CONE-VALUED LYAPUNOV FUNCTIONS

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Introduction

It is very well known that employing a single Lyapunov function and the theory of scalar differential inequality offers a useful mechanism to study a variety of qualitative problems of differential equations in a unified way [10]. Nevertheless, when using this powerful technique for concrete problems, the main difficulty we face is the lack of a general method of constructing a Lyapunov function. This naturally beads to the development of the method of vector Lyapunov functions which utilizes several Lyapunov-like functions and the theory of vector differential inequalities in a fruitful manner [5,8-12]. This method offers a more flexible mechanism to discuss qualitative properties of nonlinear systems. Also, it provides an effective tool to investigate the properties of large scale interconnected dynamical and control systems whose multivariability, composite structure, multi-connection and the variety of the nature of subsystems make the construction of a single Lyapunov function much more difficult. Moreover, several Lyapunov functions result in a natural way in the study of such systems by the decomposition and aggregation method [1,3,5,6,13-15].

However, an unpleasant fact in this approach is the requirement of quasimonotone nondecreasing property of the comparison system. Since comparison systems with a desired property like stability exist without satisfying the quasimonotone property, the limitation of this general and effective method in applying to concrete problems is obvious [4,9]. It was observed in [9] that this difficulty is due to the choice of the cone relative to the comparison system, namely, the cone of nonnegative elements of R^n i.e., R^n_+ and a possible answer lies in choosing a suitable cone other than R^n_+ to work in a given situation. It is precisely this idea that is investigated in this paper by systematically developing the method of cone-valued Lyapunov functions. Examples are presented to illustrate the usefulness of the method developed.

2. *Comparison Principle*

Let R^n denote the n-dimensional Euclidean space with the Euclidean norm $||\cdot||$ and the scalar product $(\ ,\ )$. A proper subset K of R^n is called a cone if (i) $\lambda K \subset K$, $\lambda \geq 0$, (ii) $K + K \subset K$, (iii) $K = \overline{K}$, (iv) $K \cap (-K) = \{0\}$ and (v) K^o is non-empty. Here $\overline{K}$ denotes the closure of K and K^o, the interior of K. We shall also denote by ∂K the boundary of K. The cone K induces the order relations on R^n defined by

$$x \underset{K}{\leq} y \quad \text{iff} \quad y - x \in K$$

and

$$x \underset{K^o}{<} y \quad \text{iff} \quad y - x \in K^o.$$

The set K^* defined by

$$K^* = \{\phi \in R^n : (\phi, x) \geq 0 \quad \text{for all} \quad x \in K\}$$

satisfies the properties (i) to (v) and is called the adjoint cone. We note that

$$K = (K^*)^*,$$

$$x \in K^o \quad \text{iff} \quad (\phi,x) > 0 \quad \text{for all} \quad \phi \in K_o^*$$

and $x \in \partial K$ iff $(\phi,x) = 0$ for some $\phi \in K_o^*$, where $K_o = K - \{0\}$.

Let $f \in C[D,R^n]$, D being a subset in R^n. Following Elsner [2], we define the function $f(x)$ to be quasimonotone in x relative to the cone k if x, $y \in D$, $y - x \in \partial K$ implies that there exists a $\phi \in K_o^*$ such that $(\phi,y - x) = 0$ and $(\phi,f(y) - f(x)) \geq 0$.

If f is linear, that is, $f(x) = Ax$ where A is a $n \times n$ matrix, the quasimonotonicity condition of f reduces to the following:

$x \in \partial K$ implies that there exists a $\phi \in K_o^*$ such that $(\phi,x) = 0$ and $(\phi,Ax) \geq 0$.

Let us start with the following basic result on differential inequalities.

<u>*THEOREM 2.1.*</u> Let $g \notin C[R_+ \times R^n,R^n]$ and $g(t,u)$ be quasimonotone in u relative to the cone K for each $t \in R_+$. Let $u,v \in C[R_+,R^n]$ and for $t > t_0$,

$$D_-u(t) \underset{K}{\leq} g(t,u(t)),$$

$$g(t,v(t)) \underset{K^o}{<} D_-v(t).$$

Then,

$$u(t_0) \underset{K^o}{<} v(t_0) \quad \text{implies} \quad u(t) \underset{K^o}{<} v(t), \quad t \geq t_0.$$

<u>*PROOF.*</u> Suppose that the assertion of the theorem is false. Then, there exists a $t_1 > t_0$ such that

$$v(t_1) - u(t_1) \in \partial K,$$

and

$$v(t) - u(t) \in K^o, \quad t \in [t_0,t_1).$$

Since $g(t,u)$ is quasimonotone in u with respect to K for each $t \in R_+$, there exists a $\phi \in K_0^*$ such that

$$(\phi, v(t_1) - u(t_1)) = 0$$

and $(\phi, g(t_1, v(t_1)) - g(t_1, u(t_1))) \geq 0$.

Setting

$$w(t) = (\phi, v(t) - u(t)), \quad t \in [t_0, t_1),$$

it is clear that $w(t) > 0$ for $t \in [t_0, t_1)$ and $w(t_1) = 0$. Hence $D_- w(t_1) \leq 0$. But

$$D_- w(t_1) = (\phi, D_- v(t_1) - D_- u(t_1))$$
$$> (\phi, g(t_1, v(t_1)) - g(t_1, u(t_1))) \geq 0$$

which is a contradiction. The proof is therefore complete.

The next result gives the existence of the maximal solution of

(2.1) $$u' = g(t,u), \quad u(t_0) = u_0,$$

relative to K.

<u>*THEOREM 2.2.*</u> Let $g \in C[R_0, R^n]$, where

$$R_0 = \{(t,u) : t_0 \leq t \leq t_0 + a, \quad ||u - u_0|| \leq b\}.$$

Let $||g(t,u)|| \leq M$ on R_0. Assume that $g(t,u)$ is quasimonotone in u with respect to K for each $t \in [t_0, t_0 + a]$. Then, there exists a maximal solution of (2.1) relative to K on $[t_0, t_0 + \alpha]$ where $\alpha = min(a, \frac{b}{2M + b})$.

<u>*PROOF.*</u> Let $\eta \in K^0$ with $||\eta|| = 1$ and let $0 < \varepsilon \leq \frac{b}{2}$. Consider the system

$$u' = g(t,u) + \varepsilon\eta, \quad u(t_0) = u_0 + \varepsilon\eta.$$

The rest of the proof is very much similar to the proof of the corresponding result with $K = R_+^n = \{x \in R^n : x_i \geq 0, \quad i = 1,2,\ldots,n\}$, with appropriate modifications. See Theorem 1.6.1 in [10]. Of course, we make use of the differential inequalities Theorem 2.1 in the present situation.

We now merely state the comparison principle through the cone K, the proof of which is again similar to the corresponding result with $K = R^n_+$ and necessary changes.

THEOREM 2.3. Assume that

(a) $g \in C[R_+ \times R^n, R^n]$, $g(t,u)$ is quasimonotone in u relative to K for each $t \in R_+$ and $[t_0,\infty)$, $t_0 \in R_+$, is the largest interval of existence for the maximal solution $r(t,t_0,u_0)$ of (2.1), relative to K;

(b) $m \in C[R_+, R^n]$ and

(2.2) $$D_-m(t) \underset{K}{\leq} g(t,m(t)), \quad t \geq t_0.$$

Then, $m(t_0) \underset{K}{\leq} u_0$ implies

(2.4) $$m(t) \underset{K}{\leq} r(t,t_0,u_0), \quad t \geq t_0.$$

The following corollary which concerns with the positivity of solutions and the flow-invariance of the cone K, is an important result in itself.

COROLLARY 2.1. Let the assumption (a) of Theorem 2.3 hold. Suppose that $g(t,0) \equiv 0$. Then, $u_0 \in K$ implies that $r(t,t_0,u_0) \in K$. If further the uniqueness of solutions of (2.1) is assumed, then the cone is flow-invariant, that is, $u_0 \in K$ implies that $u(t,t_0,u_0) \in K$, $t \geq t_0$.

For the proof, it is enough to choose $m(t) \equiv 0$ so that (2.2) is trivially satisfied. The conclusion (2.4) proves the first assertion of the corollary. The second statement is a direct consequence of the uniqueness assumption.

A special case of Corollary 2.1 with $K = R^n_+$ is given in [6].

REMARK 2.1. Observe that the quasimonotonicity of $g(t,u)$ in u relative to the cone P need not imply the quasimonotonicity of $g(t,u)$ in u relative to the cone Q, whenever $P \subset Q$. However, the order relations relative to P imply the same order relations relative to Q, if $P \subset Q$.

From this observation results the following Corollary which is very useful in applications.

COROLLARY 2.2. Let P,Q be two cones in R^n such that $P \subset Q$. Let the assumptions (a) and (b) of Theorem 2.3 hold with K replaced by P. Then, $m(t_0) \underset{P}{\leq} u_0$ implies

$$m(t) \underset{Q}{\leq} r(t,t_0,u_0), \quad t \geq t_0.$$

3. Stability Theory

Consider the differential system

$$x' = f(t,x), \quad x(t_0) = x_0, \tag{3.1}$$

where $f \in C[R_+ \times S(\rho), R^N]$ and $S(\rho) = \{x \in R^N : ||x|| < \rho\}$. Let K be a cone in R^n, $n \leq N$ and let $V \in C[R_+ \times S(\rho), K]$. Define for $(t,x) \in R_+ \times S(\rho)$,

$$D^+V(t,x) \equiv \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)].$$

The following comparison theorem plays a prominent role whenever we employ cone-valued Lyapunov functions.

THEOREM 3.1. Assume that

(i) $V \in C[R_+ \times S(\rho), K]$, $V(t,x)$ satisfies a local Lipschitz condition in x relative to K and for $(t,x) \in R_+ \times S(\rho)$, (3.2) $$D^+V(t,x) \underset{K}{\leq} g(t,V(t,x));$$

(ii) $g \in C[R_+ \times K, R^n]$ and $g(t,u)$ is quasimonotone in u with respect to K for each $t \in R_+$.

If $r(t,t_0,u_0)$ is the maximal solution of (2.1) relative to K and $x(t,t_0,x_0)$ is any solution of (3.1) such that $V(t_0,x_0) \underset{K}{\leq} u_0$, then, on the common interval of existence, we have

(3.3) $$V(t,x(t,t_0,x_0)) \underset{K}{\leq} r(t,t_0,u_0).$$

PROOF. Let $x(t) = x(t,t_0,x_0)$ be any solution of (3.1) such that $V(t_0,x_0) \underset{K}{\leq} u_0$. Set $m(t) = V(t,x(t))$. Then, for small $h > 0$, we have, using the fact that $V(t,x)$ is locally Lipschitzian in x relative to K,

$$m(t+h) - m(t) \underset{K}{\leq} L||x(t+h) - x(t) - hf(t,x(t))||$$

$$+ V(t+h, x(t) + hf(t,x(t))) - V(t,x(t)).$$

From this, follows the differential inequality

$$D^+m(t) \underset{K}{\leq} g(t,m(t))$$

in view of the condition (3.2). Now, applying Theorem 2.3, we get (3.4).

The next theorem which is a variant of Theorem 3.1 is more flexible in applications. We merely state the result, as its proof follows from Corollary 2.2.

THEOREM 3.2. Let P and Q be two cones in R^n such that $P \subset Q$. Suppose that

(i) $V \in C[R_+ \times S(\rho), Q]$, $V(t,x)$ satisfies a local Lipschitz condition in x relative to P and

$$D^+V(t,x) \underset{P}{\leq} g(t,V(t,x)), \quad (t,x) \in R_+ \times S(\rho);$$

(ii) $g \in C[R_+ \times Q, R^n]$ and $g(t,u)$ is quasimonotone in u with respect to P for each $t \in R_+$.

If $r(t,t_0,u_0)$ is the maximal solution of (2.1) relative to P and $x(t,t_0,x_0)$ is any solution of (3.1) such that $V(t_0,x_0) \underset{P}{\le} u_0$, then

$$V(t,x(t,t_0,x_0)) \underset{Q}{\le} r(t,t_0,u_0), \tag{3.5}$$

on the common interval of existence of $r(t,t_0,u_0)$ and $x(t,t_0,x_0)$. If, in particular, $Q = R^n_+$, then the relation (3.5) implies the componentwise inequalities

$$V(t,x(t,t_0,x_0)) \le r(t,t_0,u_0).$$

REMARK 3.1. The comparison Theorem 3.1 in the special case $K = R^n_+$ has been fruitfully employed in connection with the use of vector Lyapunov functions. See [6,7,10,13-15]. In this situation, the comparison system $g(t,u)$ is required to satisfy the quasimonotone nondecreasing property in u for each $t \in R_+$, that is, for each $i = 1,2,\ldots,n$, the function $g_i(t,u_1,\ldots,u_i,\ldots,u_n)$ is nondecreasing in u_j, $i \ne j$. This comparison technique which is called "the method of vector Lyapunov functions" is known to be a very flexible mechanism to study the qualitative properties of differential systems as well as to effectively discuss interconnected dynamical systems or large scale competitive systems [1,3,9,13-15]. However, the requirement that $g(t,u)$ be quasimonotone is too restrictive for many applications. If $g(t,u) = Au$ where A is a $n \times n$ matrix, this condition implies that the off-diagonal elements of A must be non-negative. Since this property of A is not a necessary condition for a matrix to be a stable matrix, the limitation of this technique is clear. An idea to get around this difficulty was proposed in [9]. Here we exploit another important idea that was mentioned in [9], namely, choosing an appropriate cone other than R_+^n to work with in a given situation.

Having established comparison theorems 3.1 and 3.2, it is now easy to discuss various qualitative properties including stability results by the method of cone-valued Lyapunov functions. We shall state a typical stability result.

THEOREM 3.3. Let the assumptions (i) and (ii) of Theorem 3.1 hold. Suppose further that

(a) $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$;

(b) for some $\phi_0 \in K_0^*$ and $(t,x) \in R_+ \times S(\rho)$,

$$(3.6) \qquad b(||x||) \leq (\phi_0, V(t,x)) \leq a(t,||x||),$$

where $a \in C[R_+ \times [0,\rho), R_+]$, $b \in C[[0,\rho), R_+]$, $a(t,0) \equiv 0$, $b(0) = 0$ and $a(t,u)$, $b(u)$ are increasing in u;

(c) the trivial solution $u \equiv 0$ of (2.1) is ϕ_0-equistable, that is, given $\varepsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(t_0,\varepsilon) > 0$ such that

$$(\phi_0, u_0) < \delta \quad \text{implies} \quad (\phi_0, r(t,t_0,u_0)) < \varepsilon, \quad t \geq t_0.$$

Then, the trivial solution $x \equiv 0$ of (3.1) is equistable.

The following version of Theorem 3.3 is in a more flexible setting so as to be useful in applications.

THEOREM 3.4. Let the assumptions of Theorem 3.2 and assumption (a) of Theorem 3.3 hold. Assume further that the condition (3.6) is satisfied for some $\phi_0 \in Q_0^*$ and that the trivial solution $u \equiv 0$ of (2.1) is ϕ_0-equistable, with $\phi_0 \in Q_0^*$. Then, the trivial solution $x \equiv 0$ of (3.1) is equistable.

The proofs of the above theorems follow the standard pattern of

Lyapunov theory with appropriate modifications. They depend crucially on Theorems 3.1 and 3.2 respectively. We have avoided the indication of proofs to avoid monotony.

Few remarks are now in order.

REMARK 3.2. If $K = R^n_+$, $\phi_0 = (1,1,\ldots,1)$. Theorem 3.3 is the well known result in the method of vector Lyapunov functions [5,10]. In this case, the condition (3.6) reduces to the familiar assumption

$$b(||x||) \leq \sum_{i=1}^{n} V_i(t,x) \leq a(t,||x||).$$

REMARK 3.3. One could also use other measures in place of $(\phi_0, V(t,x))$. For example, let $\Phi \in C[K,R_+]$, $\Phi(u)$ is nondecreasing in u relative to K, i.e., $u \underset{K}{\leq} v$ implies $\Phi(u) \leq \Phi(v)$. The condition (3.6) now becomes

$$b(||x||) \leq \Phi(V(t,x)) \leq a(t,||x||).$$

Correspondingly, assumption (c) of Theorem 3.3 has to be modified in terms of Φ.

REMARK 3.4. Let $P \subset Q = R^n_+$ in Theorem 3.4. Then, the unpleasant fact concerning the quasimonotonicity of $g(t,u)$, mentioned in Remark 3.1 can be removed. This, of course, means that we have to choose an appropriate cone P which necessarily depends on the nature of $g(t,u)$.

Let us demonstrate this by means of examples.

EXAMPLE 1. Consider the system

$$(3.7) \qquad \begin{cases} u_1' = a_{11}u_1 + a_{12}u_2 \equiv g_1(t,u_1,u_2), & u_1(t_0) = u_{10}, \\ u_2' = a_{21}u_1 + a_{22}u_2 \equiv g_2(t,u_1,u_2), & u_2(t_0) = u_{20}. \end{cases}$$

Let $Q = R_+^2$. Suppose that we do not demand a_{21} and a_{12} to be nonnegative. Then the function $g(t,u)$ violates the quasimonotone nondecreasing condition in $u = (u_1,u_2)$ relative to Q. Hence, the differential inequalities

$$(3.8) \qquad \begin{aligned} D^+V_1(t,x) &\leq g\ (t,V_1\ (t,x),V_2(t,x)), \\ D^+V_2(t,x) &\leq g_2(t,V_1(t,x),\ V_2(t,x)) \end{aligned}$$

do not yield the componentwise estimates of $V(t,x(t))$ in terms of the solution of (3.7).

Suppose now that there exist two numbers α, β such that $0 < \beta < \alpha$ and

$$(3.9a) \qquad \alpha^2 a_{21} + \alpha a_{22} \geq \alpha a_{11} + a_{12},$$

$$(3.9b) \qquad \beta^2 a_{21} + \beta a_{22} \geq \beta a_{11} + \beta_{12}.$$

These conditions can hold with no restriction of nonnegativity of a_{21} and a_{12}. We shall now choose the cone $P \subset Q = R_+^2$ defined by

$$P = \{u \in R_+^2:\ \ \beta u_2 \leq u_1 \leq \alpha u_2\}.$$

This cone has two boundaries $\alpha u_2 = u_1$, and $\beta u_2 = u_1$. On the boundary $\alpha u_2 = u_1$, we take $\phi = (-\frac{1}{\alpha}, 1)$ so that $(-\frac{1}{\alpha}, 1)$ $(u_1, \frac{u_1}{\alpha}) = 0$ and

$$(-\frac{1}{\alpha}, 1)(a_{11}u_1 + a_{12}\frac{u_1}{\alpha},\ a_{21}u_1 + a_{22}\frac{u_1}{\alpha}) \geq 0, \quad \text{for all} \quad u \neq 0.$$

This reduces to the condition (3.9a). Similarly, we can obtain (3.9b). Thus, if the inequalities (3.8) are relative to P, we obtain the componentwise estimates on V as

$$V_i(t,x(t)) \le r_i(t,t_0,V(t_0,x_0)), \tag{3.10}$$

by Theorem 3.2. We note that the estimate (3.10) is precisely the one we would have obtained if $a_{12}, a_{21} \ge 0$, by the standard method of vector Lyapunov functions. Needless to say that since a_{12}, a_{21} need not be nonnegative in our case, that method breaks down and we cannot get (3.10).

EXAMPLE 2. Let us now consider the system

$$u_i' = \sum_{j=1}^{3} a_{ij}u_j, \quad u_i(t_0) = u_{i0}, \quad i = 1,2,3. \tag{3.11}$$

Let $Q = R_+^3$. Suppose that a_{12}, a_{13}, a_{21}, and a_{23} are nonnegative and a_{31}, a_{32} are negative. Then, clearly the function $g(t,u)$ violates the quasimonotone nondecreasing condition in $u = (u_1,u_2,u_3)$ and consequently, the componentwise estimates are not possible to obtain. Suppose we choose $P \subset Q$ such that

$$P = \{u \in R_+^3 : u_1^2 + u_2^2 \le u_3^2\}.$$

To work with this cone P and obtain componentwise estimates, it is enough to assume the relations

$$a_{11} \ge a_{33}, \quad a_{22} \ge a_{33}, \quad a_{13} \ge a_{31} \quad \text{and} \quad a_{23} \ge a_{32}. \tag{3.12}$$

For, on the boundary $u_1^2 + u_2^2 = u_3^2$, we can take

$$\phi = \left(\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, \frac{u_2}{\sqrt{u_1^2 + u_2^2}}, -1 \right) \quad \text{so that} \quad (\phi,u) = 0, \quad \text{where}$$

$u = (u_1, u_2, \sqrt{u_1^2 + u_2^2})$ and $(\phi,g(t,u)) \ge 0$. This last inequality leads to

$$(a_{11} - a_{33})u_1^2 + (a_{22} - a_{33})u_2^2 + (a_{12} + a_{21})u_1u_2$$
$$+ [(a_{13} - a_{31})u_1 + (a_{23} - a_{32})u_2]\sqrt{u_1^2 + u_2^2} \geq 0 \quad \text{for all} \quad u_1,u_2 \neq 0,$$

from which results (3.12). It is easy to obtain the nonnegativity of $a_{12}, a_{21}, a_{13}, a_{23}$ by verifying the quasimonotone property for the boundaries $u_1 = 0$ and $u_2 = 0$ with the choice of $\phi = (1,0,0)$ and $\phi = (0,1,0)$ respectively.

Even this simple example shows the complexities involved in constructing appropriate cone P for given $g(t,u)$. Thus, the general problem of constructing suitable cone P given the nature of $g(t,u)$ seems to be as difficult as that of constructing a suitable Lyapunov function. However, the fruitfulness of such a construction at least in special cases deserves merit and is worthy of further investigation.

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