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Cone Invariance and Rendezvous of Multiple Agents

Raktim Bhattacharya, Abhishek Tiwari, Jimmy Fung and Richard M. Murray

College of Engineering and Applied Science

California Institute of Technology

Pasadena, CA, 91125.

Abstract

In this paper we present a dynamical systems framework for analyzing multi-agent rendezvous problems and characterize the dynamical behavior of the collective system. Recently, the problem of rendezvous has been addressed considerably in the graph theoretic framework, which is strongly based on the communication aspects of the problem. The proposed approach is based on set invariance theory and focusses on how to generate feedback between the vehicles, a key part of the rendezvous problem. The rendezvous problem is defined on the positions of the agents and the dynamics is modeled as linear first order systems. The proposed framework however is not fundamentally limited to linear first order dynamics and can be extended to analyze rendezvous of higher order agents.

In the proposed framework, the problem of rendezvous is cast as a stabilization problem, with a set of constraints on the trajectories of the agents defined on the phase plane. We pose the n-agent rendezvous problem as an ellipsoidal cone invariance problem in the n dimensional phase space. Theoretical results based on set invariance theory and monotone dynamical systems are developed. The necessary and sufficient conditions for rendezvous of linear systems are presented in form of linear matrix inequalities. These conditions are also interpreted in the Lyapunov framework using multiple Lyapunov functions. Numerical examples that demonstrate application are also presented.

Index Terms

Multi-agent rendezvous, cooperative dynamical systems, monotone systems, cone invariance, non-negative matrices.

I. INTRODUCTION

Recently there has been considerable interest in multi-agent coordination or cooperative control [1]. This has led to the emergence of several interesting control problems. One such problem is the *rendezvous problem*. In

R. Bhattacharya is currently with Texas A&M University, Aerospace Engineering

J. Fung is currently with the Computational Analysis and Simulation Group, Los Alamos National Laboratory, Los Alamos, NM 87545

a rendezvous problem, one desires to have several agents arrive at predefined destination points *simultaneously*. Cooperative strike or cooperative jamming are two examples of the rendezvous problem. In the first scenario, multiple strikes are executed *within a time interval*, from different agents firing from different distances and traveling at different speeds. In the second scenario, one or more agents need to start jamming *slightly before* the strike vehicle enters the danger zone and sustain jamming until strike vehicle exits. In both the scenarios, it is imperative that all the agents act simultaneously else the objective is not fulfilled.

The idea of rendezvous extends beyond just convergence to a static set of destination points or the origin. Rendezvous can also entail formation flying or interception problems where the origin is effectively moving. Interception of incoming ballistic missiles is a rendezvous problem where the origin becomes a moving target and one of the agents is non cooperating. Formation flying is a type of rendezvous problem where multiple agents must coordinate position and velocity. The docking of two spacecraft is a rendezvous problem that involves the two spacecraft matching both position and velocity with the proper orientation. Air-to-air refueling is another rendezvous problem. Additional applications arise in submersibles where robotic vehicles must converge upon a set location, either moving or stationary.

In the current literature, several researchers have addressed problems related to path planning with timing constraints. In 1963, Meschler [2] investigated a time optimal rendezvous problem for linear time varying systems. He assumed that both the rendezvous point and rendezvous time are not known a priori and that determining the minimum time at which rendezvous occurred was of interest. In principle, complicated rendezvous problems can be formulated using optimal control theory [3] and solved numerically. However, for many vehicles, obstacles and threats, the resulting optimization problem becomes quite complicated and the computational time increases very rapidly with problem size. McLain *et al.* [4], [5] have proposed decomposition methods that breaks down the monolithic problem into sub-problems that can be solved efficiently in a decentralized manner. Similar decomposition methods have also being proposed in [6], [7], [8], [9], [10] that solve path planning problems with timing constraints in a decentralized manner. Rendezvous problems solved in this framework are not amenable for formal analysis that is required for the purposes of verification and validation and it is difficult to assert guarantees on stability and limits of performance.

The problem of rendezvous has also been addressed as a consensus problem in the graph theoretic framework. Lin *et al.* [11] apply consensus seeking to a rendezvous problem for a group of mobile autonomous agents, where both the synchronous case and the asynchronous case are considered. The algorithm presented is provable correct, however does not address uncertainty in communication or dynamics. Cortes *et al.* [12] proposed an iterative algorithm with guaranteed convergence and is robust with respect to communication failures. Jadbabaie *et al.* [13] developed a coordination algorithm based on nearest neighbor rules. Smith *et al.* [14] solves the rendezvous problem with fixed communication topology based on Euclidian curve shortening methods and is restricted to planar rendezvous. Ren *et al.* [15] provides a survey of multi-agent coordination problems based on graph theoretic framework. The strength of the graph theoretic framework is its ability to analyze the communication aspect of the rendezvous problem. It however does not characterize the behavior of the collective system, which is necessary to generate feedback between the vehicles. This is the prime difference between the state-of-the-art in this area and the work presented in this paper.

In the dynamical systems literature the problem of cooperation and competition have been addressed in the context of cone invariance. The cone is used to define a partial order on the system trajectories, which results in the cooperative or competitive behavior of the system. In the seminal work by Hirsch [16], [17], [18], [19], [20], [21] on systems of differential equations that are competitive or cooperative, he developed what is known as *monotone dynamical systems* theory [22]. He demonstrated that the generic solution of a cooperative and irreducible system of differential equations converges to a set of equilibria. Furthermore, the flow on a compact limit set of an n-dimensional cooperative or competitive system of differential equations is shown to be topologically conjugate to the flow of an $n-1$ dimensional system of differential equations, restricted to a compact invariant set.

Invariant sets play an important role in many situations when the behavior of the closed-loop system is constrained in some way. Blanchini [23] provides an excellent survey of set invariant control. Invariant sets that are cones have found application in problems related to areas as diverse as industrial growth [24], ecological systems and symbiotic species [25], arms race [26] and compartmental system analysis [27], [28]. In general, cone invariance is an essential component in problems involving competition or cooperation.

In this paper we formulate rendezvous problems as cone invariance problems. Theoretical results on necessary and sufficient conditions for rendezvous are developed in the ellipsoidal cone invariance framework. Similar results have also been developed using polyhedral cones [29], but is not included in this paper. The rendezvous problem is defined on the positions of the agents and the dynamics is modeled as linear first order systems. The proposed framework however is not fundamentally limited to linear first order dynamics and can be extended to analyze rendezvous of higher order agents. For applications with higher order vehicle dynamics, we assume that the rendezvous trajectories generated from the first order models will be tracked reasonable closely.

The paper is organized as follows. We first interpret rendezvous in phase plane and define the rendezvous problem along with notions of *perfect* and *approximate* rendezvous. The problem of rendezvous is then analyzed using ellipsoidal cones. This is followed by theoretical results and numerical examples.

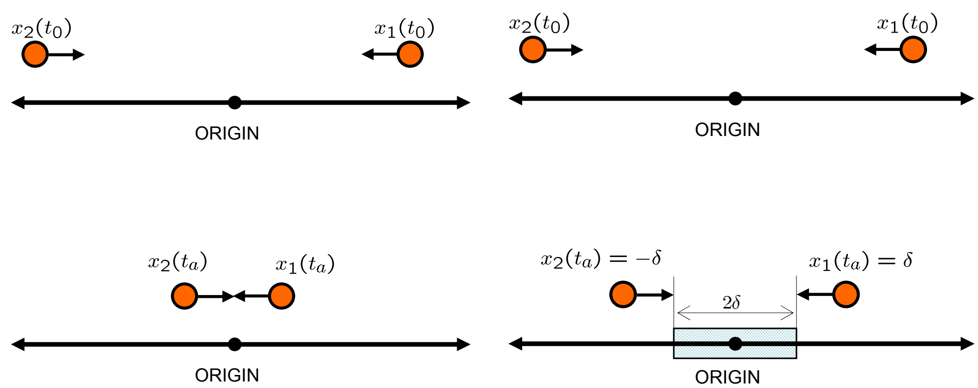
II. Rendezvous in the Phase Plane

In this paper we define the rendezvous problem to be the problem of determining a control algorithm that drives multiple agents to a desired destination point. The trajectories of the agents must be such that they visit the destination point only once and arrive at the same time.

Consider two scalar systems or agents $\mathcal{V}_1$ and $\mathcal{V}_2$, characterized by first order dynamics, as

$$\begin{aligned} \mathcal{V}_1 \ : \quad \dot{x}_1 = f_1(x_1) + g_1(x_1)u_1\,; \quad f_1(0) = 0 \\ \mathcal{V}_2 \ : \quad \dot{x}_2 = f_2(x_2) + g_2(x_2)u_2\,; \quad f_2(0) = 0, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}$ for $i \in \{1, 2\}$ and the destination point being the origin. Let x_1 and x_2 in eqn. (1) be the spatial coordinates of $\mathcal{V}_1$ and $\mathcal{V}_2$ on the real line. It is of interest to design control laws u_1 and u_2 such that $\mathcal{V}_1$ and $\mathcal{V}_2$ reach the origin of the real line at the same time. This is depicted in fig.1(a).



(a) Rendezvous at the origin. (b) Rendezvous at a region $\mathcal{R}$ about origin.

Fig. 1. Rendezvous on the real line.

Clearly agents that are exponentially stable will reach the origin as time tends to infinity. In such a case the comparison of arrival times at the origin, of two different agents becomes meaningless. Even with cooperative control in place, if the origin is exponentially stable, rendezvous at origin will occur at infinite time in theory. From a practical standpoint, it is desired that the agents achieve rendezvous in finite time. For this reason we relax the definition of rendezvous to be such that rendezvous is achieved if the agents enter a certain neighborhood around the origin, at the same time. We define this region to be the *rendezvous region* $\mathcal{R}$.

$$\mathcal{R} = \{x \in \mathbb{R} : -\delta \leq x \leq \delta\} \text{ for some } \delta > 0$$

Therefore a valid rendezvous is one in which agents enter $\mathcal{R}$ at the same time. This is illustrated in fig. 1(b). In Section II-B we will relax this definition for agents entering $\mathcal{R}$ at approximately the same time.

A. Rendezvous Interpretation on Phase Plane

Rendezvous is best visualized on the phase plane. To interpret rendezvous for first order systems in eqn.(1) in the phase plane, we define the following

$$\begin{aligned} U_1 &= \{(x_1, x_2) : -\delta \le x_1 \le \delta\} \\ U_2 &= \{(x_1, x_2) : -\delta \le x_2 \le \delta\} \\ \mathcal{S} &= U_1 \cap U_2 \\ \mathcal{F} &= (U_1 \cup U_2) - (U_1 \cap U_2) \\ \mathcal{W} &= (\mathbb{R}^2 - (U_1 \cup U_2)). \end{aligned} \tag{2}$$

We refer to $\mathcal{S}$ as the *rendezvous square* and $\mathcal{F}$ as the *forbidden region*.

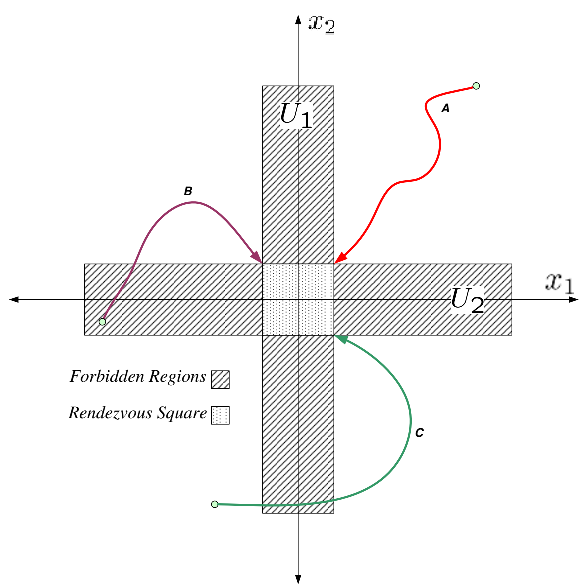


Fig. 2. Perfect rendezvous in phase plane.

With reference to fig.2, the strip on x_2-axis is U_1, the strip on x_1-axis is the region U_2 and the rendezvous square is the destination set where the trajectories must converge to. The rendezvous square $\mathcal{S}$ is the set of configurations with both agents in the rendezvous region $\mathcal{R}$. The rendezvous problem is well-posed if the initial conditions of the two agents satisfy

$$(x_1(0), x_2(0)) \in \mathcal{W}, \tag{3}$$

i.e. both the agents start far from the rendezvous region. If the condition in eqn. (3) is violated either $\mathcal{V}_1$, or $\mathcal{V}_2$, or both start from within the rendezvous region $\mathcal{R}$. In fig.2 trajectory B starts from an invalid initial point. The forbidden region is the set of points $\mathcal{F}$ where one agent enters the rendezvous region much before the other. In fig.2, trajectory C crosses the forbidden region which implies that the agent $\mathcal{V}_1$ with state x_1 comes within the rendezvous region prior to the final entry. Such trajectories are not acceptable, i.e. the trajectories must satisfy

$$(x_1(t), x_2(t)) \notin \mathcal{F} \ \ \forall t. \tag{4}$$

Trajectory A is an example of two agents, with valid initial conditions, achieving rendezvous as desired.

B. Perfect and Approximate Rendezvous

With constraint defined in eqn.(4), the only way trajectories can enter $\mathcal{S}$ is through the corners of the rendezvous square, i.e. through one of the points

$$(\delta, \delta),\ (\delta, -\delta),\ (-\delta, \delta) \text{ and } (-\delta, -\delta), \tag{5}$$

as shown in fig.2.

This implies that the agents are constrained to enter $\mathcal{S}$ at precisely the same time, which is the time the trajectory meets one of the four corners of $\mathcal{S}$. In most applications it is acceptable if agents $\mathcal{V}_1$ and $\mathcal{V}_2$ reach the rendezvous region within a certain time interval ΔT of each other. We now refer to the case when ΔT is zero as *ideal* or *perfect* rendezvous and the case when ΔT is small as *real* or *approximate* rendezvous.

Since the phase plane does not reveal time explicitly, we use a related measure ρ to characterize rendezvous. We will first define ρ, its relation to ΔT will be explained thereafter. To define ρ, we first introduce $t_{\mathcal{V}_1}$ and $t_{\mathcal{V}_2}$ to be the arrival times of agents $\mathcal{V}_1$ and $\mathcal{V}_2$ at the boundary of the rendezvous region $\mathcal{R}$, i.e.

$$\begin{aligned} t_{\mathcal{V}_1} &= \inf \ [\ t \mid x_1(t) \in U_1\ ] \\ t_{\mathcal{V}_2} &= \inf \ [\ t \mid x_2(t) \in U_2\ ]\,. \end{aligned}$$

Clearly, ΔT is given by

$$\Delta T = |t_{\mathcal{V}_1} - t_{\mathcal{V}_2}|\,. \tag{6}$$

Therefore the time t_a at which the trajectory enters the region $U_1 \cup U_2$ in the phase plane is given by

$$t_a = \min(t_{a_1}, t_{a_2}).$$

For a given trajectory $x(t) = [x_1(t)\ x_2(t)]^T$, ρ can be defined to be the maximum ratio of the distance from the origin of the two agents, after one of them has reached the rendezvous region $\mathcal{R}$. It can be expressed as

$$\rho = \frac{\max(|x_1(t_a)|, |x_2(t_a)|)}{\min(|x_1(t_a)|, |x_2(t_a)|)} = \frac{\max(|x_1(t_a)|, |x_2(t_a)|)}{\delta}. \tag{7}$$

The parameter ρ can also be defined using $||.||_1$ or $||.||_2$ as well,

$$\begin{array}{ll} \text{1-norm}: & \rho = \frac{|x_1(t_a)|+|x_2(t_a)|+\cdots+|x_n(t_a)|}{n\delta}, \\ \text{2-norm}: & \rho = \frac{\sqrt{x_1^2(t_a)+x_2^2(t_a)+\cdots+x_n^2(t_a)}}{\sqrt{n}\delta}. \end{array}$$

For the rest of the paper, rendezvous will always be specified by δ and a design measure of approximate rendezvous, ρ_{des}. In other words we will call a given rendezvous to be successful, if all the trajectories satisfy

$$\rho \;\leq\; \rho_{\text{des}}. \tag{8}$$

This notion of approximate rendezvous is illustrated in fig.3. Whenever a trajectory starting in the first quadrant enters the region $U_1 \cup U_2$ it is constrained to lie within the angle generated by joining the points

$$(\delta, \delta\rho_{\text{des}}),\ (0,0),\ \text{and}\ \left(\delta\rho_{\text{des}}, \delta\right).$$

There exists similar constraints for trajectories originating in the other quadrants. The introduction of ρ in the definition of rendezvous allows trajectories to enter the forbidden region $\mathcal{F}$ as long as they remain within the above mentioned angle set by the design constraint. By the definition of ρ in eqn. (7) it is clear that for a given trajectory $\rho \geq 1$. Therefore a specification of rendezvous is meaningful if and only if

$$\rho_{\text{des}} \geq 1. \tag{9}$$

Note that for perfect rendezvous the specification becomes $\rho_{\text{des}} = 1$.

In the worst case, at the time of entry of the first agent, t_a, the distances of the two agents from the origin can differ by $\delta(\rho_{\text{des}} - 1)$. By ensuring that the trajectories remain within the bold lines in fig.3, upon entry in the region $U_1 \cup U_2$ we can make sure that the two agents enter the rendezvous region $\mathcal{R}$ within a small time ΔT of each other. Thus the constraint in eqn.(8) helps keep ΔT small.

In fig.3 both trajectories A and B fail to achieve perfect rendezvous as they do not enter the rendezvous square $\mathcal{S}$ from its four corners. On the basis of eqn.(8), trajectory B is unacceptable. Trajectory A is acceptable since it lies within the angle defined by the bold lines.

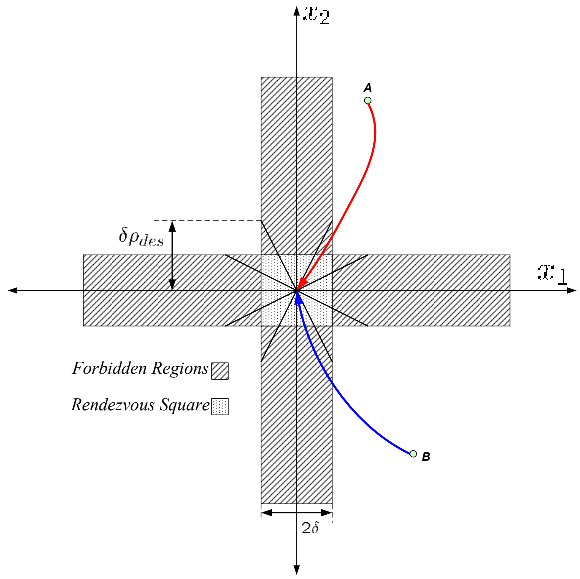


Fig. 3. Approximate rendezvous in phase plane.

III. Lyapunov Functions and Multi-Agent Rendezvous

In this section we motivate the use of control Lyapunov functions (CLFs) to solve the rendezvous problem. Consider the Lyapunov function candidate

$$V(x_1, x_2) = x_1^2 + x_2^2 + (x_1^2 - x_2^2)^2. \tag{10}$$

Ensuring $\dot{V} < 0$ guarantees that all the three terms in eqn. (10) goes to zero as time tends to infinity. If x_1 and x_2 denote the spatial coordinates of agents $\mathcal{V}_1$ and $\mathcal{V}_2$ and the origin is the rendezvous point, the first two terms ensure that they converge to the origin and the third term ensures that the agents reach the origin simultaneously. This is demonstrated by the following example.

Let the dynamics of the agents be given by

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2. \end{aligned} \tag{11}$$

It is easy to verify that $V(x)$ in eqn. (10) is a CLF. Sontag in [30] proposed a formula for producing a stabilizing controller based on the existence of a CLF $V(x)$. Because of its guarantee of stabilization and of providing a convenient relationship between closed-loop trajectories and CLF level sets, Sontag's formula is used here. For nonlinear systems with affine input such as

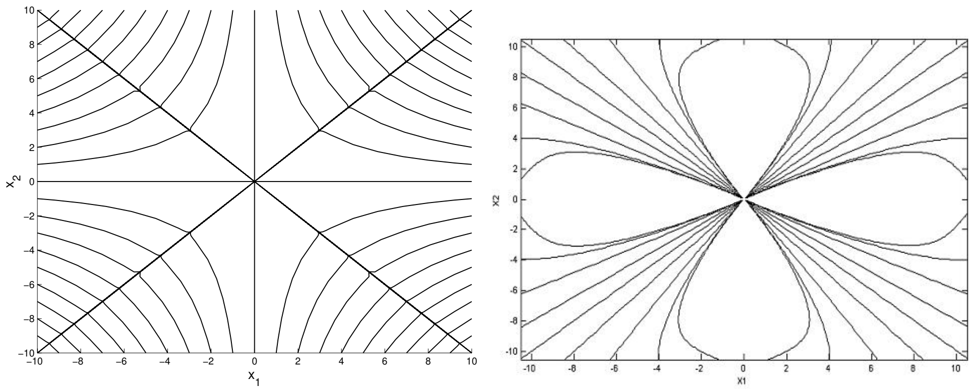
$$\dot{x} = f(x) + g(x)u,$$

Sontag's formula can be written as

$$u_s = \begin{cases} -\frac{V_x f + \sqrt{(V_x f)^2 + q(x) V_x g g^T V_x^T}}{V_x g g^T V_x^T} g^T V_x^T & V_x g \neq 0 \\ 0 & V_x g = 0 \end{cases} \tag{12}$$

where $V_x = \frac{\partial V(x)}{\partial x}$.

For the system in eqn. (11) and control derived from $V(x)$ in eqn. (10) using Sontag's formula, the phase portrait is shown in fig.4(a).



(a) (b)

Fig. 4. Rendezvous using control Lyapunov functions.

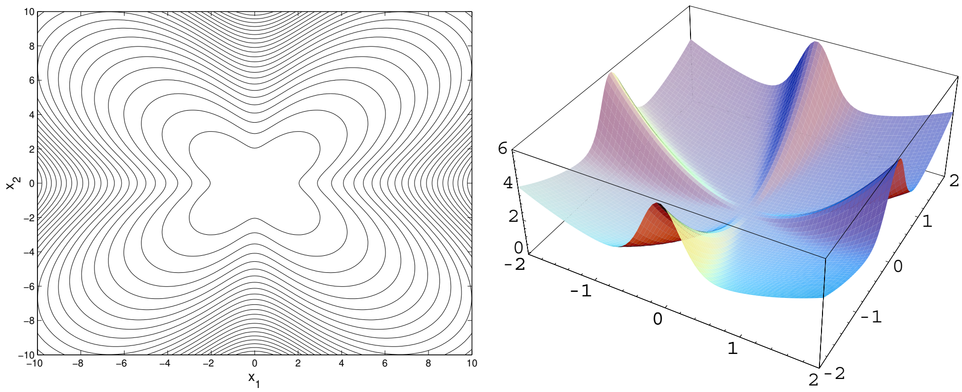
The term $(x_1^2 - x_2^2)^2$ in eqn.(10) ensures that the agents become equidistant from the origin by converging them to the lines $x_1 = \pm x_2$ prior to their arrival at the origin. In this sense, rendezvous is achieved for any ρ_{des} and δ. fig.4(b) shows the phase portrait for the same system but with Lyapunov function defined as

$$V(x_1, x_2) = (x_1^2 + x_2^2)\left[a + b e^{-8x_1^2 x_2^2 / d^2 (x_1^2 + x_2^2)^2}\right]. \tag{13}$$

Rendezvous is achieved by $\mathcal{V}_1$ and $\mathcal{V}_2$ in fig. 4(b) only under restricted values of ρ_{des} for a given δ. In one sense, however, rendezvous achieved by $\mathcal{V}_1$ and $\mathcal{V}_2$ in fig. 4(b) is "better" than that in fig. 4(a) because the agents are equidistant from the origin only locally. Rendezvous in fig. 4(a) forces the agents to be equidistant from the origin even at large distances, which may not be necessary.

Thus, it is possible to implicitly satisfy the constraints on ρ, as defined in eqn. (8), if the Lyapunov function has a certain form. For valid rendezvous, trajectories in phase plane should not cross either axes. If $\dot{V}$ is negative definite for all points in the phase plane and trajectories are constrained to be within the quadrant they start from, outside $\mathcal{S}$, the level sets are expected to have clover leaf appearance as shown in fig.5(a). Figure 5(a) shows the level sets of the Lyapunov function defined in eqn.(13) and the corresponding Lyapunov surface is shown in fig.5(b).

The level set of these control Lyapunov functions provide insight into why rendezvous is achieved for these cases.



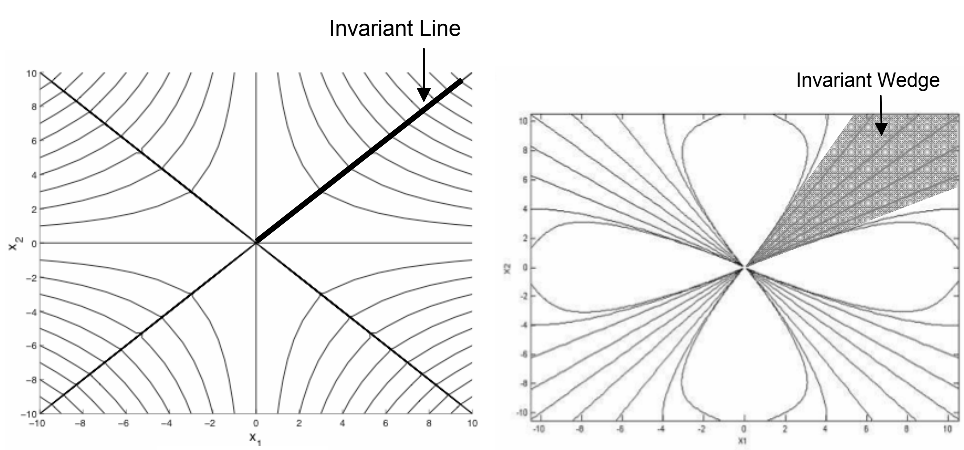
(a) (b)

Fig. 5. Desired Lyapunov surface and its level sets.

With control using Sontag's formula for the system in eqn.(11), rendezvous is achievable because trajectories are constrained to be normal to the level set contours. Controllers based on CLF's, whose level sets are similar to those in fig. 5(a), should drive agents for system eqn.(11) to a successful rendezvous.

IV. Cone Invariance and Rendezvous

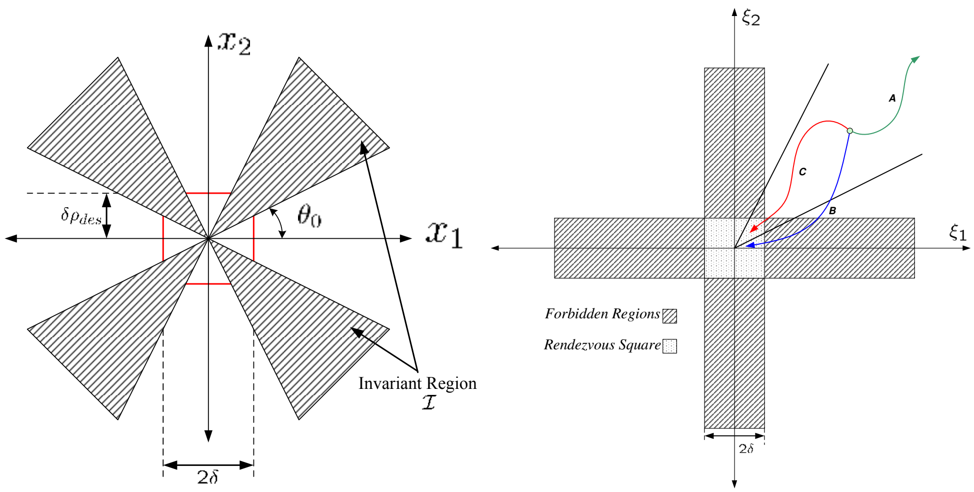
The system trajectories shown in fig.4(a) and fig.4(b) render a wedge-like region invariant. For the Lyapunov function given by eqn.(10), the invariant wedge degenerates to a line as shown in fig.6(a). For the Lyapunov function given by eqn.(13), the invariant region is a wedge as shown in fig.6(b).



(a) Invariant line. (b) Invariant wedge.

Fig. 6. Invariant regions rendered by system trajectories.

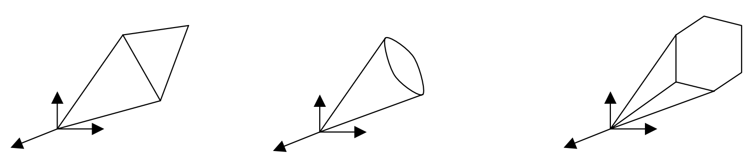
Therefore, the only admissible trajectories for approximate rendezvous are those that arrive at the origin while remaining in the wedge-like region $\mathcal{I}$ as shown in fig.7(a).



(a) Region $\mathcal{I}$ in 2 dimensional state space. (b) Possible trajectories in $\mathcal{I}$.

Fig. 7. Cone invariance and rendezvous.

For n agents achieving rendezvous, the region $\mathcal{I}$ becomes a cone in n-dimensional phase space. Depending on the norm used to define ρ in eqn.(7), the cone is either polyhedral or ellipsoidal. For ∞-norm, as is in eqn.(7), the cone is a polyhedral cone with $2^n - 2$ sides, a polyhedral cone with n sides for 1-norm or an ellipsoidal cone for 2-norm. This is shown in fig.8.



Desired region of invariance	Polyhedral Cone with n-sides	Quadratic Cone in n-dimension	Polyhedral Cone with 2^n-2 sides
Norm	$\|\|\cdot\|\|_1$	$\|\|\cdot\|\|_2$	$\|\|\cdot\|\|_\infty$
Complexity	*n* constraints	*1* constraint	2^n-2 constraints

Fig. 8. Region $\mathcal{I}$ in 3 dimensional state space.

Cone invariance alone does not guarantee that the agents reach the origin. Figure 7(b) shows trajectories A, B and C. Trajectory A achieves cone invariance but does not reach the origin. Trajectory B reaches the origin but escapes the cones. Trajectory C is the only trajectory that reaches the origin and stays within the cone. We are interested in trajectories such as C.

V. Ellipsoidal Cone Invariance and Rendezvous

In this section we analyze the rendezvous problem in the framework of ellipsoidal cone invariance. We first present mathematical preliminaries on ellipsoidal cones and related invariance theory. Formulation of the rendezvous problem as a cone invariance problem is then presented. This is followed by necessary and sufficient conditions for rendezvous in one and two dimensions. The controller synthesis problem is presented next. The section concludes with numerical examples that demonstrate application of the theory.

A. Mathematical Preliminaries

1) Ellipsoidal Cones: An ellipsoidal cone in $\mathbb{R}^n$ is the following,

$$\Gamma_n = \{\xi \in \mathbb{R}^n : K_n(\xi, Q) \le 0,\, \xi^T u_n \ge 0\}, \tag{14}$$

where $K_n(\xi, Q) = \xi^T Q \xi$, $Q \in \mathbb{R}^{n,n}$ is a symmetric nonsingular matrix with a *single* negative eigen-value λ_n and u_n is the eigen-vector associated with λ_n.

The boundary of the cone Γ_n is denoted by $\partial\Gamma_n$ and is defined by

$$0 \ne \xi \in \partial\Gamma_n \equiv \{\xi \in \Gamma_n : K_n(\xi, Q) = 0\}.$$

The outward pointing normal is the vector $Q\xi$ for $\xi \in \partial\Gamma_n$.

Theorem 1 (2.7 in [31]): If Γ_n is an ellipsoidal cone, then there exists a nonsingular transformation matrix $M \in \mathbb{R}^{n,n}$ such that

$$(M^{-1})^T Q M^{-1} = \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} = Q_n$$

where $P \in \mathbb{R}^{n-1,n-1}$, $P > 0$ and $P = P^T$.

Let the transformed state be $x = M\xi$. The ellipsoidal cone in x is therefore,

$$\Gamma_n = \{x : \begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} P & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \le 0\} \tag{15}$$

where $x = (w \;\; z)^T$, $w \in \mathbb{R}^{n-1}, z \in \mathbb{R}$.

An ellipsoidal cone in three dimension is shown in Fig.(9). The axis of the cone is the eigen-vector associated with the z axis.

2) Ellipsoidal Cone Invariance: Consider a linear autonomous system

$$\dot{\xi} = A\xi. \tag{16}$$

A cone Γ_n is said to be invariant with respect to the dynamics in eqn.(16) if $\xi(t_0) \in \Gamma_n \Rightarrow \xi(t) \in \Gamma_n$, $\forall t \ge t_0$, i.e. if the system starts inside the cone, it stays in the cone for all future time. Such a condition is also known as *exponential non-negativity*, i.e. $e^{At}\Gamma_n \in \Gamma_n$.

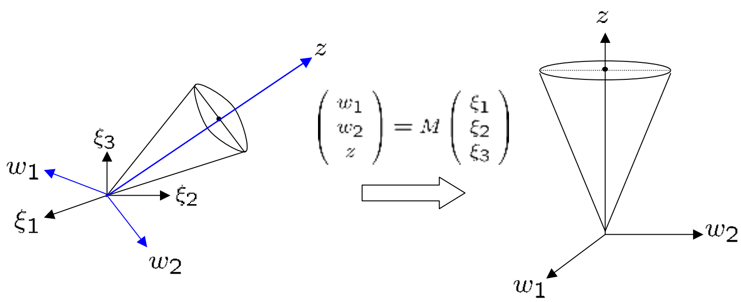


Fig. 9. Ellipsoidal cone in 3-dimension.

It is well known that certain structure in the matrix A imposes constraints on e^{At} [32]. The most well known result is the condition of non-negativity on A which states that if $A_{ij} \geq 0$ for $i \neq j$, then non-negative initial conditions yield non-negative solutions. Schneider and Vidyasagar [33] introduced the notion of *cross-positivity* of A on Γ_n which was shown to be equivalent to exponential non-negativity. Meyer *et al.* [34] extended cross-positivity to nonlinear fields.

Let us characterize $p(\Gamma_n)$ to be the set of matrices $A \in \mathbb{R}^{n,n}$ which are exponentially non-negative on Γ_n. It is defined by the following theorem.

Theorem 2 (3.1 in [31]): Let Γ_n be an ellipsoidal cone as in eqn.(15). Then,

$$p(\Gamma_n) = \{A \in \mathbb{R}^{n,n} \ :< A\xi, Q\xi > \leq 0, \ \forall \xi \in \Gamma_n\}. \tag{17}$$

Theorem 2 states that A is such that the flow of the associated vector field is directed towards the interior of Γ_n, i.e. the dot product of the outward normal of Γ_n and the field is negative at the boundary of the cone. This leads to the result on the necessary and sufficient condition for exponential non-negativity of ellipsoidal cones.

Theorem 3 (3.5 in [31]): A necessary and sufficient condition for $A \in p(\Gamma_n)$ is that there exists $\gamma \in \mathbb{R}$ such that,

$$Q_n\hat{A} + \hat{A}^T Q_n - \gamma Q_n \leq 0.$$

where Q_n is defined in Theorem 1 and $\hat{A} = MAM^{-1}$.

Proof Please refer to pg.162 of [31].

3) Monotone Dynamical Systems: A dynamical system

$$\dot{x} = f(t, x)$$

is *monotone* [22] if $x_0 \leq x_1 \Rightarrow x(t,t_0,x_0) \leq x(t,t_0,x_1)$, where $x(t,t_0,x_0)$ is the solution of the differential equation and the inequality is component-wise. For linear systems positivity (or negativity) invariance implies monotonicity [35]. Therefore, theorem 3 is also necessary and sufficient conditions for monotonicity.

We define a partial order with respect to the cone Γ_n as $\leq_{\Gamma_n}$, defined by

$$x_1 \leq_{\Gamma_n} x_2 \;\; \Leftrightarrow K_n(x_1, Q) \leq K_n(x_2, Q)$$

where K_n is defined in eqn.(14). Other relations such as $<_{\Gamma_n}, \geq_{\Gamma_n}$ and $>_{\Gamma_n}$ can be similarly defined.

For linear systems, invariance of the set Γ_n is equivalent to monotonicity with respect to Γ_n, i.e.

$$A \in p(\Gamma_n) \quad \Leftrightarrow \quad x_0 \leq_{\Gamma_n} x_1 \;\Rightarrow\; x_0 e^{A(t-t_0)} \leq_{\Gamma_n} x_1 e^{A(t-t_0)}, t \geq t_0.$$

B. Rendezvous in One Dimension

Given a cone Γ_n, as in eqn.(15) and dynamics as in eqn.(16), we present conditions for stability and invariance. We transform dynamics as

$$x = M\xi \Rightarrow \dot{x} = MAM^{-1}x = \hat{A}x.$$

With respect to the partition $x = (w \;\; z)^T$, the dynamics can be written as

$$\begin{pmatrix} \dot{w} \\ \dot{z} \end{pmatrix} = \left[\begin{array}{c|c} A_{ww} & A_{wz} \\ \hline A_{zw} & a_{zz} \end{array} \right] \begin{pmatrix} w \\ z \end{pmatrix}, \tag{18}$$

where a_{zz} is written in small case to emphasize that it is a scalar.

For cone invariance, theorem 3 implies $\exists \gamma \in \mathbb{R}$ such that

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} - \gamma P & PA_{wz} - A_{zw}^T \\ A_{wz}^T P - A_{zw} & \gamma - 2a_{zz} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} < 0.$$

For stability, consider the Lyapunov function $V(w,z) = w^T P w + z^2$. It is a valid Lyapunov function since $P > 0$. Therefore, for stability $\dot{V}(w,z) < 0$, which implies

$$\begin{pmatrix} w \\ z \end{pmatrix}^T \begin{bmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} + A_{zw}^T \\ A_{wz}^T P + A_{zw} & 2a_{zz} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix} < 0.$$

Therefore, for stability and cone invariance we have the following matrix inequalities,

$$\begin{bmatrix} A_{ww}^T P + P A_{ww} & P A_{wz} + A_{zw}^T \\ A_{wz}^T P + A_{zw} & 2a_{zz} \end{bmatrix} < 0 \quad (19)$$

$$\begin{bmatrix} A_{ww}^T P + P A_{ww} - \gamma P & P A_{wz} - A_{zw}^T \\ A_{wz}^T P - A_{zw} & \gamma - 2a_{zz} \end{bmatrix} < 0 \quad (20)$$

A simplified sufficient condition is expressed in the following theorem, which also addresses the feasibility of the LMIs in eqn.(19,20).

Theorem 4: A sufficient condition for cone invariance and stability is given by the following relations,

$$A_{ww}^T P + P A_{ww} - 2a_{zz} P < 0$$

and

$$a_{zz} < -max(||g^-||, ||g^+||),$$

where $g^- = P A_{wz} - A_{zw}^T$ and $g^+ = P A_{wz} + A_{zw}^T$.

Proof

Sufficiency for Stability

Define matrices

$$M_1 = \begin{bmatrix} A_{ww}^T P + P A_{ww} & 0 \\ 0 & 2a_{zz} \end{bmatrix}, \; M_2 = \begin{bmatrix} 0 & g^+ \\ (g^+)^T & 0 \end{bmatrix}.$$

For stability we need to show $M_1 + M_2 < 0$. Theorem 4 implies $\lambda_{max}(M_1) = 2a_{zz}$, $2a_{zz} < -||g^+||$, and $\lambda_{max}(M_2) = ||g^+||$. Therefore,

$$\begin{aligned} \lambda_{max}(M_1) + \lambda_{max}(M_2) &< 0 \\ \Rightarrow \qquad \lambda_{max}(M_1 + M_2) &< 0 \\ \Rightarrow \qquad M_1 + M_2 &< 0 \end{aligned}$$

Hence proved.

Sufficiency for Cone Invariance

Define matrices

$$M_3 = \begin{bmatrix} A_{ww}^T P + P A_{ww} - \gamma P & 0 \\ 0 & \gamma - 2a_{zz} \end{bmatrix}, \; M_4 = \begin{bmatrix} 0 & g^- \\ (g^-)^T & 0 \end{bmatrix}.$$

For cone invariance we need to show $M_3 + M_4 < 0$. Theorem 4 implies $\lambda_{max}(M_3) = 2a_{zz}$, $2a_{zz} < -||g^-||$, and $\lambda_{max}(M_4) = ||g^-||$. Following the steps in the proof for stability, we can arrive at the conclusion that $M_3 + M_4 < 0$.

Hence proved.

Theorem 4 leads to the following corollary.

Corollary 1: Trajectories originating outside the cone will enter the cone in finite time.

Proof

The cone $K_n(\xi, Q)$ can be written as $K_n(x, Q_n)$. Condition for cone invariance implies

$$\dot{K}_n(x, Q_n) < \gamma K_n(x, Q_n).$$

For x outside the cone, $K_n(x, Q_n) > 0$. Stability and cone invariance implies $\gamma < 2a_{zz} < 0$, which implies $\dot{K}_n(x, Q_n) < 0$ outside the cone. Hence proved.

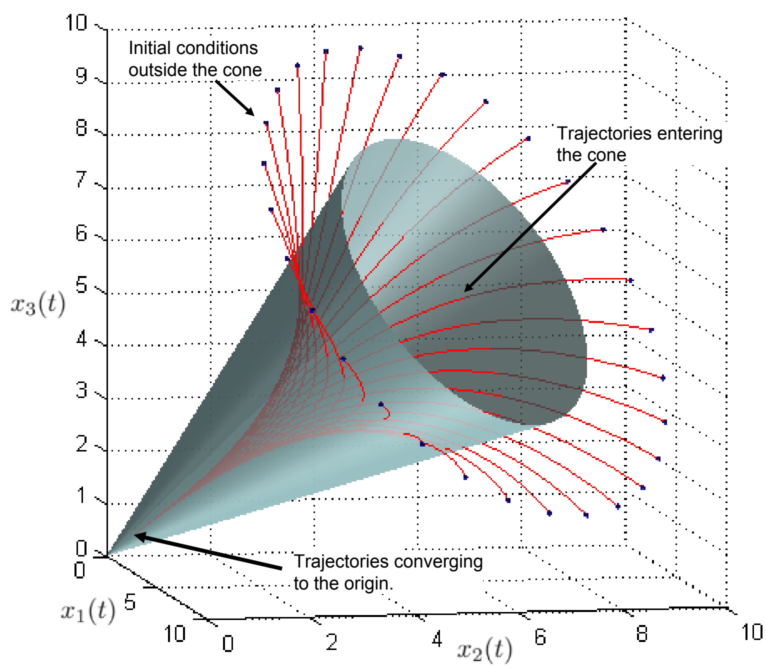


Fig. 10. Cone as an attractor. If the eigenvalues are real the trajectories will converge radially to the origin. For complex eigenvalues, the trajectories will converge spirally.

Example 1: Figure(10) illustrates trajectories for the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} -0.9713 & 0.0185 & 0.5813 \\ 0.5813 & -0.9713 & 0.0185 \\ 0.0185 & 0.5813 & -0.9713 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{21}$$

We observe that trajectories originating outside the cone, enter the cone. The eigenvalues of the system in eqn.(21) are `-0.3715, -1.2712 + 0.4874i, -1.2712 - 0.4874i`. These correspond to the dynamics of trajectories $z(t), w_1(t)$ and $w_2(t)$. The conditions for stability and cone invariance imply that the decay rate of $w(t) = [w_1(t)\ w_2(t)]$ is faster than that of $z(t)$, which is observed here.

As observed, trajectories with initial conditions outside the cone, enter the cone. Such trajectories will be valid rendezvous trajectories if they enter the cone before intersecting the *forbidden* region $\mathcal{F}$, as defined in eqn.(2), for n-dimensions.

To characterize the set of valid initial conditions for which rendezvous is achieved, let us define hyperplanes

$$\mathcal{H}_i = \{x : x_i = \delta, i = 1, \cdots, n\}$$

and half space intersections

$$\mathcal{S}_i = \{x : x_j \geq \delta,\ j \neq i,\ j = 1, \cdots, n\}.$$

Let $\mathcal{E}_i$ be the ellipse segments defined by

$$\mathcal{E}_i = \mathcal{H}_i \cap S_i \cap \partial\Gamma_n.$$

Let $\mathcal{T}$ be the closed curve obtained by the union of the ellipse segments, i.e.

$$\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{E}_i.$$

Figure 11(a) shows the curve $\mathcal{T}$ for three agents, with $\delta = 1$.

Let $\partial\Omega$ be the surface defined by

$$\partial\Omega = \{x(t_0) : x(t_0)e^{A(t-t_0)} \in \mathcal{T},\ \text{for some } t \geq t_0\}.$$

Figure 11(b) shows the surface $\partial\Omega$ for three agents, with $\delta = 1$.

Therefore, the set of all initial conditions $x_0 = x(t_0)$, for which trajectories enter Γ_n before entering $\mathcal{F}$ is given by

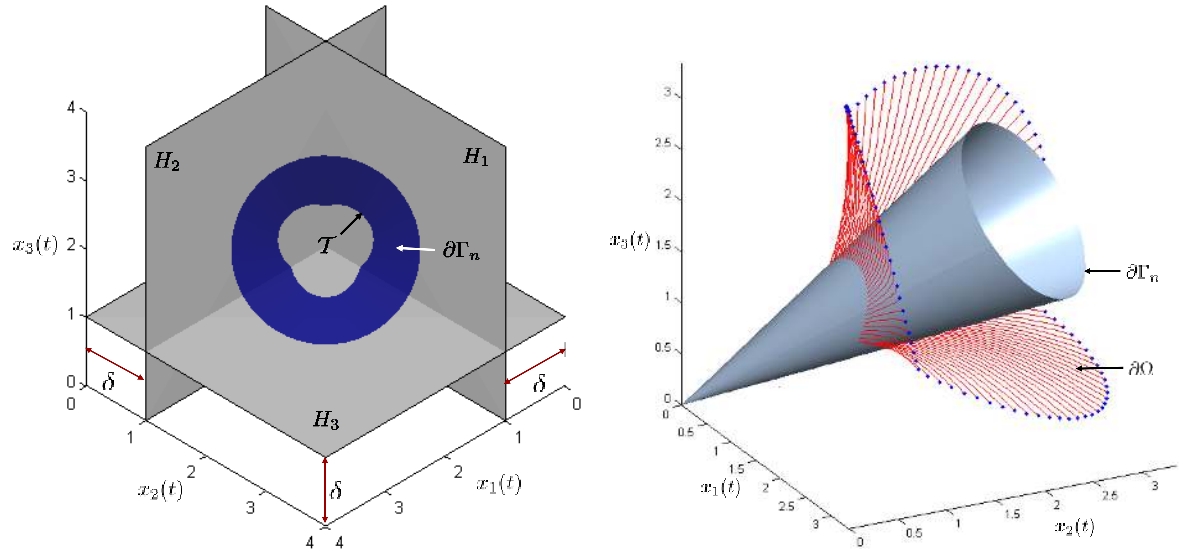
$$\Omega = \{x(t_0) : x(t_0) \leq_{\Gamma_n} \partial\Omega\}.$$

Clearly, for initial conditions outside Ω, the definition of approximate rendezvous is violated and can be demonstrated as follows. Monotonicity implies, for all $x(t_0) >_{\Gamma_n} \Omega$, the solution satisfies $x(t_0)e^{A(t-t_0)} >_{\Gamma_n} \Omega$. Let $t_a = \inf_t x(t_0)e^{A(t-t_0)} \in \mathcal{F}$. Therefore, $x(t_0)e^{(t_a-t_0)} >_{\Gamma_n} \Omega$, i.e. $x(t_0)e^{A(t-t_0)}$ never enters the cone Γ_n before entering $\mathcal{F}$.

The set Ω will include initial conditions originating from $\mathcal{F}$. Therefore, the set of *valid* initial conditions for which rendezvous is achieved is given by

$$\Omega_R = \Omega \cap \mathcal{W},$$

where $\mathcal{W}$ is defined in eqn.(2).



(a) The contour $\mathcal{T}$.

(b) The surface $\partial\Omega$.

Fig. 11. Set of initial conditions for which trajectories enter the cone before entering the forbidden region. Figure 11(a) shows the closed curve $\mathcal{T}$, which is the intersection of the hyperplanes H_i, the half space intersections $\mathcal{S}_i$ and the surface of the cone Γ_n. The surface $\partial\Omega$ is shown in fig.11(b), which defines the set of all initial conditions for which trajectories enter the cone through the closed curve $\mathcal{T}$.

Next we characterize matrices $A \in p(\partial\Gamma_n)$ where

$$p(\partial\Gamma_n) := \{A \in \mathbb{R}^{n,n} : e^{At}(\partial\Gamma_n) \in \partial\Gamma_n \forall t \geq 0\}.$$

The necessary and sufficient conditions for $A \in p(\partial\Gamma_n)$ can be derived by setting vector field tangent to the locally smooth surface of the cone, $\partial\Gamma_n/\{0\}$. As an LMI constraint this is equivalent to

$$\begin{bmatrix} A_{ww}^T P + P A_{ww} - \gamma P & P A_{wz} - A_{zw}^T \\ A_{wz}^T P - A_{zw} & \gamma - 2a_{zz} \end{bmatrix} = 0. \tag{22}$$

This leads to the following result.

Theorem 5: Sufficient condition for rendezvous, defined by invariance of $\partial\Gamma_n$ is given by the following:

$$A_{ww}^T P + P A_{ww} = 2a_{zz} P \tag{23}$$

$$A_{wz}^T P = A_{zw} \tag{24}$$

$$a_{zz} < \frac{-||A_{zw}||}{\lambda_{max}\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}} \tag{25}$$

Proof:

Sufficiency for Invariance of $\partial\Gamma_n$: It is straight forward to see eqn.(23, 24) imply eqn.(22).

Sufficiency for Stability : Given eqn.(25) is true,

$$
\begin{aligned}
\Rightarrow \quad & a_{zz}\lambda_{max}\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} + ||A_{zw}|| &<& \quad 0 \\
\Rightarrow \quad \lambda_{max}&\begin{pmatrix} a_{zz}P & 0 \\ 0 & a_{zz} \end{pmatrix} + \lambda_{max}\begin{pmatrix} 0 & A_{zw}^T \\ A_{zw}^T & 0 \end{pmatrix} &<& \quad 0 \\
\Rightarrow \quad & \begin{pmatrix} a_{zz}P & A_{zw}^T \\ A_{zw} & a_{zz} \end{pmatrix} &<& \quad 0 \\
\Rightarrow \quad & \begin{pmatrix} A_{ww}^T P + PA_{ww} & PA_{wz} + A_{zw}^T \\ A_{wz}^T P + A_{zw} & 2a_{zz} \end{pmatrix} &<& \quad 0
\end{aligned}
$$

which is the condition for stability. Hence proved.

Theorem 5 results in the following corollary.

Corollary 2: The surface of the cone $\partial\Gamma_n$ is an attractor.

Proof

Condition for invariance of $\partial\Gamma_n$ implies

$$\dot{K}_n(x, Q_n) = \gamma K_n(x, Q_n).$$

For x outside the cone, $K_n(x, Q_n) > 0$. Stability and cone invariance implies $\gamma = 2a_{zz} < 0$ which implies $\dot{K}_n(x, Q_n) < 0$ outside the cone. Similarly, for x inside the cone, $K_n(x, Q_n) < 0$. $\gamma < 0$ implies $\dot{K}_n(x, Q_n) > 0$. Hence proved.

C. Rendezvous in Two Dimensions

Here we consider rendezvous of n agents in two dimensions. Let the state of each agent be $(\xi_{x_i}, \xi_{y_i}), i = 1, \cdots, n$. Collectively their dynamics can be written as

$$\begin{pmatrix} \dot{\xi}_x \\ \dot{\xi}_y \end{pmatrix} = \begin{bmatrix} A_{\xi_{xx}} & A_{\xi_{xy}} \\ A_{\xi_{yx}} & A_{\xi_{yy}} \end{bmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}, \tag{26}$$

where $\xi_x = (\xi_{x_1} \cdots \xi_{x_n})^T$ and $\xi_y = (\xi_{y_1} \cdots \xi_{y_n})^T$ and $A_{\xi_{xx}}, A_{\xi_{xy}}, A_{\xi_{yx}}, A_{\xi_{yy}} \in \mathbb{R}^{n\times n}$.

In this work we solve the rendezvous problem in two dimension as two separate rendezvous problems in one dimension. We assume that cones $\xi_x^T Q_{\xi_x} \xi_x < 0$ and $\xi_y^T Q_{\xi_y} \xi_y < 0$, each satisfying eqn.(14), are given. We are interested in determining necessary and sufficient conditions for cone invariance and stability.

For ellipsoidal cones $\xi_x^T Q_{\xi_x}\xi_x < 0$ and $\xi_y^T Q_{\xi_y}\xi_y < 0$, there exists transformation R_x and R_y respectively such that

$$\begin{aligned} Q_x^c &= (R_x^{-1})^T Q_{\xi_x} R_x^{-1} &= \begin{bmatrix} P_x & 0 \\ 0 & -1 \end{bmatrix}, \\ Q_y^c &= (R_y^{-1})^T Q_{\xi_y} R_y^{-1} &= \begin{bmatrix} P_x & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned}$$

where $P_x, P_y > 0 \in \mathbb{R}^{(n-1)\times(n-1)}$ and the superscript "c" on Q_x and Q_y denotes cones.

Let the transformed states be

$$\begin{aligned} x &= R_x\xi_x, \\ y &= R_y\xi_y. \end{aligned}$$

The system dynamics with respect to the transformed states (x, y) can be written as

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{bmatrix} R_x & 0 \\ 0 & R_y \end{bmatrix} \begin{bmatrix} A_{\xi_{xx}} & A_{\xi_{xy}} \\ A_{\xi_{yx}} & A_{\xi_{yy}} \end{bmatrix} \begin{bmatrix} R_x^{-1} & 0 \\ 0 & R_y^{-1} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{bmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Using theorem (3), the necessary and sufficient conditions for cone invariance with respect to trajectories $x(t, t_0, x_0)$ and $y(t, t_0, y_0)$ are

$$\begin{bmatrix} A_{xx}^T Q_x^c + Q_x^c A_{xx} - \gamma_x Q_x^c & Q_x^c A_{xy} \\ A_{xy}^T Q_x^c & 0 \end{bmatrix} < 0, \tag{27}$$

$$\begin{bmatrix} 0 & A_{yx}^T Q_y^c \\ Q_y^c A_{yx} & A_{yy}^T Q_y^c + Q_y^c A_{yy} - \gamma_y Q_y^c \end{bmatrix} < 0, \tag{28}$$

for some $\gamma_x \in \mathbb{R}$ and $\gamma_y \in \mathbb{R}$.

For stability, define

$$Q_x^s = \begin{bmatrix} P_x & 0 \\ 0 & 1 \end{bmatrix}, Q_y^s = \begin{bmatrix} P_y & 0 \\ 0 & 1 \end{bmatrix},$$

where the superscript "s" on Q_x and Q_y denotes stability.

Therefore $V(x, y) = x^T Q_x^s x + y^T Q_y^s y$ is a valid Lyapunov function. Stability with respect to $V(x, y)$ implies

$$\begin{bmatrix} A_{xx}^T Q_x^s + Q_x^s A_{xx} & Q_x^s A_{xy} + A_{yx}^T Q_y^s \\ Q_y^s A_{yx} + A_{xy}^T Q_x^s & A_{yy}^T Q_y^s + Q_y^s A_{yy} \end{bmatrix} < 0 \tag{29}$$

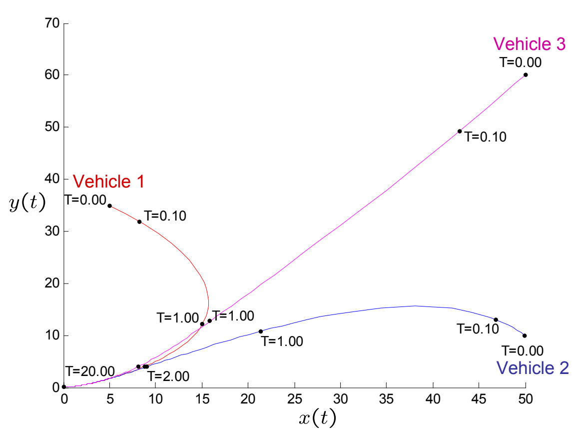
Therefore, equations (27, 28, 29) are the necessary and sufficient conditions for rendezvous in two dimensions. If the dynamics of ξ_x and ξ_y are decoupled, then the conditions simplify to the following,

$$\begin{aligned} A_{xx}^T Q_x^c + Q_x^c A_{xx} - \gamma_x Q_x^c &< 0, \\ A_{yy}^T Q_y^c + Q_y^c A_{yy} - \gamma_y Q_y^c &< 0, \\ A_{xx}^T Q_x^s + Q_x^s A_{xx} &< 0, \\ A_{yy}^T Q_y^s + Q_y^s A_{yy} &< 0. \end{aligned} \tag{30}$$

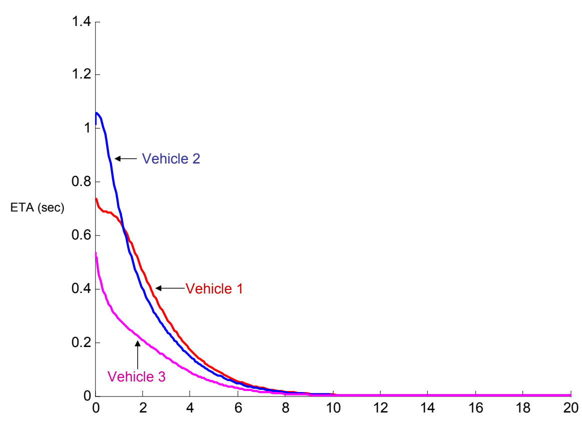
Following the treatment presented in this section, these results can be easily extended to define necessary and sufficient conditions for rendezvous in higher dimensions. Note that the approach presented, solves higher dimensional rendezvous problems as separate rendezvous problems in each dimension, which is restrictive.

Example 2: Consider the following first order dynamics in (x, y) plane of three agents,

$$\left(\begin{array}{c} \dot{x} \\ \hline \dot{y} \end{array}\right) = \left[\begin{array}{ccc|ccc} -1.4596 & 0.2140 & 0.6043 & 0.0000 & 0.0000 & 0.0000 \\ 0.6043 & -1.4596 & 0.2140 & 0.0000 & 0.0000 & 0.0000 \\ 0.2140 & 0.6043 & -1.4596 & 0.0000 & 0.0000 & 0.0000 \\ \hline 0.0000 & 0.0000 & 0.0000 & -2.3186 & 0.0439 & 1.2807 \\ 0.0000 & 0.0000 & 0.0000 & 1.2807 & -2.3186 & 0.0439 \\ 0.0000 & 0.0000 & 0.0000 & 0.0439 & 1.2807 & -2.3186 \end{array}\right] \left(\begin{array}{c} x \\ \hline y \end{array}\right).$$



(a) Initial conditions $(5, 35), (50, 10), (50, 60)$.



(b) Expected arrival times.

Fig. 12. Rendezvous of three agents in (x, y) plane. Agents modeled as first order systems in x and y.

Observe that the dynamics in x is decoupled from y. Figure 12(a) shows the trajectories of the three agents in (x, y) plane achieving rendezvous with different sets of initial conditions. The trajectories are time stamped to indicate their location with respect to time. In fig.12(a), we observe that the agents start far away from each other. Vehicles $1, 2$ and 3 start from points $(5, 35), (50, 10), (50, 60)$ respectively. At time $T = 1.00$ the trajectories are close to each other. At $T = 2.00$ the trajectories overlap. Of particular interest is the trajectory of vehicle 1, which moves away from the origin to meet the other agents so that rendezvous is possible. Figure 12(b) shows expected time of arrival (ETA) as a function of time. ETA is computed by dividing the instantaneous distance from origin by the instantaneous average velocity. Observe that the initial ETA of the vehicles further away (Vehicle 2 and 3) is lower than vehicles closer to the origin (Vehicle 1). This is due to the model of the position dynamics assumed, where the velocity of the vehicle is linearly proportional to the distance from the origin. In this example we observe that the ETA trajectories for all the vehicles begin to overlap as they approach the origin, indicating same arrival times at the origin.

D. Rendezvous in Lyapunov Framework

In this section we derive necessary and sufficient conditions for rendezvous in the Lyapunov framework. We first consider rendezvous in one dimension, followed by rendezvous in two dimensions.

1) Rendezvous in One Dimension: Consider two Lyapunov functions $V_w(w) = w^T P w, P > 0$ and $V_z(z) = z^T z$. The cone Γ_n can then be represented as

$$\Gamma_n = \{ \begin{pmatrix} w \\ z \end{pmatrix} : V_w(w) < V_z(z) \}.$$

Conditions for rendezvous in the Lyapunov framework is then given by the following theorem.

Theorem 6: Necessary and sufficient conditions for rendezvous in terms of Lyapunov functions V_w and V_z are

$$\textit{Cone Invariance} \quad : \quad \dot{V}_w - \dot{V}_z \leq \gamma(V_w - V_z),\, \gamma \in \mathcal{R} \tag{31}$$

and

$$\textit{Stability} \quad : \quad \dot{V}_w + \dot{V}_z < 0. \tag{32}$$

Equality in eqn.(31) implies invariance of $\partial\Gamma_n$

Proof: These conditions are obtained by rewriting equations (20) and (19) in terms of the Lyapunov functions and their derivatives.

2) Rendezvous in Two Dimensions: To analyze rendezvous in two dimensions in the Lyapunov framework, we first partition the states as $x = (w_x \, z_x)$ and $y = (w_y \, z_y)$. Define Lyapunov functions

$$\begin{aligned} V_{w_x} &= w_x^T P_x w_x \\ V_{z_x} &= z_x^2 \\ V_{w_y} &= w_y^T P_y w_y \\ V_{z_y} &= z_y^2 \end{aligned}$$

Theorem 7: Necessary and sufficient conditions for rendezvous in two dimensions can be written in terms of these four Lyapunov functions as follows,

$$\begin{aligned}
\dot{V}_{w_x} - \dot{V}_{z_x} &\leq \gamma_x(V_{w_x} - V_{z_x}),\ \gamma_x \in \mathbb{R}, \\
\dot{V}_{w_y} - \dot{V}_{z_y} &\leq \gamma_y(V_{w_y} - V_{z_y}),\ \gamma_y \in \mathbb{R}, \\
\dot{V}_{w_x} + \dot{V}_{z_x} &< 0, \\
\dot{V}_{w_y} + \dot{V}_{z_y} &< 0.
\end{aligned} \tag{33}$$

Proof: These conditions are obtained by rewriting the equations (27, 28, 29) in terms of the Lyapunov functions and their derivatives.

E. Controller Design Problem

Let us assume that there are n agents for which rendezvous is desired. Let us also assume that the agents are modeled as *first order* LTI systems. Collectively, they can be written as

$$\dot{\xi} = A\xi + Bu. \tag{34}$$

where A and B are matrices of appropriate dimensions and (A, B) is controllable. We also assume that we are given an ellipsoidal cone Γ_n as defined by eqn.(15), where Q depends on the specified measure of rendezvous ρ_{des}. Therefore, given a cone Γ_n and n agents modeled as first order LTI systems, we are interested in determining control $u(t)$ such that the following are true,

$$\begin{aligned}
&\xi(t_0) \in \Gamma_n \Rightarrow \xi(t) \in \Gamma_n,\ \forall t \geq t_0,\ \text{and} \\
&\xi(t) \rightarrow 0 \text{ as } t \rightarrow \infty
\end{aligned} \tag{35}$$

If we consider a *full state feedback* control framework, then

$$u = F\xi = FM^{-1}x$$

and the closed-loop system is therefore

$$\dot{x} = M(A + BF)M^{-1}x. \tag{36}$$

which can be represented in the form as in eqn.(18).

Assuming that the pair (A, B) in eqn.(34) is controllable, the controller synthesis problem is to determine F such that the LMI constraints in eqn.(19,20) are feasible. If the states are not available for feedback, the current approach can be extended to incorporate any linear observer based controller design methodology.

For higher order dynamics, the controller synthesis problem is not straight forward. Consider agents whose dynamics is represented by the linear second order differential equation

$$m_i\ddot{\xi}_i(t) + d_i\dot{\xi}_i(t) + k_i\xi_i(t) = u_i(t),$$

where m_i, d_i and k_i are mass, damping and stiffness respectively. In matrix-vector notation the dynamics can be represented by

$$\frac{d}{dt}\begin{pmatrix} \xi_i \\ \dot{\xi}_i \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}\begin{pmatrix} \xi_i \\ \dot{\xi}_i \end{pmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u_i.$$

For n agents the collective dynamics can be represented by the equation

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{bmatrix} 0 & I_N \\ A_{\eta\xi} & A_{\eta\eta} \end{bmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{bmatrix} 0 \\ B_\eta \end{bmatrix} u, \tag{37}$$

and we assume that the system is controllable.

For dynamical systems given by eqn.(37), the cone Γ_n defined on position states ξ is not closed-loop *holdable* (pg.65 [36]). A cone Γ_n is said to be closed-loop holdable if there exists control $u(t)$ such that the condition of exponential non-negativity can be enforced, i.e.

$$\exists\, u(t) : \dot{K}_n(\xi, Q) < 0, \forall \xi \in \partial\Gamma.$$

For the system in eqn.(37) and the cone in eqn.(14),

$$\begin{aligned} \dot{K}_n(\xi, Q) &= \dot{\xi}^T Q\xi + \xi^T Q\dot{\xi} \\ &= \eta^T Q\xi + \xi^T Q\eta, \end{aligned}$$

which is *independent* of u. Therefore, the condition of exponential non-negativity cannot be enforced by any choice of u.

However, it is possible to design tracking controllers, where reference $\xi^r(t)$ is first determined using first order models and then $u(t)$ is determined to track the reference $\xi^r(t)$. An example using this two degree of freedom controller design is presented in the next section.

VI. Example

In this section we consider rendezvous of three agents in the (x, y) plane. The open loop dynamics of the x and y positions of each agent are modeled as second order systems, i.e.

$$\begin{pmatrix} \dot{x}_i \\ \dot{v}_{x_i} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} x_i \\ v_{x_i} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{x_i}, \tag{38}$$

$$\begin{pmatrix} \dot{y}_i \\ \dot{v}_{y_i} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} y_i \\ v_{y_i} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{y_i}. \tag{39}$$

Since the x and y dynamics of the agents are second order systems, we first solve the rendezvous problem using the first order dynamics and eqn.(20),(19) to generate reference trajectories $x_i^r(t), y_i^r(t)$, for each agent. Full state feedback is assumed in determining the reference trajectories, i.e. every agent has position information of all the agents. The feedback structure of the outer-loop (reference generation) structure is shown in Fig.13. Observe that the reference trajectory is generated in a decentralized manner.

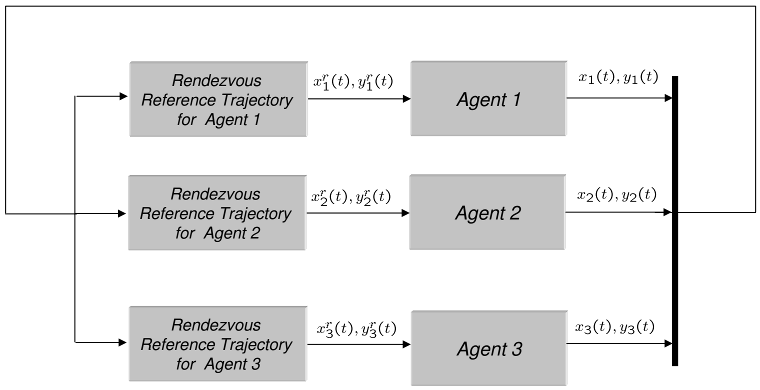


Fig. 13. Feedback structure of the outer-loop.

The reference trajectories are then tracked using a separately designed tracking controller. The inner-loop (tracking) structure is shown in fig.(14) for tracking of reference $x^r(t)$. The tracking controller is identical for both $x^r(t)$ and $y^r(t)$ and also for every agent.

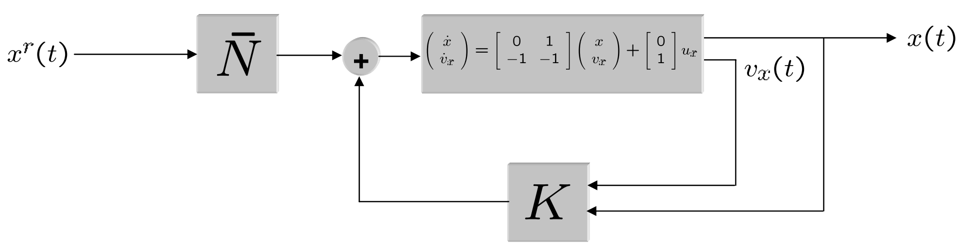
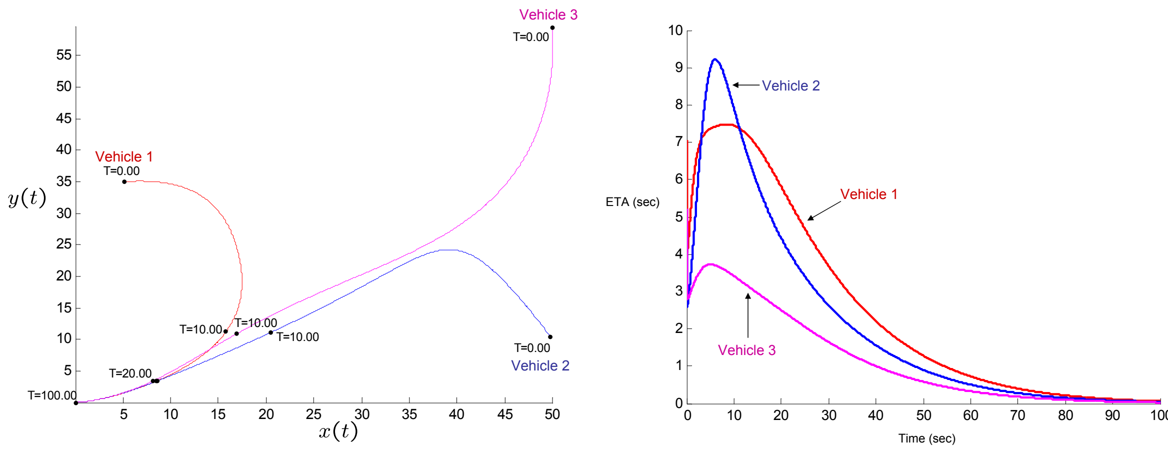


Fig. 14. Feedback structure of the inner-loop.

Example 3: Simulation with Tracking Controller

Figure 15(a) shows the trajectories of the three agents. The initial conditions for position of the three agents are $(5, 35)$, $(50, 10)$ and $(50, 60)$ respectively. The initial velocities of the three agents along x, y are $(10, 1)$, $(-10, 20)$ and $(1, -30)$ respectively. We observe that the agents achieve rendezvous with a reasonably good position tracking controller. The expected arrival times of the agents are shown in fig. 15(b). We observe that the ETA of all the vehicles increase initially. This is due to the mismatch in the velocity of the system and the required velocity for rendezvous. The ETAs become identical as the vehicles approach the origin. This is also visible from the plots in fig.12(a). We observe that the trajectories are close to each other at $T = 10s$ and become identical at $T = 20s$.



(a) Initial conditions (x, y): $(5, 35), (50, 10), (50, 60)$. Initial conditions for (v_x, v_y): $(10, 1), (-10, 20)$ and $(1, -30)$.

(b) Estimated arrival times.

Fig. 15. Rendezvous of three agents with second order dynamics in (x, y) plane. Reference position trajectories are generated using first order dynamics. Position tracking controller is then used to track the reference.

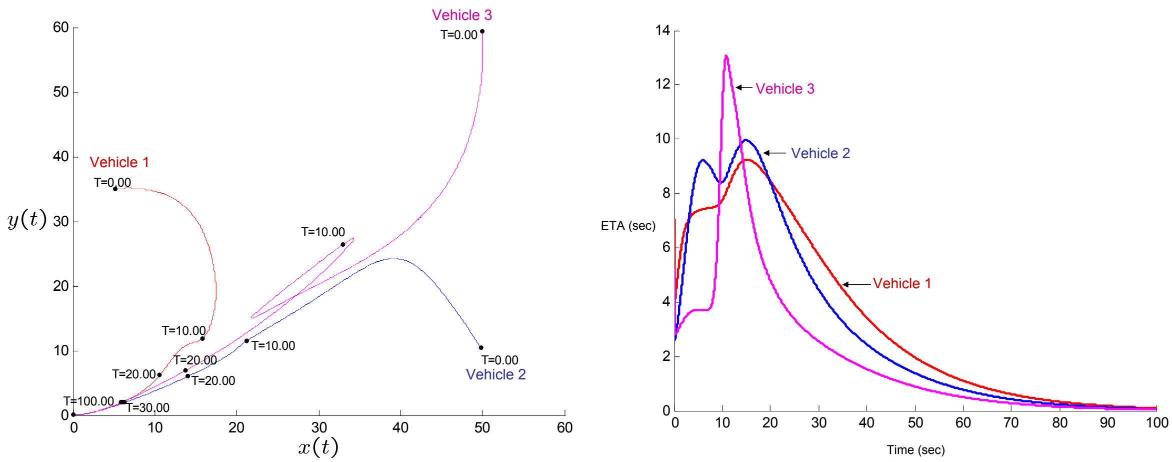
Example 4: Simulation with Tracking Controller & Uncertainty in Vehicle Behavior

Figure 16(a) shows the same simulation as the previous example, but with vehicle 3 making an unexpected loop in the time interval of $T = [5, 15]$ seconds. We observe that the other vehicles modify their trajectories accordingly to achieve rendezvous. This is particularly visible in the ETA plots as shown in fig.16(b). Due to the diversion of vehicle 3, its ETA increases considerably. ETA of the other vehicles also increase appropriately so that they achieve rendezvous. Note that the first peak in the ETA of vehicles 1 and 2 are due to the mismatch in the velocity as in the previous example. The second peak is due to the deviation of vehicle 3 from the reference trajectory. Once again the ETAs become identical as the vehicles approach the origin. Figure 16(a) shows that the vehicle trajectories come close to each other by $T = 20s$ and become identical at $T = 30s$.

The above examples demonstrate that the proposed method is also applicable to second order systems with suitably designed position tracking controller. The method is also robust to changes in the vehicle behavior.

VII. Communication Issues

In the proposed method we have assumed full state feedback for controller synthesis. In reality, the communication topology may not allow such a luxury. In such cases, state estimations are required. Recent developments on multi-agent consensus can be applied to estimate the positions of the agents. Future work along this direction is to incorporate some of the results available in multi-agent consensus into this framework.



(a) Initial conditions (x, y): $(5, 35), (50, 10), (50, 60)$. Initial conditions (v_x, v_y): $(10, 1), (-10, 20)$ and $(1, -30)$.

(b) Estimated arrival times.

Fig. 16. Rendezvous of three agents with second order dynamics in (x, y) plane and robustness with respect to uncertainty in vehicle behavior.

VIII. Summary

This paper presented our initial results on rendezvous of multiple agents. We addressed the problem in a non-graph theoretic framework. The problem was formulated as a cone invariance problem and necessary and sufficient conditions were developed using ellipsoidal cones, for systems with first order dynamics. The necessary and sufficient conditions were also presented in the Lyapunov framework using multiple Lyapunov functions. A control synthesis algorithm using full state feedback approach for first order system was also presented. Numerical examples demonstrating application of this method to higher order systems and also robustness with respect to uncertainty in vehicle behavior was also presented.

Future work along this theme is focussed on multiple directions including formal analysis of multi-agent rendezvous with higher order dynamics, addressing state estimation and consensus and extension of this framework to nonlinear systems using multiple Lyapunov functions.

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