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CONDITIONALLY COMPLETE AND CONDITIONALLY ORTHOGONALLY COMPLETE RINGS

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In what follows every ring is assumed to have the property that in it every element x has a *unique* cube root y (i.e., $x = y^3$) which, as usual, is denoted by $\sqrt[3]{x}$. For brevity, we call such a ring a *cuberoot* ring. Let x be an element of a cuberoot ring R. Then $x^2 = 0$ implies $x^3 = 0$ which implies $x = 0$. Thus, every cuberoot ring R is *reduced* (i.e., R has no nonzero nilpotent element).

It is known [1, Theorem 1, p. 46] that if R is a reduced ring then $(R, \leqq)$ is a partially ordered set where for every element x and y of R,

$$(1) \qquad x \leqq y \quad \text{if and only if} \quad xy = x^2 .$$

In what follows any reference to order is made in connection with the partial order given by (1).

Note. It is known that a cuberoot ring is not necessarily commutative. Based on the Structure Theorem [1, Theorem 3, p. 49] all the results of this paper concerning cuberoot rings can be proved *without commutativity assumption*. However, for the sake of simplicity, we assume commutativity.

Lemma 1. *Let a and b be elements of a commutative ring R. Let M as well as N be a product of $n \geqq 1$ factors each factor being equal to a or to b. Then*

$$(2) \qquad ab(a - b) = 0 \quad \textit{implies} \quad abM = abN$$

Proof. Clearly, $ab(a - b)$ implies $a^2b = ab^2$ from which (2) follows trivially. Thus, in particular, $ab(a - b) = 0$ implies

$$(3) \qquad bababa = b^3a^2b = a^3a^2b = a^2ba^2b .$$

Remark 1. If R is a reduced ring which is not necessarily commutative then (2) holds for all the permutations of the factors in abM and abN.

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Lemma 2. *Let a and b be elements of a commutative-cuberoot ring R. Then the following are pairwise equivalent*:

(4) $\{a, b\}$ *is bounded above*,

(5) $ab(a - b) = 0$,

(6) $\operatorname{lub}\{a, b\} = b + a - \sqrt[3]{(a^2b)}$,

(7) $b(a - \sqrt[3]{(a^2b)}) = a(b - \sqrt[3]{(a^2b)}) = 0$,

(8) $\operatorname{lub}\{b, a - \sqrt[3]{(a^2b)}\} = \operatorname{lub}\{a, b\} = b + a - \sqrt[3]{(a^2b)}$.

Proof. Let u be an upper bound of $\{a, b\}$. Then by (1) we have

(9) $$au = a^2 \quad \text{and} \quad bu = b^2 .$$

Hence, $aub = a^2b = ab^2$ or $ab(a - b) = 0$. Thus, (4) implies (5).

Next, let $ab(a - b) = 0$. Then (3) is valid and by taking cube roots from both sides of the equalities in (3) we obtain:

(10) $$ba = b\sqrt[3]{(a^2b)} = a\sqrt[3]{(a^2b)} = \sqrt[3]{(a^2b)}\sqrt[3]{(a^2b)} .$$

Hence $b^2 + ba - b\sqrt[3]{(a^2b)} = b^2$ or $b(b + a - \sqrt[3]{(a^2b)}) = b^2$ which by (1) implies $b \leqq (b + a - \sqrt[3]{(a^2b)})$. Similarly, by (10) we derive that $a \leqq (b + a - \sqrt[3]{(a^2b)})$. Thus, $b + a - \sqrt[3]{(a^2b)}$ is an upper bound of $\{a, b\}$. Let u be an upper bound of $\{a, b\}$. Then (9) is valid and implies $(u\sqrt{(a^2b)})^3 = u^3a^2b = a^5b$. Consequently,

(11) $$u\sqrt[3]{(a^2b)} = a\sqrt[3]{(a^2b)} .$$

But then, (9), (10), (11) imply $u(b + a - \sqrt[3]{(a^2b)}) = (b + a - \sqrt[3]{(a^2b)})^2$ which by (1) shows that $b + a - \sqrt[3]{(a^2b)} = \operatorname{lub}\{a, b\}$. Thus, (5) implies (6).

Next, let $\operatorname{lub}\{a, b\} = b + a - \sqrt[3]{(a^2b)}$. Hence, by (1) we have $b(b + a - \sqrt[3]{(a^2b)}) = b^2$ and $a(b + a - \sqrt[3]{(a^2b)}) = a^2$ which readily imply (7). Thus, (6) implies (7).

Next, let (7) hold. But then (8) is derived from (7) the same way as (6) is derived from (5).

Finally, we observe that (8) implies (4) trivially.

Thus, Lemma 2 is proved.

Remark 2. Using the terminology introduced in [2], from Lemma 2 it follows that if $\{a, b\}$ is a boundable (quasiorthogonal in the sense of [4]) subset of a commutative-cuberoot ring then $\{a, b\}$ has a least upper bound which is equal to $b + a - \sqrt[3]{(a^2b)}$.

We recall that a subset S of a ring R is called *orthogonal* if and only if for every element x and y of S if $x \neq y$ then $xy = 0$.

Remark 3. From (4) and (7) of Lemma 2 it follows that if $\{a, b\}$ is a bounded above subset of a commutative-cuberoot ring R then $\{b, a - \sqrt[3]{(a^2 b)}\}$ is an orthogonal subset of R. Moreover, by (8), these two subsets have least upper bounds and these least upper bounds are equal.

We recall also that a partially ordered set P is called *conditionally complete* if and only if every nonempty bounded above subset of P has a least upper bound (equivalently, every nonempty bounded below subset of P has a greatest lower bound).

If R is a reduced ring then $(R, \leqq)$ is called *conditionally orthogonally complete* if and only if every nonempty bounded above orthogonal subset of R has a least upper bound.

Clearly, every conditionally complete reduced ring is conditionally orthogonally complete. The Theorem below shows that the converse holds for commutative-cuberoot rings.

Theorem 1. *Let R be a commutative-cuberoot ring. Then $(R, \leqq)$ is conditionally complete if and only if $(R, \leqq)$ is conditionally orthogonally complete.*

Proof. Clearly, it is enough to prove that if $(R, \leqq)$ is conditionally orthogonally complete then $(R, \leqq)$ is conditionally complete.

Thus, we assume that every nonempty bounded above orthogonal subset of R has a least upper bound. Let A be a nonempty subset of R such that A is bounded above by p. To prove the Theorem it suffices to show that lub A exists.

Let $\overline{\overline{A}} = k$. Thus, A can be represented as a well ordered sequence $(a_i)_{i<k}$ of type k. Hence,

$$(12) \qquad A = (a_i)_{i<k} \quad \text{and} \quad a_i \leqq p \quad \text{for every} \quad i < k\,.$$

We define a sequence $(d_i)_{i<k}$ of type k of elements d_i of R by

$$(13) \qquad d_i = a_i - \sqrt[3]{(a_i^2 \operatorname*{lub}_{j<i} d_j)} \quad \text{for every} \quad i < k$$

which is justified since we prove immediately that

$$(14) \qquad (d_i)_{i<v} \text{ is a bounded above by } p \text{ orthogonal subset of } R \text{ such that}$$
$$\operatorname*{lub}_{i<v} d_i = \operatorname*{lub}_{i<v} a_i \quad \text{for every} \quad v \leqq k\,.$$

To this end it is enough to show that: (i) the truth of (14) implies the truth of (14) with v replaced by $v + 1$ and that: (ii) the truth of (14) for every ordinal v less than a limit ordinal u implies the truth of (14) with v replaced by the limit ordinal u.

(i) Since $(a_i)_{i<k}$ is bounded above by p, we see that $\{a_v, \operatorname*{lub}_{i<v} a_i\}$ is also bounded above by p and from (14) it follows that $\{a_v, \operatorname*{lub}_{i<v} d_i\}$ is bounded above. Thus, by

Lemma 2 (cf. Remark 3) we have

$$(15) \qquad \{\operatorname*{lub}_{i<v} d_i,\ a_v - \sqrt[3]{(a_v^2 \operatorname*{lub}_{i<v} d_i)}\} \quad \text{is orthogonal}$$

and

$$(16) \qquad \operatorname{lub}\{\operatorname{lub} d_i,\ a_v - \sqrt[3]{(a_v^2 \operatorname*{lub}_{i<v} d_i)}\} = \operatorname{lub}\{a_v, \operatorname*{lub}_{i<v} d_i\}\,.$$

But then from (16), (13), (14) it follows

$$\operatorname{lub}\{\operatorname*{lub}_{i<v} d_i, d_v\} = \operatorname{lub}\{a_v, \operatorname*{lub}_{i<v} a_i\}$$

which implies

$$(17) \qquad \operatorname*{lub}_{i<v+1} d_i = \operatorname*{lub}_{i<v+1} a_i \leqq p\,.$$

On the other hand, from (15) and (13) we have

$$\{\operatorname*{lub}_{i<v} d_i, d_v\} \quad \text{is orthogonal}$$

which implies $d_v \operatorname*{lub}_{i<v} d_i = 0$ which in turn, in view of [1, Theorem 2, p. 47] implies $\operatorname*{lub}_{i<v} d_v d_i = 0$. Consequently, $d_v d_i = 0$ for every $i < v$, i.e.,

$$(18) \qquad (d_i)_{i<v+1} \quad \text{is orthogonal}\,.$$

Clearly, from (17) and (18) it follows that (14) is true with v replaced by $v + 1$. Hence (i) is proved.

(ii) Let (14) be true for every ordinal v less than a limit ordinal u. But then trivially, $(d_i)_{i<u}$ is a bounded above by p orthogonal subset of R. Thus, by our assumption $\operatorname*{lub}_{i<u} d_i$ exists and in view of (14) we see that $\operatorname*{lub}_{i<u} d_i = \operatorname*{lub}_{i<u} a_i$.

Hence, (14) is established. Thus, $(d_i)_{i<k}$ is a bounded above orthogonal subset of R and therefore $\operatorname*{lub}_{i<k} d_i$ exists which by (14) implies that $\operatorname*{lub}_{i<k} a_i$ as well as $\operatorname{lub} A$ exists, as desired.

Hence, the Theorem is proved.

As mentioned earlier, in view of the Structure Theorem [1, Theorem 3, p. 49] the conclusion of Theorem 1 remains valid even if in its hypothesis the commutativity assumption is dropped. Hence, we have:

Theorem 2. *Let R be a cuberoot ring. Then $(R, \leqq)$ is conditionally complete if and only if $(R, \leqq)$ is conditionally orthogonally complete.*

In our attempt to establish criteria relating completeness to orthogonal completeness of reduced rings (cf. [2], [3]), we prove below (see Theorems 3 and 4) that the conclusions of Theorems 1 and 2 remain valid if in them "conditional completeness" and "conditional orthogonal completeness" are replaced respectively by "countable completeness" and "countable orthogonal completeness". First we introduce some definitions and prove a lemma.

As in [2], a subset B of a ring is called *boundable* if and only if every two-element subset of B is boundable. Thus, in view of Remark 2, *a subset B of a commutative-cuberoot ring is boundable if and only if every two-element subset of B has a least upper bound.*

Lemma 3. *Let R be a commutative-cuberoot ring and let S be a finite subset of R. Then* lub *S exists.*

Proof. Let $S = \{a_0, a_1, \ldots, a_n\}$. Since every two-element subset of S has a least upper bound, by Lemma 2 we see that

$$a_1 + a_0 - \sqrt[3]{(a_0^2 a_1)} = \operatorname{lub}\{a_0, a_1\} = b_1 .$$

Again, it can be readily verified that

$$a_2 + b_1 - \sqrt[3]{(b_1^2 a_2)} = \operatorname{lub}\{a_0, a_1, a_2\}$$

and so on, asserting the conclusion of the Lemma.

Following [2], we call a reduced ring R *countably complete* if and only if every countable boundable subset of R has a least upper bound. Also, we call R *countably orthogonally complete* if and only if every countable orthogonal (which, a priori is boundable) subset of R has a least upper bound.

Based on the above definitions, we prove:

Theorem 3. *Let R be a commutative-cuberoot ring. Then $(R, \leqq)$ is countably complete if and only if $(R, \leqq)$ is countably orthogonally complete.*

Proof. Since every orthogonal subset of R is a priori boundable, to prove the Theorem it is enough to show that if $(R, \leqq)$ is countably orthogonally complete then $(R, \leqq)$ is countably complete.

Thus, we assume that every countable orthogonal subset of R has a least upper bound. Let A be a countable boundable subset of R. To prove the Theorem it suffices to show that lub A exists.

The proof parallels that of Theorem 1 where k is replaced by the set ω of all natural numbers and (14) is replaced by

$$\operatorname*{lub}_{i<v} d_i = \operatorname*{lub}_{i<v} a_i \quad \text{for every} \quad v \leqq \omega .$$

For the case at hand, it is enough to verify only (i) of the proof of Theorem 1. But this amounts to verifying only that for every natural number v, the set $\{a_v, \operatorname*{lub}_{i<v} a_i\}$

is bounded above. This however, follows from Lemma 3, since $\{a_i \mid i < v + 1\}$ is a finite boundable subset of a commutative-cuberoot ring R and therefore has a least upper bound implying that $\{a_v, \operatorname{lub}_{i<v} a_i\}$ is bounded above.

Again we observe that in view of [1, Theorem 3, p. 49] in the above considerations the commutativity assumption can be dropped. Hence, we have:

Theorem 4. *Let R be a cuberoot ring. Then $(R, \leqq)$ is countably complete if and only if $(R, \leqq)$ is countably orthogonally complete.*

Remark 4. The main results of this paper remain valid if instead of cuberoot rings n-root rings are considered (as expected if $n \geqq 2$ is a fixed natural number then a ring R is called an *n-root ring* if and only if in R every element has a unique n-th root). One of the reasons that we have chosen cuberoot rings is that they are less restrictive. For instance, squareroot rings are necessarily of characteristic two.

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