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RESEARCH ARTICLE

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Summary

In this paper we consider the γ-stabilization of n^{th}-order linear time-invariant (LTI) dynamical systems using *Multiplicity-Induced-Dominancy* (MID)-based controller design in the presence of delays in the input or the output channels. A sufficient condition is given for the dominancy of a real root with multiplicity at least $n+1$ and at least n using an integral factorization of the corresponding characteristic function. A necessary condition for γ-stabilizability is analyzed utilizing the property that the derivative of a γ-stable quasipolynomial is also γ-stable under certain conditions. Sufficient and necessary conditions are given for systems with real-rooted open-loop characteristic function: the delay intervals are determined where the conditions for dominancy and γ-stabilizability are satisfied. The efficiency of the proposed controller design is shown in the case of a multi-link inverted pendulum.

KEYWORDS:
feedback system, time delay, stabilizability, characteristic equation, dominant roots

1 | INTRODUCTION

Stabilization of unstable equilibria and orbits in the presence of communication delay is an important and challenging task in engineering applications[1,2,3]. Finding the control parameters that allow stable operation for large feedback delay is not a trivial task[4]. Stability diagrams can be used to visualize stability properties in the space of control parameters[5,6,7]. When performance with respect to settling time has to be optimized in linear time-invariant systems then one has to deal with γ-stability in order to minimize the corresponding spectral abscissa or, equivalently, to maximize the decay rate of the closed-loop system's solutions. A negative spectral abscissa can also be interpreted as a kind of robustness indicator, i.e., a measure of stability reserve against parameter perturbation. Time delay in the feedback loop is generally seen as a source of unstable behavior. Typically, as the feedback delay gets larger, the stable region in the stability diagrams (charts) gets smaller. A challenging task is to find the maximum delay (critical delay), for which the system can still be stabilized by some control law, but for larger delay, it is not possible any more. For a fixed control law, such a delay is called (generalized) delay margin if there exists only one (several) stable delay interval(s). The critical delay can also be interpreted to achieve a given spectral abscissa $\gamma < 0$, i.e., to achieve γ-stabilizability.

On the other hand, delay in the feedback loop can also be considered as a control parameter. To the best of the authors' knowledge, the idea of exploiting the delay effect in controller design was first introduced in[8] where it is shown that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus it can replace

[0]**Abbreviations:** LTI, linear time-invariant; MID, multiplicity-induced-dominancy

the proportional-derivative controller. Furthermore, it was stressed in[9] that time-delay may have a stabilizing effect in the control design. Indeed, the closed-loop stability is guaranteed precisely by the existence of the delay. In the context of mechanical engineering problems, the effect of time-delay was emphasized in[6] where concrete applications are studied, such as the machine tool vibrations and robotic systems.

It is worth noting that the real part of the rightmost root for quasipolynomial function corresponding to stable time-delay systems is actually the exponential decay rate of its time-domain solution, see for instance[10] for an estimate of the decay rate for stable linear delay systems. Also, to the best of the authors' knowledge, the first time an analytical proof of the dominancy of a multiple spectral value for the scalar equation with a single delay was presented in[11]. For reduced-order models (mainly first- and second-order systems), the link between multiple spectral values and the spectral abscissa corresponding to time-delay systems were observed in several recent works, see for instance[12,13,14,15]. In particular, an analytical proof of the dominancy of multiple spectral value thanks to an integral representation of quasipolynomial functions was proposed in[16] for scalar delay equations, then extended to second-order systems controlled by a delayed proportional feedback in[17,18,19] and applied in damping active vibrations for a piezo-actuated beam in[20]. This property was named *Multiplicity-Induced-Dominancy* (MID) in[21] where the dominancy of the multiple spectral value was parametrically characterized and proven using the argument principle. It appears that the emphasized integral representation of quasipolynomials satisfying the MID property is closely related to some degenerate hypergeometric functions as proved in[22].

In this paper, we analyse the general n^{th}-order linear time-invariant dynamical system with single delay. Relying on the MID property, we propose a unified methodology to assess the critical delay associated with γ-stabilizability based on the integral representation of quasipolynomial as in the works[23,18,17,24]. Furthermore, we extend the idea of[21] to n^{th}-order setting in exploiting the root location of the open-loop characteristic polynomial in order to have the MID property of the overall system. In particular, we analyze real-rooted open-loop systems. Such systems arise in many biological applications. Trivial cases are given by first-order scalar systems, e.g, in the description of the control of blood cell dynamics[25], the pupil light reflex[26] or simple models of human postural sway[27]. Real-rooted systems typically arise when a mechanical system is set to its completely unstable position, i.e., the number of unstable characteristic roots is N for an N-degree-of-freedom system. Human balancing can be mentioned as example, where single, double or even multiple link inverted pendulum models are used to describe human standing, walking or running[28,29,30,31,32]. Balancing on rolling or pinned balance board is another example where the governing equation resembles that of a double inverted pendulum[33,34]. The ball-and-beam balancing task can also be mentioned as a special case: actually $s = 0$ is a double root of the open-loops system[35]. In human controlled tasks, the critical delay is directly linked to the human reaction delay and is therefore a crucial parameter in respect of performance. Stick balancing on the fingertip can be used to demonstrate the relation between critical delay and stabilizability. Shorter sticks are more difficult to balance on the fingertip since they fall faster than the time required for the human subject to perform a corrective movement. Actually, most humans cannot balance a stick of length shorter than 30 cm on their fingertip.

The rest of the paper is organized as follows. The problem is stated in Subsection 1.1 and Subsection 1.2 presents a motivating example. Section 2 collects some preliminary results. Subsection 2.1 provides the main ingredient of the dominancy proof, which consists in writing quasipolynomial function with multiple ($(n + 1)$-fold) real root as an integral operator. Subsection 2.2 gives a similar result for an at least n-fold real root. Subsection 2.3 reviews a necessary condition for γ-stabilizability. The main results are presented in Section 3 where sufficient conditions for the dominancy of a multiple real root and necessary conditions for γ-stabilizability are provided for a real-rooted plant. The results of the paper are illustrated through the stabilization of an N-link inverted pendulum in Section 4. Finally, an outlook to more general systems is discussed in Section 5.

1.1 | Problem statement

We consider delayed feedback systems whose characteristic function is a quasipolynomial of the form

$$D(s) = P(s) + e^{-s\tau} Q(s)\,, \tag{1}$$

where the degrees of polynomials $P(s)$ and $Q(s)$ are n and $n - 1$, respectively:

$$\begin{aligned} P(s) &= a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0\,, \\ Q(s) &= b_{n-1} s^{n-1} + b_{n-2} s^{n-2} + ... + b_1 s + b_0\,. \end{aligned} \tag{2}$$

Assume that the plant parameters a_i are known and fixed, such that $a_n > 0$. Furthermore, assume that coefficients b_i in $Q(s)$ can be considered as independently adjustable control parameters. The problem we are focusing on can be summarized as follows:

for what values of τ is system (1) *γ-stabilizable (i.e. $\exists\ b_i(a_j,\tau)$ for which the real parts of all roots of* (1) *are less than γ<0 for some given γ).* To give a sufficient condition for γ-stabilizability we utilize the MID-property: the control parameters b_i are tuned in a way that the characteristic function $D(s)$ has a real root s_0 with multiplicity $n+1$.

Note that when derivative actions are involved in the feedback loop then implementation of the derivative term may lead to an improperly-posed problem[36,37], which may destroy the internal stability. The arguments in the papers[36,37] suggest a constructive way to make the closed-loop system properly-posed.

1.2 | Motivating example: the inverted pendulum

Balancing an inverted pendulum in the presence of feedback delay is a frequently cited example in dynamics and control theory[38,39]. Different control methods are often implemented in simple inverted pendulum systems[40,41,42]. The inverted pendulum is also a basic concept in human balancing models[43,44,45]. The equation of motion of an inverted pendulum controlled by a proportional-derivative (PD) controller reads as:

$$\begin{aligned}\ddot{\varphi}(t)+a_0\varphi(t)&=u(t)\,,\\ u(t)&=-b_0\varphi(t-\tau)-b_1\dot{\varphi}(t-\tau)\,,\end{aligned}\tag{3}$$

with a feedback delay $\tau>0$ and a system parameter $a_0<0$. The characteristic function corresponding to (3) is

$$D(s)=s^2+a_0+e^{-s\tau}(b_0+b_1s)\,.\tag{4}$$

The open-loop characteristic function $P(s)=s^2+a_0$ has real roots $\pm\sqrt{-a_0}$ since $a_0<0$. This property proves to be useful in Section 3.

The critical delay of system (3) is well-known from the literature[46]:

$$\tau_{\text{crit}}=\sqrt{-\frac{2}{a_0}}\,,\tag{5}$$

that is, the trivial solution of system (3) can be asymptotically stable if and only if $\tau<\tau_{\text{crit}}$. Next, we will show that the critical delay (5) can be obtained by studying the multiple roots of the characteristic function $D(s)$.

Assume that $D(s)$ has a real root s_0 with algebraic multiplicity at least $\deg P(s)+1=3$. Then $D(s_0)=0$, $D'(s_0)=0$ and $D''(s_0)=0$ give

$$\left.\begin{aligned}s_0^2+a_0+e^{-s_0\tau}(b_0+b_1s_0)=0\,,\\ 2s_0+e^{-s_0\tau}(-\tau(b_0+b_1s_0)+b_1)=0\,,\\ 2+e^{-s_0\tau}(\tau^2(b_0+b_1s_0)-2\tau b_1)=0\,.\end{aligned}\right\}\tag{6}$$

From (6) we obtain

$$\left.\begin{aligned}b_0&=e^{s_0\tau}\left(\tau s_0^3+s_0^2+a_0\tau s_0-a_0\right)\,,\\ b_1&=-e^{s_0\tau}\left(\tau s_0^2+2s_0+a_0\tau\right)\,,\\ s_0&=\frac{-2\pm\sqrt{2-a_0\tau^2}}{\tau}=:\ s_{0,\pm}\,.\end{aligned}\right\}\tag{7}$$

It can be shown that the triple root $s_{0,+}$ is negative and dominant for every $0<\tau<\tau_{\text{crit}}$, and therefore system (3) is asymptotically stable. In particular, at the upper bound $\tau=\tau_{\text{crit}}$ the triple root is $s_{0,+}=0$ and it is the dominant (rightmost) root of (4) with control coefficients $b_0=-a_0$ and $b_1=-a_0\tau_{\text{crit}}$. Alternatively, for a given $s_0=\gamma<0$, (6) can be solved for b_0, b_1 and τ. The smallest positive solution for τ is the critical delay $\tau_{\text{crit}}(\gamma)$ associated with γ-stability. The dominancy of $s_{0,+}$ may be shown by using the argument principle, see, for instance[18,19]. In the next section, we use a different method based on an integral representation of the characteristic function.

2 | PRELIMINARY RESULTS FOR ARBITRARY PLANTS

In this section some preliminary results are discussed for arbitrary plants without restriction to real-rootedness. First, factorization and sufficient condition for dominancy is derived in the case of a real spectral value with multiplicity at least n, and with multiplicity at least $n+1$. Then, a necessary condition for stabilizability is given.

2.1 | Factorization and a sufficient condition for dominancy

We have the following result:

Proposition 1. If the quasipolynomial (1) has a real root s_0 with multiplicity at least n then it can be written as

$$D(s) = (s - s_0)^n \left(a_n + \int_0^1 e^{-(s-s_0)\tau t} \frac{\tau R_{n-1}(s_0; \tau t)}{(n-1)!} \mathrm{d}t \right), \tag{8}$$

where the family of polynomials $R_k(s; \tau)$ is defined as

$$R_k(s; \tau) = \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(s) \tau^{k-i},\ k \in \mathbb{Z}_0^+ . \tag{9}$$

Proof. The quasipolynomial $D(s)$ has a root s_0 with algebraic multiplicity at least n if and only if $D^{(k)}(s_0) = 0$, $k = 0, 1, ..., n-1$:

$$\left.\begin{aligned} P(s_0) + e^{-s_0\tau} Q(s_0) = 0\,, \\ P'(s_0) + e^{-s_0\tau} \left((-\tau) Q(s_0) + Q'(s_0) \right) = 0\,, \\ \vdots \\ P^{(k)}(s_0) + e^{-s_0\tau} \sum_{i=0}^{k} \binom{k}{i} Q^{(i)}(s_0)(-\tau)^{k-i} = 0\,, \\ \vdots \\ P^{(n-1)}(s_0) + e^{-s_0\tau} \sum_{i=0}^{n-1} \binom{n-1}{i} Q^{(i)}(s_0)(-\tau)^{n-1-i} = 0\,. \end{aligned}\right\} \tag{10}$$

Equation (10) gives a linear system of equations for the control coefficients. Solving (10) for b_i enables the integral factorization of the form (8). The system of equations (10) is equivalent to the following system of equations:

$$\left.\begin{aligned} e^{s_0\tau} P(s_0) + Q(s_0) = 0\,, \\ e^{s_0\tau} \left(\tau P(s_0) + P'(s_0) \right) + Q'(s_0) = 0\,, \\ \vdots \\ e^{s_0\tau} \sum_{i=0}^{k} \binom{k}{i} P^{(i)}(s_0) \tau^{k-i} + Q^{(k)}(s_0) = 0\,, \\ \vdots \\ e^{s_0\tau} \sum_{i=0}^{n-1} \binom{n-1}{i} P^{(i)}(s_0) \tau^{n-1-i} + Q^{(n-1)}(s_0) = 0\,. \end{aligned}\right\} \tag{11}$$

The system of equation (11) can be written as

$$\underbrace{\begin{bmatrix} 1 & s_0 & s_0^2 & \dots & s_0^{n-1} \\ 0 & 1 & 2s_0 & \dots & (n-1)s_0^{n-2} \\ 0 & 0 & 2 & \dots & (n-1)(n-2)s_0^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & (n-1)! \end{bmatrix}}_{=:\mathbf{S}(s_0)} \underbrace{\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}}_{=:\mathbf{b}} = -e^{s_0\tau} \underbrace{\begin{bmatrix} R_0(s_0; \tau) \\ R_1(s_0; \tau) \\ R_2(s_0; \tau) \\ \vdots \\ R_{n-1}(s_0; \tau) \end{bmatrix}}_{=:\mathbf{R}(s_0;\tau)} . \tag{12}$$

Since $\mathbf{S}$ is an upper triangular matrix with nonzero diagonal elements the unique solution of (12) is

$$\mathbf{b} = -e^{s_0\tau} \mathbf{S}^{-1}(s_0) \mathbf{R}(s_0; \tau)\,. \tag{13}$$

With the control coefficients (13) the polynomial $Q(s)$ and the characteristic function $D(s)$ in (1) have the form

$$Q(s) = \underbrace{\begin{bmatrix} 1 & s & s^2 & \dots & s^{n-1} \end{bmatrix}}_{=:\mathbf{s}^T} \mathbf{b}\,, \tag{14}$$

and

$$D(s) = P(s) + Q(s)e^{-s\tau} = P(s) + \mathbf{s}^T \mathbf{b} e^{-s\tau} = P(s) - \mathbf{s}^T \mathbf{S}^{-1}(s_0)\mathbf{R}(s_0;\tau)e^{-(s-s_0)\tau}\,. \tag{15}$$

It can be proved by induction that if

$$\mathbf{w}^T = \begin{bmatrix} w_0 & w_1 & \dots & w_{n-1} \end{bmatrix} := \mathbf{s}^T \mathbf{S}^{-1}(s_0) \tag{16}$$

then

$$w_k = \frac{1}{k!}(s-s_0)^k\,,\ k = 0, 1, ..., n-1\,. \tag{17}$$

Since $R_k(s_0;0) = P^{(k)}(s_0)$

$$\mathbf{s}^T \mathbf{S}^{-1}(s_0)\mathbf{R}(s_0;0) = \sum_{k=0}^{n-1} \frac{1}{k!}(s-s_0)^k P^{(k)}(s_0) = P(s) - \frac{1}{n!}(s-s_0)^n P^{(n)}(s_0) = P(s) - a_n(s-s_0)^n\,. \tag{18}$$

From (15) and (18) it can be seen that

$$\begin{aligned} D(s) =& a_n(s-s_0)^n + \mathbf{s}^T \mathbf{S}^{-1}(s_0)\mathbf{R}(s_0;0) - \mathbf{s}^T \mathbf{S}^{-1}(s_0)\mathbf{R}(s_0;\tau)e^{-(s-s_0)\tau} \\ =& a_n(s-s_0)^n - \left[\mathbf{s}^T \mathbf{S}^{-1}(s_0)\mathbf{R}(s_0;\tau t)e^{-(s-s_0)\tau t}\right]_0^1\,. \end{aligned} \tag{19}$$

Let

$$F(t) := \mathbf{s}^T \mathbf{S}^{-1}(s_0)\mathbf{R}(s_0;\tau t)e^{-(s-s_0)\tau t}\,. \tag{20}$$

Then, after a long but elementary calculation we obtain

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = -(s-s_0)^n e^{-(s-s_0)\tau t} \frac{\tau}{(n-1)!} R_{n-1}(s_0;\tau t)\,. \tag{21}$$

Using equations (19) and (21) we arrive to the desired integral factorization

$$\begin{aligned} D(s) &= a_n(s-s_0)^n - \left[F(t)\right]_0^1 = a_n(s-s_0)^n - \int_0^1 \frac{\mathrm{d}F(t)}{\mathrm{d}t}\mathrm{d}t \\ &= (s-s_0)^n \left(a_n + \int_0^1 e^{-(s-s_0)\tau t} \frac{\tau R_{n-1}(s_0;\tau t)}{(n-1)!}\mathrm{d}t \right)\,. \end{aligned} \tag{22}$$

□

Remark 1. If $D(s)$ has a real root s_0 with multiplicity at least $n+1$ then (8) holds and, in addition, $D^{(n)}(s_0) = 0$:

$$D^{(n)}(s_0) = n! \left(a_n + \int_0^1 \frac{\tau R_{n-1}(s_0;\tau t)}{(n-1)!}\mathrm{d}t \right) = R_n(s_0;\tau) = 0\,. \tag{23}$$

In this case $D(s)$ can be factorized as

$$D(s) = \frac{1}{n!}(s-s_0)^{n+1} \int_0^1 e^{-(s-s_0)\tau t} \tau R_n(s_0;\tau t)\mathrm{d}t\,. \tag{24}$$

Proposition 2. Let s_0 be a real root of the quasipolynomial (1) with multiplicity at least $n+1$. If $R_{n-1}(s_0;\tau t) \le 0$, $\forall t$, $0 < t \le 1$ then s_0 is the dominant root of (1).

Proof. Due to the Proposition 1, (1) can be written in the form of (8). To prove that there exists no root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ of (8) such that $\gamma_1 > s_0$, substitute s_1 into (8). Since $a_n > 0$ one can obtain that

$$\begin{aligned} a_n &= \left| \int_0^1 e^{-(s_1-s_0)\tau t} \frac{\tau}{(n-1)!} R_{n-1}(s_0;\tau t)\mathrm{d}t \right| \leq \int_0^1 \left| e^{-(s_1-s_0)\tau t} \frac{\tau}{(n-1)!} R_{n-1}(s_0;\tau t) \right| \mathrm{d}t \\ &= \int_0^1 e^{-(\gamma_1-s_0)\tau t} \frac{\tau}{(n-1)!} \left| R_{n-1}(s_0;\tau t) \right| \mathrm{d}t \,. \end{aligned} \tag{25}$$

Using the condition $R_{n-1}(s_0;\tau t) \leq 0$, $\forall t$, $0 < t \leq 1$ (25) can be written as

$$a_n \leq - \int_0^1 e^{-(\gamma_1-s_0)\tau t} \frac{\tau}{(n-1)!} R_{n-1}(s_0;\tau t)\mathrm{d}t =: f(\gamma_1)\,. \tag{26}$$

For $\gamma_1 = s_0$ the function $f(\gamma_1)$ takes the value

$$\begin{aligned} f(\gamma_1 = s_0) &= - \int_0^1 \frac{\tau}{(n-1)!} R_{n-1}(s_0;\tau t)\mathrm{d}t = - \int_0^1 \frac{1}{n!} \frac{\mathrm{d}R_n(s_0;\tau t)}{\mathrm{d}t}\mathrm{d}t \\ &= -\frac{1}{n!} \left[R_n(s_0;\tau t) \right]_0^1 = -\frac{1}{n!} \left(\underbrace{R_n(s_0;\tau)}_{=0} - \underbrace{R_n(s_0;0)}_{=a_n n!} \right) = a_n \,. \end{aligned} \tag{27}$$

For $\gamma_1 > s_0$ the value of the integral in (26) is $f(\gamma_1) < a_n$ since $0 < e^{-(\gamma_1-s_0)\tau t} < 1$ for $\gamma_1 > s_0$, $\tau > 0$, $0 < t \leq 1$. Therefore from (26) we obtain $a_n < a_n$ which proves the inconsistency of the hypothesis that the characteristic function (8) has a root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ with $\gamma_1 > s_0$. Consequently, (1) has no root of the form $s_1 = \gamma_1 + \mathrm{i}\omega_1$ with $\gamma_1 > s_0$. □

Similarly to Proposition 2, we can give sufficient condition for the dominancy of a root s_0 with multiplicity at least n.

2.2 | Sufficient condition for the dominancy of a root with multiplicity at least *n*

Proposition 3. Let s_0 be a real root of (1) with multiplicity at least n, and let $R_{n-1}(s_0;\vartheta)$ has k sign changes in the interval $0 < \vartheta < \tau$: $\tau_1 < \tau_2 < ... < \tau_k$, with notations $c(s_0) := \operatorname{sgn} R_{n-1}(s_0;\vartheta)$ for $0 < \vartheta < \tau_1$, $\tau_0 = 0$ and $\tau_{k+1} = \tau$. If

$$\frac{c(s_0)}{n!} \sum_{i=0}^{k} (-1)^i \left[R_n(s_0;\vartheta) \right]_{\tau_i}^{\tau_{i+1}} \leq a_n \tag{28}$$

then s_0 is the dominant root of (1).

Proof. Since s_0 is a root with multiplicity at least n, (1) can be written as

$$D(s) = (s-s_0)^n \left(a_n + \int_0^\tau e^{-(s-s_0)\vartheta} \frac{R_{n-1}(s_0;\vartheta)}{(n-1)!} \mathrm{d}\vartheta \right), \tag{29}$$

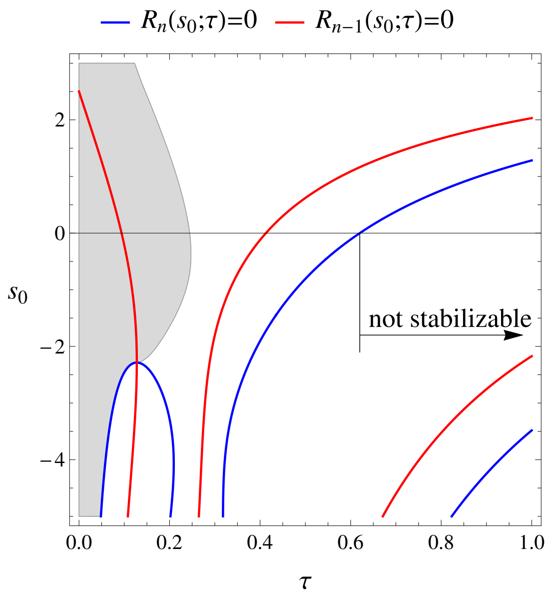

FIGURE 1 Illustration of the sufficient condition for the MID property in the sense of Proposition 3 (gray shading) and necessary condition for stabilizability according to Remark 4 for the plant $P(s) = (s-2)(s-(2-10\mathrm{i}))(s-(2+10\mathrm{i}))(s-4)$

by Proposition 1. To prove that there exists no root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ of (29) such that $\gamma_1 > s_0$, substitute s_1 into (29). Since $a_n > 0$ one can obtain that

$$
\begin{aligned}
a_n &= \left| \int_0^\tau e^{-(s_1-s_0)\vartheta} \frac{1}{(n-1)!} R_{n-1}(s_0;\vartheta)\mathrm{d}\vartheta \right| \le \int_0^\tau \left| e^{-(s_1-s_0)\vartheta} \frac{1}{(n-1)!} R_{n-1}(s_0;\vartheta) \right| \mathrm{d}\vartheta \\
&= \int_0^\tau e^{-(\gamma_1-s_0)\vartheta} \frac{1}{(n-1)!} \left| R_{n-1}(s_0;\vartheta) \right| \mathrm{d}\vartheta < \int_0^\tau \frac{1}{(n-1)!} \left| R_{n-1}(s_0;\vartheta) \right| \mathrm{d}\vartheta \\
&= c(s_0) \left(\int_0^{\tau_1} \frac{1}{(n-1)!} R_{n-1}(s_0;\vartheta)\mathrm{d}\vartheta - \int_{\tau_1}^{\tau_2} \frac{1}{(n-1)!} R_{n-1}(s_0;\vartheta)\mathrm{d}\vartheta + \ldots + (-1)^k \int_{\tau_k}^{\tau} \frac{1}{(n-1)!} R_{n-1}(s_0;\vartheta)\mathrm{d}\vartheta \right) \\
&= \frac{c(s_0)}{n!} \sum_{i=0}^{k} (-1)^i \int_{\tau_i}^{\tau_{i+1}} \frac{\mathrm{d}R_n(s_0;\vartheta)}{\mathrm{d}\vartheta} \mathrm{d}\vartheta = \frac{c(s_0)}{n!} \sum_{i=0}^{k} (-1)^i \left[R_n(s_0;\vartheta) \right]_{\tau_i}^{\tau_{i+1}} \le a_n \,.
\end{aligned}
\tag{30}
$$

From (30) we obtain $a_n < a_n$ which proves the inconsistency of the hypothesis that the characteristic function (29) has a root $s_1 = \gamma_1 + \mathrm{i}\omega_1$ with $\gamma_1 > s_0$. Therefore, (1) has no root of the form $s_1 = \gamma_1 + \mathrm{i}\omega_1$ with $\gamma_1 > s_0$. □

Remark 2. $c(s_0) = 1$ for $s_0 > s_\mathrm{a}$ and $c(s_0) = -1$ for $s_0 < s_\mathrm{a}$, where s_a is the average of the roots of $P(s)$.

Figure 1 demonstrates the MID-property for a plant of degree $n = 4$. Gray shading indicates the pairs (s_0, τ) that satisfies the sufficient condition of Proposition 3. In this region s_0 is the dominant root.

2.3 | Necessary condition for stabilizability

In this subsection the necessary condition of the stabilizability of system (1) is discussed. We start with a lemma from [47] and [48].

Lemma 1. Consider the quasipolynomial

$$h(s) = \sum_{i=0}^{n} \sum_{j=1}^{r} h_{ij} s^{n-i} e^{\tau_j s} \tag{31}$$

such that $\tau_1 < \tau_2 < ... < \tau_r$, with main term $h_{0r} \neq 0$, and $\tau_1 + \tau_r > 0$. If $h(s)$ is stable (i.e. the roots of $h(s) = 0$ are located in the open left half of the complex plane), then $h'(s)$ is also a stable quasipolynomial.

Remark 3. Lemma 1 can be generalized to γ-stability by applying a shift $z = s - s_0$ with $s_0 = \gamma$.

We can give a necessary condition for the γ-stability of (1) by the successive application of Lemma 1 and Remark 3 to the quasipolynomial $P(s)e^{s\tau} + Q(s)$.

Proposition 4. If the quasipolynomial (1) is γ-stable, then the polynomial $R_n(s;\tau)$ is γ-stable.

Proof. If (1) is γ-stable then $P(s)e^{s\tau} + Q(s)$ is γ-stable, and by Lemma 1 and Remark 3 the n^{th} derivative of $P(s)e^{s\tau} + Q(s)$ is also γ-stable. □

Remark 4. $R_n(s;\tau)$ is independent of the control coefficients b_i, therefore Proposition 4 also gives a necessary condition for the γ-stabilizability of (1).

Figure 1 also demonstrates the necessary condition for stabilizability (i.e., $\gamma = 0$). If $\tau > 0.6202$ then the system cannot be stabilized.

3 | SUFFICIENT AND NECESSARY CONDITIONS FOR DOMINANCY AND STABILIZABILITY FOR SYSTEMS WITH REAL-ROOTED OPEN-LOOP CHARACTERISTIC FUNCTION

In this section we assume that the polynomial $P(s)$ corresponding to the open-loop system has only real roots. In this case, $P(s)$ has the form $P(s) = a_n \prod_{i=1}^{n}(s - s_i)$, $s_i \in \mathbb{R}$, $s_n \leq s_{n-1} \leq ... \leq s_1$. To apply the sufficient condition in Proposition 2 and the necessary condition in Proposition 4, first, we need to characterize the properties of polynomials $R_k(s;\tau)$. These properties are outlined and discussed in the forthcoming subsections.

3.1 | Interlacing property of polynomials $R_k(s;\tau)$

The two-variable polynomials $R_k(s;\tau)$, $k \in \mathbb{Z}^+$ have the following properties:

$$R_k(s;\tau) = \tau R_{k-1}(s;\tau) + \frac{\partial R_{k-1}(s;\tau)}{\partial s}, \tag{32}$$

$$\frac{\partial R_k(s;\tau)}{\partial \tau} = k R_{k-1}(s;\tau). \tag{33}$$

Property (32) allows us to say that for a fixed τ the polynomials $R_k(s;\tau)$ and $R_{k-1}(s;\tau)$ interlace and $R_k(s;\tau)$ has only real roots for s since $R_0(s;\tau) = P(s)$ has only real roots[49]. Polynomials $R_n(s;\tau)$ and $R_{n-1}(s;\tau)$ have n distinct real roots for s if $\tau \neq 0$. Let $s_{0,k}$, $k = 1, 2, ..., n$ denote the roots of $R_n(s;\tau)$, $\tau \neq 0$ with $s_{0,n} < s_{0,n-1} < ... < s_{0,1}$.

It can also be shown that for a fixed s the polynomial $R_n(s;\tau)$ has only real roots for τ[50]. Moreover, $R_k(s;\tau)$, $k = 1, 2, ..., n-1$ has only real roots for τ, and $R_k(s;\tau)$ and $R_{k-1}(s;\tau)$ interlace which are direct consequences of property (33) and Rolle's theorem.

3.2 | Monotonicity

In the (τ, s) plane the algebraic curve $R_n(s;\tau) = 0$ has distinct branches, and every branch is strictly increasing since the derivative of the implicit function $R_n(s;\tau) = 0$ in a point (s, τ) reads

$$\frac{\mathrm{ds}}{\mathrm{d\tau}} = -\frac{\frac{\partial R_n(s;\tau)}{\partial \tau}}{\frac{\partial R_n(s;\tau)}{\partial s}} = -\frac{n R_{n-1}(s;\tau)}{R_{n+1}(s;\tau)} > 0. \tag{34}$$

The fraction $\frac{R_{n-1}(s;\tau)}{R_{n+1}(s;\tau)}$ is negative since for a fixed $\tau \neq 0$ at a root s of the polynomial $R_n(s;\tau)$ the function values $R_{n-1}(s;\tau)$ and $R_{n+1}(s;\tau)$ are nonzero and have different signs because of the interlacing property.

A similar property holds for the algebraic curve $R_{n-1}(s;\tau) = 0$. If $P(s)$ has at least two distinct roots then $R_{n-1}(s;\tau) = 0$ has distinct branches, and every branch is strictly increasing. If $P(s)$ has one root s_1 with multiplicity n (i.e. $P(s) = a_n(s - s_1)^n$) then one branch is constant and all the other branches are strictly increasing as a function of τ.

3.3 | Asymptotic properties

If $\tau \to \infty$ (or $\tau \to -\infty$) then the roots of $R_k(s;\tau)$ for s approach the roots of $P(s)$ (i.e. $s_n \leq s_{n-1} \leq \ldots \leq s_1$). Similarly, if $s \to \infty$ or $s \to -\infty$ then the roots of $R_k(s;\tau)$ for τ approach the roots of $\tau^k = 0$ (i.e. 0 with multiplicity k).

3.4 | Roots of $R_k(s;0)$

If $\tau = 0$ then $R_k(s;\tau) = R_k(s;0) = P^{(k)}(s)$. Therefore if $k = n$ then $R_n(s;0) = n!a_n$ has no roots for s. If $k = n-1$ then $R_{n-1}(s;0) = \frac{n!}{1!}a_n s + \frac{(n-1)!}{0!}a_{n-1}$ has one root for s:

$$s_a = -\frac{1}{n}\frac{a_{n-1}}{a_n} = \frac{1}{n}\sum_{i=1}^{n} s_i\,, \tag{35}$$

which is the average of the roots of $P(s)$.

3.5 | Sufficient and necessary conditions for dominancy and stabilizability

Let τ_0 denote the smallest positive root of $R_n(0;\tau) = 0$ for τ. For $\tau > 0$ the first branch of the algebraic curve $R_n(s;\tau) = 0$ corresponds to the greatest s values, and takes values in the interval $]-\infty, s_1[$. Therefore if $s_1 > 0$ then τ_0 corresponds to the first branch of $R_n(s;\tau) = 0$. If $s_1 \leq 0$ then $R_n(0;\tau) = 0$ has no positive roots: in this case we set $\tau_0 = \infty$.

Furthermore, let τ_a denote the smallest positive root of $R_n(s_a;\tau) = 0$ for τ. If $P(s) \neq a_n(s-s_1)^n$, i.e. $P(s)$ has at least two distinct roots, then τ_a corresponds to the first branch of $R_n(s;\tau) = 0$ since $s_n < s_a < s_1$. If $P(s) = a_n(s-s_1)^n$, then $s_a = s_1$ and $R_n(s_a;\tau) = 0$ has no roots: in this case we set $\tau_a = \infty$.

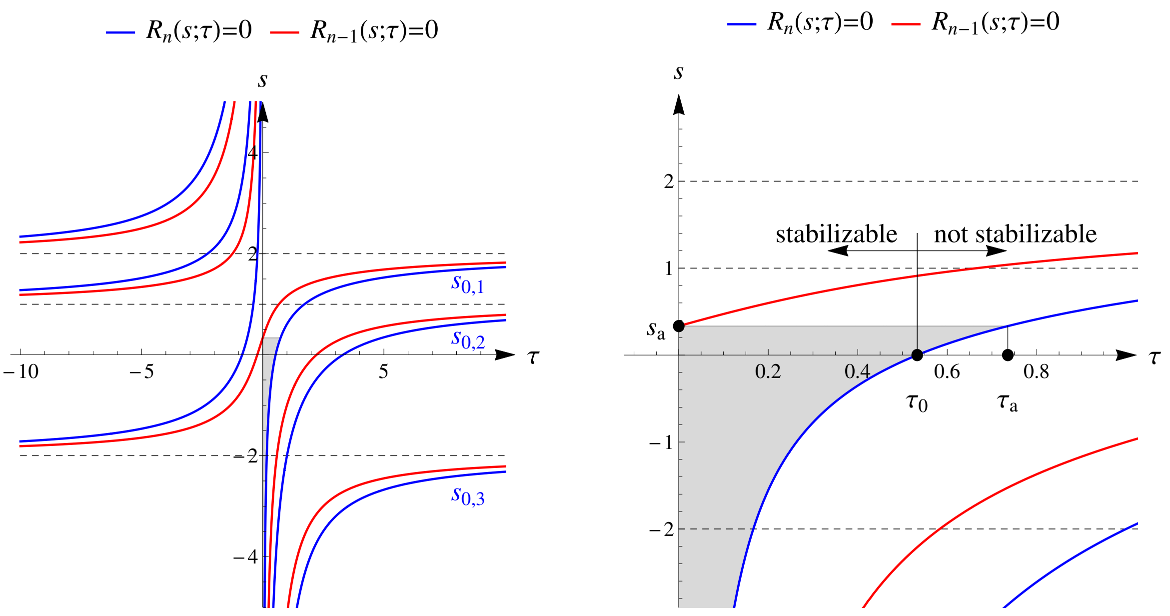

FIGURE 2 Illustration of the sufficient condition for the MID property (gray shading) and the necessary and sufficient condition for stabilizability based on the roots of the polynomials $R_n(s;\tau)$ and $R_{n-1}(s;\tau)$ on the (τ, s) plane for the plant $P(s) = (s-2)(s-1)(s+2)$ according to Theorem 1 (case $s_a \geq 0$).

The curve $R_n(s;\tau) = 0$ gives a connection between the delay τ and the possible values of the real root s_0 with multiplicity $n+1$, while the curve $R_{n-1}(s;\tau) = 0$ is needed to analyze the sufficient condition given in Proposition 2. It is clear that the condition $R_{n-1}(s_{0,k};\tau t) \leq 0$, $\forall t$, $0 < t \leq 1$ can be satisfied if and only if $k = 1$ and $0 < \tau \leq \tau_a$ (i.e. for the greatest s_0 and in a certain delay interval).

These observations are summarized in the following theorems.

Theorem 1. Let $P(s)$ be real-rooted and consider the case $s_a \geq 0$. Then

1. $\tau_0 \leq \tau_a$
2. $s_{0,1}$ is the dominant root of system (8) if $0 < \tau \leq \tau_a$
3. system (1) is stabilizable if and only if $0 < \tau < \tau_0$.

Proof. If $s_a > 0$ then there is at least 1 positive root s_1 of $P(s)$, therefore there is a finite τ_0 corresponding to the first branch of $R_n(s;\tau) = 0$. Then item 1. follows from the monotonicity of the curve $R_n(s;\tau) = 0$. If $s_a = 0$ then $\tau_0 = \tau_a$. Item 2. follows from the sufficient condition in Proposition 2. If $0 < \tau < \tau_0$ then $s_{0,1} < 0$, which gives the sufficient condition for the stabilizability of system (1) in item 3. For $\tau \geq \tau_0$, $R_n(s;\tau)$ has a root in the closed right half of the complex plane, therefore by Propostion 4, system (1) cannot be stabilized. This gives the necessary condition in item 3. □

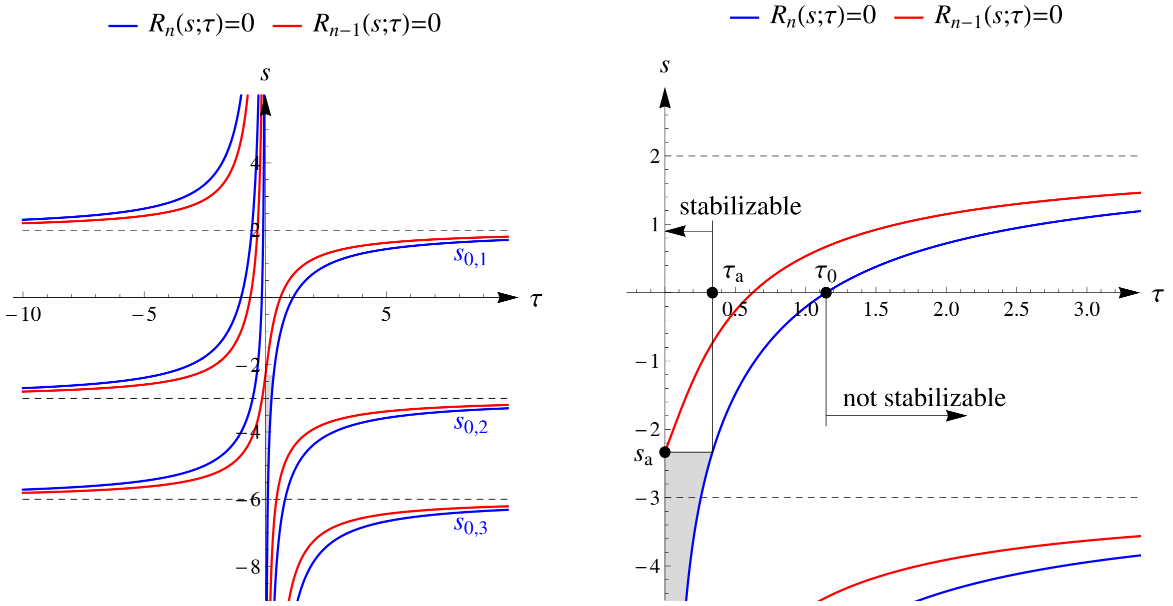

FIGURE 3 Illustration of the sufficient condition for the MID property (gray shading) and the sufficient and necessary conditions for stabilizability based on the roots of the polynomials $R_n(s;\tau)$ and $R_{n-1}(s;\tau)$ on the (τ, s) plane for the plant $P(s) = (s-2)(s+3)(s+6)$ according to Theorem 2 (case $s_a < 0$).

Figure 2 shows the branches of the algebraic curves $R_n(s;\tau) = 0$ and $R_{n-1}(s;\tau) = 0$ corresponding to the interlacing and asymptotic properties for a plant of degree $n = 3$. The roots of the open-loop system are $s_1 = 2$, $s_2 = 1$, $s_3 = -2$, thus $s_a = 1/3 > 0$ hence Theorem 1 applies. The horizontal asymptotes corresponding to the roots of $P(s)$ are indicated with dashed lines. In the gray shaded region $R_{n-1}(s_0;\tau) \leq 0$ hence Proposition 2 applies. In this example, the numerical values are $\tau_0 = 0.532$ and $\tau_a = 0.735$ and the critical delay is $\tau_{crit} = \tau_0$.

Theorem 2. Let $P(s)$ be real-rooted and consider the case $s_a < 0$. Then

1. $\tau_0 \geq \tau_a$
2. $s_{0,1}$ is the dominant root of system (8) if $0 < \tau \leq \tau_a$
3. system (1) is stabilizable if $0 < \tau \leq \tau_a$and cannot be stabilized for $\tau \geq \tau_0$.

Proof. The proof follows the same lines as the proof of Theorem 1. □

Figure 3 shows the branches of the algebraic curves $R_n(s;\tau) = 0$ and $R_{n-1}(s;\tau) = 0$ for a plant of degree $n = 3$, when the roots of the open-loop system are $s_1 = 2$, $s_2 = -3$, $s_3 = -6$. In this case $s_\mathrm{a} = -7/3 < 0$ hence Theorem 2 applies. Again, the horizontal asymptotes corresponding to the roots of $P(s)$ are indicated with dashed lines and in the gray shaded region $R_{n-1}(s_0;\tau) \leq 0$ hence Proposition 2 applies. Here, the numerical values are $\tau_\mathrm{a} = 0.337$ and $\tau_0 = 1.145$. Numerical analysis shows that we can stabilize the system by the proposed controller design (s_0 with multiplicity $n+1$) if $\tau < \hat{\tau} = 0.977$, i.e., $\tau_\mathrm{a} < \hat{\tau} < \tau_0$. Furthermore it can be shown that s_0 is the dominant root if $\tau < 0.831$.

Remark 5. Let $P(s)$ be real-rooted, and let s_0 be a real root of (1) with multiplicity at least $n+1$. Then $D^{(n+1)}(s_0) = R_{n+1}(s_0;\tau) \neq 0$ because of the interlacing property, therefore the maximal multiplicity of a real root s_0 is $n+1$.

Remark 6. It can be seen by applying Proposition 4 that the real root $s_{0,1}$ gives a lower bound on the spectral abscissa of the quasipolynomial (1), and this lower bound is independent of the control parameters b_i. Thus, if $s_{0,1}$ is dominant, then it gives the minimum of the spectral abscissa with respect to the control parameters.

This remark implies the following proposition.

Proposition 5. Let $P(s)$ be real-rooted. Then system (1) is γ-stabilizable with $\gamma \leq s_\mathrm{a}$ if and only if $\tau \in]0,\tau_\gamma[$, where τ_γ is the smallest positive root of $R_n(\gamma;\tau)$ for τ.

4 | MULTI-DEGREE-OF-FREEDOM MECHANICAL EXAMPLE: N-LINK INVERTED PENDULUM

Consider an N-link inverted pinned pendulum with rods of equal mass m and length l moving in the vertical plane. The control torque is applied at the first (lowest) rod:

$$M = -\sum_{i=1}^{N} p_i \varphi_i(t-\tau) - \sum_{i=1}^{N} d_i \dot{\varphi}_i(t-\tau)\,. \tag{36}$$

4.1 | Derivation of the equation of motion

The equation of motion can be determined using the Euler–Lagrange equations. The generalized coordinates are chosen to be the absolute pendulum angles φ_i (i.e. angular displacement of the rods from the vertically upward position). The equation of motion linearized around the unstable equilibrium has the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}\,, \tag{37}$$

where the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ can be written as

$$\mathbf{M} = \frac{1}{6}ml^2 \begin{bmatrix} 6(N-1)+2 & 6(N-2)+3 & \dots & 21 & 15 & 9 & 3 \\ 6(N-2)+3 & 6(N-2)+2 & \dots & 21 & 15 & 9 & 3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 21 & 21 & \dots & 20 & 15 & 9 & 3 \\ 15 & 15 & \dots & 15 & 14 & 9 & 3 \\ 9 & 9 & \dots & 9 & 9 & 8 & 3 \\ 3 & 3 & \dots & 3 & 3 & 3 & 2 \end{bmatrix}, \tag{38}$$

$$\mathbf{K} = -\frac{1}{2}mgl \begin{bmatrix} 2(N-1)+1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 2(N-2)+1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 7 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 5 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 3 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{39}$$

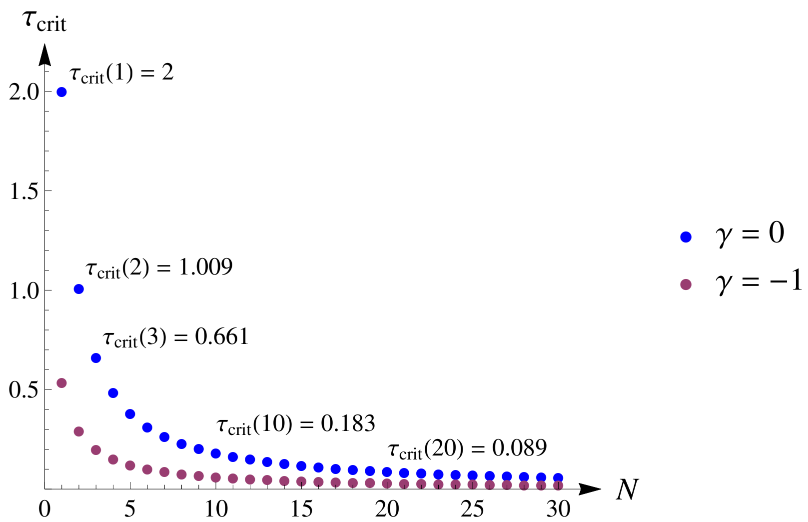

FIGURE 4 Critical delay $\tau_{\text{crit}}(N, \gamma)$ of an N-link inverted pendulum if $3g/l = 1$

that is

$$\begin{aligned} m_{ij} &= \frac{1}{6} ml^2 (6(N-j)+3), \quad i < j\,, \\ m_{ij} &= \frac{1}{6} ml^2 (6(N-i)+2), \quad i = j\,, \\ m_{ij} &= \frac{1}{6} ml^2 (6(N-i)+3), \quad i > j\,, \end{aligned} \tag{40}$$

and

$$\begin{aligned} k_{ij} &= -\frac{1}{2} mgl(2(N-i)+1), \quad i = j\,, \\ k_{ij} &= 0, \quad i \neq j\,. \end{aligned} \tag{41}$$

For more details see Appendix A. The generalized force $\mathbf{Q}$ reads

$$\mathbf{Q} = -\begin{bmatrix} p_1 & p_2 & \dots & p_N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \mathbf{q}(t-\tau) - \begin{bmatrix} d_1 & d_2 & \dots & d_N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \dot{\mathbf{q}}(t-\tau)\,. \tag{42}$$

Therefore, the characteristic function has the form (1) where the open-loop characteristic function $P(s)$ reads

$$P(s) = \det\left(s^2\mathbf{M} + \mathbf{K}\right). \tag{43}$$

4.2 | Stabilizable delay interval

The mass matrix $\mathbf{M}$ is positive definite and the stiffness matrix $\mathbf{K}$ is negative definite therefore $P(s)$ has only real roots. Furthermore, the roots occur in real pairs $\pm s_i$, $i = 1, \dots, N$, therefore the average of the roots is $s_{\text{a}} = 0$. Thus, we can apply the results of Theorem 1: system (37) is stabilizable if and only if $0 < \tau < \tau_0$, where τ_0 can be calculated if the system parameters are known. Figure 4 shows the critical delay $\tau_{\text{crit}}(N) = \tau_0$ as a function of N if $3g/l = 1$. The case $N = 1$ gives the single inverted pendulum when $\tau_{\text{crit}}(1) = 2$ (see Subsection 1.2). This figure also shows the largest delays for which we can reach a given spectral abscissa γ. That is, we use a slightly generalized notion of the critical delay: $\tau_{\text{crit}}(N, \gamma)$ denotes the critical delay corresponding to a given degree of freedom N and spectral abscissa γ (with $\tau_{\text{crit}}(N) = \tau_{\text{crit}}(N, 0)$ corresponding to stabilizability).

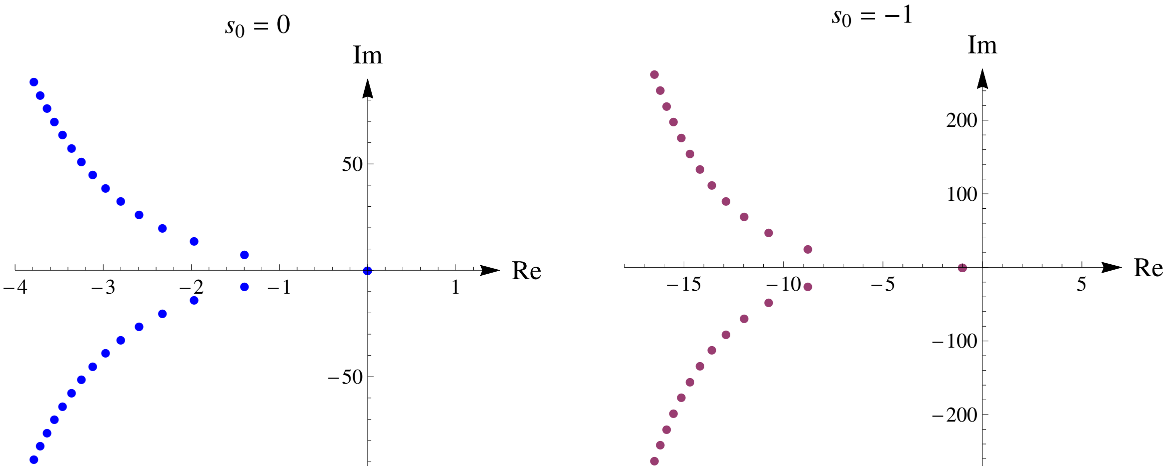

FIGURE 5 Illustration of the root location of the characteristic function if $N = 2$ and $\tau = \tau_{\text{crit}}(N, s_0)$ with quintuple roots $s_0 = 0$ and $s_0 = -1$ yielding $\gamma = 0$ and $\gamma = 1$, respectively.

5 | ON BEYOND OF REAL-ROOTED PLANTS: PERSPECTIVES FOR CONTROL DESIGN APPROACH IN A MORE GENERAL SETTING

The previous sections show the interest in exploiting the MID property in the design of stabilizing delayed-controllers for real-rooted plants. However, in recent studies concerned with reduced-order plants such as[21], it is shown that the MID property still applies for open-loop plants with pairs of complex conjugate roots.

Based on the results of this paper, as well as on other recent results on the MID property for systems with time-delays such as[21,17,24,51,22], a Python software for the parametric design of stabilizing feedback laws with time-delays, called "Partial Pole Placement via Delay Action" (P3δ for short), has been developed. P3δ also implements other features, which are detailed in[52]. The software is freely available for download on https://cutt.ly/p3delta, where installation instructions, video demonstrations, and the user guide are also available.

By this section, we provide an illustrative example generated using P3δ showing the validity of the MID property even for plants with pairs of complex conjugate roots. Let us revisit the problem of controlling the standard oscillator:

$$\ddot{\zeta}(t) + 2\,\xi\,\omega_0\,\dot{\zeta}(t) + \omega_0^2\,\zeta(t) = c(t) \tag{44}$$

where $\omega_0 > 0$ and $0 < \xi < 1$ stand respectively for the oscillator natural frequency and the damping factor. Let us define the controller c as a *proportional-derivative* delayed-controller; that is

$$c(t) = -b_0\,\zeta(t-\tau) - b_1\,\dot{\zeta}(t-\tau).$$

Thus, the closed-loop characteristic function is given by:

$$D(s,\tau) = s^2 + 2\,\xi\,\omega_0\,s + {\omega_0}^2 + (b_0 + b_1\,s)\,\mathrm{e}^{-\tau\,s}.$$

Assume that the natural frequency $\omega_0 = 1$ and the damping factor $\xi = 1/2$, which corresponds to an open-loop plant with a complex-conjugate pair $s_{OL}^{\pm} = -1/2 \pm \mathrm{i}\,\sqrt{3}/2$. Then, the closed-loop plant corresponds to the following characteristic quasipolynomial function:

$$D(s,\tau) = s^2 + s + 1 + (b_0 + b_1\,s)\,\mathrm{e}^{-\tau\,s}. \tag{45}$$

Forcing the existence of a triple spectral value suggests that $s_{CL}^{\pm} = -1/2 - 2\,\tau^{-1} \pm 1/2\,\tau^{-1}\sqrt{-3\,\tau^2 + 8}$ are the only admissible roots. As a matter of fact, those triple spectral values are defined if, and only if, the controller's gains are such that:

$$b_0 = \frac{\left(6 + \left(2 + s_{CL}^{\pm}\right)\tau^2 + \left(10\,s_{CL}^{\pm} + 6\right)\tau\right)\mathrm{e}^{s_{CL}^{\pm}\tau}}{\tau^2},\; b_1 = \frac{\mathrm{e}^{s_{CL}^{\pm}\tau}\left(2\,s_{CL}^{\pm}\tau + \tau + 2\right)}{\tau} \tag{$\star_\pm$}$$

It follows that if $(\star_+)$ is satisfied then, s_{CL}^{+} which is a triple root is also the dominant spectral value as illustrated in Figure 6.

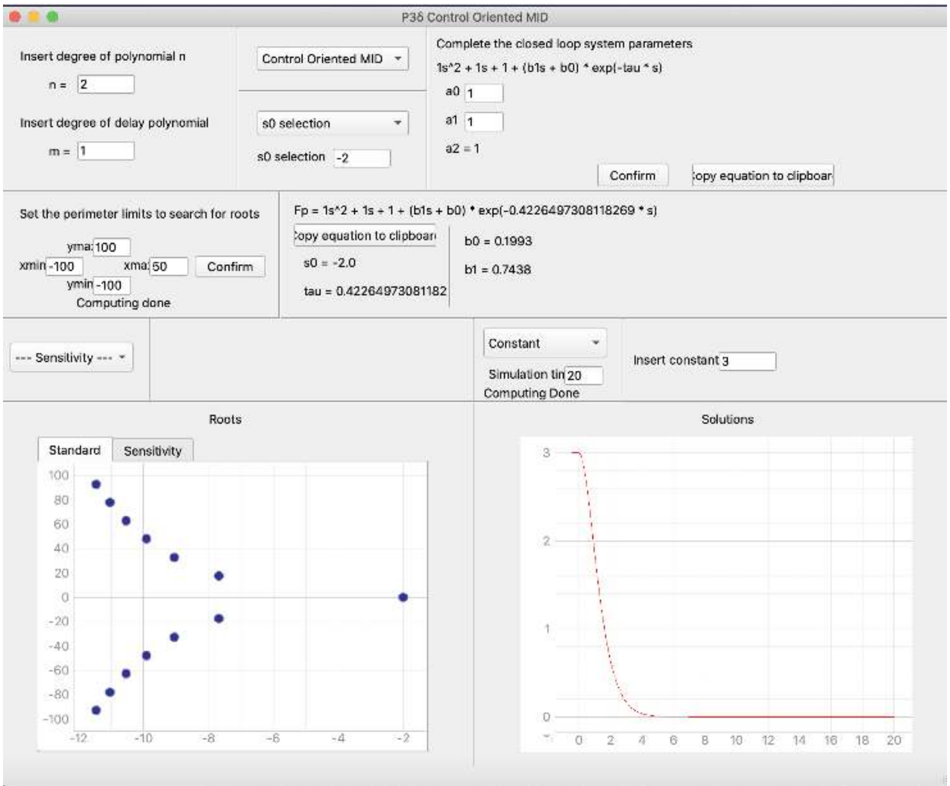

FIGURE 6 The P3δ interface exhibiting the design of a stabilizing delayed PD controller in the case of the characteristic function (45). (Left) Illustration of the root location in the case $\tau = \tau_{-2} \approx 0.422649$ which corresponds to $s^+_{CL} = -2$. (Right) The closed-loop response corresponding to the history function $\varphi(t) = 3$ for all $t \in [-\tau_{-2}, 0]$.

Notice that the assignment of the triple root $s = s^+_{CL}$ is possible only in the admissibility region $s \in (-\infty, s_M]$ which corresponds necessarily to $\tau \in (0, \tau_M]$ where $\tau_M = 2/3\sqrt{6}$ and $s_M = -1/2\,(1 + \sqrt{3})$, see for instance[21]. This fact is illustrated in Figure 7.

6 | CONCLUSION

We have provided sufficient conditions for the MID property in the case of a real root s_0 with multiplicity at least $n+1$ and with multiplicity at least n. Necessary condition for γ-stabilizability was investigated based on[47] and[48]. As a main result, sufficient and necessary condition for the MID property and γ-stabilizability was derived for systems with real-rooted open-loop characteristic function. The result was applied to an N-link inverted pendulum subjected to delayed state feedback. One advantage of the results is that only roots of polynomials should be found in order to check sufficient and necessary conditions for γ-stabilizability.

Although the main results were derived for systems with real-rooted open-loop characteristic function they can be generalized to systems with $P(s)$ having not only real roots. If $R_n(s;\tau)$ and $R_{n-1}(s;\tau)$ have single dominant real roots $s_{0,1}(\tau)$ and $\tilde{s}_{0,1}(\tau)$ for $\tau > 0$, respectively, such that $\tilde{s}_{0,1}(\tau) > s_{0,1}(\tau)$, and $s_{0,1}(\tau)$ and $\tilde{s}_{0,1}(\tau)$ are increasing as a function of τ and, furthermore, $R_{n-1}(s(\tau);\tau) \neq 0$ for $s_{0,1}(\tau) < s(\tau) < \tilde{s}_{0,1}(\tau)$ then similar statements can be made.

If the coefficients a_i of an arbitrary plant are known, root location of polynomials $R_n(s;\tau)$ and $R_{n-1}(s;\tau)$ can be accessed easily using numerical techniques, hence the results in Section 2 can be applied. However, in general, it is difficult to parametrically characterize the root location of these polynomials. Although, it could be done by exploiting the structure of the open-loop characteristic polynomial, as we saw in the case of real-rooted plants in Section 3.

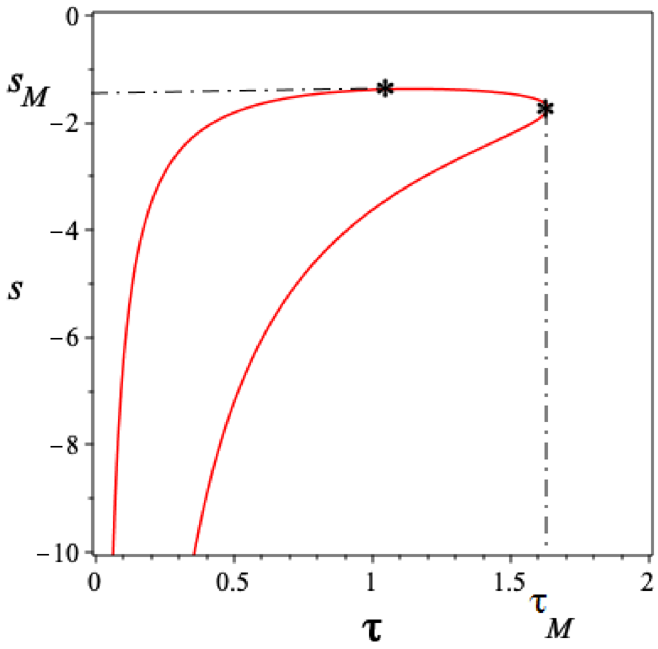

FIGURE 7 The MID stabilizability region is defined by $(-\infty,\ s_M]$, in which the assignment of a triple negative dominant root of (45) is possible. The value τ_M corresponds to an upper-bound for the corresponding delay.

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APPENDIX

A N-LINK INVERTED PENDULUM: EQUATION OF MOTION

The position and the velocity of the center of mass of the k^{th} rod ($k = 1, ..., N$) can be written as

$$\mathbf{r}_{C_k} = \begin{bmatrix} l\sum_{i=1}^{k}\sin\varphi_i - \frac{l}{2}\sin\varphi_k \\ l\sum_{i=1}^{k}\cos\varphi_i - \frac{l}{2}\cos\varphi_k \\ 0 \end{bmatrix}, \tag{A1}$$

$$\mathbf{v}_{C_k} = l\begin{bmatrix} \sum_{i=1}^{k}\cos\varphi_i\dot{\varphi}_i - \frac{1}{2}\cos\varphi_k\dot{\varphi}_k \\ -\sum_{i=1}^{k}\sin\varphi_i\dot{\varphi}_i + \frac{1}{2}\sin\varphi_k\dot{\varphi}_k \\ 0 \end{bmatrix}. \tag{A2}$$

Using (A1) and (A2) the kinetic energy is

$$T = \sum_{k=1}^{n}\left(\frac{1}{2}m\mathbf{v}_{C_k}^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\varphi}_k^2\right) = \frac{1}{2}ml^2\sum_{k=1}^{n}\left(\frac{\dot{\varphi}_k^2}{3} + \sum_{j=1}^{k-1}\sum_{i=1}^{k}\cos(\varphi_i - \varphi_j)\dot{\varphi}_i\dot{\varphi}_j\right), \tag{A3}$$

and the potential energy is

$$U = \sum_{k=1}^{n} mgr_{C_k,y} = mgl\left(\sum_{k=1}^{n}\sum_{i=1}^{k}\cos\varphi_i - \frac{1}{2}\sum_{k=1}^{n}\cos\varphi_k\right). \tag{A4}$$

From (A3) and (A4) we obtain the matrix elements (40) and (41) by:

$$\begin{aligned} m_{\alpha\beta} &= \frac{\partial^2 T}{\partial\dot{\varphi}_\alpha\dot{\varphi}_\beta}(\varphi_i = 0, \dot{\varphi}_i = 0), \\ k_{\alpha\beta} &= \frac{\partial^2 U}{\partial\varphi_\alpha\varphi_\beta}(\varphi_i = 0, \dot{\varphi}_i = 0). \end{aligned} \tag{A5}$$