COLLOQUIUM MATHEMATICUM

VOL. LXIII 1992 FASC. 2

CONDITIONS WHICH ENSURE THAT A SIMPLE MAP DOES NOT RAISE DIMENSION

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The present paper deals with those continuous maps from compacta into metric spaces which assume each value at most twice. Such maps are called here, after Borsuk and Molski (1958) and as in our previous paper (1990), *simple*.

We investigate the possibility of decomposing a simple map into essential and elementary factors, and the so-called splitting property of simple maps which raise dimension. The aim is to get insight into the structure of those compacta which have the property that simple maps from them do not raise dimension.

In what follows a *map* means a continuous map, unless explicitly stated otherwise. A space is, except in some general lemmas, understood to be metrizable. A *compactum* means a compact metric space.

1. Outline of the problem. If $f : X \to Y$ is a simple map from a compactum, then the dimension of $f(X)$ can exceed that of X by at most 1. This follows from the formula

$$\dim f(X) - \dim X \le k - 1,$$

established by Hurewicz (1933) for maps from compacta assuming each value at most at k points.

There are compacta no simple map from which raises dimension. For instance, as proved by Hurewicz (1933), this property is enjoyed by any compactum satisfying the following condition:

(α) *subcompacta of full dimension have non-empty interiors.*

This condition is rather restrictive. It is satisfied by manifolds, in particular by intervals of the reals.

If a one-dimensional compactum satisfies (α), its non-degenerate subcontinua cannot be nowhere dense. If it is connected, it is hereditarily locally

Supported by Polish Scientific Grant RP.I.10.

connected. It follows that a one-dimensional continuum satisfying (α) is the union of open arcs, which form a dense (and open) subset, and of a totally disconnected (closed) subset. Maximal open arcs form, in view of hereditary local connectedness, a null family. This implies that a one-dimensional continuum satisfying (α) is *regular in the sense of ramification*, i.e. has a base of open sets with finite boundaries.

Dendrites are regular in this sense. However, not all dendrites have property (α). None the less, there are no simple maps from dendrites which raise dimension. This is an unpublished result of Sieklucki, announced by Lelek (1962); its proof will be given here ([1]).

As far as we know, the problem of characterizing the compacta with the property that simple maps from them do not raise dimension is not solved even in the case of one-dimensional continua. We will call such one-dimensional continua *thin*.

The property of being small in the sense of ramification does not imply the property of being thin: there are regular continua which are not thin; see e.g. our paper (1990). It remains an open problem whether or not the Sierpiński triangular curve is thin. In the paper quoted above we showed that there are no dimension-raising simple maps from the Sierpiński triangular curve into the plane.

On the other hand, as we shall see, the Anderson–Choquet curve (1959) and the Andrews chainable continuum (1961), which are big in the sense of order of ramification, happen to be thin.

Let $f : X \to Y$ be a map. We say that it has the *splitting property* if there exists a subcontinuum C of Y, with $\dim C = \dim X$, for which $f^{-1}(C)$ is the union of disjoint subsets C' and C'' of X such that $f|C'$ and $f|C''$ are homeomorphisms onto C.

The statement that simple dimension-raising maps from compacta have the splitting property appears implicitly in the literature; see e.g. Sieklucki (1969). It is—in full generality—a consequence of a theorem of Freudenthal (1932) from general dimension theory. As we shall show, in some particular cases, e.g. in the case of maps from regular continua, the proof can be performed in an elementary way.

2. Essential and elementary factors of simple maps. If f is a simple map from a space X, then a pair of continua C' and C'' in X will be called a *pair of twins* if

$$C' \cap C'' = \emptyset \quad \text{and} \quad f(C') = f(C''). \tag{1}$$

([1]) Professor Sieklucki kindly informed us that the original proof can be found in his doctoral dissertation.

For brevity, a pair of points x' and x'' will be regarded as a pair of twins if $f(x') = f(x'')$ ([2]).

The family of pairs of twins is in a natural manner partially ordered by inclusion. A simple map for which there do not exist maximal pairs of twins will be called *essentially simple*.

The function $f(x) = x^2$ on the reals is an example of an essentially simple map. The map of the interval $0 \le t \le 1$ onto the circle $|z| = 1$ given by $f(t) = e^{2\pi it}$ is simple but not essentially simple, since $\{0, 1\}$ is a pair of maximal twins.

Let f be a simple map from a compact Hausdorff space X into a Hausdorff space. Consider the collection of all pairs $\{x', x''\}$ of point twins which cannot be enlarged to maximal twins. Extend this collection to a partition of X by assuming that other elements of the partition are one-point sets. Let $P(f)$ stand for this partition.

A partition of a space is said to be *upper semicontinuous at* its element A if for each open set U containing A and each point a in A there exists a neighbourhood V of a such that each element of the partition intersecting V is contained in U. A partition is called *upper semicontinuous* if it is upper semicontinuous at each of its elements.

LEMMA 1. *The partition $P(f)$ is upper semicontinuous.*

Proof. The upper semicontinuity of $P(f)$ at elements being the counter-images of values of f follows easily from the fact that the partition into the counter-images of values of a closed (continuous) map is upper semicontinuous.

An element of $P(f)$ which is not a counter-image of a value of f is a one-point set $\{a\}$ such that $f^{-1}(f(a)) = \{a, b\}$, where $a \ne b$. Let $\{K, L\}$ be a maximal pair of twins such that $a \in K$ and $b \in L$.

Let $J = f(K) = f(L)$. Take neighbourhoods, G of K and H of L, with disjoint closures. Then J is a connected component of $f(\overline{G}) \cap f(\overline{H})$, since K and L form a maximal pair of twins. Let $W \subset \mathrm{int}(f(G \cup H))$ be an open neighbourhood of J with boundary disjoint from $f(\overline{G}) \cap f(\overline{H})$, and let V be a neighbourhood of a such that

$$V \subset G, \quad f(V) \subset W, \quad f^{-1}(f(W)) \subset G \cup H\,. \tag{2}$$

We shall show that

(3) if $x' \in V$ and $\{x', x''\} \in P(f)$, then $x'' \in G$.

By (2), there is nothing to prove if $x'' = x'$, so assume $x' \ne x''$. We have $f(x') = f(x'')$, since $\{x', x''\} \in P(f)$. Suppose $x' \notin G$. Then $x'' \in H$, as $f^{-1}(f(x'')) \subset G \cup H$, by (2). Thus, the common value of f at x' and x''

([2]) Whenever we call a set $\{u, v\}$ a pair, we assume u and v to be distinct.

belongs to $f(G) \cap f(H)$. Let C be the connected component of $f(x') = f(x'')$ in $f(\overline{G}) \cap f(\overline{H})$. Since $\overline{G}$ and $\overline{H}$ are disjoint, $f^{-1}(C)$ splits into disjoint continua C' and C'', each mapped homeomorphically under f onto C. Since $f(x') = f(x'') \in W$ by (2), and the boundary of W is disjoint from $f(\overline{G}) \cap f(\overline{H})$, and hence from C, we have $C \subset W$. Thus, C' and C'' form a maximal pair of twins such that $x' \in C'$ and $x'' \in C''$. This contradicts our assumptions on $P(f)$, and (3) is proved.

To finish the proof, let U be a neighbourhood of a. Since (2) is satisfied for sufficiently small neighbourhoods V of a, it is enough to show that the elements of $P(f)$ intersecting those V are contained in U.

Suppose to the contrary that for arbitrarily small neighbourhoods V of a satisfying (2) there are elements of $P(f)$ intersecting V and not contained in U. These elements are of the form $\{x', x''\}$, where $x' \in V$ and $f(x') = f(x'')$. By continuity and compactness, there exists $c \notin U$ (an accumulation point of the x'') such that $f(c) = f(a)$. We have $c \neq a$. Also, $c \neq b$, since, by (3), if an element $\{x', x''\}$ of $P(f)$ intersects V, it cannot intersect H (in which b lies). A contradiction, as f is simple.

Let $g : X \to P(f)$ be the quotient map corresponding to the partition $P(f)$. It follows from Lemma 1 that $P(f)$ is (compact) Hausdorff in the quotient topology. Since $g(x') = g(x'')$ implies $f(x') = f(x'')$, we get a factorization

$$f = h \circ g \tag{4}$$

of f, where g is essentially simple by construction.

The map h has the property opposite to the preceding one, namely, as is easy to see, each pair of points which are identified under h can be enlarged to a maximal pair of twins with respect to h. A simple map having this property will be called *elementarily simple*.

Thus, we get the following

THEOREM 1. *Each simple map f from a compact Hausdorff space into a Hausdorff space admits a factorization $f = h \circ g$ into a surjective essentially simple map g and an elementarily simple map h.*

3. Simple maps from regular compacta. Let X be a compact Hausdorff space and let $f : X \to Y$ be a continuous map into a Hausdorff space. If $y \in Y$ and if W is a neighbourhood of $f^{-1}(y)$ in X, then $f(W)$ is a neighbourhood of y in Y. The interiors $\operatorname{int} f(W)$ for all such y and W form an open base in Y, provided that f is surjective. But, in order to get a base, it suffices to consider only those W which are unions of finitely many sets from a given base in X.

The above statements from general topology will be used in the proof of the following

Splitting Lemma I. *Let X be a compact Hausdorff space regular in the sense of ramification, and let $f : X \to Y$ be a simple map onto a Hausdorff space of dimension ≥ 2. There exists a non-degenerate nowhere dense subcontinuum C of Y such that $f^{-1}(C) = C' \cup C''$, where C' and C'' are disjoint and $f|C'$ and $f|C''$ are homeomorphisms onto C.*

Proof. Let $y \in Y$. Consider neighbourhoods W of $f^{-1}(y)$ which are unions of finitely many elements of a base in X consisting of sets with finite boundaries. The sets W have finite boundaries, too. The interiors of $f(W)$ for all y in Y form a base in Y, according to the above remarks. Since $\dim Y \geq 2$, the boundary of some $\operatorname{int} f(W)$ contains a non-degenerate continuum, say C. We can assume that C is disjoint from the (finite) image of the boundary of W. The continuum C is nowhere dense in Y, since it lies on the boundary of an open subset of Y.

Let $c \in C$. Then $f^{-1}(c)$ consists of two points. One of them, say c', lies in W, since $C \subset f(W)$, by our last assumption. To find the other one, observe that c being a point of the boundary of $f(W)$, in each neighbourhood of c there are values of f at points in the complement of $\overline{W}$. Thus, by continuity and compactness, c is the value of f at a point c'' in the closure of $X - \overline{W}$, in fact in $X - \overline{W}$, as C (in which c lies) is disjoint from the image of $\overline{W} - W$ (the boundary of W). Clearly, $c'' \neq c'$. The set $\{c', c''\}$ is the full counter-image of c, as f is simple.

It follows that $f^{-1}(C)$ is the union of

$$C' = W \cap f^{-1}(C) \quad \text{and} \quad C'' = (X - \overline{W}) \cap f^{-1}(C),$$

which are disjoint and open, and therefore closed, in $f^{-1}(C)$. Each of them is mapped onto C continuously and in one-to-one way. Since C is compact, C' and C'' are compact. Thus, $f|C'$ and $f|C''$ are homeomorphisms onto C.

Corollary (contained in Theorem II of Hurewicz (1933) and containing the theorem of Hahn (1913) and Mazurkiewicz (1915)). *Simple maps do not raise the dimension of one-dimensional continua having property* (α); in other words, *one-dimensional continua having property* (α) *are thin.*

Proof. Assume to the contrary that f is a simple map from a one-dimensional continuum X having property (α) onto a Hausdorff space of dimension ≥ 2. The Splitting Lemma I can be applied, since X, having property (α), is regular in the sense of ramification. By that lemma, there exists a nowhere dense non-degenerate subcontinuum C of Y such that $f^{-1}(C)$ splits into disjoint continua C' and C'' mapped homeomorphically under f onto C. Both C' and C'', being non-degenerate, are of dimension 1 everywhere. Hence, the interiors D' of C' and D'' of C'' with respect to X are dense in C' and C'', respectively, since X has property (α). Thus $f(D')$ and $f(D'')$ are open and dense subsets of C. Take $y \in f(D') \cap f(D'')$. Then

$D' \cup D''$ is a neighborhood of $f^{-1}(y)$ in X. Thus, $f(D' \cup D'')$ is a neighbourhood of y in Y. But $f(D' \cup D'')$, being contained in C, is nowhere dense in Y. A contradiction.

4. Elementarily simple maps from regular compacta. In the sequel the following lemma on twins will be needed.

LEMMA. *Let f be a simple map between Hausdorff compact spaces. The elements of maximal twins of f form a disjoint family.*

Proof. It suffices to show that if $\{C', C''\}$ and $\{D', D''\}$ are distinct pairs of twins of f and $\{C', C''\}$ is maximal, then $C' \cap D' = \emptyset$. Suppose that this is not true and take $x \in C' \cap D'$. Let U' and U'' be open and such that $C' \subset U'$, $C'' \subset U''$ and $\overline{U'} \cap \overline{U''} = \emptyset$. Let E' be the component of x in $\overline{U'}$. We have $C' \subsetneq E'$. The inclusion is strict since E' has, according to the Janiszewski lemma, a point on the boundary of U'. Then $\{E', E''\}$, where E'' is the component of $f^{-1}(f(E))$ intersecting C'', is a pair of twins larger than $\{C', C''\}$, contrary to the maximality of the latter.

A map is called *monotone* if the counter-images of points are all connected. The property of being regular compact Hausdorff is preserved under monotone (continuous) maps into Hausdorff spaces; see e.g. Whyburn (1942), p. 138.

A family of subsets of a metric space is called a *null family* if for every $\varepsilon > 0$ only finitely many sets in this family have diameters $\le \varepsilon$.

THEOREM 2. *Elementarily simple maps from regular compacta do not raise dimension.*

Proof. Let X be a regular compactum and let f be an elementarily simple map from X onto a Hausdorff space Y; then, in fact, Y is a compactum.

Let $\mathcal{F}$ be the family of elements of all maximal twins of f. By the lemma, the elements of $\mathcal{F}$ are pairwise disjoint. Since X is regular, and thus hereditarily locally connected, $\mathcal{F}$ is a null family. Identify each element of $\mathcal{F}$ to a point. The corresponding quotient map $r : X \to X_*$ is monotone. The quotient space X_* is Hausdorff, since the partition into continua which form a null family is upper semicontinuous. Thus, X_* is a regular compactum, since monotone maps preserve regularity.

Identify the images of elements of $\mathcal{F}$ to single points. Let $s : Y \to Y_*$ be the corresponding quotient map. The space Y_* is a compactum, the identification being upper semicontinuous, as the images of elements of F form a null family.

We get the following diagram of commuting maps:

$$\begin{array}{ccc} X & \xrightarrow{r} & X_* \\ f\downarrow & & \downarrow g \\ Y & \xrightarrow[s]{} & Y_* \end{array}$$

where g is induced—in view of the properties of quotients—by the maps formerly defined.

The map g is elementarily simple. By construction, it has no pair of non-degenerate twins.

In particular, g does not have the splitting property. Since X_* is regular, from the Splitting Lemma I applied to g it follows that $\dim Y_* \leq \dim X_* \leq 1$.

Arrange the images under f of members of $\mathcal{F}$ into a sequence

$$J_1,\ J_2,\ \ldots \tag{5}$$

Let Y_k be obtained from Y by identifying each of the sets $J_k,\ J_{k+1}, \ldots$ from (5) to a single point. Then $Y_* = Y_1$. Let $s_k : Y \to Y_k$ be the corresponding quotient map; we have $s_1 = s$. Consider the inverse system

$$\ldots \to Y_{k+1} \to Y_k \to \ldots \to Y_2 \to Y_1 = Y_*, \tag{6}$$

where the bonding maps are induced by the quotient maps described above.

The maps $s_k : Y \to Y_k$ induce a map s_∞ from Y into the limit space Y_∞ of (6). The map s_∞ is one-to-one, which follows from our description. Thus, it is a homeomorphism. In consequence, the dimensions of Y and Y_∞ are equal.

Recall that $\dim Y_* = \dim Y_1 \leq 1$. Also $\dim Y_2 \leq 1$, since Y_2 can be viewed as a compactification of Y_1 with a point replaced by a one-dimensional remainder homeomorphic to J_1; this conclusion on the dimension of compactification follows easily from the sum theorem in dimension theory (see e.g. Hurewicz and Wallman (1941), p. 30). For the same reason, $\dim Y_3 \leq 1$, and hence $= 1$. Thus, the limit space Y_∞, and in consequence Y, have dimension ≤ 1. A contradiction.

5. The structure of essentially simple maps from dendrites. A continuum is said to be *hereditarily unicoherent* if any two of its subcontinua have a connected intersection.

If f is a simple map then the points of the domain of f will be called *relative* (with respect to f) if there exists a pair of twins such that the points belong to the same element of the pair.

The proof of the following lemma is straightforward and will be omitted.

LEMMA 2. *Let X be hereditarily unicoherent. Let f be a simple map from X. The relation of being relative* (*with respect to f*) *is an equivalence.*

The equivalence classes of relativeness are *semicontinua*, i.e. any two points in an equivalence class can be joined by a continuum. The equivalence class *corresponding* to an equivalence class S' is the equivalence class S'' which is the union of twins C'' from all pairs $\{C', C''\}$, where C' is contained in S'. We have $S' \cap S'' = \emptyset$, and $f|S'$ and $f|S''$ are continuous one-to-one maps onto the same image.

If S' is closed, and hence a continuum, then S'' is closed; in this case $\{S', S''\}$ is a pair of twins (in fact, a maximal pair of twins).

If f is an essentially simple map, the equivalence classes of relativeness are never closed.

If two classes S' and S'' corresponding to each other are not closed, then

$$\overline{S'} \cap \overline{S''} \neq \emptyset, \tag{7}$$

and, by hereditary unicoherence, the intersection (7) is connected; it is not excluded that one of the closures is contained in the other, as is the case when X is a hereditarily indecomposable continuum.

Now we add local connectedness to the assumptions on X, i.e. we consider the *dendrites*, that is, locally connected uniquely arcwise connected continua.

All properties of dendrites needed here can be found in, or easily deduced from, the material contained in Kuratowski (1950). Note, for instance, that dendrites are regular in the sense of ramification.

We leave without proof the following—perhaps less known—property of dendrites: *if M and N are connected and disjoint subsets of a dendrite*, *then $\overline{M} \cap \overline{N}$ consists of at most one point.*

Taking into account (7), we get

LEMMA 3. *For dendrites the intersections* (7) *are one-point sets.*

COROLLARY. *If a map f from a dendrite into a Hausdorff space is essentially simple and ab is an arc in the dendrite* (*the unique arc with ends a and b*) *such that $f(a) = f(b)$*, *then there exists a point c on ab such that $f|ac$ and $f|cb$ are homeomorphisms onto the same image.*

Proof. Since f is essentially simple, the classes of relativeness, S' of a and S'' of b, are non-closed semicontinua. Since $f(a) = f(b)$, we easily deduce that S' and S'' correspond to each other, i.e. $f|S'$ and $f|S''$ are one-to-one continuous maps onto the same image, where S' and S'' are disjoint. By Lemma 3, $\overline{S'}$ and $\overline{S''}$ intersect in a single point c. Take an arc joining a and b in $\overline{S'} \cup \overline{S''}$. Then c lies on this arc, which in view of the uniqueness of arcs in dendrites is the given arc ab. The arcs ac and cb lie, up to the point c, in S' and S'', respectively. They are mapped under f homeomorphically onto arcs joining $f(c)$ and $f(a) = f(b)$ (recall that $f|S'$ and $f|S''$ are one-to-one and extend continuously to $S' \cup \{c\}$ and $S'' \cup \{c\}$). The images of $f|ac$

and $f|cb$ coincide, since otherwise the counter-images of $f(ac)$ and $f(cb)$ would give two distinct arcs joining a and b in the dendrite.

The converse implication is also true, but it will not be used in the sequel.

6. A theorem on dendrites. A value y of a map $f : X \to Y$ is said to be *open* if $y \in \operatorname{int}(f(V))$ for each neighbourhood U of any x in $f^{-1}(y)$. If f is a continuous map between compacta, the set of open values of f is a dense G_δ subset of the set of all values. This follows from the Baire theorem.

Single values are always open. A value y of a simple map f (assume that $y = f(a) = f(b)$) is open if and only if $\lim a' = a$, $f(a') = f(b')$, $a' \neq b'$, implies $\lim b' = b$.

THEOREM 3. *The image of a dendrite under an essentially simple map is a dendrite.*

Proof. Let f be an essentially simple map from a dendrite X onto a Hausdorff space Y. Let u and v be distinct points of Y. It suffices to show—in view of the known characterization of dendrites (see Whyburn (1942), p. 88)—that there exists a point disconnecting Y between u and v.

Let $u = f(a)$ and $v = f(b)$. Let ab be the unique arc in X with ends a and b. Consider maximal open arcs on ab at whose ends f assumes the same values. By the Corollary to Lemma 3, the map f folds any such arc around an inner point. Thus, any two such maximal arcs, including their ends, are disjoint. Remove these maximal open arcs from ab. We get a compactum K such that f assumes the same values only at those pairs of points of K which are the ends of a removed open arc.

The ends of ab are not removed, and we conclude from the construction that $f(K)$ is an arc with ends $f(a)$ and $f(b)$.

According to the aforementioned consequence of the Baire theorem, the points of $f(K)$ which are open values of both $f|K$ and $f|f^{-1}(f(K))$ form a dense G_δ subset of $f(K)$. Since the set of branch points of a dendrite is at most countable (Kuratowski (1950), p. 227; Whyburn (1942), p. 60), choose $p \in f(K)$ distinct from $f(a)$ and $f(b)$ in such a way that p is an open value for $f|K$ and $f|f^{-1}(f(K))$ and all points of $f^{-1}(p)$ are of order ≤ 2 on X in the sense of ramification.

We shall show that p is the desired point, i.e.

(∗) p disconnects Y between $f(a)$ and $f(b)$.

We consider two cases.

Case 1: $f^{-1}(p)$ is a one-point set, say $f^{-1}(p) = \{c\}$. Since c is an inner point of the arc ab, the order of X at c is 2. Thus, $X - \{c\} = M \cup N$, where M and N are continua and $M \cap N = \{c\}$. Since a and b lie in the different sets M and N, $f(a)$ and $f(b)$ lie in the different sets $f(M)$ and $f(N)$.

It remains to show that $f(M) \cap f(N) = \{p\}$.

Suppose that there is one more point in this intersection. This means that there exist $x \in M$ and $y \in N$ such that $f(x) = f(y)$ (then $x \neq a$ and $y \neq b$). Then c is an inner point of the arc xy (recall that $M \cap N = \{c\}$). Since c is a single value of f, it is precisely the point of xy around which the arc xy is folded in view of the Corollary to Lemma 3. Since the order of X at c is 2, the arcs ab and xy coincide in a neighbourhood of c. Thus, f assumes the same value at the ends of an arc around c lying on ab. But such arcs were removed from ab in order to get K. A contradiction, as $c \in K$.

Case 2: $f^{-1}(p) = \{c, d\}$ and $c \neq d$. Either c or d lies in K, say $c \in K$. Then $d \notin K$, since p is an open value of $f|K$ and, as is easy to see, open values of $f|K$ are single values of $f|K$.

Since p is an open value of $f|f^{-1}(f(K))$, there is a subarc of $f(K)$, containing p as an inner point, whose counter-image splits into two arcs in X, one passing through c and the other through d. Since $d \notin K$, we can assume that the arc passing through d is disjoint from K. In consequence, the arc passing through c is contained in K. From the existence of these arcs it follows that c and d are of order 2 in X (recall that they are of order ≤ 2, as they belong to $f^{-1}(p)$). Thus, the dendrite is the union of three continua M, N' and N'' such that $N' \cap M = \{c\}$, $M \cap N'' = \{d\}$ and $N' \cap N'' = \emptyset$.

We shall show that

$$(8) \qquad f(M) \cap f(N') = \{p\} \quad \text{and} \quad f(M) \cap f(N'') = \{p\}.$$

By symmetry, it suffices to show one of these equalities. We shall show the first.

Suppose to the contrary that there is one more point, besides p, in $f(M) \cap f(N')$. This means the existence of $x \in M$ and $y \in N'$, different from c and d and such that $f(x) = f(y)$. From $M \cap N' - \{c\}$ it follows that c is an inner point of the arc xy. Since $M \cup N'$ is connected, xy is contained in $M \cup N'$. But xy cannot touch the point d in M (d cannot be the end x of xy, since this would imply $f(x) = f(c)$ and, in consequence, $y = c$; it cannot be an inner point of xy, since d is of order 2 in X and therefore an arc passing through d must have points outside M). Thus, $f|xy$ assumes the value p only at c.

From $f(x) = f(y)$ it follows, in view of the Corollary to Lemma 3, that f folds the arc xy around c. Since the order of X at c is 2, the arcs xy and cd coincide in a neighbourhood of c. From $f(c) = f(d)$ it follows that, in view of the Corollary mentioned above, f folds the arc cd around an inner point of cd. As xy and cd coincide in a neighbourhood of c, there are three points, two in a neighbourhood of c and the third in a neighbourhood of d, such that the values of f at them are the same. We get a contradiction, since f is simple. This concludes the proof of (8).

From (8) it follows that $f(M) \cap f(N' \cup N'') = \{p\}$.

Clearly, $f(M) \cup f(N' \cup N'') = Y$. So, in order to prove $(*)$, it suffices to show that neither term of the union reduces to $\{p\}$. The latter will be proved if we show that a and b belong to the different sets M and $N' \cup N''$.

To this end, observe that since $c \in K \subset ab$, the point c is an inner point of ab. Thus, a and b lie in the different sets M and $N' \cup N''$, since c is of order 2 in X.

COROLLARY (Sieklucki). *There are no dimension-raising simple maps from dendrites*; in other words, *dendrites are thin.*

Proof. Let f be a simple map from a dendrite X onto a Hausdorff space Y. Let $f = h \circ g$ be the decomposition of f into an essential factor g and an elementary factor h. By Theorem 3, the image $g(X)$ is a dendrite. Dendrites are regular, so we can apply Theorem 2 to the elementary (surjective) factor $h : g(X) \to Y$. It follows that $\dim Y \le \dim X$.

7. Splitting lemma in a general setting. A general background for the splitting property of simple maps is contained in a result of Hurewicz (1933) (derived from a more general theorem of Freudenthal (1932)), stating that the set of multiple values of a dimension-raising map from a compactum onto a compactum Y has dimension $\ge \dim Y - 1$.

We shall derive from that result of Hurewicz the following

SPLITTING LEMMA II. *If $f : X \to Y$ is a dimension-raising simple map from a compactum onto a compactum, then Y contains a continuum C with $\dim C = \dim X$ such that $f^{-1}(C) = C' \cup C''$, where C' and C'' are disjoint and $f|C'$ and $f|C''$ are homeomorphisms onto C.*

Proof. Since a compactum contains a connected component of full dimension (X is finite-dimensional), it suffices to show the existence of a compactum satisfying the conclusions.

For any positive integer k, let F_k be the set of those y in Y for which $\operatorname{diam} f^{-1}(y) \ge 1/k$. The sets F_k are closed, thus compact, and their union is the set of all multiple values of f. By the theorem of Hurewicz quoted above, this set has dimension $\ge \dim Y - 1$, hence one of the summands has the same property.

So, fix a compact subset B of Y and $\eta > 0$ such that $\dim B \ge \dim Y - 1$ and $\operatorname{diam} f^{-1}(y) \ge \eta$ for $y \in B$.

Let $y \in B$ and let $f^{-1}(y) = \{a, b\}$ (recall that f is simple). Take neighbourhoods U_a of a and U_b of b of diameters $< \eta/2$, and therefore disjoint. Let V_y be an open neighbourhood of y in B such that $f^{-1}(V_y) \subset U_a \cup U_b$; its existence follows from the continuity and compactness.

If L is a compact subset of V_y, then $f^{-1}(L)$ splits into two compacta $L' = U_a \cap f^{-1}(L)$ and $L'' = U_b \cap f^{-1}(L)$, since $f|f^{-1}(B)$ is strictly two-to-one. It follows that $f|L'$ and $f|L''$ are homeomorphisms onto L.

The sets V_y, $y \in B$, form an open cover of B. Let $\delta > 0$ be a Lebesgue number of that cover. Decompose B into a finite number of compacta of diameters $\le \delta$. One of the summands, say C, is of full dimension, hence of dimension $\ge \dim Y - 1$. Since $\operatorname{diam} C \le \delta$, the compactum C is contained in one of the sets V_y. Thus, $f^{-1}(C)$ splits into compacta C' and C'' such that $f|C'$ and $f|C''$ are homeomorphisms onto C (in view of the previous paragraph). Since $\dim X \le \dim Y$, we have $\dim C \ge \dim X$ (in fact we have equality, as X contains homeomorphic copies of C).

Corollary (Hurewicz 1933; cf. Kazhdan 1949, Sieklucki 1969). *There do not exist dimension-raising simple maps from compacta having property* (α).

We omit the proof as it is essentially the same as the proof of an analogous Corollary to the Splitting Lemma I given in Section 3.

From the Splitting Lemma II it follows that if there exists a dimension-raising simple map from a compactum, then this compactum contains two disjoint homeomorphic non-degenerate subcontinua (of full dimension).

This implies, for instance, that there do not exist dimension-raising simple maps from the Anderson–Choquet curve (1959), since no two of its non-degenerate subcontinua are homeomorphic; in other words, the Anderson–Choquet curve is thin. Note that this curve is far from being regular in the sense of ramification. For the same reasons the chainable continuum constructed by Andrews (1961) is thin.

Among one-dimensional continua which are not thin, an obvious example is the Sierpiński universal curve; that it is not thin follows from its universality, since thinness is inherited by subcompacta of full dimension, and there are an abundance of one-dimensional continua which are not thin. For instance, the Knaster simplest indecomposable (chainable) continuum is not thin, allowing simple maps onto the plane square, e.g. that suggested by its standard position in the plane.

Less obvious is the pseudo-arc, the unique chainable hereditarily indecomposable continuum; the authors are indebted to Dr. Janusz Prajs for this example. The reason for its not being thin is that the product of the pseudo-arc and the Cantor set can be embedded into the pseudo-arc. This product can be mapped, via the Cantor step function from the Cantor set onto a closed interval of the reals, onto the product of the pseudo-arc and the interval by means of a map which is simple. The image obviously has dimension 2 (everywhere). It is non-planable. The question of the existence of a dimension-raising simple map from the pseudo-arc into the plane remains open.

The authors are indebted to the referee for his constructive criticism.

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Reçu par la Rédaction le 3.1.1990;
en version modifiée le 7.12.1990