# Condorcet Cycles?
# A Model of Intertemporal Voting*

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**Abstract**

An intertemporal voting model is examined where, at each date, there is a pairwise majority vote between the existing chosen state and some other state, chosen randomly. Intertemporal voting simplifies the strategic issues and the agenda setting is as unrestricted as possible. The possibility of cycles is examined, both in the intertemporal extension to the Condorcet paradox and in more general examples. The set of possibilities is rich, as is demonstrated by an exhaustive study of a three person, three state world. Equilibrium in pure strategies may fail to exist but a weakening of the equilibrium concept to admit probabilistic voting allows a general existence result to be proved. The analysis leads to the development of a dominant state which extends the notion of a Condorcet winner.

**JEL Codes**: C73, D72, D78

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# 1 Introduction

Although the main emphasis of Condorcet (1785) was on the probability of making a 'correct' choice, his name is now most associated with the well-known paradox of majority voting. In its simplest, symmetric, form the paradox can be explained as follows: three individuals (1,2,3) have preferences over three alternatives $(x, y, z)$ as follows

$$\begin{array}{llll} 1: & x & y & z \\ 2: & z & x & y \\ 3: & y & z & x \end{array}$$

so that 1, for instance, most prefers outcome $x$, then $y$, and then $z$. A majority prefer outcome $x$ to $y$ (individuals 1 and 2), $y$ to $z$ (individuals 2 and 3), and $z$ to $x$ (individuals 2 and 3). Thus the majority voting rule gives rise to a ranking of alternatives that exhibits a cycle and there is no majority or Condorcet winner.

This is by far the most famous example in the collective choice literature. Principally, it is used to demonstrate the shortcomings of the majority voting rule. But it is more central that this. The preferences underlying the example - Condorcet preferences - are not only an example of preferences giving rise to cycles, they are also the only example (Inada (1969)). Specifically, if, over three alternatives, there are two individuals with preferences like individuals 1 and 2 in the example then, if it is never the case that there is an individual with 3's preferences, majority rule will not exhibit cycles and there will be a majority winner amongst any set of alternatives. More generally, Condorcet preferences play a crucial role in Arrow's (1963) proof of his impossibility theorem. In the proof, the existence of Condorcet preferences is used to show that the smallest group of individuals whose preferences are respected when they agree about a pairwise decision and everybody else disagrees with them, the so-called smallest almost decisive groups, consists of only one individual. This is the important step in Arrow's proof to show that the only collective choice rule satisfying 'reasonable' assumptions is a dictatorship.

It is common to suggest that the Condorcet paradox also implies that, in some circumstances, the process of decision making based upon majority rule will lead to a never ending series of decisions as individuals continually to vote to upset any proposed status quo. Whilst this is an interesting proposition, it does not follow from the Condorcet paradox example. Formally, equilibrium is defined to be a state which is a majority or Condorcet winner and, as there is no such equilibrium in the example, there is no implication about 'what will happen'. The main purpose of this paper is to extend the voting problem to an intertemporal setting and admit cycles as an equilibrium phenomenon. In particular, if there is sufficient discounting of the future then, with such myopia, short-term gains will dominate any long-term losses. In this case, cycles will be induced in the Condorcet example. But with less myopic preferences, the structure of equilibrium is less clear. It is the purpose of this paper to investigate this issue.

In voting situations, it is widely recognised that individuals may not vote for outcomes that seem to give them higher reward. The problem is usually approached as a game played between voters. For Farquharson (1969) and most work since, there is a decision mechanism which may have several stages, with voting at each stage, but there is a fixed conclusion to the procedure and then implementation of an outcome. One interpretation of this is that voters do not discount the future and it is the eventual outcome which is all important.

With several stages of voting, the agenda is crucial for the outcome. The agenda may be set exogenously (as in Farquharson), or endogenously (Banks (1985), Austen-Smith (1987)), but it is important that the process is finite so that an outcome can be implemented. One implication of this is that either the possibility of voting is a scarce good or it is made scarce by monopoly provision by an agenda setter. Adopting an alternative approach which makes the possibility of voting plentiful in supply raises the spectre of inconclusive decision making.

If the voting process takes place in real time then there is no need to reach a fixed outcome. Voters experience a path of outcomes and this, in principle, can last forever. For very impatient voters, it is only the direct outcome of any vote that is important; for patient voters, it will be where the voting process leads that will be important. For an example of repeated voting over time, see Banks and Duggan (2002).

The model examined in this paper assumes that a pairwise majority vote is taken every period between the status quo (the outcome implemented in the previous period) and some other state, chosen randomly with equal probability across all states. This process goes on forever so that even if no change occurs after some finite time, the then status quo will be subject to pairwise scrutiny against all other states. With a low rate of impatience, the possibility of voting is not a scarce resource - the rate of impatience can be viewed as a measure of scarcity of voting. It will be assumed that there is complete information and, in equilibrium each individual will be able to infer what will happen in the future, conditional on what is chosen in the present period and on the path of alternatives that will offered in the future. Thus, voters will be able to infer their expected utility from staying at the status quo or their expected utility from a specific change in outcome. With only one pairwise vote per period, there must be a majority winner each period. Equilibrium will require that voters beliefs concerning what will happen in the future be confirmed in equilibrium (voters cannot hold beliefs incompatible with the equilibrium).

We start by laying down the model of intertemporal choice and investigating equilibrium in a simple extension of the Condorcet example which gives rise to the paradox of voting. Specifically, we look at a three person, three state example with symmetry in states and voters. With such symmetry, the equilibrium *set* must be symmetric - it is possible that choosing state $y$ forever may be an equilibrium (an equilibrium is not required to be symmetric), but then choosing state $x$ forever or $z$ forever would also be equilibria. We also examine the possibility of cycles and steady states as equilibrium phenomena and determine the set of equilibria as a function of parameters. This analysis is conducted in section 3.

Section 4 examines equilibrium in all three person, three state examples where a Condorcet winner exists. It is shown that, in an intertemporal model, the Condorcet winner is not always selected as the eventual steady state. In particular, it is possible for equilibrium to involve cycles in this case. More interestingly, it is shown that when there is a Condorcet winner in the atemporal problem it is possible that no intertemporal equilibrium exists. Thus, it is possible that the behaviour induced by beliefs will contradict those beliefs, so ruling out behaviour based upon correct beliefs.

Section 5 shifts the focus away from specific examples towards the intertemporal voting problem with general preferences. Section 6 concentrates on a weakening of the concept of equilibrium which permits a general existence theorem to be proved. Section 7 develops a definition of attractiveness of a social state – a Generalized Condorcet Winner – with the property that there are equilibria where there is convergence to such a state. Concluding remarks are offered in Section 8.

# 2 The Basic Set-up

Time is discrete ($t = 0, 1, ....$) and there is an infinite horizon. At each date, social state $x_t$ must be chosen from some finite set $X$. Let $\mid X \mid = m + 1$. There is a finite set of voters N and the preferences of voter $i \in N$ can be expressed by an intertemporal utility function

$$U_i = \sum_0^{\infty} \beta^t u_i(x_t) \tag{1}$$

where $\beta, 0 < \beta < 1$, is the discount factor.

At the start of each period there is a status quo state, $\overline{x}$ at date 0, and $x_{t-1}$ otherwise. Assume that there is the possibility through pairwise majority voting, of changing the state. Specifically, assume that, at each date $t$, voters get to choose between the status quo and some other state, each other state being offered with equal probability 1/m. The new state is implemented for period $t$ if a strict majority vote for a change. The new state becomes the status quo for $t + 1$. This process treats all states other than the status quo symmetrically.

As all that matters from the past is the current status quo, it is reasonable to assume that individual behaviour is markovian. A strategy for voter $i$ is a function $s_i : X \times X \to \{0, 1\}$ determining voting intention - if $s_i(x, y) = 1$, voter $i$ votes for $y$ when the status quo is $x$. Given everybody's strategy, $U_i(x, \{s\})$ is $i$'s discounted future expected utility, starting from $x$ as the outcome at date 0. The set of strategies $\{s\}$ will be an equilibrium if, for all, $i$, $x$, $y$,

$$s_i(x, y) = 1 \quad \text{iff} \quad U_i(x, \{s\}) < U_i(y, \{s\}) \tag{2}$$

We are therefore looking at markovian (perfect Bayesian) equilibrium strategies under a weak dominance requirement - agent $i$ votes for $y$ over $x$ if he prefers the consequences starting from $y$, irrespective of the fact that his vote will 'count' only when his vote is pivotal.

# 3 The Condorcet Example

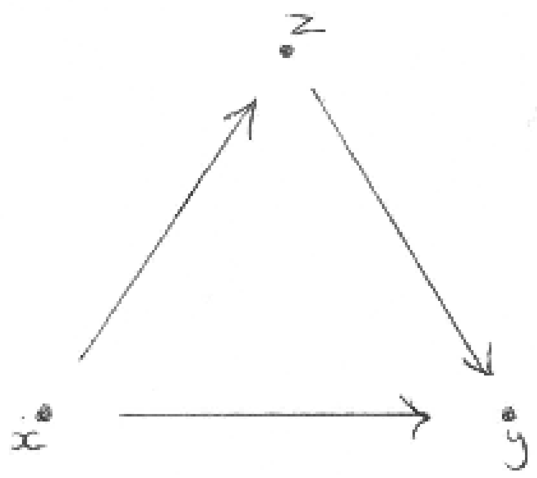
We first investigate equilibrium in an intertemporal version of the Condorcet example. Thus there are three states $(x, y, z)$ and three individuals (1,2,3). Individual $i$'s preferences are given by equation (1). The function $u_i$ is a cardinal function which permits any affine transformation. The utility of the worst (best) alternative can therefore be normalized to 0 (1), and it assumed that there is no indifference. If the three individuals are exactly symmetric then the instantaneous utility $u_i(x)$ will be given as in Table 1:

**Table 1**

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| 1 : | 1 | $m$ | 0 |
| 2 : | $m$ | 0 | 1 |
| 3 : | 0 | 1 | $m$ |

where $0 < m < 1$. These preferences imply the Condorcet preferences of the introduction. Incorporating these instantaneous utilities into the intertemporal utility function (1) allows us to see that preferences are dictated by two parameters, the discount factor $\beta$ and $m$, a parameter which is a measure of the preference for intertemporal variation. Abstracting from discounting, an agent prefers variation with equal weight on all three states to the constant median outcome if $\frac{1+m+0}{3} > m$ or $m < \frac{1}{2}$. This is akin to convexity of $u_i$ and we will refer to preferences being convex or concave depending upon whether $m$ falls short or exceeds $\frac{1}{2}$.

We now investigate equilibrium strategies. The optimal behaviour of a voter depends upon the future which is induced by the choice of a particular state in the present. If $y$ is chosen when $x$ is the status quo, then a strict majority prefers the path of states starting at $y$ rather than $x$. This implies that when $y$ is the status quo, $x$ will not be chosen over it. Thus the movement between states induced through voting is a directed graph over the set of states. This may be incomplete. Figure 1 is one such example




where, eventually, state $y$ will be chosen and then it will become a steady state outcome.

With three alternatives there are $3^3 = 27$ possible directed graphs though many will fail to be compatible with equilibrium. To determine equilibrium, assume that there is an equilibrium where two states, $y$ and $z$ say, are both steady states. Consider what happens when $z$ is the status quo and $y$ is proposed as an alternative. If $y$ is chosen, individual 1 will receive, applying Table 1, $m$ forever ($U_1(y) = \frac{m}{1-\beta}$); if $z$ is chosen then he will receive 0 forever ($U_1(z) = 0$). Thus he will vote to change to $y$. Individual 3 will also gain. Thus, $z$ cannot be a steady state: there is at most one steady state in any equilibrium. If $y$ is a steady state then individuals 1 and 3 will always vote for $y$ in a contest between $y$ and $z$. If $z$ does not beat $x$ then the transfer from $x$ must be directly towards $y$. However, individuals 1 and 2 will not vote for this change. Thus, the only possible voting outcomes which sustain

$y$ as a steady state are as in Figure 1 (we have yet to show that $y$ must positively beat $x$ in a contest between the two).

To determine the conditions under which this is an equilibrium, it is necessary to ensure that individuals have an incentive to induce these voting outcomes. Consider the vote between $x$ and $y$. If $y$ is a steady state, individual 2 will prefer to stay at $x$ (moving to $y$ gives the worse possible future path of outcomes of utility 0 forever) and 3 will vote for $y$ (thus giving the best possible future path). What about 1? Suppressing the strategies from the discounted utility functions gives,

$$U_1(x) = 1 + \beta(\tfrac{1}{2}U_1(y) + \tfrac{1}{2}U_1(z)) \tag{3}$$
$$U_1(y) = m + \beta(\tfrac{1}{2}U_1(y) + \tfrac{1}{2}U_1(y)) \tag{4}$$
$$U_1(z) = 0 + \beta(\tfrac{1}{2}U_1(z) + \tfrac{1}{2}U_1(y)) \tag{5}$$

which then solve to give:

$$U_1(x) = 1 + \frac{m\beta/2}{(1-\beta)(1-\beta/2)} \tag{3'}$$
$$U_1(y) = \frac{m}{(1-\beta)} \tag{4'}$$
$$U_1(z) = \frac{m\beta/2}{(1-\beta)(1-\beta/2)} \tag{5'}$$

so that $y$ will be weakly preferred to $x$ if

$$m \geq 1 - \beta/2 \tag{6}$$

If (6) is not strict then there will be no direct transfer from $x$ to $y$.

It is still necessary to confirm that, between $x$ and $z$, $z$ will be chosen. Individual 1 will vote against the change, irrespective of what happens in a ballot between $x$ and $y$. Individual 2 will vote for the change if and only if $y$ wins in a ballot between $x$ and $y$. This requires the inequality in (6) to be strict. Individual 3 always votes for the change (it is better to spend time in state $z$ rather than state $x$ before transfer to ($y$).

We have thus shown:

**Proposition 1** *If $m > 1 - \beta/2$ (preferences are sufficiently concave), there is an equilibrium which involves voting transfers as in Figure 1 and state $y$ being reached as a steady state.*

Through symmetry, there are two other equilibria with $x$ being reached as a steady state and $z$ being reached as a steady state.

If the opportunities to change the status quo are not scarce then voting can occur often and the discount factor will be close to unity. In this case, reaching a steady state can occur as an equilibrium whenever preferences are strictly concave ($m > \frac{1}{2}$).

We now investigate the possibility of an equilibrium with cycles. When $\beta \to 0$, we have seen that the motivation behind the Condorcet paradox is applicable. We are interested more in the case where $\beta$ is closer to unity. Consider a voting outcome as in Figure 2.

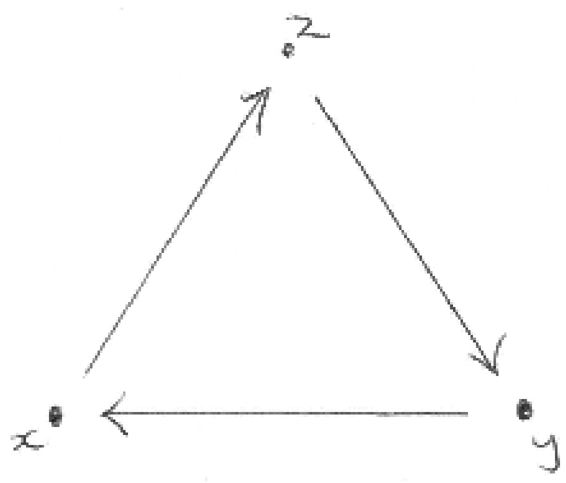



Figure 1:

To examine when this can occur as an equilibrium, consider voting intentions over $\{x, y\}$. For 2, anything is better than having state $y$ for the next period; for 3, state $y$ gives the highest flow return for one period and delays the path of (lower) returns for one period. Thus, individual 1 is pivotal and expected discounted utility is given by

$$U_1(x) = 1 + \beta(\tfrac{1}{2}U_1(x) + \tfrac{1}{2}U_1(z)) \tag{7}$$

$$U_1(y) = m + \beta(\tfrac{1}{2}U_1(y) + \tfrac{1}{2}U_1(x)) \tag{8}$$

$$U_1(z) = 0 + \beta(\tfrac{1}{2}(U_1(z) + \tfrac{1}{2}U_1(y)) \tag{9}$$

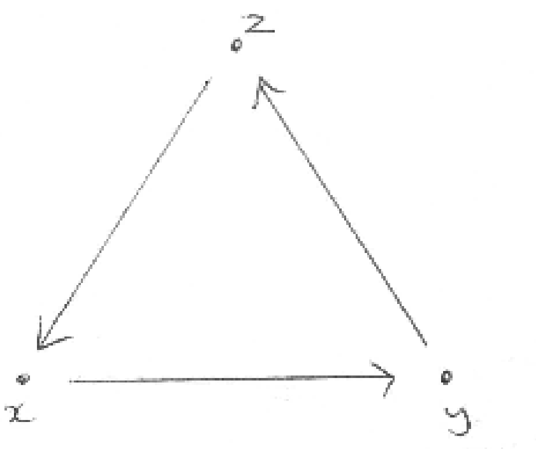
These conditions can be solved to give expected discounted utilities. Individual 1 is more likely to vote for $x$ over $y$, the lower is $m$. Indifference occurs when $U_1(x) = U_1(y) = \frac{m}{1-\beta}, U(z) = \frac{m\beta/2}{(1-\beta)(1-\beta/2)}$ and this occurs when $m = 1 - \beta/2$. As voting intentions over $\{y, z\}$ and $\{z, x\}$ are symmetric, we have

**Proposition 2** *If $m < 1 - \beta/2$ (preferences are sufficiently convex), there is a cyclic equilibrium which involves voting transfers as in Figure 2.*

Note that, as $m < 1$, a cyclic equilibrium exists when there is sufficient discounting.

A gap is left by Propositions 1 and 2 when $m = 1 - \beta/2$. Both classes of equilibria depend upon the pivotal voter wishing to vote for change. If $m = 1 - \beta/2$, the pivotal voter is indifferent about the outcome, but other voters must believe that a particular outcome will obtain to sustain their behaviour in different ballots. If a strategy can be specified for indifferent voters then there can be equilibria of the type described by Propositions 1 and 2 when $m = 1 - \beta/2$. This is a measure zero possibility.

Propositions 1 and 2 do not exhaust all the possibilities of potential equilibria. Consider the voting outcome in Figure 3 which describes a perverse cycle.




This possibility is perverse because, when there is a vote for change, a majority of voters see their flow payoff reduce. Can perverse cycles arise as an equilibria? Consider voting intentions over $\{x, y\}$. Individual 3 prefers $y$ because it hastens a path of returns which dominates the current flow return of 0; individual 1, on the other hand, will vote for $x$ because it sustains a flow return of unity which dominates the future path of returns. Thus, individual 2 is pivotal (recall that with the belief by voters in a normal cycle (Figure 2), it was individual 1 who was pivotal). For 2, expected discounted utility is given by:

$$U_2(x) = m + \beta(\tfrac{1}{2}U_2(x) + \tfrac{1}{2}U_2(y)) \quad (10)$$

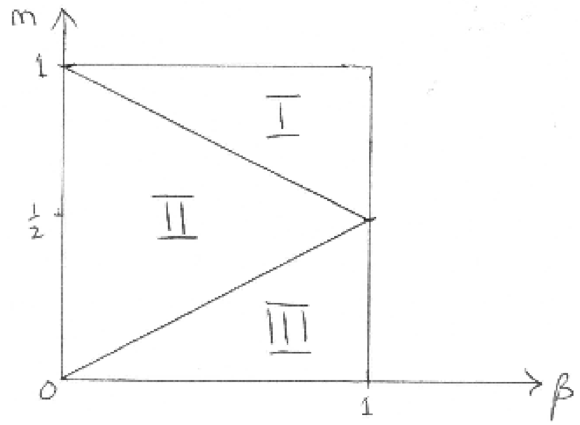
$$U_2(y) = 0 + \beta(\tfrac{1}{2}U_2(y) + \tfrac{1}{2}U_2(z)) \quad (11)$$

$$U_2(z) = 1 + \beta(\tfrac{1}{2}(U_2(z) + \tfrac{1}{2}U_2(x)) \quad (12)$$

Individual 2 is more likely to prefer $y$ over $x$ when $m$ is small. When there is indifference, we have $U_2(x) = U_2(y) = \frac{m}{1-\beta}$ and $U_2(z) = m(1 - \beta/2)/(1-\beta)(\beta/2)$ and this occurs when $m = \beta/2$. As voting intentions over the other two possible pairs are symmetric, we have

**Proposition 3** *If $m < \beta/2$, there is a perverse cyclic equilibrium which involves voting transfers as in Figure 3.*

We can collect together the results of these three propositions. Figure 4 divides the parameter space into three regions. In I, there are three voting equilibria, each involving a move towards a single steady state; in II, the only equilibrium is the intertemporal extension of the standard Condorcet cycle; in III, there are two equilibria, the Condorcet cycle and a perverse cycle. The possible existence of perverse cycles demonstrates that behaviour in intertemporal voting problems can be driven predominantly by the beliefs of what will happen in the future, rather than by short-term pay-offs. As the figure makes clear:




when $\beta \to 0$, behaviour is myopic and normal Condorcet cycles can be expected to obtain; when $\beta \to 1$, which occurs when the time between ballots is short, perverse cycles are as likely as normal cycles and steady state equilibria can obtain for a range of parameter values.

To complete this section, we consider an asymmetric version of the model where ordinal preferences are the same as in the above model but cardinal preferences differ across individuals. In particular, we take the case where each agent may have a different median state utility value $m_i$. When can a voting outcome as portrayed in Figure 1 arise? Our previous analysis showed that this depended upon $m_1 > 1 - \beta/2$. Similarly, $x$ ($z$) will be the steady state if $m_2 > 1 - \beta/2$ ($m_3 > 1 - \beta/2$). If $m_1, m_2, m_3 < 1 - \beta/2$ then there are no steady state equilibria but this is exactly the condition needed for a normal Condorcet cycle.

This demonstrates that, apart from on the boundaries between the different classes of equilibria, an equilibrium always exists. This is a result in stark contrast to the atemporal version of the model built upon Condorcet winners as equilibria. We also note that, in this asymmetric model, perverse cycles can arise when $m_1, m_2, m_3 < \beta/2$.

Finally, we note that the asymmetric version of the model favours the existence of steady state equilibria over cycles (steady state equilibria require that the relevant inequality be satisfied by at least one agent, cycles require the relevant inequality to be satisfied by all agents).

# 4 Non-Condorcet Examples

This section investigates equilibria when underlying preferences do not imply a Condorcet cycle. We again concentrate on the case of three states and three individuals as examples with more states or individuals must always embody components of three state, three person cases within them. For simplicity, we again rule out individual indifference between states (see the next section).

Consider first the case where there is unanimity of view over some pair of states $x$ and $y$, say. If everybody prefers $x$ to $y$ then two individuals, call them 1 and 2, either prefer $z$ to $x$ or $x$ to $z$. In the first case, 1 and 2 share the same preference of $z$ over $x$ over $y$ and they will never vote for a move from $z$, they will always vote for $z$ in a pairwise ranking and, given this, they will always vote for $x$ over $y$. We therefore have:

**Proposition 4** *If two individuals have the same preferences over the triple of alternatives then their preferences are respected in the intertemporal equilibrium.*

Next , consider, the second case where 1 and 2 prefer $x$ to $z$. Now, 1 and 2 will never vote for a move from $x$, given this they will always vote for $x$ in a pairwise ballot. Voting over $\{y, z\}$ will depend upon individual preference but, whatever, state $x$ will always be reached. Putting together both cases gives

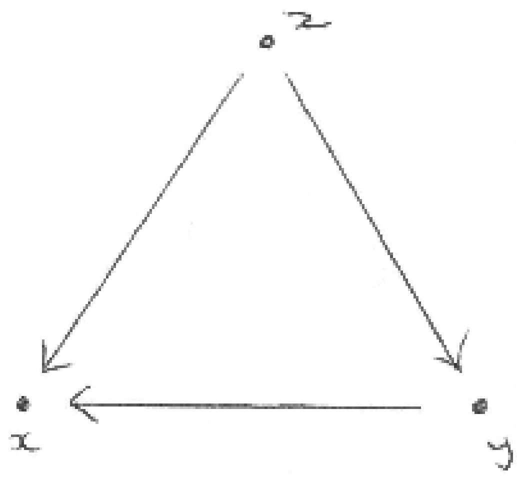
**Proposition 5** *If there is unanimity in preference over some pairwise ranking then there is a unique intertemporal equilibrium which involves a steady state.*

The final case to consider arises when there is no pairwise unanimity but there is a Condorcet winner. Without loss of generality, let $x$ be the Condorect winner with individuals 1 and 2 preferring $x$ to $y$ and 1 and 3 preferring $x$ to $z$. By suitable of labelling of states, assume that 1 prefers $y$ to $z$. If there is no pairwise unanimity then 2 must prefer $z$ to $x$ and 3 must prefer $y$ to $x$. Cardinal preferences must therefore be as in Table 2, where $0 < m_1, m_2, m_3 < 1$.

**Table 2**

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| 1 : | 1 | $m_1$ | 0 |
| 2 : | $m_2$ | 0 | 1 |
| 3 : | $m_3$ | 1 | 0 |

The first question to be addressed is to ask when outcome $x$ will win in any pairwise ranking. If $x$ always wins, the voting over $\{y, z\}$ will be determined by purely short-term interests and $y$ will be the chosen outcome. This, equilibrium must be as in Figure 5




Individuals 1 and 3 will vote for $y$ over $z$. Consider the ballot over $\{x, y\}$. As $y$ is chosen from $\{y, z\}$, the choice is between $x$ forever and $y$ forever so individuals 1 and 2 will vote for $x$ and $y$. Finally, consider the ballot over $\{x, z\}$. Individual 1 prefers outcome $x$ forever to any other intertemporal path so will vote for $x$ and $z$. For individual 2, we have

$$U_2(x) = \frac{m_2}{1-\beta} \tag{13}$$

$$U_2(y) = 0 + \beta(\tfrac{1}{2}U_2(x) + \tfrac{1}{2}U_2(y)) \tag{14}$$

$$U_2(z) = 1 + \beta(\tfrac{1}{2}(U_2(x) + \tfrac{1}{2}U_2(y)) \tag{15}$$

which gives

$$U_2(z) = 1 + \frac{\beta/2}{1-\beta/2}\left(\frac{m_2}{1-\beta}\right). \tag{16}$$

Thus, 2 will vote for $x$ over $z$ if $m_2 > 1 - \beta/2$. For individual 3, we have

$$\begin{aligned} U_3(x) &= \tfrac{m_3}{1-\beta} & (17)\\ U_3(y) &= 1 + \beta(\tfrac{1}{2}U_3(y) + \tfrac{1}{2}U_3(x)) & (18)\\ U_3(z) &= 0 + \beta(\tfrac{1}{2}(U_3(x) + \tfrac{1}{2}U_3(y)) & (19) \end{aligned}$$

which gives

$$U_3(z) = \frac{\beta/2}{1-\beta/2}\left(1 + \frac{m_3}{1-\beta}\right). \tag{20}$$

Thus, 3 will vote for $x$ over $z$ if $m_3 > \beta/2$; if 2 and 3 have the same preferences then individual 3 is more likely to vote for $x$ and $z$. We have

**Proposition 6** *If there is a Condorcet winner, but no pairwise unanimity, then preferences are as in Table 2 (for some labelling of individuals and states). The Condorcet winner is the steady state equilibrium if either $m_2 > 1 - \beta/2$ or $m_3 > \beta/2$. If $m_2 < 1 - \beta/2$ and $m_3 < \beta/2$ then the Condorcet winner cannot be the steady state.*

Proposition 6 shows that the Condorcet winner can fail to be the steady state if preferences are sufficiently convex. Indeed, if all three individuals have sufficiently convex preferences then, with large enough $\beta$, it is easy to show that it is possible to support cycles as equilibrium phenomena. Similarly, different configurations of preferences can lead to a steady state away from the Condorcet winner. Thus, embedding the decision making into an intertemporal voting problem strips the Condorcet winner of its position as the natural outcome (but see Section 7 below).

In the last section it was shown that, with Condorcet preferences, an equilibrium exists generically. When preferences imply the existence of a Condorcet winner, Proposition 6 tells us that, when $\beta$ is sufficiently small, $m_3 > \beta/2$ will be assured and equilibrium exists (with the Condorcet winner as steady state). However, when $\beta$ is close to unity, the intertemporal path dictates individual preference. In particular, if the individuals have different preferences towards intertemporal variability, there may be no equilibrium over a range of parameter values.

**Proposition 7** *If preferences are as in Table 2,* $m_1 > \frac{1}{2}, m_2, m_3 < \frac{1}{2}$*, then, as* $\beta \to 1$*, there is no intertemporal equilibrium.*

The proof is given in the appendix. This non-existence result implies that there are no beliefs that individuals can hold which are confirmed in equilibrium. At any date, agents hold some belief about the future. This allows them to determine which of some pair of states that they would prefer. This determines the voting outcome at that date. Thus, the model as laid down has the feature that, given beliefs about the future, an outcome is determined at every date. Non-existence does not relate to the inability to choose an outcome at each date; instead, it says that agents' beliefs will determine outcomes and the outcomes so determined cannot be compatible with the beliefs. It is interesting to note that the generic non-existence only arises when there is a Condorcet winner: when there is no Condorcet winner, the structure biases towards the existence of a cycle - depending upon the preferences for variability, cycles and/or steady states will be supported as equilibria.

# 5 The General Case

This section investigates possibilities when there are many individuals and many states. Assuming that the number of states is finite, there are a finite number of pairwise-state dependent voting outcomes that can describe an equilibrium. Generically, any individual will have a strict preference in any pairwise ballot so that, if there are an odd number of individuals, each ballot will be decisive: any equilibrium configuration will be a directed graph with all states connected.

The implication of equilibria involving all states being connected rules out an equilibrium configuration with two steady states at $x$ and at $y$. When a ballot occurs between $x$ and $y$, one of them will be the winner, so ruling out the other as a steady state. The other possibility of equilibrium requires the existence of a subset of states $Z$ which recur infinitely often with a 'cycle' taking place between the states - each state wins in a pairwise ballot with some other state in the subset, any pairwise ballot between $x$ and $y$ where

$x \in Z$ and $y \notin Z$ is won by $x$. States that tend to win in more ballots against other elements of $Z$ will recur more often in the cycle. Clearly, there can be no equilibrium with two 'cycles' defined by unconnected subsets $Z_1$ and $Z_2$ as the loser in the ballot between $z_1$ and $z_2$, $z_1 \in Z_1, z_2 \in Z_2$, is ruled out as a possible candidate for a cycle subset. Similarly, it is not possible to have an equilibrium with both a steady state and a cycle.

Assume that equilibrium involves a cycle. Within the cycle, the shortest sub-cycle must, generically, be of length three - if $x$, $y$, $z$ are part of the shortest sub-cycle, assume that transfer is from $x$ to $y$ and $y$ to $z$. If $x$ is chosen over $z$ then there is a 3-cycle, if $z$ is chosen over $x$ then the shortest sub-cycle is one link less, excluding $y$, which is a contradiction. Thus the motivation for equilibrium cycles is similar to that we have already studied, the value to an individual of continuing to support a cycle relating, when $\beta \to 1$, to the frequency of occurrence of different outcomes within the cycle.

# 6 An Existence Theorem

The model as laid down ensures that, with some beliefs concerning the future, an outcome is determined in every period. The troublesome result is that it may be impossible for beliefs to be confirmed (Proposition 7). In these situations, the analysis is mute because the defined equilibrium concept does not apply. There are a number of ways of relaxing the equilibrium concept which permits beliefs to be confirmed. Firstly, if behaviour can be time dependent, with equilibrium strategies fixed for $t+1$ forwards, strategies and equilibrium for date $t$ are determined and backward induction defines equilibrium. This process is straightforward, at least when the model has finite time. Secondly, it is possible that there are mixed strategies where individuals determine the probability of voting for a particular outcome in any pairwise ballot. Equilibrium then relates to the probability that an outcome will emerge as the majority winner in a pairwise ranking with any other outcome.

To formalize this, let the strategy of individual $i$ be a function $\widetilde{s}_i : X \times X \to [0,1]$ where $\widetilde{s}_i(x,y)$ denotes the probability that $i$ votes for $y$ over $x$

when the status quo is $x$ and $\{x, y\}$ is the pairwise ballot. The probability that the vote is for $x$ is assumed to be $1 - \widetilde{s}_i$ though it would be possible to introduce a probability of abstention by the inclusion of another strategy function. Strategies determine stochastic intertemporal transfers between states and it is possible to compute $i$'s discounted future expected utility, starting from $x$ as the outcome at date 0. As earlier, this can be expressed as $U_i(x, \{\widetilde{s}\})$. To be more specific, fix all other individuals' strategies at $\{\widetilde{s}_{-i}\}$. Given this, let $P_i(x, y, \{\widetilde{s}_{-i}\})$ be the probability that $y$ is chosen in the pairwise ranking of $\{x, y\}$ when $i$ votes for $x$ over $y$ and let $Q_i(x, y, \{\widetilde{s}_{-i}\})$ be the probability $y$ is chosen when $i$ votes for $y$ over $x$. The function $U_i$ will satisfy (where other individuals strategies are suppressed as arguments).

$$\begin{aligned} U_i(x, \{\widetilde{s}\}) &= u_i(x) + \frac{\beta}{m} \sum_{y \in X|x} [P_i(x, y)(1 - \widetilde{s}_i(x, y)) + Q_i(x, y)\widetilde{s}_i(x, y)] U_i(y, \{\widetilde{s}\}) \\ &\quad + \frac{\beta}{m} \sum_{y \in X|x} [1 - P_i(x, y)(1 - \widetilde{s}_i(x, y)) - Q_i(x, y)\widetilde{s}_i(x, y)] U_i(x, \{\widetilde{s}\}) \end{aligned} \tag{21}$$

For fixed behaviour of other individuals and behaviour in the future optimized, $i$ chooses $\widetilde{s}_i(x, \cdot)$ to maximize (21) and, as this is linear in the strategy vector, the objective function is quasi-concave. Thus, the optimal $\widetilde{s}_i$ is an interval. As (21) is also continuous in its arguments, and $\widetilde{s}_i$ is chosen from a closed interval, an optimal $\widetilde{s}_i$ always exists. A standard existence theorem (Fudenberg and Tirole (1991), Theorem 1.1), based upon an application of Kakutani's fixed point theorem ensures existence - essentially, we have a mixed strategy equilibrium where agents are indexed by an individual $i$ and a pair of states $\{x, y\}$.

**Proposition 8** *If individuals vote probabilistically, an intertemporal equilibrium exists.*

If such a probabilistic equilibrium involves cycles then the consequences of randomization is to slow down the speed of the cycle and to ensure that the outcome incorporates a mixture of the elements that drive both a steady state or a cyclic equilibrium. For instance, taking the preferences underlying the non-existence result given in Proposition 7, it is possible for a stochastic cycle to exist $\{x \leftarrow y,\ y \leftarrow z,\ z \leftarrow x\}$. For instance, if $m_2 = m_3$ then an equilibrium exists with individual 1 being pivotal over $\{x, y\}$ and voting stochastically for $x$ and individual 3 being pivotal over $\{x, z\}$ and voting stochastically for $z$. The movement from $x$ to $z$ is slowed, allowing 1 to consider $x$ to be a satisfactory move from $y$ - he will be indifferent - and the extra delay that occurs when $y$ is attained allows 3 to consider the move from $x$ to $z$ to be satisfactory, again with indifference.

# 7 Generalized Condorcet Winners

In a static model, a state that is a Condorcet winner is, almost by definition, an equilibrium state. However, we have seen that this does not apply in the intertemporal setting. In such a setting, a state may be a Condorcet winner so that, as a steady state, it dominates any other path which is a steady state - a particular state chosen forever. However, it may fail to be preferred by an majority to a path of states (leading to a 'cycle') which can be supported through majority voting. A strengthening of the conditions for a Condorcet winner gives:

**Generalized Condorcet Winner.** *Let $P_X$ be the set of all probability distributions defined over the set of states $X$. State $x$ is a generalized Condorcet winner (GCW) if for all $p \in P_x$ such that $p_x \neq 1$, for all $y \in X \mid x$, the inequality*

$$u_i(x) > \sum_{y \in X} p_y u_i(y)$$

*holds for a strict majority of the population (this majority group being dependent on $y$).*

Thus, if $x$ is a GCW then a majority prefer it to any probability mixing of other states. We have:

**Proposition 9** *Let $x$ be a GCW. There exists a probabilistic voting equilibrium with $x$ as a steady state.*

This result is proved in the appendix. The result does not rule out the existence of other equilibria where $x$ is not a steady state: if the belief is that $x$ is not a steady state then a majority may not vote for $x$ over some other state $y$ leading to a path of states because $x$ itself will lead to a path of states.

# 8 Concluding Remarks

The standard approach to the investigation of voting with 'foresight' looks at an atemporal voting problem and demands of the equilibrium concept, e.g. sophisticated voting, that agents recognise their strategic role. By looking at an intertemporal problem, one can ensure that the voting problem at each point of time is sufficiently simple so that the equilibrium concept and optimal behaviour are uncontroversial. The cost that is paid is that it is necessary to specify an extensive form game - an agenda - which, through its construction, will in part determine the type of outcome reached. In this paper, an attempt has been made to ensure that the agenda is flexible in the sense that, over time, every chosen outcome will be faced by every other outcome repeatedly. In addition, every finite agenda path will recur infinitely often. As long as individuals are sufficiently patient, they can recognise the flexibility of the agenda - in particular, equilibria are not determined by a restrictive agenda *per se*.

Within the intertemporal model, the nature of possible equilibria is rich, even when the problem is simple. This paper has provided an exhaustive analysis of the three state, three agent model. In this model with preferences as in the Condorcet paradox, it is possible to have a equilibrium Condorcet cycle as suggested by the paradox, a perverse cycle where a majority lose from every change, or a steady state. When preferences give rise to a Condorcet winner, the set of possibilities is further 'enriched' to include the possibility of no equilibrium.

Finally, it has been shown that general models must have, embedded within them, the features of three state, three agent models. Pure strategy equilibrium may take the form of a steady state, a cycle, or there may be non-existence. However, a probabilistic equilibrium will always exist. If states are sufficiently attractive - they are Generalized Condorcet Winners - then equilibria always exist with these states as steady states of the system.

# Appendix

### Proof of Proposition 7

We show that with preferences as specified, there are eight possible equilibrium configurations and none of them are supportable as equilibrium (no pairwise ranking involves a ballot outcome that is sensitive to the status quo).

Let configurations be denoted so that the Figure 5 configuration is presented as $\{x \leftarrow y, y \leftarrow z, z \rightarrow x\}$. Consider the eight configurations in turn:

1. $\{x \leftarrow y,\ y \leftarrow z,\ z \rightarrow x\}$. As $m_2$, $m_3 < \frac{1}{2}$ and $\beta$ is large, individuals 2 and 3 will vote for $z$ over $x$. #
2. $\{x \rightarrow y,\ y \leftarrow z,\ z \rightarrow x\}$. Individuals 1 and 2 will vote for $x$ over $y$. #
3. $\{x \leftarrow y,\ y \rightarrow z,\ z \rightarrow x\}$. Individuals 1 and 3 will vote for $y$ over $z$. #
4. $\{x \rightarrow y,\ y \leftarrow z,\ z \leftarrow x\}$. Individuals 1 and 3 will vote for $x$ over $z$. #
5. $\{x \rightarrow y,\ y \rightarrow z,\ z \leftarrow x\}$. Individuals 1 and 3 will vote for $y$ over $z$. #
6. $\{x \leftarrow y,\ y \rightarrow z,\ z \leftarrow x\}$. Individuals 1 and 3 will vote for $y$ over $z$. #
7. $\{x \leftarrow y,\ y \leftarrow z,\ z \leftarrow x\}$. As $m_1 > \frac{1}{2}$ and $\beta$ is large, individuals 1 and 3 will vote for $y$ over $x$. #
8. $\{x \rightarrow y,\ y \rightarrow z,\ z \rightarrow x\}$. As $m_1 > \frac{1}{2}$ and $\beta$ is large, individuals 1 and 3 will vote for $y$ over $z$. #

As this exhausts the configurations, there is no equilibrium.

### Proof of Proposition 9

To prove this result, we construct an equilibrium with $x$ as a steady state. Fixing intra period utility functions, consider equilibrium in the model when individuals have the discount factor $\left(\frac{m}{m+1}\right)\beta$ and the state space is $X|\,x$, i.e. the GCW $x$ is excluded from consideration and the number of states is

$m$. By Proposition 8, an equilibrium exists in the truncated model. In this equilibrium, let $\widetilde{p}(y, z, \tau)$ be the probability that, starting at y, state z will be chosen after $\tau$ periods ($\widetilde{p}(y, y, 0) = 1$). Individual $i$'s preference for $w$ over $y$ in a pairwise ranking is judged by comparing

$$\widetilde{U}_i(w) = \sum_{\tau \geq 0} \sum_{z \in X|x} \left( \left( \frac{m}{m+1} \right) \beta \right)^{\tau} \widetilde{p}(w, z, \tau) u_i(z) \tag{A1}$$

with

$$\widetilde{U}_i(y) = \sum_{\tau \geq 0} \sum_{z \in X|x} \left( \left( \frac{m}{m+1} \right) \beta \right)^{\tau} \widetilde{p}(y, z, \tau) u_i(z) \tag{A2}$$

Individual $i$ will definitely vote for $w$ over $y$ if $\widetilde{U}_i(w) > \widetilde{U}_i(y)$, he may randomize his vote if $\widetilde{U}_i(w) = \widetilde{U}_i(y)$.

Now consider the model with state space $X$ where individuals have discount factor $\beta$. We postulate a candidate equilibrium where, between $w$ and $y$, $w, y \in X| x$, individuals vote as in the truncated model and between $w$, $w \in X| x$, and $x$, the GCW, a strict majority vote for $x$. With such voting $x$ is a steady state.

We must check that individuals are maximizing their utility by supporting this candidate equilibrium. When $x$ is offered it will be chosen and then it will be chosen forever. Thus, starting at state $w$, the probability that $x$ is chosen after $\tau$ periods is given by

$$p(w, x, \tau) = 1 - \left( \frac{m}{m+1} \right)^{\tau} \tag{A3}$$

Here $\left(\frac{m}{m+1}\right)^{\tau}$ is the probability that $x$ has not yet arisen on the agenda.

If $w$ is the initial state then the probability that $z, z \in X| x$, is chosen after $\tau$ periods is given by

$$p(w, z, \tau) = \widetilde{p}(w, z, \tau) \left( \frac{m}{m+1} \right)^{\tau} \tag{A4}$$

(A3) holds because the probability that $x$ has not been chosen after $\tau$ periods is $\left(\frac{m}{m+1}\right)^{\tau}$ and, conditional on the fact that $x$ has not yet arisen on the agenda, behaviour and probability of outcomes is the same as in the truncated model. Individual $i$'s expected utility, starting from state $w$, is given by

$$U_i(w) = \sum_{\tau \geq 0} \sum_{z \in X} \beta^{\tau} p(w, z, \tau) u_i(z) \tag{A5}$$

Using (A3) and (A4), this can be written as

$$\begin{aligned} U_i(w) &= \sum_{\tau\geq 0}\sum_{z\in X|x} \beta^\tau \left(\frac{m}{m+1}\right)^\tau \widetilde{p}(w,z,\tau)u_i(z) + \\ & \sum_{\tau\geq 0}\left(1-\left(\frac{m}{m+1}\right)^\tau\right)\beta^\tau u_i(x) \end{aligned} \tag{A6}$$

Thus,

$$U_i(w) = \widetilde{U}_i(w) + \sum_{\tau\geq 0}\left(1-\left(\frac{m}{m+1}\right)^\tau\right)\beta^\tau u_i(x) \tag{A7}$$

The second term in (A7) is independent of $w$ so that decisions in the truncated model, based upon $\widetilde{U}_i(\cdot)$, remain optimal in the untruncated model, based upon, $U_i(\cdot)$.

We now need only check that in the ballot between $w$ and $x$, the GCW, a strict majority will vote for $x$. If $x$ is chosen then we have

$$U_i(x) = \sum_{\tau\geq 0}\beta^\tau u_i(x) \tag{A8}$$

Using (A5), $x$ will be strictly prefered to $w$ by $i$ if

$$u_i(x) > (1-\beta) + \sum_{\tau\geq 0}\sum_{z\in X}\beta^\tau p(w,z,\tau)u_i(z). \tag{A9}$$

Define $p(z)$ by

$$p(z) = (1-\beta)\sum_{\tau\geq 0}\beta^\tau p(w,z,\tau) \tag{A10}$$

As $p(w,z,\tau)\geq 0$ and $\sum_{z\in X} p(w,z,\tau) = 1$, we have

$$p(z) \geq 0 \quad \text{for all } z \tag{A11}$$

and

$$\begin{aligned} \sum_{z\in X} p(z) &= \sum_{z\in X}(1-\beta)\sum_{\tau\geq 0}\beta^\tau p(w,z,\tau) \\ &= \sum_{\tau\geq 0}(1-\beta)\beta^\tau \sum_{z\in X} p(w,z,\tau) \\ &= \sum_{\tau\geq 0}(1-\beta)\beta^\tau \\ &= 1 \end{aligned} \tag{A12}$$

Also, $p(w, x, 0) \neq 1$ so $p(x) \neq 1$. Thus, (A9) reduces to

$$u_i(x) > \sum_{z \in X} p(z) u_i(z) \tag{A13}$$

and, from the definition of a GCW, this will be satisfied for a strict majority of the population. We have now shown that the candidate equilibrium is, indeed, supported by optimal behaviour and the result is proved.

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