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Electroosmotic flow is fluid motion driven by an electric field acting on the net fluid charge produced by charge separation at a fluid-solid interface. Under many conditions of practical interest, the resulting fluid velocity is proportional to the local electric field, and the constant of proportionality is everywhere the same. Here we show that the main conditions necessary for this similitude are a steady electric field, uniform fluid and electric properties, an electric Debye layer that is thin compared to any physical dimension, and fluid velocities on all inlet and outlet boundaries that satisfy the Helmholtz-Smoluchowski relation normally applicable to fluid-solid boundaries. Under these conditions, the velocity field can be determined directly from the Laplace equation governing the electric potential, without solving either the continuity or momentum equations. Three important consequences of these conditions are that the fluid motion is everywhere irrotational, that fluid velocities in two-dimensional channels bounded by parallel planes are independent of the channel depth, and that such flows exhibit no dependence on the Reynolds number.

Introduction

Microfluidic systems are finding increasing use in the separation, identification and synthesis of a wide range of chemical and biological species [1,2,3]. Employing transverse channel dimensions in the range from several to a few hundred microns, such systems may permit the large-scale integration of wet analytical methods in a manner analogous to that already achieved in microelectronics. Applications for microfluidics now under development include such diverse processes as DNA sequencing, immunochromatography, and the identification of explosives and chemical and biological warfare agents. The analytical methods used in these processes include traditional chromatography, electrochromatography and electrophoresis.

One promising method for driving fluid motion in microfluidic systems is electroosmosis [4]. Here, motion is induced by applying an electric field to a fluid that is bounded by an insulating solid. Since charge separation generally occurs at a fluid-solid boundary, a layer of fluid near the interface carries a net electric charge. The applied field acts on this net fluid charge to produce a body force that drives fluid motion. The region of net charge is usually confined to a thin Debye layer immediately adjacent to the surface, so the body force is nearly coincident with the bounding surface. The resulting boundary or sheath velocities impart a plug-like motion to the remaining neutral fluid inside a channel.

Electroosmotic flows offer several important advantages over pressure-driven flows for the small physical dimensions characteristic of microfluidic systems. First, electroosmosis provides a direct means of transporting fluids in microchannel networks using only applied electric fields. Further, fluid speeds in electroosmotic flows are independent of the transverse tube or channel dimension over a wide range of conditions, making this technique extensible to extremely small physical scales. In contrast, pressure driven flows require that the pressure gradient increase inversely with the square of the transverse channel dimension to maintain a given fluid speed. Finally, the profile of the fluid velocity across a channel is essentially flat whenever the Debye layer thickness is small compared to the channel width. Because of this, analyte samples can be transported over long ranges with little hydrodynamic dispersion [5].

Direct numerical simulation of electroosmotic flow is a challenging task [6]. In addition to the usual Navier-Stokes and species transport equations, the electric field equation must also be solved, and these equations are generally coupled through the unknown charge density. Further, these solutions must resolve length scales ranging from a Debye layer thickness of perhaps 10 nm, to channel widths on the order of 100 μm, and to device lengths of nearly 10 cm. These widely disparate length scales, spanning roughly seven orders of magnitude, make traditional numerical meshes impractical for all but the very simplest geometries. Moreover, these disparate length scales persist in the problem even when the fluid transport and electric potential are not coupled, necessitating either a highly nonuniform computational mesh or the use of multiple meshes to address separately the Navier-Stokes and Poisson equations. One means for avoiding this problem is to assume that the Debye layer thickness is very small compared to any channel dimension. In this case, the electric potential is governed by the much simpler Laplace equation, but the Navier-Stokes equations describing fluid motion must still be solved [7]. This complexity in solving the most general equations governing electroosmotic flow motivates our present interest in one unique aspect of these flows: under fairly broad conditions of practical interest, the steady velocity field of an electroosmotic flow is uniformly proportional to the electric field.

Such similitude between the fluid velocity and electric field was previously revealed by Morrison [7] for the motion of a single particle in an unbounded fluid. In that work, he showed that the electrophoretic particle speed was independent of the particle shape and that fluid motion outside the Debye layer was irrotational provided that the particle was free of external forces and moments other than those induced by the applied electric field. Here we examine internal flows bounded by the walls of channel or capillary networks and derive the conditions necessary and sufficient for similitude. When these conditions are met, both the electric potential and fluid velocity fields can be computed by solving only the Laplace equation.

Preliminary Observations

Many of the conditions necessary for similitude between the electroosmotic fluid velocity and applied electric field can be discerned by counter example. For instance, body forces acting on the fluid appear only near fluid-solid boundaries when the Debye layer is thin, so an electric field suddenly applied will produce a fluid transient as the neutral fluid accelerates. Such a transient cannot exhibit similitude with the steady electric field, thus we see that the velocity and electric fields must be at least quasi-steady. Applied electric fields ramped at sufficiently low rates can, of course, satisfy this condition.

Similarly, any applied pressure difference between the ends of a channel will produce a velocity profile that is at least in part parabolic. This nonuniform fluid velocity is clearly not similar to a uniform electric field, so a uniform pressure on all inlet and outlet boundaries is generally required for similitude. Flows induced by gravity likewise cannot resemble an applied uniform electric field.

When the thickness of a channel is smaller than the Debye length, a net charge is present everywhere in the fluid. In this case, an applied uniform electric field produces a uniform body force resulting in a parabolic velocity profile like that of flows driven by pressure or gravity. Only when Debye layers are thin can the velocity field in the neutral fluid be uniformly proportional to the applied electric field. Thus thin Debye layers are necessary for similitude.

Similitude of the two fields also requires several less stringent conditions. These include uniform density, viscosity and conductivity of the neutral fluid, a uniform surface potential at the fluid-solid interface, and a Debye layer conductance in the direction of fluid motion that is very small compared to that of the neutral fluid. In all but very specialized cases, the channel walls must also be impermeable, and the conductivity of the solid must be negligible compared to that of the neutral fluid. Counter examples supporting these requirements are apparent from the results that follow.

Finally, the fluid velocity on all inlet and outlet boundaries must also be proportional to the electric field. That this condition is necessary follows directly from the definition of similitude: the flow is uniformly proportional to the electric field. Here we show that this condition, along with the necessary conditions above, is sufficient for global similitude between the fluid velocity and electric field.

Equations Describing Electroosmotic Flow

To derive the conditions sufficient for similitude, we consider a simply or multiply connected fluid volume, bounded in its entirety by a mix of two surface types. The surfaces S_1 describe the interface between the fluid and an impermeable insulating solid, while the surfaces S_2 describe inlet or outlet boundaries.

The electric potential within this volume is governed by the Poisson equation relating the divergence of the electric field to the local charge density,

$$\nabla \cdot (\epsilon \nabla \phi) = -\rho_e, \quad (1)$$

where ϵ is the dielectric constant of the fluid and ρ_e is the local charge density. The local charge density may be related to the electric potential through the Boltzmann distribution or similar relations.

Boundary conditions for the electric potential on the charged surface may be specified either through a prescribed density of the surface charge in conjunction with the governing equation (1) or, alternatively, as a prescribed wall potential with respect to the potential of the adjacent neutral fluid. The latter is generally preferred since the wall or zeta potential is often known or can be obtained from simple experiments. Finally, the applied electric field is generally specified by prescribed potentials on inlet and outlet boundaries.

Restricting our attention to liquid flows, the working fluid may be considered incompressible and the continuity equation reduces to

$$\nabla \cdot \mathbf{u} = 0. \quad (2)$$

Under the additional assumptions that the flow is steady and that the fluid viscosity, μ, is constant, the momentum equation becomes

$$\rho (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \rho_e \nabla \phi + \mu \nabla^2 \mathbf{u}. \quad (3)$$

where ρ is the uniform fluid density. Using the vector identities of Eqs. (A1) and (A7), the momentum equation can also be written as

$$-\rho \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla (p + \frac{\rho}{2} \mathbf{u} \cdot \mathbf{u}) + \rho_e \nabla \phi - \mu \nabla \times (\nabla \times \mathbf{u}). \quad (4)$$

This form of the equation is again generally applicable only to an incompressible fluid having constant viscosity.

Boundary conditions on the fluid velocity follow directly from the nature of the fluid-solid interface. Since no flow crosses the impermeable solid boundaries,

$$\mathbf{u} \cdot \hat{\mathbf{n}} = 0 \quad \text{on} \quad S_1, \quad (5)$$

where $\hat{\mathbf{n}}$ is a unit vector locally normal to the interface. The velocity tangential to these impermeable boundaries must also obey the no-slip condition at the fluid-solid interface,

$$\mathbf{u} \cdot \hat{\tau} = 0 \quad \text{on} \quad S_1, \quad (6)$$

where $\hat{\tau}$ is any unit vector lying in the plane of the boundary. We will next consider an alternate form of Eq. (6) describing the fluid velocity at the outer edge of the Debye layer.

Thin Debye Layer Limit

Assuming that the Debye layer thickness, λ, is much smaller than the channel width, a, a boundary-layer approximation may be used to sequentially solve for the velocity fields in the regions within and outside the Debye layer. Under all foreseeable conditions, the Reynolds number based on the Debye layer thickness is extremely small, so the inner Debye-layer solution can be constructed by considering only the balance between electric and viscous forces. The maximum velocity at the outer edge of the layer is then used as a boundary condition in calculating the larger-scale outer flow field. Sequential solution procedures of this type are best justified using the formalism of matched asymptotic expansions [9]. Here we present only an outline of the matching procedure for first-order terms.

To identify nonessential terms, the governing equations are first rewritten in terms of normal and tangential coordinates rescaled by the Debye layer thickness and the geometric length scale, respectively. The electric potential is then split into an applied field $\phi_a(\tau)$ having no normal gradients and an intrinsic component $\phi_i(\eta)$ having no tangential gradients or at most a tangential gradient induced by linear polarization that is proportional to the applied field. In the latter case, polarization of the intrinsic field can be accounted for by a suitable choice of the dielectric constant. After dropping those terms of order λ/a and smaller, the resulting

Poisson and momentum equations are combined to obtain a balance between viscous and body forces,

$$\mu \frac{d^2\mathbf{u}}{d\eta^2} = -\epsilon \frac{d^2\phi_i}{d\eta^2} \nabla\phi_a, \tag{7}$$

where η is the scaled coordinate normal to the surface. Since the intrinsic field exists only within the Debye layer and gradients within this layer are much steeper than those outside, ϕ_i, $d\mathbf{u}/d\eta$, and $d\phi_i/d\eta$ all must vanish as $\eta \to \infty$. Subject to these boundary conditions, Eq. (7) can be integrated twice to obtain the Helmholtz-Smoluchowski equation [4]

$$\mathbf{u}_\infty = -\frac{\epsilon\zeta}{\mu} \nabla\phi_a, \tag{8}$$

relating the velocity at the outer edge of the Debye layer, $\mathbf{u}_\infty$, to the applied field gradient, $\nabla\phi_a$, and the wall potential or zeta potential, ζ. Later, in solving for the bulk fluid motion, this relationship will be used as a boundary condition on all S_1 surfaces.

In reality, the viscosity and dielectric constant vary in crossing the Debye layer. Equation (8) remains valid in such cases provided that these properties are evaluated at their nominal or bulk values and the zeta potential is viewed as a constitutive parameter. In this sense, the zeta potential accounts for the property variations, as well as the usual influences of surface chemistry, buffer pH, and the like. For present purposes it is only important that the zeta potential be the same on all surfaces, in keeping with our earlier assumption of uniform fluid and surface properties.

Outside the Debye layer, the charge density and intrinsic field ϕ_i both vanish. The applied electric potential in this region is thus governed by the Laplace equation

$$\nabla^2\phi = \nabla^2\phi_a = 0, \tag{9}$$

subject to prescribed potentials on S_2 boundaries and a zero-flux condition on all S_1 boundaries. For any non-zero fluid conductivity, the latter is equivalent to

$$\nabla\phi \cdot \hat{\mathbf{n}} = 0 \quad \text{on} \quad S_1, \tag{10}$$

where again $\hat{\mathbf{n}}$ is a unit vector locally normal to the interface. Fluid velocities on S_1 are given by the Helmholtz-Smoluchowski relation, so it remains only to specify velocity conditions on the inlet and outlet boundaries, S_2.

In splitting the inner and outer regions we have neglected terms of order λ/a and smaller. One such term involves convective transport normal to the surface. As a result, no account is made of charge redistribution resulting from convective charge transport. However, these polarization effects are unimportant provided that the Peclet number based on the Debye layer thickness, electroosmotic speed and ion diffusivity remains small. This is expected in most practical applications. Convective charge transport tangential to the surface cannot influence charge distribution within the Debye layer so long as this distribution does not vary along the surface.

Conditions for Similitude

From continuity and the Laplace equation governing the electric potential in the neutral fluid outside the Debye layer we can write

$$\nabla \cdot (\mathbf{u} + \alpha\nabla\phi) = 0, \tag{11}$$

where $\alpha \equiv \epsilon\zeta/\mu$ is a constant. This expression applies without any loss of generality since $\nabla^2\phi = 0$ describes the electric field in this region and $\nabla \cdot \mathbf{u} = 0$ describes continuity. The term $\mathbf{u} + \alpha\nabla\phi$ is thus solenoidal [10], and hence there exists some vector function $\boldsymbol{\psi}$ such that

$$\mathbf{u} = -\alpha\nabla\phi + \nabla\times\boldsymbol{\psi}. \tag{12}$$

The condition for similitude between the local electric field and the local fluid velocity is therefore equivalent to the condition that $\nabla\times\boldsymbol{\psi} = \mathbf{0}$. When this latter condition is satisfied at all points in the flow, the steady fluid velocity is proportional to the electric field and the constant of proportionality is everywhere the same.

Since the fluid outside the Debye layer is electrically neutral, the body force $\rho_e\nabla\phi$ in this region vanishes. Dropping this term, the momentum equation (4) may be rewritten in terms of the vorticity,

$$-\rho\,\mathbf{u}\times\boldsymbol{\omega} = -\nabla(p + \frac{\rho}{2}\,\mathbf{u}\cdot\mathbf{u}) - \mu\nabla\times\boldsymbol{\omega}, \tag{13}$$

where the vorticity $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega} \equiv \nabla\times\mathbf{u} = \nabla\times(\nabla\times\boldsymbol{\psi}). \tag{14}$$

Note that the far right side of this result is obtained by dropping terms of the form $\nabla\times\nabla\phi$ since by Eq. (A3) they are identically zero. Taking the curl of Eq. (13) likewise eliminates the gradient of the total pressure on the right-hand side, yielding

$$\rho\nabla\times(\omega\times\mathbf{u}) = \mu\nabla^2\omega. \qquad (15)$$

The corresponding elliptic equation governing the vector function ψ is given directly by the definition of the vorticity in Eq. (14). The coupled equations (12), (14) and (15) thus generally describe incompressible flow within any three-dimensional volume. There are no restrictions on the shape of this volume, and the domain of the volume may be either simply or multiply connected.

Boundary conditions for the function ψ are derived from the primitive conditions on the normal and tangential components of the fluid velocity. Because the boundary condition on the tangential fluid velocity is automatically satisfied by the first term on the right of Eq. (12), the remaining portion of the velocity must satisfy

$$\nabla\times\psi = 0 \quad \text{on} \quad S_1. \qquad (16)$$

Note that this condition also satisfies the requirement that the normal component of the fluid velocity vanish everywhere on the impermeable fluid-solid boundary. Also note the Eq. (16) makes no statement about the fluid vorticity on the S_1 surface. Despite the fact that $\mathbf{u} = -\alpha\nabla\phi$ everywhere on this boundary, it is not necessarily the case that $\omega = -\alpha\nabla\times(\nabla\phi)$ vanishes. Although this appears to be the curl of a gradient, which would ordinarily be identically zero, such is not the case since $\mathbf{u}=-\alpha\nabla\phi$ applies only on the boundary and therefore conveys nothing of how the velocity varies moving into the fluid. Thus it is impossible to make any *a priori* claim that vorticity vanishes at the fluid-solid boundaries of an electroosmotic flow.

To show the conditions under which $\nabla\times\psi$ is identically zero, we now substitute Eq. (14) into Eq. (15) yielding

$$\rho\nabla\times[(\nabla\times\xi)\times\mathbf{u})] = \mu\nabla^2(\nabla\times\xi), \qquad (17)$$

where

$$\xi\equiv\nabla\times\psi. \qquad (18)$$

From Eq. (16), the boundary condition for ξ is

$$\xi = 0 \quad \text{on} \quad S_1. \qquad (19)$$

Given this homogeneous governing equation and boundary condition on S_1, we see that a necessary and sufficient condition for $\xi=0$ everywhere is that $\xi = 0$ on the inlet and outlet boundaries S_2. By Eq. (12), this is equivalent to requiring

$$\mathbf{u} = -\alpha\nabla\phi \quad \text{on} \quad S_2, \qquad (20)$$

Thus the fluid velocities on all inlet and outlet boundaries must satisfy the Helmholtz-Smoluchowski relation normally applicable to fluid-solid boundaries. This result is the most general condition for similitude between the fluid velocity and electric field. When this condition is satisfied, along with the other restrictions previously described, the fluid velocity is uniformly proportional to the gradient of the electric potential.

Consequences

Similitude between the fluid velocity and electric field has several important consequences. Firstly, the flow is everywhere irrotational outside the Debye layer, regardless of the geometry of bounding surfaces. All fluid motion within the domain is simply potential flow, and the vorticity defined by Eq. (14) is exactly zero. Such flows cannot manifest closed cells of fluid recirculation. Thus, electroosmotic flows satisfying the conditions for similitude cannot contain any vortex, regardless of how complex the geometry may be. The absence of vorticity further implies, according to Eq. (13), that the total pressure is everywhere uniform and is therefore the same on all inlet and outlet boundaries. This automatically holds when the condition $\mathbf{u}=-\alpha\nabla\phi$ is satisfied on these boundaries.

Secondly, the velocity field under the conditions for similitude exhibits no dependence on the Reynolds number. This is because any potential flow satisfies the full Navier-Stokes equations for all Reynolds numbers. These solutions thus remain valid even when inertial forces are not negligible.

Finally, under the conditions for similitude, the velocity field in any two-dimensional channel is independent of the channel depth, provided that the depth is uniform and the electric field is vertically uniform on all inlet and outlet boundaries. For similitude to hold, the top and bottom boundaries of these two-dimensional channels must be electrically insulated planes. Given these restrictions the electric field can have no vertical component, and all horizontal planes through the channel are equivalent. The electric field, and hence the fluid velocity, are then truly two-dimensional and will show no dependence on the channel depth.

Practical Realization of Similitude

The condition $\mathbf{u} = -\alpha\nabla\phi$ on the surfaces S_2 is effectively satisfied in many practical devices employing electroosmotic flow. This condition naturally arises for flow in any long channel between fluid reservoirs maintained at fixed potentials by embedded electrodes. Such arrangements are common in electrochromatographic systems. Although this S_2 boundary condition may not be satisfied very near an electrode, and in fact cannot be satisfied on an electrode surface, the flow field in a long narrow channel should relax to satisfy this condition within a few channel widths of its ends. This condition may thus be satisfied on the interior of a complex channel network, even if it is not satisfied everywhere.

In considering conditions on the boundaries S_2, we must keep in mind that these inlet and outlet surfaces need not be the physical ends of a channel or channel network. The surfaces S_2 are simply defined as open boundaries that may be crossed by the fluid, so their positions can be specified rather arbitrarily. Thus any portion of a device bounded in its entirety by S_1 and S_2 surfaces will exhibit similitude between the velocity and electric field when the necessary conditions are satisfied and $\mathbf{u}=-\alpha\nabla\phi$ on all specified S_2 surfaces. It follows immediately that any subset of such a domain will also exhibit similitude since the velocities on all S_2 boundaries of the subset must satisfy $\mathbf{u}=-\alpha\nabla\phi$.

One of the necessary conditions for similitude is that all S_1 surfaces are electrically nonconductive relative to the working fluid. While this condition is readily satisfied by many materials of practical importance, it contains a hidden and somewhat subtle constraint. No electrodes or conductive elements may appear within the domain bounded by S_1 and S_2 if similitude is to hold in a rigorous sense. Thus, even if there exists a set of S_2 boundaries satisfying the sufficient condition for similitude, an imbedded electrode within the domain will violate the necessary condition that no electric currents cross the S_1 boundaries. From a practical view, this means that some devices, such as electrokinetic pumps, may never exhibit similitude.

As described above, one consequence of similitude is that the total pressure is the same on all inlet and outlet boundaries. This uniformity may thus also be viewed as a necessary condition for similitude to hold. At low fluid speeds, this new condition is readily satisfied by holding the static pressures fixed and uniform on the inlet and outlet boundaries. However, the required equality of the total pressures becomes more difficult to ensure as the dynamic pressure increases. For example, a difference in cross-section area between a channel inlet and outlet yields differing fluid speeds at the two ends. If the dynamic portion of the total pressure is not negligible, this produces a difference in the total pressures. In such a case the static pressures must be adjusted to ensure that the total pressures are once again equal. This, of course, would not be very practical. Fortunately, the fluid speeds in most real applications are sufficiently small that dynamic contributions to the total pressure are negligible from a practical point of view.

Even in cases where the conditions for similitude are locally violated, the region of nonzero ξ may be small. If we assume that $|\xi| \ll |\mathbf{u}|$, the left term in Eq. (4) can be approximated by $(\mathbf{u}\cdot\nabla)\xi + (\alpha\nabla\phi\cdot\nabla)\alpha\nabla\phi$. The momentum equation can then be decomposed into two components:

$$(\alpha\nabla\phi\cdot\nabla)\alpha\nabla\phi + \frac{\nabla p_i}{\rho} = 0, \tag{21}$$

where p_i is the static pressure field of the potential flow satisfying all boundary conditions on S_1. This component vanishes because the total pressure is uniform. The second component is

$$(\mathbf{u}\cdot\nabla)\xi + \frac{\nabla p_a}{\rho} - \frac{\mu\nabla^2\xi}{\rho} = 0, \tag{22}$$

where p_a represents a pressure mismatch induced by ξ or imposed by the boundary conditions on S_2. The left-most term of Eq. (22) is the steady convective derivative of ξ, hence if U is a characteristic speed, a is a characteristic dimension transverse to the direction of motion, L is the length of the streamline, and ℓ is the distance traveled along a streamline, we may write

$$U\frac{d|\xi|}{d\ell} \sim -\frac{\Delta p_a}{\rho L} - \frac{\mu|\xi|}{\rho a^2}. \tag{23}$$

Note the term $\nabla^2\xi$ is approximated as $-|\xi|/a^2$ in keeping with the condition that $\nabla\times\xi$ vanishes on S_1. The solution to Eq. (23) for $|\xi|$ is

$$|\xi(\ell)| \sim |\xi_0|e^{-\ell/a\mathrm{Re}_a} + \frac{a^2\Delta p_a}{\mu L}, \tag{24}$$

where $\mathrm{Re}_a = \rho U a/\mu$ is the Reynolds number based on the transverse dimension. Hence, ξ decays rapidly at low Reynolds numbers if no applied pressures appear on S_2. Pressure gradients produce a persistent ξ field. Similar dimensional analysis shows that unsteady, purely electroosmotic flow produced by a step in applied voltage decays to the similitude solution with a time constant $\sim \rho a^2/\mu$, during which the flow proceeds a distance $\sim a\mathrm{Re}_a$. Hence steps in the applied potentials can also produce a Reynolds number dependence as the bulk fluid relaxes to the steady irrotational flow. Based on these observations, flows may exhibit a Reynolds number dependence when the boundary conditions are unsteady or are incompatible with the potential-flow solution. This may explain the evidence of a weak dependence on the Reynolds number in previous numerical analyses of electroosmotic flow [6].

Summary

In many cases of practical interest, the fluid velocity of an electroosmotic flow is proportional to the applied electric field and the constant of proportionality is everywhere the same. Here we have shown that necessary conditions for such similitude include a quasi-steady electric field, uniform fluid density, and uniform viscosity of the neutral fluid outside the Debye layer. The condition of uniform viscosity may be relaxed in the special case in which gradients of the viscosity are everywhere orthogonal to gradients of the fluid velocity. Further, the Debye layer thickness must be small compared to any channel dimension, and the conductance of the layer in the direction of fluid motion must be small compared to that of the neutral fluid. In addition, all solid surfaces bounding the fluid must have a uniform surface charge or surface potential, must be impermeable to flow, and must be electrically nonconducting relative to the fluid. The last of these implies that conductors and electrodes may not appear anywhere within the domain of similitude.

To derive the remaining less-obvious conditions required for similitude, the velocity field was expressed as the sum of a component uniformly proportional to the electric field and a residual component of unknown value. Using the full three-dimensional stream-function and vorticity formulation of the steady Navier-Stokes equation, we showed that the unknown residual component vanishes everywhere if and only if the velocity on all inlet and outlet boundaries satisfies the Helmholtz-Smoluchowski relation normally applicable to the fluid-solid interface. This condition, along with the necessary conditions above, is both necessary and sufficient for similitude between the velocity and electric field. The resulting electroosmotic flows are irrotational and therefore have a uniform total pressure. This proof employs no assumptions regarding the geometry of the channel, so the results are equally applicable to simple channels, channel networks, and complex three-dimensional geometries such as those in the interior of packed beds.

When the conditions for similitude are fully satisfied, the resulting fluid motion is simply a potential flow. As a result, the flow is irrotational, the total pressure is uniform, and the fluid velocity field exhibits no dependence on the Reynolds number. However, if dynamic pressures are large, care must be taken to ensure that the total pressures on all inlet and outlet boundaries remain equal. Fortunately, dynamic pressures are rarely significant in systems of practical interest.

The principal benefit of similitude is that modeling these flows is greatly simplified. Both the electric field and the velocity field are obtained from a single solution of the Laplace equation. The Navier-Stokes and Poisson equations need not be solved, and thin boundary layers on the channel walls need not be resolved. Moreover, under the conditions for similitude, flow in two-dimensional channel networks bounded by parallel planes is independent of the channel depth provided that the depth is uniform and the electric potential is vertically uniform on the inlet and outlet boundaries. In this case the flow is strictly two-dimensional and two-dimensional models may be employed. Thus, recognition of similitude is useful in modeling for the design and evaluation of channel junctions, nozzles, injectors, flow dividers, control devices and other such components of microfluidic systems.

Beyond these mathematical benefits, similitude provides useful physical insight into the design and operation of electroosmotic systems. For example, the absence of vorticity generally impedes mixing processes. Thus long range species transport with little mixing or dispersion benefits from the conditions for similitude. In contrast, irrotational flows may not be desirable when mixing is the goal. The conditions for similitude in this case also provide guidance in the means to produce vorticity by violating one or more of the requirements.

Nomenclature

U	characteristic speed
a	characteristic transverse dimension
ℓ	distance traveled along a streamline
L	length of streamline within domain
S	boundary surface

T	temperature
$\mathbf{u}$	local fluid velocity
α	velocity constant: $\alpha = \epsilon\zeta/\mu$
ϵ	dielectric constant
ζ	effective surface electric potential
λ	Debye length
μ	kinematic viscosity
ω	vorticity vector
ρ_e	charge density
ϕ	electric potential
ψ	stream function vector

Subscripts and Superscripts

1	on fluid-solid boundary
2	on inlet and outlet boundaries
i	intrinsic
a	applied

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Appendix: Useful Vector Identities

Advective dyad:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u}) \qquad \text{(A1)}$$

Commutation of cross product:

$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u} \qquad \text{(A2)}$$

Curl of a gradient:

$$\nabla\times\nabla s = \mathbf{0} \qquad \text{(A3)}$$

Divergence of a curl:

$$\nabla\cdot(\nabla\times\mathbf{u}) = 0 \qquad \text{(A4)}$$

Curl of a cross product:

$$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\nabla\cdot\mathbf{v} - \mathbf{v}\nabla\cdot\mathbf{u} + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v} \qquad \text{(A5)}$$

Gradient of a dot product:

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{u}) \qquad \text{(A6)}$$

Laplacian of a vector:

$$\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u}) \qquad \text{(A7)}$$

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