TAIWANESE JOURNAL OF MATHEMATICS
Vol. 21, No. 2, pp. 305–318, April 2017
DOI: 10.11650/tjm/7814
This paper is available online at `http://journal.tms.org.tw`

Conditions of Parallelism of *-Ricci Tensor of Three Dimensional Real Hypersurfaces in Non-flat Complex Space Forms

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Abstract. This paper focuses on the study of three dimensional real hypersurfaces in non-flat complex space forms whose *-Ricci tensor satisfies conditions of parallelism. More precisely, results concerning real hypersurfaces with vanishing, semi-parallel and pseudo-parallel *-Ricci tensor in complex hyperbolic space are provided. Furthermore, new results concerning ξ-parallelism of *-Ricci tensor of real hypersurfaces in non-flat complex space forms are presented.

1. Introduction

A *complex space form* is an n-dimensional Kähler manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space form is complex analytically isometric to complex projective space $\mathbb{C}P^n$ if $c > 0$ or to complex Euclidean space $\mathbb{C}^n$ if $c = 0$ or to complex hyperbolic space $\mathbb{C}H^n$ if $c < 0$. The complex projective and hyperbolic spaces are called *non-flat complex space forms* and the symbol $M_n(c)$, $c \neq 0$, is used to denote them if it is not necessary to distinguish them. The complex projective space $\mathbb{C}P^n$ is of constant holomorphic sectional curvature $c = 4$ and the complex hyperbolic space $\mathbb{C}H^n$ is of constant holomorphic sectional curvature $c = -4$. In this paper we are focused on the study of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$, so $c \neq 0$. The case of $c = 0$ is not investigated in this study, but it would be of high interest, since the formulas will change and different methods are used.

A real hypersurface M is an immersed submanifold with real co-dimension one in $M_n(c)$. The Kähler structure (J, G), where J is the complex structure and G is the Kähler metric of $M_n(c)$, induces on M an almost contact metric structure (φ, ξ, η, g), which consists of the tensor field of type $(1, 1)$ φ called *structure tensor*, the 1-form η, the vector field ξ called *structure vector field* and the induced Riemannian metric g (for more details on the definitions of the latter see Section 2). A real hypersurface is called *Hopf*

Received May 10, 2016; Accepted October 19, 2016.
Communicated by Sai-Kee Yeung.
2010 *Mathematics Subject Classification.* Primary: 53C40; Secondary: 53C15, 53D15.
Key words and phrases. Real hypersurfaces, *-Ricci tensor, ξ-parallel, Semi-parallel, Pseudo-parallel, Non-flat complex space forms.
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hypersurface when the structure vector field ξ is an eigenvector of the shape operator A at every point of the real hypersurface M with corresponding pointwise eigenvalue $\alpha = g(A\xi, \xi)$.

The study of real hypersurfaces M in $M_n(c)$ was initiated by Takagi, who classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ and divided them into six types, namely (A_1), (A_2), (B), (C), (D) and (E) in [17]. These real hypersurfaces are Hopf ones with constant principal curvatures. In the case of $\mathbb{C}H^n$ the study of real hypersurfaces with constant principal curvatures was initiated by Montiel in [11] and completed by Berndt in [1]. They are divided into two types, namely (A) and (B), depending on the number of constant principal curvatures and they are homogeneous and Hopf hypersurfaces.

Real hypersurfaces in non-flat complex space forms in terms of certain geometric conditions have been studied by many geometers. An important condition is that of the shape operator A commuting with the structure tensor field φ. The following Theorem has been proved and is due to Okumura [14] for the case of $\mathbb{C}P^n$ and to Montiel and Romero [12] for the case of $\mathbb{C}H^n$.

Theorem 1.1. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\varphi = \varphi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely, in case of $\mathbb{C}P^n$,*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$,*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$, ($1 \leq k \leq n-2$), where $0 < r < \pi/2$.*

In case of $\mathbb{C}H^n$,

(A_0) *a horosphere in $\mathbb{C}H^n$, i.e., a Montiel tube,*

(A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,*

(A_2) *a tube over a totally geodesic $\mathbb{C}H^k$, ($1 \leq k \leq n-2$).*

Generally, the Ricci tensor S of a Riemannian manifold is given by the relation

$$S(X, Y) = \mathrm{trace}\,\{Z \to R(Z, X)Y\}\,,$$

where X, Y are tangent vectors on M. The definition is the same for real hypersurfaces in non-flat complex space forms. Real hypersurfaces in $M_n(c)$, $n \geq 2$, in terms of their Ricci tensor satisfying geometric conditions such as parallelism and commutativity with other tensor fields of real hypersurfaces, have been studied. A review of known results concerning the Ricci tensor of the real hypersurfaces can be viewed in [13].

In [3] Hamada, motivated by Tachibana's work in [16], where the **-Ricci tensor* of almost Hermitian manifolds was defined, introduced the latter notion in case of real hypersurfaces in non-flat complex space forms. Therefore, the *-Ricci tensor S^* of real hypersurfaces in non-flat complex space forms is given by

$$S^*(X, Y) = \frac{1}{2}\,\mathrm{trace}(Z \to R(X, \varphi Y)\varphi Z)$$

for any vector fields X, Y tangent to M (φ is defined in Section 2).

Due to the work that has been done in case of studying real hypersurfaces in terms of their Ricci tensor, the authors have started studying real hypersurfaces in non-flat complex space forms in terms of their *-Ricci tensor. More precisely, in [8] real hypersurfaces in $M_2(c)$, $c \neq 0$ with parallel *-Ricci tensor, i.e., $(\nabla_X S^*)Y = 0$, for any tangent vectors X, Y to M were classified. In [7] real hypersurfaces in complex projective space $\mathbb{C}P^2$ whose *-Ricci tensor is (1) semi-parallel, i.e., $(R(X, Y) \cdot S^*)Z = 0$, and (2) pseudo-parallel i.e., $(R(X, Y) \cdot S^*)Z = L\{[(X \wedge Y) \cdot S^*]Z\}$, where L is a nowhere vanishing function, have been studied.

It has been proved for the case of real hypersurfaces with semi-parallel *-Ricci tensor it has been proved

Theorem 1.2. *There do not exist real hypersurfaces M in $\mathbb{C}P^2$, whose *-Ricci tensor is semi-parallel.*

In the case of real hypersurfaces with pseudo-parallel *-Ricci tensor it has been proved

Theorem 1.3. *Every real hypersurface M in $\mathbb{C}P^2$, whose *-Ricci tensor is pseudo-parallel is a Hopf hypersurface. More precisely, M is locally congruent*

- *either to a geodesic hypersphere of radius r, where $0 < r < \pi/2$, and $L = \cot^2(r)$,*
- *or to a non-homogeneous real hypersurface, which is considered as a tube of radius $\pi/4$ over a holomorphic curve and $L = 1$.*

In this paper first we complete the work in [7] by studying real hypersurfaces in complex hyperbolic space $\mathbb{C}H^2$ with (1) semi-parallel *-Ricci tensor, and (2) pseudo-parallel *-Ricci tensor. More specifically, we provide a detailed proof of Propositions 2 and 3 in [7] and we study Hopf hypersurfaces in $\mathbb{C}H^2$ satisfying the above geometric conditions. The combination of the new results with Theorems 1.2 and 1.3 implies the following results.

Theorem 1.4. *The only real hypersurface in any $M_2(c)$, $c \neq 0$, with semi-parallel *-Ricci tensor is the geodesic hypersphere of radius r satisfying $\coth(r) = 2$ in $\mathbb{C}H^2$.*

Theorem 1.5. *Every real hypersurface in $M_2(c)$, $c \neq 0$, with pseudo-parallel *-Ricci tensor is a Hopf hypersurface. Furthermore, M is locally congruent to either a real hypersurface of type (A) or to a Hopf hypersurface satisfying relation $A\xi = 0$, with L constant.*

Furthermore, in this paper it is first examined if there are three-dimensional real hypersurfaces in $M_2(c)$, $c \neq 0$, whose *-Ricci tensor is *ξ-parallel*, i.e.,

$$(\nabla_\xi S^*)X = 0 \quad \text{for any tangent vector } X \text{ on } M. \tag{1.1}$$

The following theorem is proved.

Theorem 1.6. *Every real hypersurface in $M_2(c)$, $c \neq 0$, with ξ-parallel *-Ricci tensor is a Hopf hypersurface. Moreover, M is locally congruent to* (i) *a real hypersurface of type (A) or* (ii) *to a real hypersurface of type (B) or* (iii) *to a Hopf hypersurface whose principal curvatures corresponding to the holomorphic distribution are non-constant and the derivative of them in the direction of ξ is equal to zero.*

This paper is organized as follows: In Section 2 basic relations and results about real hypersurfaces in $M_2(c)$ are given. In Section 3 analytic proofs of Theorems 1.4 and 1.5 are presented. Finally, in Section 4 proof of Theorem 1.6 is provided.

2. Preliminaries

Throughout this paper all manifolds, vector fields, etc. are assumed to be of class C^∞ and all manifolds are assumed to be connected.

Let M be a real hypersurface without boundary immersed in a non-flat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c. Let N be a locally defined unit normal vector field on M and $\xi = -JN$ be the structure vector field of M. The *shape operator* A of the real hypersurface M in $M_n(c)$ with respect to N is defined by

$$\overline{\nabla}_X N = -AX.$$

For any vector field X tangent to M relation

$$JX = \varphi X + \eta(X)N$$

holds, where φX and $\eta(X)N$ are respectively the tangential and the normal component of JX. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M satisfy the relation

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,$$

where g is the Riemannian metric induced from the metric G and for any vector fields X, Y on M.

The real hypersurface is equipped with an almost contact metric structure (φ, ξ, η, g), that is induced by J of $M_n(c)$, where φ is a tensor field of type $(1, 1)$ and η is a 1-form.

The following relations hold:

$$g(\varphi X, Y) = G(JX, Y), \quad \eta(X) = g(X, \xi) = G(JX, N),$$
$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y).$$

Moreover, J being parallel implies $\overline{\nabla} J = 0$ and this leads to

$$\nabla_X \xi = \varphi AX \quad \text{and} \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

The ambient space $M_n(c)$ is of constant holomorphic sectional curvature c and this results in Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\big[&g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ &- 2g(\varphi X, Y)\varphi Z\big] + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.1}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\left[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\right], \tag{2.2}$$

where R denotes the Riemannian curvature tensor on M and X, Y, Z are any vector fields on M.

The tangent space $T_P M$ at every point $P \in M$ is decomposed as

$$T_P M = \operatorname{span}\{\xi\} \oplus \mathbb{D},$$

where $\mathbb{D} = \ker \eta = \{X \in T_P M : \eta(X) = 0\}$ and is called *(maximal) holomorphic distribution* (if $n \geq 3$). Due to the above decomposition the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U,$$

where $\beta = |\varphi \nabla_\xi \xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi \xi \in \ker(\eta)$ is a unit vector field.

If $\beta = 0$ at every point P of M then ξ is an eigenvector of the shape operator and M is a Hopf hypersurface. If $\beta \neq 0$ in the neighborhood of every point P of M then M is called non-Hopf real hypersurface.

Let M be a non-Hopf real hypersurface in $M_2(c)$ and P a point of M with local orthonormal basis $\{U, \varphi U, \xi\}$. Then the following lemma holds.

Lemma 2.1. *Let M be a non-Hopf real hypersurface in $M_2(c)$. The following relations hold in the neighborhood of P:*

$$\begin{aligned} AU &= \gamma U + \delta\varphi U + \beta\xi, & A\varphi U &= \delta U + \mu\varphi U, & A\xi &= \alpha\xi + \beta U. \\ \nabla_U \xi &= -\delta U + \gamma\varphi U, & \nabla_{\varphi U}\xi &= -\mu U + \delta\varphi U, & \nabla_\xi \xi &= \beta\varphi U, \\ \nabla_U U &= \kappa_1\varphi U + \delta\xi, & \nabla_{\varphi U} U &= \kappa_2\varphi U + \mu\xi, & \nabla_\xi U &= \kappa_3\varphi U, \\ \nabla_U \varphi U &= -\kappa_1 U - \gamma\xi, & \nabla_{\varphi U}\varphi U &= -\kappa_2 U - \delta\xi, & \nabla_\xi \varphi U &= -\kappa_3 U - \beta\xi, \end{aligned} \tag{2.3}$$

where α, β, γ, δ, μ, κ_1, κ_2, κ_3 are smooth functions on M.

Remark 2.2. The proof of Lemma 2.1 is included in [15].

The Codazzi equation (2.2) for $X \in \{U, \varphi U\}$ and $Y = \xi$ because of Lemma 2.1 implies

$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2, \tag{2.4}$$

$$(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu, \tag{2.5}$$

$$(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu, \tag{2.6}$$

and for $X = U$ and $Y = \varphi U$,

$$U\delta - (\varphi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu. \tag{2.7}$$

Since in Gauss and Codazzi equation there is $c/4$ instead of c and $n = 2$, the *-Ricci tensor of M in $M_2(c)$ becomes

$$S^*X = -[c\varphi^2 X + (\varphi A)^2 X] \quad \text{for } X \in TM. \tag{2.8}$$

If M is a non-Hopf real hypersurface in $M_2(c)$ and $\{U, \varphi U, \xi\}$ is a local orthonormal basis of it at some point P, the *-Ricci tensor for $X \in \{U, \varphi U, \xi\}$ due to (2.3) and (2.8) takes the form

$$S^*\xi = \beta\mu U - \beta\delta\varphi U, \quad S^*U = (c + \gamma\mu - \delta^2)U \quad \text{and} \quad S^*\varphi U = (c + \gamma\mu - \delta^2)\varphi U. \tag{2.9}$$

Finally, the following theorem, which in the case of $\mathbb{C}P^n$ is owed to Maeda [10] and in the case of $\mathbb{C}H^n$ is owed to Ki and Suh [9] (also Corollary 2.3 in [13]), is provided.

Theorem 2.3. *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 2$. Then*

(i) $\alpha = g(A\xi, \xi)$ *is constant.*

(ii) *If W is a vector field which belongs to $\mathbb{D}$ such that $AW = \lambda W$, then*

$$\left(\lambda - \frac{\alpha}{2}\right) A\varphi W = \left(\frac{\lambda\alpha}{2} + \frac{c}{4}\right)\varphi W.$$

(iii) *If the vector field W satisfies $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ then*

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{2.10}$$

Remark 2.4. In case of three-dimensional Hopf hypersurfaces we can always consider a local orthonormal basis $\{W, \varphi W, \xi\}$ at some point $P \in M$ such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ (see [4,6]). Thus, relation (2.10) holds. Furthermore, the *-Ricci tensor for $X \in \{W, \varphi W, \xi\}$ satisfies the relation

$$S^*\xi = 0, \quad S^*W = (c + \lambda\nu)W \quad \text{and} \quad S^*\varphi W = (c + \lambda\nu)\varphi W. \tag{2.11}$$

3. Proofs of Theorems 1.4 and 1.5

Before proving Theorems 1.4 and 1.5 the extension of Theorem 5 in [7] in case of real hypersurfaces in $\mathbb{C}H^2$ is given. More precisely, the following theorem is obtained.

Theorem 3.1. *The only real hypersurface in $M_2(c)$ with vanishing *-Ricci tensor is the geodesic hypersphere of radius r satisfying* $\coth(r) = 2$ *in* $\mathbb{C}H^2$.

In order to prove that every real hypersurface in $M_2(c)$, with vanishing *-Ricci tensor, i.e., $S^*X = 0$, for any $X \in TM$ is a Hopf one, we follow the same steps as in the proof of Theorem 5 in [7]. The case of Hopf hypersurfaces in $\mathbb{C}P^2$ with vanishing *-Ricci tensor is also included in the above proof. Hence, the case of Hopf hypersurfaces in $\mathbb{C}H^2$ remains to be examined, so that the proof of Theorem 3.1 of the present paper is completed.

Since M is a Hopf hypersurface in $M_2(c)$ Theorem 2.3 and Remark 2.4 hold. Since $S^* = 0$ relation (2.11) implies that

$$c + \lambda\nu = 0.$$

The above relation, taking into account relation (2.10), yields that the real hypersurface has constant principal curvatures and this leads to the conclusion that a three-dimensional Hopf hypersurface with vanishing *-Ricci tensor is locally congruent to a real hypersurface of type (A) or type (B).

The eigenvalues which correspond to three-dimensional Hopf hypersurfaces in $\mathbb{C}H^2$ according to [1,2], after making the necessary adjustments since instead of c we have $c/4$, are displayed in the following table. The type ($A_{1,1}$) refers to a geodesic hypersphere and the type ($A_{1,2}$) refers to a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^1$. Moreover, α, λ and ν are the principal curvatures corresponding to ξ and the holomorphic distribution respectively, and m_α, m_λ and m_ν are their multiplicities.

Type	α	λ	ν	m_α	m_λ	m_ν
(A_0)	2	1	–	1	2	–
($A_{1,1}$)	$2\coth(2r)$	$\coth(r)$	–	1	2	–
($A_{1,2}$)	$2\coth(2r)$	$\tanh(r)$	–	1	2	–
(B)	$2\tanh(2r)$	$\tanh(r)$	$\coth(r)$	1	1	1

Substitution of the above eigenvalues in relation $c+\lambda\nu = 0$ and because of $c = -4$ leads to the conclusion that only the eigenvalues of the geodesic hypersphere satisfies the latter. Furthermore, the radius r of the geodesic hypersphere satisfies the relation $\coth(r) = 2$.

3.1. Semi-parallel *-Ricci tensor and proof of Theorem 1.4

The *-Ricci tensor is called semi-parallel when $(R(X,Y)\cdot S^*)Z = 0$, where R is the Riemannian curvature, which acts as derivation on S^*. More analytically, the above relation is written

$$R(X,Y)S^*Z = S^*(R(X,Y)Z) \tag{3.1}$$

for any vector fields X, Y and Z tangent to M.

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

The inner product of relation (3.1) for $X = U$, $Y = \varphi U$ and $Z = U$ with φU, due to (2.1) and (2.9) yields

$$g(AU, \varphi U) = g(A\varphi U, U) = \delta = 0,$$

and relation (2.9) becomes

$$S^*\xi = \beta\mu U, \quad S^*U = (c+\gamma\mu)U \quad \text{and} \quad S^*\varphi U = (c+\gamma\mu)\varphi U. \tag{3.2}$$

Furthermore, relation (3.1) for $X = \varphi U$, $Y = \xi$ and $Z = \varphi U$ due to (2.1) and (3.2) implies

$$\mu\left(\frac{c}{4}+\alpha\mu\right) = 0 \quad \text{and} \quad (c+\gamma\mu)\left(\frac{c}{4}+\alpha\mu\right) = 0.$$

Suppose that $c/4 \neq -\alpha\mu$ then the first of the above relations implies that $\mu = 0$ and the second due to the latter results in $c = 0$, which is a contradiction.

Therefore, on $\mathcal{N}$ relation $c/4 + \alpha\mu = 0$ holds. The inner product of the relation (3.1) for $X = U$, $Y = \xi$ and $Z = U$ with U because of (2.1) and (3.2) yields

$$\mu\left(\frac{c}{4}+\alpha\gamma-\beta^2\right) = 0.$$

If $c/4 + \alpha\gamma \neq \beta^2$ then we obtain $\mu = 0$ and relation $c/4 + \alpha\mu = 0$ leads to $c = 0$, which is a contradiction. So on $\mathcal{N}$ relation $c/4 + \alpha\gamma = \beta^2$ holds.

The structure Jacobi operator $l = R_\xi$ of a real hypersurface in $M_n(c)$, $n \geq 2$, is given by

$$lX = R_\xi X = R(X,\xi)\xi.$$

In the case of non-Hopf hypersurfaces M in $M_2(c)$, taking into account relations (2.1) and (2.3), the structure Jacobi operator is given by

$$lU = \left(\frac{c}{4}+\alpha\gamma-\beta^2\right)U + \alpha\delta\varphi U, \quad l\varphi U = \alpha\delta U + \left(\frac{c}{4}+\alpha\mu\right)\varphi U \quad \text{and} \quad l\xi = 0.$$

Since $\delta = 0$, $c/4 + \alpha\mu = 0$ and $c/4 + \alpha\gamma = \beta^2$ we obtain

$$lU = l\varphi U = l\xi = 0.$$

It is known that real hypersurfaces do not exist in $M_n(c)$, $n \geq 2$, with vanishing structure Jacobi operator (see Lemma 9 in [4]). Thus, $\mathcal{N}$ is empty and the following proposition is proved.

Proposition 3.2. *Every real hypersurface in $M_2(c)$ whose *-Ricci tensor is semi-parallel is a Hopf hypersurface.*

Since M is a Hopf hypersurface, Theorem 2.3 and Remark 2.4 hold. The case of Hopf hypersurfaces in $\mathbb{C}P^2$ with semi-parallel *-Ricci tensor has been studied in [7]. It remains to examine if there are Hopf hypersurfaces in $\mathbb{C}H^2$ with semi-parallel *-Ricci tensor. Relation (3.1) for $X = W$, $Y = \xi$ and $Z = W$ and for $X = \varphi W$, $Y = \xi$ and $Z = \varphi W$ because of relations (2.1) and (2.11) implies

$$(\lambda\nu - 4)(\alpha\lambda - 1) = 0 \quad \text{and} \quad (\lambda\nu - 4)(\alpha\nu - 1) = 0. \tag{3.3}$$

Combination of the above relations implies that

$$\alpha(\lambda - \nu)(4 - \lambda\nu) = 0.$$

Suppose that $\alpha(\lambda - \nu) = 0$, then we have two cases either $\alpha = 0$ or $\lambda = \nu$. If $\alpha = 0$ then relation (2.10) implies $\lambda\nu = -1$. Substitution of the latter relation in the first of (3.3) leads to $-5 = 0$, which is a contradiction. If $\lambda = \nu$, then the shape operator A commutes with the structure tensor φ and because of Theorem 1.1 M is locally congruent to a real hypersurface of type (A). Moreover, combining relations (2.10) and the first of (3.3) results in $\lambda^2(\lambda^2 - 4) = 0$. Because of the table in Section 3 we conclude that $\lambda^2 = 4$. This occurs in case of geodesic hypersphere in $\mathbb{C}H^2$.

Finally, if $\lambda\nu = 4$ then relation (2.11) implies that the *-Ricci tensor vanishes and because of Theorem 3.1 it is concluded that M is a geodesic hypersphere and this integrates the proof of Theorem 1.4.

3.2. Pseudo-parallel *-Ricci tensor and proof of Theorem 1.5

The *-Ricci tensor is called pseudo-parallel when $(R(X,Y) \cdot S^*)Z = L\{[(X \wedge Y) \cdot S^*]Z\}$, where R is the Riemannian curvature and acts as derivation on S^* and L is a nowhere vanishing function. More analytically, the above relation is written as

$$\begin{aligned} &R(X,Y)S^*Z - S^*(R(X,Y)Z) \\ &= L\{g(Y,S^*Z)X - g(X,S^*Z)Y - S^*[g(Y,Z)X - g(X,Z)Y]\} \end{aligned} \tag{3.4}$$

for any vector fields X, Y and Z tangent to M.

We consider $\mathcal{N}$ to be the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

The inner product of relation (3.4) for $X = U$, $Y = \varphi U$ and $Z = U$ with φU because of (2.1) and (2.9) yields

$$\delta = 0,$$

and relation (2.9) becomes

$$S^*\xi = \beta\mu U, \quad S^*U = (c + \gamma\mu)U \quad \text{and} \quad S^*\varphi U = (c + \gamma\mu)\varphi U. \tag{3.5}$$

Relation (3.4) for $X = U$, $Y = \varphi U$ and $Z = \xi$ because of (2.1) and (3.5) yields

$$\mu = 0.$$

Moreover, relation (3.4) for $X = \varphi U$, $Y = \xi$ and $Z = \varphi U$ due to (2.1) and (3.5) implies

$$\frac{c}{4} = L.$$

Relation (3.4) for $X = U$, $Y = \xi$ and $Z = U$ due to (2.1), (3.5), $\mu = 0$ and $c/4 = L$ yields

$$\alpha\gamma = \beta^2.$$

On $\mathcal{N}$ relation (2.4), (2.5), (2.6) and (2.7) because of $\delta = \mu = 0$ become

$$\gamma\kappa_3 = \beta\kappa_1 + \frac{c}{4}, \quad (\varphi U)\alpha = \beta(\alpha + \kappa_3), \quad (\varphi U)\beta = \beta^2 + \beta\kappa_1 + \frac{c}{2}, \quad (\varphi U)\gamma = \kappa_1\gamma + \beta\gamma.$$

Differentiation of $\alpha\gamma = \beta^2$, with respect to φU and taking into account all the above relations, results in $c = 0$, which is a contradiction.

Thus, $\mathcal{N}$ is empty and the following proposition is proved.

Proposition 3.3. *Every real hypersurface in $M_2(c)$ whose *-Ricci tensor is pseudo-parallel is a Hopf hypersurface.*

Since M is a Hopf hypersurface, Theorem 2.3 and Remark 2.4 hold. The case of Hopf hypersurfaces in $\mathbb{C}P^2$ with pseudo-parallel *-Ricci tensor has been studied in Theorem 3 in [7]. It remains the case of Hopf hypersurfaces in $\mathbb{C}H^2$. Relation (3.4) for $X = W$, $Y = \xi$ and $Z = W$ because of relations (2.1) and (2.11) implies

$$(\lambda\nu - 4)(\alpha\lambda - 1 - L) = 0.$$

Suppose that $\lambda\nu = 4$, then relation (2.11) yields $S^*X = 0$ for any vector field X tangent to M. The only real hypersurface with vanishing *-Ricci tensor because of Theorem 3.1 is the geodesic hypersphere in $\mathbb{C}H^2$ with $\coth(r) = 2$.

Next the case $L = \alpha\lambda - 1$ is examined. Relation (3.4) for $X = \varphi W$, $Y = \xi$ and $Z = \varphi W$ because of (2.1) and (2.11) implies

$$(\lambda\nu - 4)(\alpha\nu - 1 - L) = 0.$$

Suppose that $\lambda\nu = 4$, then relation (2.11) implies that $S^* = 0$ and due to Theorem 3.1 M is geodesic hypersphere. Secondly, if $L = \alpha\nu - 1$ combination of the latter relation with $L = \alpha\lambda - 1$ results in

$$\alpha(\lambda - \nu) = 0.$$

Thus, on M either $\alpha = 0$ or $\lambda = \nu$. If $\alpha = 0$ then M is locally congruent to a real hypersurface in $\mathbb{C}H^2$ with $A\xi = 0$ (for the construction of these real hypersurfaces see [5]). If $\lambda = \nu$ it implies that the shape operator A commutes with the structure tensor φ and because of Theorem 1.1 it is concluded that M is locally congruent to a real hypersurface of type (A) in $\mathbb{C}H^2$.

Conversely, it is easily proved that the latter real hypersurfaces in $\mathbb{C}H^2$ have pseudo-parallel *-Ricci tensor and that L is constant given by $L = \alpha\lambda - 1$. Furthermore, substitution of the eigenvalues of table in Section 3 implies the following

- if M is locally congruent to a horosphere then $L = 1$,
- if M is locally congruent to geodesic hypersphere then $L = \coth^2(r)$, where $r > 0$,
- if M is locally congruent to tube over $\mathbb{C}H^1$ then $L = \tanh^2(r)$, where $r > 0$,
- if M is locally congruent to Hopf hypersurface with $\alpha = 0$ then $L = -1$.

4. Proof of Theorem 1.6

Let M be a real hypersurface in $M_2(c)$ whose *-Ricci tensor is ξ-parallel. More analytically, relation (1.1) is written as

$$\nabla_\xi(S^*X) = S^*(\nabla_\xi X) \quad \text{for any } X \in TM. \tag{4.1}$$

Let $\mathcal{N}$ be the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0 \text{ in a neighborhood of } P\}.$$

On $\mathcal{N}$ the inner product of relation (4.1) for $X = \xi$ with ξ and φU because of (2.9) and relations of Lemma 2.1 implies respectively

$$\delta = 0 \quad \text{and} \quad \mu\kappa_3 = c + \gamma\mu. \tag{4.2}$$

Hence, relation (2.9) becomes

$$S^*\xi = \beta\mu U, \quad S^*U = (c+\gamma\mu)U \quad \text{and} \quad S^*\varphi U = (c+\gamma\mu)\varphi U. \tag{4.3}$$

The inner product of relation (4.1) for $X = \varphi U$ with U due to relation (4.3) and relations of Lemma 2.1 yields

$$\mu = 0.$$

Substitution of the above relation in the second of (4.2) results in $c = 0$ which is a contradiction. Therefore, the following proposition has been proved.

Proposition 4.1. *Every real hypersurface in $M_2(c)$ with ξ-parallel *-Ricci tensor is a Hopf hypersurface.*

Since M is a Hopf hypersurface Theorem 2.3 and Remark 2.4 hold. Using the fact stated in Remark 2.4 and taking notice that

$$\nabla_\xi W = \kappa\varphi W \quad \text{and} \quad \nabla_\xi \varphi W = -\kappa W,$$

relation (2.2) for $X = \xi$ and $Y = W$ and for $X = \xi$ and $Y = \varphi W$ implies respectively

$$\xi(\lambda) = 0 \quad \text{and} \quad \xi(\nu) = 0.$$

Relation (2.4) taking into account (2.10) yields

$$(\lambda - \nu)\left(\frac{\alpha}{2} - \kappa\right) = 0.$$

If $\lambda = \nu$, then $A\varphi = \varphi A$. The last relation because of Theorem 1.1 implies that M is locally congruent to a real hypersurface of type (A).

If $\lambda \neq \nu$, then since M is a three-dimensional real hypersurface we have two cases:

Case 1*: λ or ν is constant.* In this case, if one of them is constant, then relation (2.10) implies that the other one is also constant. So, the real hypersurface is locally congruent to a real hypersurface of type (B) both in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. So, the *-Ricci tensor of real hypersurfaces of type (B) is ξ-parallel.

Case 2*: both λ and ν are non-constant.* In this case the Hopf hypersurface satisfies the relations

$$\xi(\lambda) = 0, \quad \xi(\nu) = 0 \quad \text{and} \quad \kappa = \frac{\alpha}{2}.$$

Furthermore, the inner product of relation (2.2) for $X = W$ and $Y = \varphi W$ with W and φW results in respectively

$$\varphi W(\lambda) = (\lambda - \nu)g(\nabla_W W, \varphi W) \quad \text{and} \quad W(\nu) = (\lambda - \nu)g(\nabla_{\varphi W} W, \varphi W).$$

Therefore, the *-Ricci tensor of Hopf hypersurfaces with non-constant principal curvatures λ and ν, which satisfy all the above relations, have ξ-parallel *-Ricci tensor. Specific examples of such real hypersurfaces are Hopf hypersurfaces with $A\xi = 0$, which in the case of $\mathbb{C}P^2$ is a tube of radius of radius $r = \pi/4$ over a holomorphic curve and in the case of $\mathbb{C}H^2$ it was first constructed in [5]. This integrates the proof of Theorem 1.6.

Acknowledgments

The authors would like to express their gratitude to the referee for valuable comments on improving the whole paper.

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