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CONDITIONS FOR THE POSITIVITY OF DETERMINANTS

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Abstract

Consider a matrix with positive diagonal entries, which is similar via a positive diagonal matrix to a symmetric matrix, and whose signed directed graph has the property that if a cycle and its symmetrically placed complement have the same sign, then they are both positive. We provide sufficient conditions so that A be a P–matrix, that is, a matrix whose principal minors are all positive. We further provide sufficient conditions for an arbitrary matrix A whose (undirected) graph is subordinate to a tree, to be a P–matrix. If, in addition, A is sign symmetric and its undirected graph is a tree, we obtain necessary and sufficient conditions that it be a P–matrix. We go on to consider the positive semi–definiteness of symmetric matrices whose graphs are subordinate to a given tree and discuss the convexity of the set of all such matrices.

1 INTRODUCTION

In a paper on the question of unicity of best spline approximations for functions having a positive second derivative, the following result is proved:

Proposition 1.1 ([1, Proposition 1]) Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be a tridiagonal matrix with positive diagonal entries. If

$$a_{i,i-1}a_{i-1,i} \;\leq\; \frac{1}{4}a_{ii}a_{i-1,i-1}\left(1+\frac{\pi^2}{1+4n^2}\right), \quad i = 2,\ldots,n, \qquad (1.1)$$

then $\det(A) > 0$.

The proof given in [1] (see pp.1136–1138) rests on arguments involving second order finite differences and Green's functions. This paper grew out of an attempt to find a matrix–theoretic proof for Proposition 1.1. The question of the existence of such a proof was raised by a colleague of one of the present authors, Professor Joseph P. McKenna.

We will extend Proposition 1.1 to matrices whose undirected graphs are subordinate to a tree and whose diagonal entries are all positive. We will also consider the class of matrices which are similar via a positive diagonal matrix to a symmetric matrix, and whose signed directed graphs have the property that if a cycle and its symmetrically placed complement have the same sign, then they are both positive. For matrices in this class we will provide sufficient conditions (inequalities) on the off–diagonal entries for the positivity of the determinant and all other principal minors as well.

It is interesting to note that the conditions on A in Proposition 1.1 are inherited by all the principal submatrices of A showing that, in fact, all principal minors of A are positive and hence A is, by definition, a P–matrix. A similar situation occurs in our extensions of Proposition 1.1, permitting us to conclude that the matrix of interest is a P–matrix (see Theorems 2.1 and 2.2). Then Proposition 1.1 and an improvement of it follow from our results.

It is also worth noting that the verification of each of the $n-1$ inequalities for the contiguous 2×2 principal submatrices of A in (1.1) can be carried out independently of each other. Thus it is possible to check the validity of condition (1.1) in parallel. This will remain true for all our results.

In Section 3 we present conditions under which a symmetric matrix whose graph is subordinate to a tree is positive semi–definite. We prove a particular geometric feature of the set of all positive semi–definite matrices whose graphs are subordinate to a given tree, namely, that no proper convex combination of distinguished boundary points (see Section 3 for definitions) is a boundary point.

We continue with notation, definitions, and useful preliminary results. We denote the set of all $n \times n$ real matrices by $M_n(\mathbb{R})$. For a matrix $A \in M_n(\mathbb{R})$ we let $\sigma(A)$ and $\rho(A)$ denote the spectrum and the spectral radius, respectively. We write $A \geq 0$ when A is entrywise nonnegative and consider the induced partial order in $M_n(\mathbb{R})$. The entrywise absolute value of A is denoted by $|A|$. Also $\mathrm{diag}(A)$ denotes the diagonal matrix whose diagonal entries coincide with the diagonal entries of A.

Given $A = (a_{ij}) \in M_n(\mathbb{R})$, the (undirected) **graph** of A, $G(A)$, has vertices $1, 2, \ldots, n$ and edges $\{i, j\}$ if and only if $i \neq j$, and $a_{ij} \neq 0$ or $a_{ji} \neq 0$. Notice that $G(A)$ is by definition loopless. We say A is **subordinate** to a graph G on n vertices and we write that $G(A) \preceq G$ if $G(A)$ is a subgraph of G (with the same vertex set.)

By $T \equiv T_n$ we denote a **tree** on the n vertices $1, 2, \ldots, n$, with edges $\{i, j\}$, namely, a connected, acyclic graph.

We say that $A = (a_{ij}) \in M_n(\mathbb{R})$ is **combinatorially symmetric** if $a_{ij} \neq 0$ implies $a_{ji} \neq 0$. The matrix A has the **positive complementary cycle** property if whenever $i_1, i_2, \ldots, i_k$ are distinct indices and

$$a_{i_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_{k-1} i_k} \cdot a_{i_k i_1} \ \cdot \ a_{i_2 i_1} \cdot a_{i_3 i_2} \cdot \ldots \cdot a_{i_k i_{k-1}} \cdot a_{i_1 i_k} > 0,$$

then

$$a_{i_1 i_2} \cdot a_{i_2 i_3} \cdot \ldots \cdot a_{i_{k-1} i_k} \cdot a_{i_k i_1} > 0 \ \text{ and } \ a_{i_2 i_1} \cdot a_{i_3 i_2} \cdot \ldots \cdot a_{i_k i_{k-1}} \cdot a_{i_1 i_k} > 0.$$

Notice that if A has the positive complementary cycle property, then A is **sign symmetric**, namely, if $a_{ij} a_{ji} \neq 0$, then $a_{ij} a_{ji} > 0$.

The matrix $A \in M_n(\mathbb{R})$ is called **diagonally symmetrizable** if there exists a diagonal matrix D with positive diagonal entries, such that $D^{-1} A D$ is symmetric.

The following results are classic but we state them here for the sake of reference in forthcoming proofs. Recall that a **signature** matrix is a real diagonal matrix S such that $|S| = I$.

First, the results in [6] readily imply that:

Lemma 1.2 Let T be a tree on n vertices and A be a combinatorially symmetric matrix with $G(A) \preceq T$. Then $|A|$ is diagonally symmetrizable.

Details regarding the following lemma can be found in [2] and in [3].

Lemma 1.3 Let $A \in M_n(\mathbb{R})$ be an irreducible matrix. Then A is signature similar to a nonnegative matrix if and only if all the cycles in the directed graph of A are nonnegative.

2 MAIN RESULTS

Our first objective is to show that any real matrix with positive diagonal entries, whose irreducible components have the positive complementary cycle property and are diagonally symmetrizable, has positive principal minors whenever a certain derived matrix E satisfies $\rho(E) < 1$. This leads to sufficient conditions of the same form as in Proposition 1.1.

Theorem 2.1 Suppose that $A = (a_{ij}) \in M_n(\mathbb{R})$ has positive diagonal entries and that the irreducible components of A have the positive complementary cycle property and are diagonally symmetrizable. If there is a nonnegative symmetric matrix $E = (e_{ij})$ with $\rho(E) < 1$ such that for every edge $\{i, j\}$ of $G(A)$,

$$a_{ij}a_{ji} \leq a_{ii}a_{jj}e_{ij}^2, \tag{2.1}$$

then A is a P–matrix.

Proof:
We can assume without loss of generality that A is irreducible, as the arguments we shall make hold for each irreducible component of A. By replacing

A by $(\mathrm{diag}(A))^{-1/2}A(\mathrm{diag}(A))^{-1/2}$, without violating (2.1), we can further assume that $a_{ii} = 1,\ i = 1, 2, \ldots, n$. Let D be a diagonal matrix with positive diagonal entries such that

$$\hat{A} \ = \ D^{-1}AD - I \ = \ (\hat{a}_{ij})$$

is symmetric. Notice that $\hat{A}$ still has the positive complementary cycle property and hence all cycles in the directed graph of $\hat{A}$ are nonnegative. So by Lemma 1.3, $\hat{A}$ is signature similar to $|\hat{A}|$. But $|\hat{A}|$ satisfies

$$|\hat{a}_{ij}|^2 \ = \ \hat{a}_{ij}\hat{a}_{ji} \ = \ a_{ij}a_{ji} \ \leq \ e_{ij}^2 \quad (i \neq j)$$

because of the symmetry of $\hat{A}$ and because of (2.1). Thus $|\hat{A}| \leq E$. By the Perron–Frobenius theory for nonnegative matrices, the latter inequality implies that $\rho(\hat{A}) = \rho(|\hat{A}|) < 1$. But then $\hat{A} + I$ is positive definite, which implies that $A = D\left(\hat{A} + I\right)D^{-1}$ is a P–matrix, as it is a matrix each of whose principal submatrix is similar to a positive definite matrix. □

The following theorem extends Proposition 1.1 to matrices whose undirected graphs are subordinate to a tree.

Theorem 2.2 Let T be a tree on n vertices and let $A = (a_{ij}) \in M_n(\mathbb{R})$, with $G(A) \preceq T$, have positive diagonal entries. If there exists a nonnegative symmetric matrix $E = (e_{ij})$ with $\rho(E) < 1$ such that for every edge $\{i, j\}$ of $G(A)$,

$$a_{ij}a_{ji} \leq a_{ii}a_{jj}e_{ij}^2, \tag{2.2}$$

then A is a P–matrix.

Proof:
We can begin by assuming that A is irreducible, for otherwise it is sufficient to show that the irreducible components of A are P–matrices. Furthermore, as in the previous theorem, we can without loss of generality assume that the diagonal entries of A are all equal to 1.

Consider a diagonal matrix with positive diagonal entries D, whose existence is assured by Lemma 1.2, such that $D^{-1}|A|D$ is symmetric, and let

$$\hat{A} \ = \ D^{-1}AD - I, \quad \tilde{A} = \frac{1}{2}(\hat{A} + \hat{A}^T) \tag{2.3}$$

By construction, $G(\tilde{A}) \preceq T$ and hence, by Lemma 1.3, we have that $\tilde{A}$ is signature similar to $|\tilde{A}|$. But $|\tilde{A}|$ satisfies

$$|\tilde{a}_{ij}\tilde{a}_{ji}| \leq e_{ij}^2$$

because of (2.2) and (2.3), so that $|\tilde{A}| \leq E$. By the Perron–Frobenius theory for nonnegative matrices, the latter inequality implies that $\rho(\tilde{A}) = \rho(|\tilde{A}|) < 1$. But then $\tilde{A} + I$ is positive definite, which implies that $\hat{A} + I$, and thus A, is a P–matrix. □

Remark 2.3 We note that while every real H–matrix (see Chapter 2 in [4]) with positive diagonal entries is a P–matrix, not every matrix that satisfies the assumptions of the above theorem is an H–matrix as the following example illustrates:

$$\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}.$$

The following result includes a converse to the above theorem in the case A is sign symmetric.

Theorem 2.4 Let T be a tree on n vertices and let $A = (a_{ij}) \in M_n(\mathbb{R})$ be a sign symmetric matrix with positive diagonal entries and such that $G(A) \preceq T$. Define $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} \left(\frac{a_{ij}a_{ji}}{a_{ii}a_{jj}}\right)^{1/2}, & \text{if } \{i,j\} \text{ is an edge of } T \\ 0, & \text{otherwise.} \end{cases}$$

Then A is a P–matrix if and only if $\rho(E) < 1$.

Proof:
Assume, without loss of generality, that $G(A) = T$, otherwise the problem reduces to one of a lesser dimension. The sufficiency of $\rho(E) < 1$ is contained in the previous theorem. Let $D = \operatorname{diag}(A)$. For the necessity part, notice that, since A is a P–matrix, so is $D^{-1/2}AD^{-1/2}$. But, by Lemma 1.2, $I - E$

is diagonally similar to $D^{-1/2}AD^{-1/2}$ and is, thus, positive definite, which implies that $\rho(E) < 1$. □

We can now use our results to give a matrix–theoretic proof as well as state an improvement of Proposition 1.1.

Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be tridiagonal with positive diagonal entries. The graph of A is subordinate to a tree (the line graph). Suppose that (1.1) holds and consider the tridiagonal matrix

$$E = (e_{ij}) = \begin{pmatrix} 0 & x & 0 & \dots & 0 \\ x & 0 & x & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & x & 0 & x \\ 0 & \dots & 0 & x & 0 \end{pmatrix}$$

in which $x = \frac{1}{2}\sqrt{1 + \frac{\pi^2}{1+4n^2}}$. According to a well known formula for the eigenvalues of a symmetric tridiagonal matrix with constant diagonals (see e.g., [5]), the largest (in modulus) eigenvalue of E is $2x \cos\frac{\pi}{n+1}$ (since if λ is an eigenvalue of E then so is $-\lambda$). Starting with the fundamental inequality $\tan\theta > \theta$, for all $\theta \in (0, \frac{\pi}{2})$, we have that:

$$\tan^2 \frac{\pi}{n+1} > \frac{\pi^2}{(n+1)^2} > \frac{\pi^2}{4n^2+1} \implies$$

$$\sin^2 \frac{\pi}{n+1} > \frac{\pi^2}{4n^2+1} \cos^2 \frac{\pi}{n+1} \implies$$

$$(1 + \frac{\pi^2}{4n^2+1}) \cos^2 \frac{\pi}{n+1} < 1 \implies$$

$$2x \cos \frac{\pi}{n+1} < 1.$$

Thus $\rho(E) < 1$. Now, by Theorem 2.2, $\det A > 0$, proving Proposition 1.1.

It is also clear from our approach that the conclusion of Proposition 1.1 can now be strengthened as follows:

Proposition 2.5 Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be a tridiagonal matrix with positive diagonal entries. If

$$a_{i,i-1}a_{i-1,i} \ < \ \frac{1}{4}a_{ii}a_{i-1,i-1}\frac{1}{(\cos\frac{\pi}{n+1})^2}, \quad i = 2, \ldots, n,$$

then $\det(A) > 0$.

3 POSITIVE DEFINITE TREES

We continue with some results on symmetric matrices and on positive semi–definite matrices whose graph is subordinate to a tree.

We denote the symmetric matrices with zero diagonal entries subordinate to T by

$$\mathcal{H}_T = \{X \in M_n(\mathbb{R}) \mid \mathrm{X}^{\mathrm{T}} = \mathrm{X},\ \mathrm{diag}(\mathrm{X}) = 0,\ \mathrm{G}(\mathrm{X}) \preceq \mathrm{T}\},$$

and also consider the set

$$\mathcal{X}_T = \{X \in \mathcal{H}_T \mid I + X \text{ is positive semi–definite}\}.$$

The **boundary** of $\mathcal{X}_T$ is defined by

$$\partial\mathcal{X}_T = \{X \in \mathcal{X}_T \mid I + X \text{ is singular}\}.$$

We call two nonzero matrices $X_1, X_2 \in \mathcal{X}_T$ **distinguished** if the irreducible components of X_1 and X_2 corresponding to the same index set (if any such components exist) are linearly independent. For example, if X_1 is irreducible, then X_1, X_2 are distinguished provided that X_2 is not a scalar multiple of X_1.

We will show that $\mathcal{X}_T$ is a convex set of matrices whose spectral radii are less than or equal to one. Moreover, no proper convex combination of two distinguished boundary points of $\mathcal{X}_T$ is again a boundary point.

Lemma 3.1 Let T be a tree on n vertices and $X \in \mathcal{H}_T$. Then:

(i) $\sigma(X) = \sigma(|X|)$.

(ii) If $\lambda \in \sigma(X)$ then $-\lambda \in \sigma(X)$.

(iii) $X \in \mathcal{X}_T$ if and only if $\rho(X) \leq 1$. Moreover,

$$\partial \mathcal{X}_T = \{X \in \mathcal{H}_T \mid I + X \text{ is positive semi–definite with } \rho(X) = 1\} \tag{3.1}$$

(iv) If $X \in \mathcal{X}_T,\ Y = Y^T$ and if $|Y| \leq |X|$, then $Y \in \mathcal{X}_T$.

Proof:

(i) By Lemma 1.3, X and $|X|$ are similar by a diagonal matrix.

(ii) This follows from (i), or by observing that the directed graph of X contains no loops and no cycles of length greater than 2. Thus, if n is odd (even) the characteristic polynomial of X is an odd (even) function.

(iii) Since by (ii) $\sigma(X) = \sigma(-X)$, and since $I + X$ is positive semi–definite, X has no eigenvalues < -1 and no eigenvalues > 1. That is, $\rho(X) \leq 1$. In particular, if $\rho(X) = 1$, then $-1 \in \sigma(X)$ and so $I + X$ is singular and conversely, proving (3.1).

(iv) If $|Y| \leq |X|$ then $\mathrm{diag}(Y) = 0$ and $G(Y) \preceq T$, thus $Y \in \mathcal{H}_T$. Also by the Perron–Frobenius theory of nonnegative matrices and by parts (i) and (iii),

$$\rho(Y) = \rho(|Y|) \leq \rho(|X|) = \rho(X) \leq 1,$$

showing that $Y \in \mathcal{X}_T$. □

We also need the following lemma. Recall that, in a tree, a vertex of degree 1 (i.e., incident with exactly one edge) is called a pendant vertex.

Lemma 3.2 Let T be a tree on n vertices and $A \in M_n(\mathbb{R})$ be a symmetric matrix with zero diagonal entries such that $G(A) \preceq T$. If all the entries of $x \in \mathbb{R}^n$ are nonzero, then the entries of x and Ax completely determine A.

Proof:
Without of loss of generality, assume that $G(A) = T$, otherwise the problem reduces to one of a lesser dimension. We proceed with induction on n. For $n = 1$ the result is trivially true. Assume the result is true for all positive

integers less than n. Without loss of generality let $1, 2, \ldots, k,\ 1 \le k \le n-1$, be the pendant vertices of T. Then A has the block form

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

in which $A_{22} \in M_{n-k}(\mathbb{R})$ and in which each row of A_{12} has exactly one nonzero entry. Let $x \in \mathbb{R}^n$ with all its entries $x_i \neq 0$. Then for every $i \le k$, there exists a unique $j > k$ such that

$$a_{ij} x_i = (Ax)_i. \tag{3.2}$$

Thus, by (3.2), A_{12} is completely determined from x and Ax. By symmetry, A_{21} is also then determined. If $x = [u, v]^T$ and $Ax = [w, z]^T$ are partitioned conformally with A, then

$$A_{22} v = z - A_{21} u.$$

By the inductive hypothesis, since A_{22} has a graph subordinate to the tree induced from T by $\{k+1, k+2, \ldots, n\}$, the entries of A_{22} are completely determined by v and $z - A_{21}u$, and hence, in turn, all the entries of A are completely determined from x and Ax. □

Remark 3.3 It follows from the above result and the Perron–Frobenius theory that if A is an irreducible nonnegative matrix with zero diagonal entries and such that $G(A) = T$, then $\rho(A)$ and the corresponding Perron eigenvector x completely determine A.

Theorem 3.4 $\mathcal{X}_T$ is a convex set. Moreover, no proper convex combination of two distinguished boundary points of $\mathcal{X}_T$ is a boundary point of $\mathcal{X}_T$.

Proof:
Let $X_i \in \mathcal{X}_T,\ i = 1, 2,\ a \in [0, 1]$, and $Y := aX_1 + (1-a)X_2$. Clearly, Y is symmetric and its graph is subordinate to T. Also

$$I + Y = a(I + X_1) + (1-a)(I + X_2)$$

is positive semi–definite, showing that $\mathcal{X}_T$ is convex. To show the second part of the theorem, suppose $X_1, X_2 \in \partial\mathcal{X}_T$ are distinguished. By Lemma

3.1 part (iii) we have that $\rho(X_1) = \rho(X_2) = 1$. We need to show that if $a \in (0,1)$, then

$$Y \notin \partial\mathcal{X}_T$$

or equivalently, again by Lemma 3.1 part (iii), that $\rho(Y) < 1$. Suppose that Y is irreducible, otherwise, since Y is symmetric and because X_1, X_2 are distinguished, the problem reduces to one of a lesser dimension. By way of contradiction, assume $\rho(Y) = 1$. From Lemma 3.1 part (i), $\rho(Y) = \rho(|Y|) \in \sigma(|Y|)$. Let x be the eigenvector of $|Y|$ corresponding to $\rho(|Y|) = 1$ and recall that the Perron–Frobenius theorem says that all entries of x are positive. By symmetry and Rayleigh's principle applied to $|Y|$ and $|X_1|, |X_2|$, we then have that

$$1 = \rho(Y) = \frac{x^T|aX_1 + (1-a)X_2|x}{x^Tx} \leq \frac{ax^T|X_1|x + (1-a)x^T|X_2|x}{x^Tx}$$
$$\leq a\rho(|X_1|) + (1-a)\rho(|X_2|) = 1. \tag{3.3}$$

Hence all inequalities in (3.3) hold as equalities. This means that X_1, X_2 have the same sign pattern, and that $|X_1|$ and $|X_2|$ share the eigenvector x corresponding to the eigenvalue 1. Thus, by Remark 3.3, $X_1 = X_2$, a contradiction. This proves that $\rho(Y) < 1$. □

In general, it is not true that $\mathcal{X}_T$ is a strictly convex set. The following example shows that in the case of the line graph, a proper convex combination of two linearly independent boundary points can be a boundary point.

Example 3.5 Let T be the line graph on four vertices and consider the matrices in $\partial\mathcal{X}_T$ given by

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}.$$

It can be easily checked that $\frac{1}{2}(A_1 + A_2) \in \partial\mathcal{X}_T$, showing that $\mathcal{X}_T$ is not strictly convex. Notice that A_1 and A_2 are not distinguished because they have equal irreducible components corresponding to the index set $\{1, 2\}$.

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