
				Without diversification enforcement					
Minimax	−54.36	30.05	17.57	337.19	0.0614	0.0270	0.0493	0.0148	0.0749
CVaR(0.05)	−52.75	28.77	10.66	332.93	0.0596	0.0264	0.0477	0.0145	0.0720
CVaR(0.1)	−52.43	30.57	12.24	347.98	0.0611	0.0263	0.0486	0.0141	0.0713
CVaR(0.25)	−49.68	34.27	26.58	307.67	0.0591	0.0237	0.0454	0.0110	0.0718
CVaR(0.5)	−53.10	31.12	32.34	282.05	0.0603	0.0267	0.0474	0.0128	0.0788
CVaR(0.75)	−59.86	27.80	29.34	381.36	0.0668	0.0325	0.0512	0.0161	0.0945
GMD	−58.64	27.85	26.21	315.79	0.0639	0.0304	0.0495	0.0151	0.0870
WCVaR(AGD)	−72.84	92.27	48.52	3625.59	0.1153	0.0427	0.0913	0.0218	0.1148
WCVaR(AGS)	−71.69	94.00	47.57	3634.52	0.1152	0.0415	0.0907	0.0209	0.1136
WCVaR(AGT)	−72.17	94.15	49.47	3631.79	0.1151	0.0418	0.0908	0.0209	0.1146
WCVaR(TG2)	−52.14	30.52	27.86	378.51	0.0614	0.0259	0.0473	0.0134	0.0719
WCVaR(TG3)	−49.50	33.09	25.23	321.26	0.0602	0.0243	0.0458	0.0116	0.0719
				With diversification enforcement					
Minimax	−53.80	31.38	25.49	337.79	0.0607	0.0263	0.0486	0.0139	0.0735
CVaR(0.05)	−53.27	28.73	13.78	329.62	0.0592	0.0264	0.0473	0.0143	0.0731
CVaR(0.1)	−52.17	29.88	11.94	334.94	0.0596	0.0259	0.0474	0.0138	0.0712
CVaR(0.25)	−46.93	33.00	26.57	302.67	0.0581	0.0232	0.0444	0.0109	0.0678
CVaR(0.5)	−53.50	30.49	32.33	289.09	0.0607	0.0268	0.0470	0.0129	0.0793
CVaR(0.75)	−60.23	28.56	29.34	384.64	0.0682	0.0333	0.0522	0.0163	0.0948
GMD	−59.19	27.50	27.78	323.99	0.0646	0.0308	0.0497	0.0153	0.0877
WCVaR(AGD)	−63.43	35.21	38.88	439.76	0.0742	0.0337	0.0582	0.0171	0.0947
WCVaR(AGS)	−62.91	36.75	36.00	426.09	0.0737	0.0327	0.0574	0.0162	0.0944
WCVaR(AGT)	−62.67	36.96	38.21	426.26	0.0735	0.0328	0.0575	0.0162	0.0945
WCVaR(TG2)	−52.04	30.19	27.86	366.65	0.0608	0.0257	0.0466	0.0132	0.0719
WCVaR(TG3)	−49.36	31.62	25.44	322.26	0.0600	0.0246	0.0452	0.0119	0.0718

sification enforcement). These performances are produced by single security portfolios with very high returns. In general, the models are too risky as demonstrated by all the dispersion measures. When we consider the models with diversification enforcement, the Wide WC-VaR models are still characterized by the highest average returns and the largest dispersion parameters but the differences from the other models are not very large. One may notice that the GMD model, which is the computationally most complex, may be easily outperformed (in terms of average returns and dispersion) by the simpler Tail WCVaR models or even by CVaR(0.5).

We have also analyzed each model performance with respect to a long-run portfolio management. Each of the portfolios selected by a specific model in the 13 instances has been evaluated ex-post in the three months period following the date of selection. It turned out that single period ex-post returns quite perfectly represent the upward and downward movements of the market. For instance, all the models showed negative results from April to July 1996 and then again in October when the market was falling down. However, in such periods some models (such as CVaR(0.1) and CVaR(0.25)) find portfolios with a better performance with respect to the market index MIB30. Similarly, many models find higher returns with respect to the MIB30 index at the beginning of the 1998 when the market showed a positive trend (see Fig. 2). Full results of this analysis can be found in our technical report (Mansini, Ogryczak and Speranza, 2003c). Further, we cumulated the returns over the horizon up to 13 periods (39 months) to better analyze each model achievements. The figures shown in Table 7 are the cumulative returns of the portfolios selected by each model in the case without diversification enforcement and with diversification enforcement, respectively. Each column of these tables refers to a period and provides the cumulative returns of the portfolios selected over the preceding periods. For a better understanding of these figures let us consider the first line of Table 7 which refers to the model Minimax. Each of the 13 portfolios selected by the Minimax model in the 13 instances has been evaluated ex-post in the three months investment period following the date of its selection. Let us define as $r_1, r_2, \ldots, r_{13}$ the ex-post returns of these 13 portfolios. Then, the first column of Table 7 gives the ex-post return (after 3 months) of the first portfolio selected, i.e. r_1. The second column of Table 7 gives the cumulative return of the portfolio selected in the first period and then modified after three months with the portfolio selected in the second period: the value is computed as $(1 + r_1)(1 + r_2) - 1$. Similarly, for all the other columns of the table. These results have been computed to simulate a multi-period setting where, at no transaction cost, the portfolio changes over time. Rates are expressed on a yearly basis.

Table 7 shows extremely high cumulative returns of the Wide WCVaR models (without diversification enforcement). These performances are due to the single security portfolios selected in the first 6 periods which resulted in dramatically high returns. Actually, when ignoring these 6 periods and focusing on the remaining horizon of the last 21 months, the cumulative returns of the Wide WCVaR models considerably shrink as it is evident when comparing the first part of Table 8 with the last seven columns of Table 7. Table 8 shows the ex-post cumulative returns over the last 21 months (7 periods) both for the case without and with diversification enforcement. Moreover, as in column (7–13) of the first part of Table 8, it can be noticed that the Wide WCVaR models perform much worse than all the other models except for the Minimax and the extremal CVaR models ($\beta = 0.05$ or $\beta = 0.1$). Note that both the Tail WCVaR models and the CVaR(0.5) here have the best cumulative performances.

When the models with diversification enforcement are considered, the Wide WCVaR models are still characterized by the highest cumulative returns but the differences from the other models are not very large. When ignoring the first 6 periods and focusing on the last 21 months (second part of Table 8), one may see again the Wide WCVaR models performing

Table 7 Out-of-sample results for MSPs: Cumulative returns

Models	3 m.	6 m.	9 m.	12 m.	15 m.	18 m.	21 m.	24 m.	27 m.	30 m.	33 m.	36 m.	39 m.
Without diversification enforcement													
Minimax	54.09	32.04	13.41	9.23	18.20	20.59	23.16	29.63	48.80	38.97	24.00	21.90	20.97
CVaR(0.05)	63.87	35.30	15.27	10.57	19.36	21.58	24.03	30.42	49.60	39.65	27.33	24.49	23.27
CVaR(0.1)	64.54	35.63	17.87	11.72	20.91	22.00	24.33	30.04	49.37	39.25	27.28	23.95	25.11
CVaR(0.25)	73.05	34.33	19.67	14.30	25.23	27.39	26.33	31.22	51.98	47.19	32.25	32.22	32.43
CVaR(0.5)	96.96	54.35	23.46	15.71	22.35	25.29	23.00	26.93	48.93	44.15	29.62	30.43	32.43
CVaR(0.75)	148.85	81.32	41.84	23.07	29.40	32.38	29.23	30.24	54.89	48.60	32.46	36.35	33.60
GMD	128.21	71.94	36.35	19.84	26.98	34.21	30.41	31.19	53.25	48.36	33.61	34.71	32.66
WCVaR(AGD)	322.73	647.41	264.67	205.56	213.21	208.53	167.22	146.28	168.85	144.28	110.52	103.96	94.55
WCVaR(AGS)	322.73	647.41	264.67	205.56	213.21	208.53	165.32	144.96	167.49	142.49	107.74	102.99	93.71
WCVaR(AGT)	322.73	647.41	264.67	205.56	213.21	208.53	167.21	144.12	167.38	144.33	109.71	102.87	94.55
WCVaR(TG2)	53.17	24.90	11.50	8.02	19.80	21.46	22.41	27.66	48.08	39.59	27.80	26.27	32.71
WCVaR(TG3)	78.91	41.59	20.94	13.41	23.99	26.85	26.05	29.07	50.19	45.29	31.02	30.92	33.35
With diversification enforcement													
Minimax	67.81	37.87	11.83	8.13	17.25	19.78	22.45	28.97	48.13	38.41	23.16	19.87	18.16
CVaR(0.05)	57.11	35.26	10.42	7.11	16.36	19.03	21.79	28.36	47.50	37.88	24.77	22.18	20.88
CVaR(0.1)	48.16	28.73	11.86	7.42	17.17	18.85	21.57	27.52	46.79	37.08	25.23	22.09	22.38
CVaR(0.25)	71.10	33.50	15.86	11.56	22.82	25.34	24.59	29.64	50.35	45.76	31.25	31.31	31.58
CVaR(0.5)	100.55	52.22	21.84	14.37	21.21	24.32	22.18	26.19	48.16	43.48	29.07	29.93	31.95
CVaR(0.75)	145.52	78.16	40.21	21.44	27.98	30.95	28.78	29.84	54.46	48.23	31.59	35.63	33.35
GMD	127.84	65.51	30.75	15.54	23.25	30.90	27.84	28.93	50.90	46.31	32.10	33.31	31.39
WCVaR(AGD)	184.78	143.24	65.70	43.49	50.95	56.21	49.09	47.81	70.77	62.37	45.22	45.12	42.03
WCVaR(AGS)	170.49	132.87	61.94	41.25	49.33	55.95	48.38	47.31	70.22	61.44	43.65	44.75	41.88
WCVaR(AGT)	173.36	134.87	62.66	41.11	50.59	57.68	50.35	47.60	70.96	63.37	45.44	45.06	42.74
WCVaR(TG2)	50.35	27.01	11.46	7.99	19.78	21.44	22.39	27.64	48.06	39.58	27.75	26.23	32.16
WCVaR(TG3)	81.68	39.85	18.18	11.49	22.31	25.41	24.83	27.97	49.05	44.30	29.50	29.52	32.04

Table 8 Out-of-sample computational results: Cumulative returns over the latest 21 months (7 periods)

Periods	Without diversification							With diversification						
Models	7	7–8	7–9	7–10	7–11	7–12	7–13	7	7–8	7–9	7–10	7–11	7–12	7–13
Minimax	39.77	60.99	115.24	71.91	28.23	23.22	22.83	39.77	60.99	115.24	71.91	27.35	19.96	16.79
CVaR(005)	39.77	60.99	115.24	71.91	34.59	27.47	24.74	39.77	60.99	115.24	71.91	32.02	25.42	22.49
CVaR(01)	39.23	57.50	112.92	69.81	33.93	25.93	27.84	39.23	57.50	112.92	69.81	33.34	25.42	25.48
CVaR(025)	20.16	43.44	106.15	8.28	38.32	37.24	36.90	20.16	43.44	106.15	8.28	38.70	37.55	37.17
CVaR(05)	10.14	32.00	100.87	77.91	35.02	35.79	38.88	10.14	32.00	100.87	77.91	35.02	35.79	38.88
CVaR(075)	11.85	24.04	102.30	76.72	32.56	40.44	34.65	16.47	26.58	104.88	78.51	32.37	40.48	35.43
GMD	9.75	22.53	91.36	72.42	32.90	35.21	31.34	10.92	23.18	92.00	72.88	33.55	35.76	31.81
WCVaR(AGD)	12.80	25.27	95.24	72.10	33.08	34.84	29.74	12.68	25.20	95.17	72.05	33.05	34.81	30.91
WCVaR(AGS)	7.30	22.59	92.48	68.97	29.23	33.56	29.98	10.06	24.15	94.01	70.04	30.16	34.36	30.84
WCVaR(AGT)	12.76	20.93	92.24	72.19	31.94	33.39	31.03	13.00	21.06	92.37	72.28	32.00	33.44	31.07
WCVaR(TG2)	28.27	48.21	109.51	71.99	35.84	31.28	43.17	28.27	48.21	109.51	71.99	35.76	31.21	42.10
WCVaR(TG3)	21.41	35.97	100.98	78.10	36.20	35.12	39.19	21.41	35.97	100.98	78.10	34.59	33.78	38.00

much worse than all the other models except for the Minimax and the extremal CVaR models. It is interesting to notice that, except for the Minimax and the extremal CVaR models, all the other models resulted in similar cumulative return over the entire horizon of 39 months with (annual) rate of return exceeding 30%. Also the GMD model is outperformed by simple Tail WCVaR models and the CVaR models for larger tolerance levels.

To better capture the models behavior over small periods with possibly different market trends we have analyzed the ex-post cumulative returns over subperiods of length 4, that is we computed the cumulative returns over the periods 7–10, 8–11, 9–12 and 10–13. In Table 9 we have shown the minimum, the average and the maximum cumulative return ($rc_{\min}$, rc_{av}, and $rc_{\max}$, respectively) for each model over these subperiods. Note that during the periods with negative market trend, as during subperiod 10–13, all the models have negative average cumulative returns. However, GMD and CVaR(0.5) and CVaR(0.25) show the best, although negative, average performance. Moreover, when the maximum cumulative return is considered, the three best models are GMD, CVaR(0.25) and WCVaR(TG3), respectively. On the contrary, in positive market trend periods such as in the subperiod 9-12 (corresponding to the first part of the year 1998), the CVaR(0.75) has the largest average and maximum cumulative return while the largest minimum cumulative return is obtained by GMD.

Finally, to show how consistently the composition of the portfolios selected by the same model over the different periods may change, we have reported, as an example, Table 10 which provides the portfolios composition changes from one period to the other for the portfolios selected by the different models in the case without diversification enforcement. For instance, the second line of Table 10 refers to model CVaR(0.05) and can be interpreted as follows. The first column gives the number of securities selected by this model in the first period (in this case 14 securities). The second column says that, with respect to the previous portfolio, the one selected in the second period contains 2 new securities and no securities have been eliminated from those already selected. Similarly for the other models.

3.4 Models behavior in a strong downward trend period: The years 2000–2002

The Markowitz type models, used without any additional forecasting procedure applied prior to portfolio selection process itself, do not recognize any market trends and therefore they are generally not appropriate tools for investment situations with a long lasting market trend. Nevertheless, due to commonly observed negative trends during recent years, both researchers and practitioners become more interested in the models behavior under such circumstances. Therefore, in order to provide a better analysis and comparison of the proposed models when the market trend is negative and thus the risk control may be relevant, we have decided to add some computational results on the period (2000–2002). During this period the Italian market has shown an impressive and continuous down-turn (see Fig. 2) with the MIB30 index reaching its highest level 50467 on 10.03.2000 and its lowest level 21546 on 4.10.2002. The following tables provide the relevant results on the analysis of the Maximum Safety Portfolios (MSPs) selected by the different models with and without diversification enforcement using the years 2000–2001 (104 weekly returns) as in-sample period and the year 2002 as out-of-sample. The data set consists of 178 securities quoted with continuity from 2000 to 2002. The meaning of tables entries is identical to those described in the former sections. Due to low weekly values, the mean return z has been converted on a yearly basis.

Notice that all the models but CVaR(0.75), GMD and the Wide WCVaR models have a mean return equal to zero. Thus, as for previous experiments, the Wide WCVaR models are still among those models with larger mean returns. The use of constraints to enforce diversification improves, on average, the portfolios performance in terms of mean return:

Table 9 Cumulative return statistics in subperiods of length 4: Case without diversification

Models	7–10			8–11			9–12			10–13		
	$rc_{\min}$	rc_{av}	$rc_{\max}$	$rc_{\min}$	rc_{av}	$rc_{\max}$	$rc_{\min}$	rc_{av}	$rc_{\max}$	$rc_{\min}$	rc_{av}	$rc_{\max}$
Minimax	39.77	71.98	115.24	24.42	93.88	188.38	3.55	110.99	348.50	−46.32	−33.69	−24.86
CVaR(0.05)	39.77	71.98	115.24	30.15	95.31	188.38	10.70	114.35	348.50	−41.26	−29.46	−22.76
CVaR(0.1)	39.23	69.87	112.92	31.90	92.20	183.96	11.93	116.40	352.58	−38.73	−27.90	−18.73
CVaR(0.25)	8.28	44.51	106.15	43.77	101.49	190.28	33.07	148.21	392.11	−28.79	−8.21	10.28
CVaR(0.5)	10.14	55.23	100.87	42.07	97.63	190.87	34.39	161.67	434.77	−30.61	−8.27	7.51
CVaR(0.75)	16.47	56.61	104.88	36.67	90.73	191.93	33.75	188.27	519.56	−36.02	−11.41	2.30
GMD	10.92	49.75	92.00	36.80	84.56	169.64	37.94	163.64	431.47	−27.42	−7.01	10.76
WCVaR(AGD)	12.68	51.28	95.17	38.69	85.57	174.63	35.75	163.56	442.15	−29.96	−10.83	3.11
WCVaR(AGS)	10.06	49.57	94.01	35.73	84.87	175.25	31.87	161.36	440.93	−33.07	−12.24	0.26
WCVaR(AGT)	13.00	49.68	92.37	29.69	81.21	167.96	36.93	168.97	453.65	−29.73	−9.19	8.57
WCVaR(TG2)	28.27	64.50	109.51	37.70	94.87	188.31	23.46	133.65	385.45	−34.24	−17.60	2.35
WCVaR(TG3)	21.41	59.12	100.98	38.10	90.32	177.28	31.27	150.53	404.90	−31.22	−9.27	7.78

Table 10 Out-of-sample computational results: Changes in portfolios composition

	Without diversification enforcement												
Models	3 m.	6 m.	9 m.	12 m.	15 m.	18 m.	21 m.	24 m.	27 m.	30 m.	33 m.	36 m.	39 m.
Minimax	14	3, −1	14, −11	9, −7	10, −5	7, −6	11, −11	9, −7	13, −13	3, −6	3, −23	8, −1	3, −8
CVaR(0.05)	14	2, 0	12, −9	9, −7	10, −5	7, −6	11, −11	9, −7	13, −13	3, −6	10, −19	8, −11	7, −5
CVaR(0.1)	18	9, −9	12, −7	5, −4	8, −6	13, −12	8, −8	15, −13	2, −7	0, −24	11, −16	12, −13	8, −9
CVaR(0.25)	11	2, −1	10, −4	10, −6	8, −8	6, −6	10, −6	8, −8	12, −9	6, −5	2, −15	9, −6	6, −6
CVaR(0.5)	17	2, −3	14, −14	17, −14	9, −4	7, −5	8, −12	8, −9	10, −4	10, −10	5, −12	6, −6	4, −5
CVaR(0.75)	17	3, −5	4, −5	4, −3	7, −2	7, −6	1, −10	8, −2	6, −2	8, −10	6, −11	7, −8	3, −4
GMD	17	1, −6	7, −5	4, −3	10, −3	5, −7	4, −8	10, −4	6, −2	8, −11	5, −9	6, −5	3, −7
WCVaR(AGD)	1	0,0	0,0	0,0	0,0	0,0	14, 0	6, −4	7, −3	6, −6	4, −9	5, −4	3, −6
WCVaR(AGS)	1	0,0	0,0	0,0	0,0	0,0	15, 0	9, −4	8, −6	8, −9	5, −10	6, −5	2, −6
WCVaR(AGT)	1	0,0	0,0	0,0	0,0	0,0	16, 0	8, −5	9, −4	5, −9	5, −10	6, −2	2, −6
WCVaR(TG2)	16	7, −9	9, −9	7, −6	10, −3	7, −6	12, −9	7, −9	12, −13	1, −1	7, −13	12, −15	7, −10
WCVaR(TG3)	15	5, −2	5, −5	9, −6	9, −5	6, −5	8, −5	7, −8	7, −8	6, −5	2, −9	9, −7	4, −6

Table 11 In-sample MSPs characteristics: Strong downward trend period

Models	Without diversification					With diversification				
	Obj. 10^{-2}	z %	Div. #	Shares min	Shares max	Obj. 10^{-2}	z %	Div. #	Shares min	Shares max
Minimax	−0.212	0.00	7	$2.06\ 10^{-4}$	0.813	−0.281	0.00	14	$6.01\ 10^{-4}$	0.2
CVaR(0.05)	−0.189	0.00	6	$2.17\ 10^{-3}$	0.563	−0.231	0.00	11	$8.83\ 10^{-4}$	0.2
CVaR(0.1)	−0.158	0.00	8	$2.38\ 10^{-4}$	0.450	−0.193	0.00	16	$7.62\ 10^{-5}$	0.2
CVaR(0.25)	−0.107	0.00	13	$1.61\ 10^{-4}$	0.469	−0.129	0.11	15	$2.47\ 10^{-4}$	0.2
CVaR(0.5)	−0.065	0.00	13	$1.69\ 10^{-4}$	0.459	−0.079	0.16	13	$5.48\ 10^{-5}$	0.2
CVaR(0.75)	−0.031	1.83	15	$1.70\ 10^{-3}$	0.463	−0.033	3.27	19	$1.83\ 10^{-3}$	0.2
GMD	−0.050	0.10	13	$2.97\ 10^{-4}$	0.464	−0.058	0.60	15	$1.48\ 10^{-3}$	0.2
WCVaR(AGD)	−0.039	0.50	15	$2.15\ 10^{-4}$	0.384	−0.044	1.16	17	$7.25\ 10^{-5}$	0.2
WCVaR(AGS)	−0.044	0.25	15	$1.46\ 10^{-4}$	0.432	−0.050	0.81	16	$1.24\ 10^{-3}$	0.2
WCVaR(AGT)	−0.043	0.34	15	$1.10\ 10^{-5}$	0.495	−0.049	1.08	15	$1.42\ 10^{-3}$	0.2
WCVaR(TG2)	−0.131	0.00	12	$7.28\ 10^{-5}$	0.429	−0.158	0.00	15	$1.15\ 10^{-4}$	0.2
WCVaR(TG3)	−0.093	0.00	14	$1.59\ 10^{-4}$	0.444	−0.112	0.06	12	$3.50\ 10^{-3}$	0.2

CVaR(0.75) shows an increase from 1.83% to 3.27%, while some models as CVaR(0.5) and CVaR(0.25) move from null to positive values. The same effect was not evident in the previous experiments.

Table 11 shows that with respect to portfolio diversification, the introduction of enforcement constraints produces an evident effect only for the Minimax model and the extremal CVaR models (CVaR(0.05) and CVaR(0.1)). As before (see Tables 3–4), the Minimax model generates some low diversified portfolios (7 securities). In contrast to previous results the basic CVaR models are not the most diversified: the CVaR(0.05) model has selected only 6 securities, while the CVaR(0.1) portfolio has 8 securities. Moreover, note that the Wide WCVaR models have selected rather diversified portfolios if we compare these results with those shown in Tables 3–4. During the period 2000–2002 no security has shown realizations with dramatically high returns, thus justifying this diversification. Finally, in only one case, i.e. for the WCVaR(TG3) model, the diversification without enforcement is larger than that obtained with additional forcing constraints. In the first part of Table 11 it is worth noticing that all the selected portfolios have a maximum share exceeding 40% (but for the model WCVaR(AGD) with 38%) and that, in two cases, namely for the models Minimax and CVaR(0.05), the maximum share is larger than 80% and 55%, respectively. For all such models we may recommend to apply the enforcement constraints (22) giving as result portfolios always containing at least 11 securities.

For the out-of-sample analysis the behavior of all the MSPs is examined in the 52 weeks following the date of each portfolio selection. The nine parameters reported in Table 12 have the same meaning defined for previous similar tables. Again, $r_{\min}$, r_{av}, r_{med} and $r_{\max}$ are expressed on a yearly basis and as per cent returns. Due to general market downward trend, all the portfolios show ex-post negative average returns: the model CVaR(0.75) has the worst performance. One may notice in Table 12 that, on average, the introduction of diversification enforcement may result in portfolios with larger ex-post dispersion. This is especially true for those models whose dispersion was already high without enforcement (see, for instance, the models CVaR(0.75) and CVaR(0.5)). When we consider the models with diversification enforcement, the model WCVaR(TG3) has the highest average return. As for previous computational results, the GMD model (which is the computationally most

Table 12 Out-of-sample results for MSPs: Strong downward trend period.

Models	$r_{\min}$	r_{av}	r_{med}	$r_{\max}$	std	s-std	MAD	s-MAD	D-DEV
				Without diversification enforcement					
Minimax	−14.18	−0.93	−0.68	14.01	0.0009	0.0007	0.0006	0.0004	0.0029
CVaR(0.05)	−12.81	−0.73	−0.79	13.66	0.0009	0.0006	0.0007	0.0004	0.0026
CVaR(0.1)	−7.81	−0.58	−1.19	10.70	0.0008	0.0006	0.0006	0.0004	0.0016
CVaR(0.25)	−10.16	−0.88	−1.11	12.93	0.0008	0.0006	0.0006	0.0004	0.0021
CVaR(0.5)	−8.75	−0.68	−0.96	13.25	0.0008	0.0006	0.0006	0.0004	0.0018
CVaR(0.75)	−48.42	−1.49	0.67	42.82	0.0031	0.0026	0.0021	0.0012	0.0126
GMD	−9.97	−0.76	−0.61	13.21	0.0009	0.0007	0.0007	0.0004	0.0020
WCVaR(AGD)	−15.44	−1.01	−0.76	14.91	0.0011	0.0009	0.0008	0.0005	0.0032
WCVaR(AGS)	−12.77	−0.80	−0.33	12.44	0.0010	0.0008	0.0008	0.0005	0.0026
WCVaR(AGT)	−17.13	−0.94	−0.42	14.57	0.0012	0.0010	0.0009	0.0005	0.0036
WCVaR(TG2)	−7.61	−0.63	−0.59	11.56	0.0008	0.0006	0.0006	0.0004	0.0015
WCVaR(TG3)	−8.98	−0.77	−1.18	12.42	0.0008	0.0006	0.0006	0.0004	0.0018
				With diversification enforcement					
Minimax	−22.28	−0.96	−0.50	20.97	0.0013	0.0010	0.0009	0.0005	0.0048
CVaR(0.05)	−12.44	−1.23	−1.27	9.63	0.0008	0.0007	0.0006	0.0004	0.0025
CVaR(0.1)	−9.78	−0.97	−1.29	9.13	0.0008	0.0006	0.0006	0.0004	0.0020
CVaR(0.25)	−9.40	−0.61	−0.46	7.70	0.0008	0.0006	0.0006	0.0004	0.0019
CVaR(0.5)	−12.04	−0.75	−0.59	7.89	0.0008	0.0007	0.0006	0.0004	0.0025
CVaR(0.75)	−62.78	−1.96	−0.07	68.23	0.0044	0.0036	0.0030	0.0017	0.0188
GMD	−13.35	−0.89	−0.32	13.77	0.0011	0.0009	0.0008	0.0005	0.0027
WCVaR(AGD)	−24.45	−1.31	−0.57	22.34	0.0016	0.0013	0.0012	0.0007	0.0054
WCVaR(AGS)	−15.93	−0.95	−0.31	16.85	0.0013	0.0010	0.0009	0.0006	0.0033
WCVaR(AGT)	−23.55	−1.22	−0.19	21.53	0.0016	0.0013	0.0011	0.0007	0.0051
WCVaR(TG2)	−10.16	−0.77	−0.55	7.96	0.0008	0.0006	0.0006	0.0004	0.0021
WCVaR(TG3)	−9.82	−0.57	−0.61	8.30	0.0007	0.0006	0.0006	0.0003	0.0020

complex) is outperformed in terms of average returns and dispersion by the Tail WCVaR models.

Table 13 shows the ex-post cumulative portfolio returns for the case without and with the diversification enforcement, respectively. Additionally, the cumulative performances of the index MIB30 has been introduced in both tables. The weighted CVaR models show a very stable ex-post performance always outperforming GMD model. Moreover, the Wide WCVaR models also outperform extremal CVaR models when diversification enforcement is introduced. Actually, except for a short period of strong increase of the MIB30 index, all the models outperform the index. From Table 13 it is evident how diversification enforcement has, on average, positively contributed to improve all the portfolios performance with the only exception of extremal CVaR portfolios whose performances are worsen with respect to the case without diversification.

The additional experimental analysis over the period 2000-2002 has allowed us to draw the following main conclusions. First, during strongly negative market trend the weighted CVaR models have, on average, performed better than the GMD, the Minimax and the extremal CVaR models. Actually, in terms of cumulative returns all the models have beaten the MIB30 index performance. Second, during strongly negative market trend the Wide WCVaR models show a more stable behavior than in positive trend period. In the latter case they may need diversification enforcement. Generally, the diversification enforcement turns out to be necessary and effective for all the models rather during unstable market trends (typically characterized by quick changes of market directions, as for previous experiments) than during strong downward periods where for some models, as for the extremal CVaR models, diversification enforcement has made performances even worse.

4 Concluding remarks

In this paper we have studied LP solvable portfolio optimization models based on extensions of the Conditional Value at Risk (CVaR) measure. The models use multiple CVaR measures thus allowing for more detailed risk aversion modeling. All the studied models are SSD consistent and may be considered some approximations to the Gini's mean difference with the advantage of being computationally much simpler than the GMD model itself. Our analysis has been focused on the weighted CVaR measures defined as simple combinations of a very few CVaR measures. We have introduced two specific types of weight-settings which relate the WCVaR measure to the Gini's mean difference (the Wide WCVaR) and its tail version (the Tail WCVaR). This allows us to use a few tolerance levels as only parameters specifying the entire WCVaR measures while the corresponding weights are automatically predefined by the requirements of the corresponding Gini's measures.

Our experimental analysis of the models performance on the real-life data from the Milan Stock Exchange has confirmed their attractiveness. The WCVaR models have usually performed better than the GMD, the Minimax or the extremal CVaR models. These promising results show a need for further comprehensive experimental studies analyzing practical performances of the WCVaR models within specific areas of financial applications. It is important to notice that although the quantile risk measures (VaR and CVaR) were introduced in banking as extreme risk measures for small tolerance levels (like $\beta = 0.05$), for the portfolio optimization good results have been provided by rather larger tolerance levels. Additional experimental analysis over the period with strongly negative market trend has confirmed good achievements of the WCVaR models. In terms of cumulative returns all the models have outperformed the MIB30 index.

While the Tail WCVaR models have always generated well diversified portfolios, the Wide WCVaR models require some diversification enforcement to avoid too small portfolios. Our experiments have also confirmed effectiveness of our CVaR based technique for a direct diversification enforcement. Although, the diversification enforcement turns out to be necessary and effective rather during unstable market trends (typically characterized by quick changes of market directions) than during strong downward periods.

Appendix

The spectral risk measures have been shown to be coherent in the sense of Artzner et al. (1999). These coherence axioms are based on the standard 'monotonicity' ($X \geq Y$ a.s. $\Rightarrow \rho(X) \leq \rho(Y)$). Since the measures represent the dual theory of choice (Yaari, 1987), they should also satisfy stronger monotonicity related to the SSD dominance. Indeed, the following SSD consistency results can be shown.

Theorem 3. *If a nonnegative real function ϕ on the interval* $[0, 1]$ *is weakly decreasing (i.e., $\alpha_1 < \alpha_2$ implies $\phi(\alpha_1) \geq \phi(\alpha_2)$), then the spectral measure $M_\phi(\mathbf{x}) = \int_0^1 \phi(\alpha) F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha$ is*

Table 13 Cumulative out-of-sample results: Strong downward trend period

Models	4 w.	8 w.	12 w.	16 w.	20 w.	24 w.	28 w.	32 w.	36 m.	40 w.	44 w.	48 w.	52 w.
Without diversification enforcement													
Minimax	3.35	−3.41	4.23	1.55	0.74	−3.25	−7.25	−7.97	−7.81	−10.04	−5.94	−2.94	−0.93
CVaR(0.05)	2.51	−2.99	5.89	2.72	2.21	-0.70	−4.58	−6.96	−6.81	−9.18	−5.22	−2.12	−0.73
CVaR(0.1)	0.43	−4.33	5.44	2.86	2.63	−0.18	−4.44	−6.40	−6.42	−8.71	−5.17	−1.88	−0.58
CVaR(0.25)	−2.68	−8.19	−3.22	−3.62	−4.73	−5.08	−8.33	−9.31	−9.43	−13.40	−8.57	−3.29	−0.88
CVaR(0.5)	−2.13	−4.90	0.22	0.22	−1.60	−2.78	−5.85	−7.69	−7.74	−12.29	−7.66	−2.79	−0.68
CVaR(0.75)	17.33	15.44	45.49	41.16	59.95	38.90	24.21	10.89	4.31	−15.59	−5.71	−1.24	−1.51
GMD	−0.66	−5.09	2.02	0.76	0.76	−0.26	−4.38	−6.71	−7.36	−11.91	−7.14	−2.48	−0.76
WCVaR(AGD)	1.27	−4.44	4.99	4.13	6.31	2.31	−2.74	−5.48	−6.84	−14.40	−8.10	−3.13	−1.01
WCVaR(AGS)	1.08	−3.04	5.50	4.14	4.98	2.20	-2.73	−5.46	−6.43	−12.11	−6.90	−2.40	−0.80
WCVaR(AGT)	0.47	−3.79	4.66	3.65	5.22	3.65	−2.24	−5.68	−7.03	−13.90	−7.90	−2.69	−0.94
WCVaR(TG2)	−0.36	−5.68	1.84	−0.16	−0.22	−1.59	−4.64	−6.43	−6.91	−9.72	−5.92	−2.09	−0.63
WCVaR(TG3)	−2.21	−7.20	−1.09	−2.04	−2.94	−3.48	−6.82	−8.46	−8.57	−12.29	−7.67	−2.84	−0.78
With diversification enforcement													
Minimax	−0.75	−4.82	12.72	11.80	10.55	1.85	−4.21	−7.25	−6.69	−11.67	−5.87	−2.31	−0.96
CVaR(0.05)	−4.84	−9.82	−1.42	−0.97	−4.50	−8.01	−11.31	−12.39	−11.18	−16.41	−11.07	−5.25	−1.23
CVaR(0.1)	−5.40	−10.72	0.10	0.42	−2.35	−8.70	−10.86	−11.93	−10.62	−15.59	−9.66	−4.33	−0.97
CVaR(0.25)	−1.62	−6.07	2.01	4.28	1.91	−3.73	−6.71	−7.62	−7.01	−12.35	−6.88	−2.79	−0.61
CVaR(0.5)	−2.31	−5.92	0.14	2.34	−0.97	−3.33	−6.26	−7.56	−7.33	−13.61	−8.25	−3.54	−0.75
CVaR(0.75)	29.69	22.69	82.34	78.72	117.34	63.31	43.86	23.26	11.48	−16.54	−4.62	−0.52	−2.01
GMD	0.80	−2.78	6.71	7.48	7.35	2.54	−1.51	−4.45	−5.77	−13.97	−7.52	−3.08	−0.89
WCVaR(AGD)	4.66	0.53	12.61	12.57	15.39	7.67	2.73	−2.26	−4.96	−16.82	−8.60	−3.70	−1.31
WCVaR(AGS)	2.80	−0.74	11.03	11.14	11.93	5.67	0.88	−2.62	−4.92	−14.34	−7.47	−2.93	−0.95
WCVaR(AGT)	4.61	0.71	13.05	13.34	17.08	9.66	3.25	−2.18	−4.68	−16.98	−8.45	−3.41	−1.23
WCVaR(TG2)	−3.09	−7.69	1.66	3.54	−0.08	−5.82	−8.53	−9.56	−8.50	−14.06	−8.42	−3.61	−0.78
WCVaR(TG3)	−1.93	−5.78	1.83	3.83	0.72	−1.63	−4.97	−6.59	−6.44	−11.68	−6.82	−2.69	−0.57
MIB30	−26.71	−95.69	174.00	181.75	−45.95	−98.92	−99.43	−99.17	−98.68	−99.47	−94.31	−64.90	−26.45

SSD consistent, i.e.,

$$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''} \quad \Rightarrow \quad M_\phi(\mathbf{x}') \geq M_\phi(\mathbf{x}'').$$

Proof: Note that $M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') = \int_0^1 \phi(\alpha)(F_{\mathbf{x}'}^{(-1)}(\alpha) - F_{\mathbf{x}''}^{(-1)}(\alpha))d\alpha$ and integrating by parts one gets

$$M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') = \phi(1)\big(F_{\mathbf{x}'}^{(-2)}(1) - F_{\mathbf{x}''}^{(-2)}(1)\big) - \int_0^1 \big(F_{\mathbf{x}'}^{(-2)}(\alpha) - F_{\mathbf{x}''}^{(-2)}(\alpha)\big)d\phi(\alpha) \quad (22)$$

If $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$, then $F_{\mathbf{x}'}^{(-2)}(\alpha) \geq F_{\mathbf{x}''}^{(-2)}(\alpha)$ for all $\alpha \in [0, 1]$ and, in particular, $F_{\mathbf{x}'}^{(-2)}(1) - F_{\mathbf{x}''}^{(-2)}(1) = \mu(\mathbf{x}') - \mu(\mathbf{x}'') \geq 0$. Hence, due to assumed properties of function ϕ, one gets $M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') \geq 0$. □

Theorem 4. *If a nonnegative real function ϕ on the interval $[0, 1]$ is strictly decreasing (i.e., $\alpha_1 < \alpha_2$ implies $\phi(\alpha_1) > \phi(\alpha_2)$), then the spectral measure $M_\phi(\mathbf{x}) = \int_0^1 \phi(\alpha)F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha$ is strictly SSD consistent, i.e.,*

$$R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''} \quad \Rightarrow \quad M_\phi(\mathbf{x}') > M_\phi(\mathbf{x}'').$$

Proof: If $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''}$, then $F_{\mathbf{x}'}^{(-2)}(\alpha) \geq F_{\mathbf{x}''}^{(-2)}(\alpha)$ for all $\alpha \in [0, 1]$ with at least one strict inequality for some $\alpha_0 \in (0, 1]$. Moreover, $F_{\mathbf{x}}^{(-2)}$ are continuous functions (Ogryczak and Ruszczyński, 2002a). Therefore, there exist $\varepsilon > 0$ such that $F_{\mathbf{x}'}^{(-2)}(\alpha) > F_{\mathbf{x}''}^{(-2)}(\alpha)$ for all $\alpha \in (\alpha_0 - \varepsilon, \alpha_0]$. Hence, due to assumed properties of function ϕ, using (22) one gets $M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') > 0$, which completes the proof. □

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