CONDITIONS THAT THE ROOTS OF A POLYNOMIAL BE LESS THAN UNITY IN ABSOLUTE VALUE

By Paul A. Samuelson

Massachusetts Institute of Technology

1. Introduction. In econometric business cycle analysis, probability theory, and numerical mathematical computation the problem of convergence of repeated iterations arises. The solution of the difference equations defining such a process can in a wide variety of cases be shown to be stable in the sense of converging to a limit if a certain associated polynomial

$$f(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n = 0, \tag{1}$$

has roots whose moduli are all less than unity.

Thus, for "timeless" linear difference equation systems of the most general type, convertible into normal form,

$$Q_i(t + 1) = \sum_{j=1}^{n} a_{ij}Q_j(t), \qquad (i = 1, \cdots, n), \tag{2}$$

the polynomial is the characteristic or determinantal equation,

$$f(x) = | a_{ij} - x\delta_{ij} | = 0, \tag{3}$$

which when expanded out is of the form (1). The roots of this equation, when multiplied by suitable polynomials in t, give the exact solution of the problem in the form

$$Q(t) = \sum_{i=1}^{m} g_i(t)x_i^t, \tag{4}$$

where m is the number of distinct roots, and the g's are polynomials of degree one less than the multiplicity of the respective root. If complex roots occur, they do so in conjugate pairs and can be combined to form damped, undamped, or anti-damped harmonic terms. All terms go to zero as t approaches infinity if, and only if, the absolute value of each x is less than unity.

For non-linear systems the exact solution does not take this form, but in the neighborhood of an equilibrium point the roots of an associated polynomial, except in singular cases, do determine the stability of the system.

As far as the writer is aware, there does not appear in the literature an account of necessary and sufficient conditions for the roots of a polynomial to be less than unity in absolute value. This is in contrast to a related problem which arises in connection with the investigation of stability of dynamical systems defined by differential equations. These have associated with them a polynomial whose roots provide solutions in the form

$$g_i(t)e^{x_it}, \tag{5}$$

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or for non-linear systems infinite power series in such terms. It is required, therefore, to determine complete conditions under which the *real parts* of all roots must be negative.

This problem has been solved by Routh[1] in a manner which leaves little to be desired. Determinantal expression of his conditions in a slightly modified form was made by Hurwitz[2] who apparently was unaware of Routh's work, and by Frazer and Duncan[3] who were unaware of the Hurwitz results. A brief outline of Routh's mode of attack will prove instructive in dealing with the problem at hand.

2. Routhian analysis of sign of real parts of roots. Routh realized that the condition that all coefficients be positive—the leading coefficient having been made so—was necessary, but not sufficient unless all the roots were real. But a "derived" equation of degree $n(n-1)/2$ whose roots equal the sums of the roots of the original equation taken two at a time has real roots which are simple sums of the real parts of those of the original equation. In consequence, it is necessary and sufficient that the coefficients of the original and the "derived" equation all be positive.

Thus, valid necessary and sufficient conditions are presented. However, they are disadvantageous from two points of view. First, they are not all independent, being $n(n+1)/2$ conditions in number, whereas only n are necessary. Secondly, despite several ingenious methods devised by Routh, it is not easy to compute them in the general case.

Recognizing these difficulties, he therefore began anew from an entirely different angle. Utilizing a theorem of Cauchy concerning the relationship between the behavior of a polynomial on a closed contour in the complex domain and the number of roots within that closed curve, he derived necessary and sufficient conditions, which may be written in the slightly more convenient determinantal form of Hurwitz and Frazer and Duncan as follows:

$$T_0 = p_0 > 0, \qquad T_1 = p_1 > 0, \qquad T_2 = \begin{vmatrix} p_1 & p_3 \\ p_0 & p_2 \end{vmatrix} > 0,$$

$$(6) \qquad T_3 = \begin{vmatrix} p_1 & p_3 & p_5 \\ p_0 & p_2 & p_4 \\ 0 & p_1 & p_3 \end{vmatrix} > 0, \quad \cdots \quad T_s = \begin{vmatrix} p_1 & p_3 & \cdots & p_{2s-1} \\ p_0 & p_2 & \cdots & p_{2s-2} \\ 0 & p_1 & \cdots & p_{2s-3} \\ 0 & p_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & p_s \end{vmatrix} > 0.$$

[1] E. J. Routh, *A Treatise on the Stability of a Given State of Motion*, (London, 1877), Chaps. 2 and 3; *Advanced Rigid Dynamics*, 6th ed., London, 1905, Chap. 6.

[2] Hurwitz, *Math. Ann.*, Vol. 46 (1895), p. 521.

[3] R. A. Frazer and W. J. Duncan, *Royal Soc. Proc.*, Series A, Vol. 124 (1929), p. 642. Also R. A. Frazer, W. J. Duncan, and A. R. Collar, *Elementary Matrices*, Cambridge University Press, 1938, pp. 151–155.

The law of formation of these determinants is obvious. In the first row the odd p's starting with the first are listed. Within each column the p's diminish one unit at a time. Any p with negative subscript derived by this formula is treated as zero, and all p's of subscript higher than the degree of the equation are set equal to zero. With this convention, for p_0 made positive, complete and independent necessary conditions are that all principal minors of T_n formed by deleting successively the last row and column must be positive. These conditions are n in number and are independent.

3. Complete, independent, necessary and sufficient conditions. Corresponding to Routh's first attack on the problem, we might consider an equation of degree $n(n - 1)/2$ whose roots equal the *products* two at a time of the original equation's. If this equation and the original equation have *real* roots less than unity in absolute value, our problem is solved. This is guaranteed if, and only if, two further transformed equations with roots equal to the squares minus unity of the roots of the original and derived equations respectively all have positive coefficients. These conditions are necessary and sufficient, but not independent, and cannot be easily computed in the general case. Therefore, I follow Routh's example and approach the problem from a different point of view.

When the roots of $f(x) = 0$ are plotted in the complex plane, they must all lie within the unit circle if their absolute values are to be less than unity, and conversely. We might therefore attempt to apply Cauchy's theorem. However, it is not necessary to do so. Routh has shown what the conditions are that there be no roots in the right-hand half-plane. Can we find a complex transformation of variables which carries the unit circle into the left-hand half-plane?

The answer is in the affirmative. The linear complex transformation

$$x = \frac{z + 1}{z - 1}, \qquad z = \frac{x + 1}{x - 1} \tag{7}$$

will accomplish this. But after substituting for x its value in terms of z, we cease to have a polynomial but rather a rational function of z as follows:

$$f(x) = f\left(\frac{z + 1}{z - 1}\right) = \frac{\sum_{i=0}^{n} p_i(z + 1)^{n-i}(z - 1)^i}{(z - 1)^n} = 0. \tag{8}$$

We need only consider the polynomial in the numerator, i.e.,

$$\varphi(z) = \sum_{0}^{n} \pi_i z^{n-i} = 0. \tag{9}$$

In order that the roots of the original equation be less than unity, in absolute value, it is necessary and sufficient that the real parts of the roots of equation (9) *be negative.* Once we determine the coefficients (π_i) in terms of the original p's, we can easily apply Routh's theorems. This yields $n + 1$ necessary and sufficient conditions, all of which are independent.

Expanding the numerator of the right-hand side of (8) and collecting terms, the following explicit formulas for the π's are directly obtained:

$$\pi_i = \sum_{j=0}^{n} p_j \sum_{k=0}^{m(i,j)} {}_{n-j}C_{i-k}(-1)^k \; {}_jC_k, \tag{10}$$

where

$${}_vC_w = \frac{v!}{(v-w)!\,w!},$$

and

$m(i, j) =$ the smaller of i and j.

For fourth and higher degree equations literal substitution, while always possible, results in complicated expressions. It is preferable, therefore, to compute the π's numerically and then apply the conditions of (6) directly.

Other necessary conditions can be easily derived, but they will be dependent upon these. Thus, each π must be positive; but this is not, by itself, sufficient. Or, adding π_0 and π_n we find

$$\pi_0 + \pi_n = p_0 + p_2 + p_4 + \cdots > 0, \tag{11}$$

i.e., the sum of the even p's must be positive. Similarly, still other linear sums of other π's will result in cancellation of certain of the p's. Except on special occasions there is probably no labor saved by utilizing conditions derived in this way.

One obvious but useful necessary condition will be stated without proof. If one forms polynomials from subsets of the coefficients of a given "stable" polynomial formed by arbitrary "cuts" which leave adjacent coefficients in unchanged order and introduce no gaps within each set, then the resulting polynomials will all be stable.

Special sufficiency conditions also can be developed. Carmichael[4] presents certain inequalities between the absolute values of the largest root and the coefficients of the original equation. For special problems these may be fruitfully applied.

4. Example. In conclusion I apply the conditions derived here to a well-known numerical equation determined statistically by Tinbergen[5] in the analysis of economic fluctuations. It is a fourth order difference equation with constant coefficients,

$$Z_t - .398Z_{t-1} + .220Z_{t-2} - .013Z_{t-3} - .027Z_{t-4} = 0 \tag{12}$$

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[4] R. D. Carmichael, *Amer. Math. Soc. Bull.*, Vol. 24 (1918), pp. 286–296.

[5] J. Tinbergen, *Business Cycles in the United States, 1919–1932*, League of Nations, 1939, p. 140.

with the associated indicial equation

$$f(x) = x^4 - .398x^3 + .220x^2 - .013x - .027 = 0. \tag{13}$$

Its roots have been computed and are known to be less than unity in absolute value. This may be verified by computing

$$\begin{aligned} \pi_0 &= 0.782 > 0 \\ \pi_1 &= 3.338 > 0 \\ \pi_2 &= 5.398 > 0 \\ \pi_3 &= 4.878 > 0 \\ \pi_4 &= 1.604 > 0 \\ T_2 &= 14.204 > 0 \\ T_3 &= 43.177 > 0 \end{aligned} \tag{14}$$

To compute the same results by cross-multiplication the work is arranged as follows:

(15)

π_0	π_2	π_4
.782	5.398	1.604
π_1	π_3	
3.338	4.878	
$\pi_1\pi_2 - \pi_0\pi_3$	$\pi_3\pi_4 - 0$	
14.204	7.824	
$\pi_3(\pi_1\pi_2 - \pi_0\pi_3) - \pi_1\pi_3\pi_4$		
43.177		

It may be remarked that the presence of a negative coefficient anywhere in the table is an immediate indication of instability, and that there is no necessity to continue the computation until a negative sign appears in a leading coefficient. This fact often saves much labor.

VALUES OF MILLS' RATIO OF AREA TO BOUNDING ORDINATE AND OF THE NORMAL PROBABILITY INTEGRAL FOR LARGE VALUES OF THE ARGUMENT

By Robert D. Gordon

Scripps Institution of Oceanography

A pair of simple inequalities is proved which constitute upper and lower bounds for the ratio R_x[1], valid for $x > 0$. The writer has failed to encounter these inequalities in the literature, hence it seems worthwhile to present them for whatever value they may have.

[1] J. P. Mills, "Table of ratio: area to bounding ordinate, for any portion of the normal curve." *Biometrika Vol. 18* (1926) pp. 395–400. Also Pearson's tables, Part II, Table III.