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ABSTRACT

Stochastic differential equations of Itô-type are considered and the theory of stochastic differential inequalities is systematically developed. Sufficient conditions for stability in probability, with probability one and in the mean of the Itô-type stochastic differential equations are given, using the method of cone-valued Lyapunov functions. Necessary conditions for the construction of stochastic cone-valued Lyapunov functions are obtained for the cases where the Itô-type stochastic differential equations have uniform asymptotic stability in probability and uniform asymptotic stability in the mean.

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1 Introduction

Many biological, physical and social phenomena can be described by a system of stochastic differential equations of Itô-type. It is well known [1] that the method of vector Lyapunov functions provides an effective tool for the discussion of the stability analysis of Itô-type stochastic differential equations. However, this method crucially depends on the requirement of quasimonotone non-decreasing property of the comparison system. It is also well known [3] that the requirement of quasimonotone non-decreasing property is restrictive. In [1,3], it was shown that the theory of differential inequalities through cones together with the comparison principle of the method of cone-valued Lypunov functions removes this unpleasant restriction of the method of vector Lyapunov functions and provides an effective and flexible tool for the investigation of the stability behaviour of the solutions of ordinary differential equations.

In this paper, we are concerned with the idea of extending the theory developed in [1,3] for the ordinary differential equations to the stochastic differential equations of Itô-type. We shall systematically develop the theory of stochastic differential inequality through cones, and obtain various stability results for the stochastic differential equations of Itô-type in the framework of cone-valued Lyapunov functions.

2 Stochastic Differential Inequalities

In this section we develop the theory of differential inequalities through cones for the Itô-type stochastic differential equations. We give sufficient conditions for the existence of maximal solution process of the Itô-type comparison differential equation relative to a cone $K \subset R^n$. We also state comparison theorems for the Itô-type stochastic differential equation in the framework of cone-valued Lypunov functions.

Consider the Itô-type stochastic differential equations

$$dx = f(t,x)dt + \sigma(t,x)\, dz(t), \qquad x(t_0) = x_0 \tag{2.1}$$

where $f \in C[R_+ \times R^N, R^N], \sigma \in C[R_+ \times R^N, R^{N^2}]$ and $z \in R[\Omega, R^N]$ is a normalized N-vector Wiener process. Consider the Itô-type comparison stochastic differential equation

$$du = g(t,u)dt + G(t,u)dz(t), \qquad u(t_0) = u_0 \tag{2.2}$$

where $g \in C[R_+ \times R^n, R^n], G \in C[R_+ \times R^n, R^{n^2}], n \le N, z \in R[\Omega, R^n]$ is a normalized n-vector Wiener process. We assume that f, g, σ, G are smooth enough to guarantee the existence of the solution process of (2.1) and (2.2).

Let R^n denote the n-dimensional Euclidean space with any convenient norm $\|\cdot\|$ and scalar product $(\cdot,\cdot), R_+ = [0,\infty), R^n_+ = \{u \in R^n : u_i \ge 0, i = 1,\dots,n\}$. Define S_ρ by, $S_\rho = \{x \in R^N : \|x\| < \rho, \rho > 0\}$.

Definition 2.1. A proper subset $K \subset R^n$ is called a *cone* if (i) $\lambda K \subset K, \lambda \ge 0$ (ii) $K + K \subset K$ (iii) $K = \bar{K}$, (iv) $K \cap (-K) = \{0\}$ (v) K^0 is non empty; where $\bar{K}$ denotes the closure of K and K^0 denotes the interior of K. We also denote the boundary of K by ∂K.

The *order relation* on R^n induced by the cone K is defined as follows: Let $x, y \in K$, then $x \underset{K}{\le} y$ if and only if $y - x \in K$ and $x \underset{K^0}{<} y$ if and only if $y - x \in K^0$.

Definition 2.2 The set K^* is called the *adjoint cone* if $K^* = \{\phi \in R^n : (\phi, x) \geq 0$ for all $x \in K\}$ satisfies properties (i)–(v) of Definition 2.1.

Definition 2.3. A function $g : D \to R^n$, $D \subset R^n$, is said to be *quasimonotone relative to the cone* K if $x, y \in D$ and $y - x \in \partial K$ imply that there exists $\phi_0 \in K_0^*$ such that $(\phi_0, y - x) = 0$ and $(\phi_0, g(y) - g(x)) \geq 0$, where $K_0^* = K^* - \{0\}$.

Definition 2.4. (Property A). Let $h \in C[K, R_+^n]$ be defined by $h(x) = h_i(x_i), i = 1, 2, \ldots, n$; so that $h(x) = (h_1(x_1), h_2(x_2), \ldots, h_n(x_n))^T$. Let $\Phi \in C[R_+^n, K]$ with the following properties:

(i) $\Phi(0) = 0$, (ii) $\Phi(v) \underset{K}{\leq} \Phi(u)$ if and only if $v_i \leq u_i, i = 1, 2, \ldots, n$,

(iii) $\|\Phi\| \geq \|v\|$, (iv) $\Phi((\phi_0, \|G(t,u)\|^2 \|w\|_{\hat{G}})\hat{e}) \underset{K}{\leq} \Phi(h(\|u\|_{\hat{G}}))$

(v) $\displaystyle\int_{0+} \frac{ds}{\|\Phi(h(u(s)))\|} = \infty$

where $G \in C[R_+ \times K, R^{n^2}]$, $\|\cdot\|$ is the matrix or vector norm in R^n and $\|\cdot\|_{\hat{G}}$ is the generalized norm of vectors defined in Definition 4.5 in [1]. $\phi_0 \in K_0^*, w \in K, (t, u) \in R_+ \times K$ and $\hat{e}$ is a unit vector in K.

Let $V \in C[R_+ \times S_\rho, K]$, V_t, V_x, V_{xx} exist and are continuous for $(t, x) \in R_+ \times S_\rho$ and $K \subset R^n$, where V_x is an $N \times n$ Jacobian matrix of $V(t,x)$ and V_{xx} is an $n \times n$ Hessian matrix whose elements $(\partial^2/\partial x_i \partial x_j)V(t,x)$ are N-dimensional vectors. By Itô's formula we obtain

$$dV(t,x) = L\,V(t,x)\,dt + V_x(t,x)\,\sigma(t,x)\,dz(t)\ . \tag{2.3}$$

where

$$L\,V(t,x) = V_t(t,x) + V_x(t,x)\,f(t,x) + \frac{1}{2}\,tr\{V_{xx}(t,x)\,\sigma(t,x)\,\sigma^T(t,x)\} \tag{2.4}$$

Theorem 2.1. *Assume that*

(i) *Property A holds,*

(ii) $m(t)$ *is a solution of*

$$dm = g(t,m)dt + G(t,m)\,dz(t), m(t_0) = m_0 \tag{2.5}$$

where $m \in C[R_+, K], g \in C[R_+ \times K, R^n], G \in C[R_+ \times K, R^{n^2}], K \subset R^n, z \in R[\Omega, R^n]$ *is a normalized* n*-vector Wiener process. Then*

$$E[\|m(t)\|_{\hat{G}}] \underset{K}{\leq} E[\|m_0\|_{\hat{G}}] + E\left[\int_{t_0}^t \|g(s, m(s))\|_{\hat{G}}\,ds\right], t \geq t_0\ .$$

Proof. Let $q_n \underset{K^0}{<} q_{n-1}$, where q_n, q_{n-1} are two points in $\hat{K} \subset R_+^n$, such that $q_n \to 0$ as $n \to \infty$ and $q_0 = e = (1, 1, \ldots, 1)^T$. Let $\tau_n = \|q_n\|$ and $\tau_{n-1} = \|q_{n-1}\|$ and define for $n = 1, 2, \ldots$

$$\int_{\tau_n}^{\tau_{n-1}} \frac{ds}{\|\Phi(h(u(s)))\|} = n\ .$$

Let $A = \{u \in \hat{K} : 0 \leq \|u\| \leq \tau_n\}, B = \{u \in \hat{K} : \tau_n < \|u\| < \tau_{n-1}\}, C = \{u \in \hat{K} : \|u\| \geq \tau_{n-1}\}$. Then there exists a twice continuously differentiable cone-valued function $T_n(u)$

defined on $\hat{K} \subset R_+^n$ such that $T_n(0) = 0$ and

$$T_n(u) = \begin{cases} 0 & \text{for } u \in A \\ \text{between 0 and } u & \text{for } u \in B \\ u & \text{for } u \in C \end{cases}$$

$$T_n'(u) = \begin{cases} 0 & \text{for } u \in A \\ \text{between 0 and } e & \text{for } u \in B \\ e & \text{for } u \in C \end{cases}$$

$$T_n''(u) = \begin{cases} 0 & \text{for } u \in A \\ \text{between 0 and } \dfrac{e}{n\|\Phi(h(\|u\|_{\hat{G}}))\|} & \text{for } u \in B \\ 0 & \text{for } u \in C \end{cases}$$

We can then extend $T_n(u)$ appropriately as a twice continuously differentiable cone-valued function to the largest cone $K \subset R^n$. That is $T_n(u) = T_n(\|u\|_{\hat{G}})$ so that as $n \to \infty$ we have $T_n(\|u\|_{\hat{G}}) = \|u\|_{\hat{G}}$. Now applying Itô's formula on $T_n(\|m(t)\|_{\hat{G}})$, integrating and taking expectation of both sides we obtain

$$\begin{aligned} E[T_n(\|m(t)\|_{\hat{G}})] &= E[T_n(\|m_0\|_{\hat{G}})] \\ &+ E\left[\int_{t_0}^t T_n'(\|m(s)\|_{\hat{G}})\ g(s, m(s))\ ds\right] \\ &+ E\left[\int_{t_0}^t T_n'(\|m(s)\|_{\hat{G}})\ G(s, m(s))\ dz(s)\right] \\ &+ E\left[\frac{1}{2}\int_{t_0}^t tr\{T_n''(\|m(s)\|_{\hat{G}})\ G(s, m(s))\ G^T(s, m(s))\}ds\right]\ . \end{aligned}$$

By the property of stochastic integral we have

$$E\left[\int_{t_0}^t T_n'(\|m(s)\|_{\hat{G}})\ G(s, m(s))\ dz(s)\right] = 0\ ,$$

Also for some $\phi_0 \in K_0^*, w \in K$ and using property A we have that

$$\begin{aligned} &E\left[\int_{t_0}^t \frac{1}{2}\ tr\{T_n''(\|m(s)\|_{\hat{G}})\ G(s, m(s))\ G^T(s, m(s))\}ds\right] \\ &= E\left[\int_{t_0}^t \frac{1}{2}\ tr\{T_n''(\|m(s)\|_{\hat{G}})\ \|G(s, m(s))\|^2\}ds\right] \\ &\leq E\left[\int_{t_0}^t \frac{1}{2} \max_{m \in B}\ tr\{\|T_n''(\|m(s)\|_{\hat{G}})\|\ \|\Phi(\|G(s, m(s))\|^2\ \hat{e})\|\}ds\right] \\ &\leq E\left[\int_{t_0}^t \frac{1}{2} \max_{m \in B}\ tr\{T_n''(\|m(s)\|_{\hat{G}})\|\ \|\Phi((\phi_0, \|G(s, m(s))\|^2\ \|w\|_{\hat{G}})\ \hat{e})\|\}ds\right] \\ &= E\left[\int_{t_0}^t \frac{1}{2} \max_{m \in B}\ tr\{\|T_n''(\|m(s)\|_{\hat{G}})\|\ \|\Phi(h(m(s)))\|\}ds\right] \\ &= E\left[tr\left\{\frac{e}{n\|\Phi(h(m(s)))\|} \cdot \|\Phi(h(m(s)))\|\right\} \frac{t - t_0}{2}\right] \\ &= E\left[\frac{t - t_0}{2} \cdot \frac{e}{n}\right] \to 0 \quad \text{as} \quad n \to \infty\ . \end{aligned}$$

Since $\|T_n'(\|m(t)\|_{\hat G} \leq 1$ we have that

$$\begin{aligned} E&\left[\int_{t_0}^t T_n'(\|m(s)\|_{\hat G})\; g(s,m(s))ds\right] \\ &\underset{\hat K}{\leq} E\left[\int_{t_0}^t \|T_n'(\|m(s)\|_{\hat G})\|\;\|g(s,m(s))\|ds\right] \\ &\underset{\hat K}{\leq} E\left[\int_{t_0}^t \|g(s,m(s))\|ds\right] . \end{aligned}$$

By the definition of $T_n(u)$, as $n \to \infty$ we have

$$T_n(\|m(t))\|_{\hat G} = \|(m(t))\|_{\hat G} \quad \text{and} \quad T_n(\|m_0\|_{\hat G}) = \|m_0\|_{\hat G}$$

and so the conclusion of the Theorem follows.

It remains now to show that $T_n(u)$ can indeed be extended as a twice continuously differentiable cone-valued function to the largest cone $K \subset R^n$. Let $K_1 \subset R^n$ such that $\hat K \subseteq K_1 \subset R^n$. Let $T_{n_1} : K_1 \to K_1$ be a function defined on K_1 with values in K_1 such that $D(T_n) \subseteq D(T_{n_1})$, where $D(T_n)$ denotes the domain of T_n. Let q_{n_1}, q_{n_1-1} be any points in K_1 such that $q_{n_1} \underset{K_1^0}{\leq} q_{n_1-1}$ and $q_{n_1} \to 0$ as $n \to \infty$ and $q_0 = a^1$, a fixed point in K_1 such that if $a^1 \in \hat K$, then $a^1 = e$. Let $A_1 = \{u \in K_1 : 0 \leq \|u\| \leq \tau_{n_1}\}, B_1 = \{u \in K_1 : \tau_{n_1} < \|u\| < \tau_{n_1-1}\}, C_1 = \{u \in K_1 : \|u\| \geq \tau_{n_1-1}\}$, where $\tau_{n_1} = \|q_{n_1}\|$ and $\tau_{n_1-1} = \|q_{n_1-1}\|$. Define a function $T_{n_1} : K_1 \to K_1$ such that $T_{n_1}(0) = 0$ and

$$T_{n_1}(u) = \begin{cases} 0 & \text{for } u \in A_1 \\ \text{between 0 and } \|a^1\|u & \text{for } u \in B_1 \\ \|a^1\|u & \text{for } u \in C_1 \end{cases}$$

clearly $T_{n_1}(u)$ is a twice continuously differentiable function defined on K_1, for,

$$T_{n_1}'(u) = \begin{cases} 0 & \text{for } u \in A_1 \\ \text{between 0 and } \|a^1\|\; e & \text{for } u \in B_1 \\ \|a^1\|\; e & \text{for } u \in C_1 \end{cases}$$

and

$$T_{n_1}''(u) = \begin{cases} 0 & \text{for } u \in A_1 \\ \text{between 0 and } \dfrac{\|a^1\|\; e}{n\|\Phi(h(\|u\|_{\hat G}))\|} & \text{for } u \in B_1 \\ 0 & \text{for } u \in C_1 \end{cases} .$$

If $u \in \hat K$, then $a^1 = e$ and so $T_{n_1}(u) = T_n(u)$.

Generally by choosing points $q_{n_r}, q_{n_r-1} \in K_r$, $\hat K \subseteq K_1 \subseteq K_2 \subseteq \ldots \subseteq K_r \subset R^n$ such that $q_{n_r} \underset{K_r^0}{\leq} q_{n_r-1}$ and $q_{n_r} \to 0$ as $n_r \to \infty$ and $q_0 = a^r$, a fixed point in K_r such that if $a^r \in \hat K$, then $a^r = e$, we can define a function $T_{n_r} : K_r \to K_r$ which is twice continuously differentiable on K_r such that $D(T_n) \subseteq D(T_{n_r})$ and if $u \in \hat K$ then $T_{n_r}(u) = T_n(u)$. If K_r is the largest cone in R^n and $D(T_{n_r}) = K_r$, then T_{n_r} is the required extension of T_n. If K_r is not the largest cone in R^n, then we take a collection S of all twice continuously differentiable functions f defined on the subset of the largest cone in R^n such that for all $f \in S, D(T_n) \subseteq D(f)$ and $f(u) = T_n(u)$ for $u \in \hat K$.

Introduce a partial ordering in S as follows: If $D(f_1) \subseteq D(f_2)$ and $f_1(u) = f_2(u)$ for $u \in D(f_1)$ then write $f_1 \subset f_2$ for $f_1, f_2 \in S$. Let W be a totally ordered subset of S. Define a function g by

$$D(g) = \bigcup_{f \in W} D(f), \quad g(u) = f(u), \quad u \in D(f) .$$

g is uniquely defined, for if $f_1, f_2 \in W$ and since W is totally ordered, then $f_1 \subset f_2$ or $f_2 \subset f_1$ and if $u \in D(f_1) \cap D(f_2)$, then $f_1(u) = f_2(u)$.

From the definition of g, it follows that g is a twice continuously differentiable function defined on some subset of the largest cone in R^n, and so $g \in S$. Since $D(g) = \bigcup_{f \in W} D(f)$, then g is an upper bound for W. Thus a partially ordered set S is such that every totally ordered subset of S has an upper bound which is in S. Then by Zorn's lemma, there exists a maximal element $\hat g \in S$.

We claim that the domain of $\hat g$ is the largest cone in R^n. Suppose this claim were false, then there would exist an element v in the largest cone in R^n with $v \notin D(\hat g)$ in which case $\hat g$ would have an extension $\hat g^*$ defined on $D(\hat g) + \{v\}$. This then contradicts the fact that $\hat g$ is the maximal element in S. Therefore $D(\hat g)$ is equal to the largest cone in R^n and so $\hat g$ is the required extension of T_n, and so the proof of Theorem 2.1 is now complete.

Theorem 2.2. *Let the assumptions of Theorem 2.1 hold and suppose that $g(t,u) \underset{\hat K}{\geq} 0$ a.s. Then $m_0 \in K \subset R_+^n \Rightarrow m(t) \in K$ a.s. for $t \geq t_0$.*

Proof. Since $m(t)$ is a solution of (2.5), then for all $t \geq t_0$, we have

$$m(t) = m_0 + \int_{t_0}^t g(s,m(s))\; ds + \int_{t_0}^t G(s,m(s))\; dz(s)$$

and so

$$E[m(t)] = E[m_0] + E\left[\int_{t_0}^t g(s,m(s))\; ds\right] + E\left[\int_{t_0}^t G(s,m(s))\; dz(s)\right] .$$

Since $E\left[\int_{t_0}^t G(s,m(s))\; dz(s)\right] = 0$, by the property of stochastic integral, then we have

$$E[m(t)] = E[m_0] + E\left[\int_{t_0}^t g(s,m(s))\; ds\right] \tag{2.6}$$

Also from Theorem 2.1 we have

$$E(\|m(t)\|_{\hat G}) \underset{\hat K}{\leq} E(\|m_0\|_{\hat G}) + E\left(\int_{t_0}^t \|g(s,m(s))\|_{\hat G}\; ds\right) \text{a.s.} . \tag{2.7}$$

Since $m_0 \in K$, then $m_0 - 0 \in K \Rightarrow m_0 \underset{\hat K}{\geq} 0$ and also by hypothesis $g(t,u) \underset{\hat K}{\geq} 0$ then $\|m_0\|_{\hat G} = m_0$ and $\|g(t,m(t))\|_{\hat G} = g(t,m(t))$ and so (2.7) becomes

$$E(\|m(t)\|_{\hat G} \underset{\hat K}{\leq} E(m_0) + E\left(\int_{t_0}^t g(s,m(s))\right)\; ds \text{ a.s.} \tag{2.8}$$

From (2.6) and (2.8) we have

$$\begin{aligned} E(\|m(t)\|_{\hat G} &\underset{\hat K}{\leq} E(m(t)) \text{ a.s.} \\ \Rightarrow \|m(t)\|_{\hat G} &\underset{\hat K}{\leq} m(t) \text{ a.s.} \end{aligned}$$

It is obvious that only the equality is admissible. Therefore $\|m(t)\|_{\hat{G}} = m(t) \Rightarrow m(t) \underset{K}{\geq} 0 \Rightarrow m(t) - 0 \in K \Rightarrow m(t) \in K$.

Theorem 2.3. *Assume that*

(i) $G \in C[R_+ \times K, R^{n^2}], g_1, g_2 \in C[R_+ \times K, R^n], g_i(t,u) \underset{K}{\geq} 0$ *a.s.* $i = 1,2$, *are quasi-monotone in* u *relative to* K *for each* $t \in R_+$ *and* $z(t)$ *is a normalized* n*-vector Wiener process.*

(ii) *Property A holds with condition (iv) replaced by*

$$\Phi((\phi_0, \|G(t,v) - G(t,v)\|^2 \|w\|_{\hat{G}})\hat{e}) \underset{K}{\leq} \Phi(h(\|v - u\|_{\hat{G}}))$$

(iii) $v(t), u(t)$ *are solutions of*

$$(2.9)\qquad \begin{aligned} du &= g_1(t,u)dt + G(t,u)dz(t), \quad u(t_0) = u_0 \\ dv &= g_2(t,u)dt + G(t,v)dz(t), \quad v(t_0) = v_0 \end{aligned}$$

respectively and $g_1(t,u) \underset{K^0}{<} g_2(t,u)$ for $(t,u) \in R_+ \times K$. Then $u_0 \underset{K}{\leq} v_0 \Rightarrow u(t) \underset{K}{\leq} v(t)$ a.s., $t \geq t_0$.

Proof. Define $m(t) = v(t) - u(t)$ so that

$$(2.10)\qquad dm = g^*(t,m)dt + G^*(t,m)dz(t), \quad m(t_0) = m_0$$

where

$$(2.11)\qquad \begin{aligned} G^*(t,m) &= G(t,v) - G(t,u) \\ g^*(t,m) &= g_2(t,v) - g_1(t,u) \end{aligned}$$

Obviously (2.10) is an Itô-type stochastic differential equation and $m(t)$ satisfies (2.10). Since $v(t), u(t)$ are solutions of (2.9), then $m(t) = v(t) - u(t)$ is Γ_t-measurable and sample continuous, where Γ_t is a sub-σ-algebra of Γ defined on R_+ and Γ is the σ-algebra of subsets of the sample space Ω. Also

$$\int_{t_0}^{t} [\|g_1(s,u(s))\| + \|G(s,u(s))\|^2]\, ds < \infty\ , \quad \text{w.p.1}$$

and

$$\int_{t_0}^{t} [\|g_2(s,v(s))\| + \|G(s,v(s))\|^2]\, ds < \infty, \quad \text{w.p.1}$$

Now

$$\begin{aligned} &\int_{t_0}^{t} \{\|g^*(s,m(s))\| + \|G^*(s,m(s))\|^2\}\, ds \\ &= \int_{t_0}^{t} \{\|g_2(s,v(s)) - g_1(s,u(s))\| + \|G(s,v(s)) - G(s,u(s))\|^2\}\, ds \\ &\leq \int_{t_0}^{t} \{\|g_1(s,u(s))\| + \|G(s,u(s))\|^2\}\, ds + \int_{t_0}^{t} \{\|g_2(s,v(s))\| + \|G(s,v(s))\|^2\}\, ds \\ &< \infty \quad \text{w.p.1}\ . \end{aligned}$$

Thus $m(t)$ is a solution process of (2.10). Now let $u_0 \underset{K}{\leq} v_0$, then $v_0 - u_0 = m_0 \in K$. Then by Theorem 2.2, $m(t) \in K$ a.s. $\Rightarrow m(t) \underset{K}{\geq} 0 \Rightarrow u(t) \underset{K}{\leq} v(t)$ a.s.

Theorem 2.4. *Let the conditions (i) and (ii) of Theorem 2.3 hold. Assume further that*

(a) $\|g(t,u)\| + \|G(t,u)\| \leq L + M\|u\|$ *for some constants* $L > 0$ *and* $M > 0$.

(b) $u(t_0) = u_0 > 0$ *is independent of* $z(t)$ *and for a positive constant* c, $E[\|u_0\|^4] \leq c$.

Then there exists a maximal solution process of (2.2) *relative to* K *for each* $t \in R_+$.

Theorem 2.5. *Assume that*

(i) $V \in C[R_+ \times S_\rho, K], V_t, V_x$ *and* V_{xx} *exist and are continuous for* $(t,x) \in R_+ \times S_\rho$ *and for each* $(t,x) \in R_+ \times S_\rho, LV(t,x) \underset{K}{\leq} g(t, V(t,x))$, *where* L *is the operator defined in* (2.4).

(ii) $g \in C[R_+ \times K, R^n], g(t,u)$ *is concave and quasimonotone in* u *relative to* K *for each* $t \in R_+$ *and* $r(t)$ *is the maximal solution of deterministic comparison system*

$$(2.12)\qquad u' = g(t,u), \qquad u(t_0) = u_0$$

relative to K.

(iii) *For the solution process* $x(t)$ *of* (2.1) $E[V(t,x)]$ *exists for* $t \geq t_0$. *Then*

$$E[V(t_0,x_0)] \underset{K}{\leq} u_0 \Rightarrow E[V(t,x)] \underset{K}{\leq} r(t), \quad t \geq t_0\ .$$

Remark 2.1. The proofs of Theorems 2.4 and 2.5 follow similar reasoning as in the proofs of Theorems 4.6.1 and 4.8.1 in [2] respectively, with appropriate modifications and so are omitted here.

3 Stability Theory

Definition 3.1. The trivial solution $u = 0$ of (2.12) is said to be *ϕ_0-equistable* if given $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon)$ which is continuous in t_0 for each ε such that the inequality $(\phi_0, u_0) < \delta$ implies $(\phi_0, r(t)) < \varepsilon, t \geq t_0$ where $\phi_0 \in K_0^*$.

Other ϕ_0-stability notions can be similarly defined.

Definition 3.2. The trivial solution $x = 0$ of (2.1) is said to be *stable in probability* if for each $\varepsilon, \eta > 0, t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon, \eta)$ that is continuous in t_0 for each ε and η such that the inequality

$$P\{\|x_0\| > \delta\} < \eta \Rightarrow P\{\|x(t)\| > \varepsilon\} < \eta, \quad t \geq t_0\ .$$

Definition 3.3. The trivial solution $x = 0$ of (2.1) is said to be *stable with probability one* (w.p.1) if for each $\varepsilon > 0, t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality

$$\|x_0\| < \delta \quad \text{w.p.1} \Rightarrow \|x(t)\| < \varepsilon \quad \text{w.p.1}, \quad t \geq t_0\ .$$

Definition 3.4. The trivial solution $x = 0$ of (2.1) is said to be *stable in the mean* if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ continuous in t_0 for each ε such that

$$(E(\|x_0\|^p))^{1/p} < \delta \Rightarrow (E(\|x(t)\|^p))^{1/p} < \varepsilon, \quad t \geq t_0; \quad p \geq 1\ .$$

Other notions of stability in probability, stability with probability one and stability in the mean can be similarly defined, (see [2]).

Theorem 3.1. *Let the conditions of Theorem 2.5 hold. Assume further that $f(t,0) = 0, g(t,0) = 0$ and for some $\phi_0 \in K_0^*, (t,x) \in R_+ \times S_\rho$,*

$$b(\|x\|^p) \leq (\phi_0, V(t,x)) \leq a(t, \|x\|^p) \tag{3.1}$$

$p \geq 1, a, b \in \mathcal{K}$, a is concave and b convex, and $a(t,r) = a(r)$.
Then, the trivial solution $x = 0$ of (2.1) satisfies each one of the stability notions of Definition 3.2 if the trivial solution $u = 0$ of (2.2) satisfies each one of the corresponding stability notions of Definition 3.1.

Remark 3.1. For the definition of $\mathcal{K}$-class functions see [1].

Proof. (i) Assume that the trivial solution $u = 0$ of (2.2) is ϕ_0-equistable. Then given $b(\varepsilon^p\eta) > 0 \;\; \eta > 0, \varepsilon > 0$, there exists $\delta_1 = \delta_1(t_0, \varepsilon, \eta)$ such that

$$(\phi_0, u_0) < \delta_1 \Rightarrow (\phi_0, r(t)) < b(\varepsilon^p\eta), \quad \phi_0 \in K_0^* .$$

Now given η and for any δ, Markov's inequality gives

$$P(\|x_0\| > \delta) \leq \frac{E[\|x_0\|^p]}{\delta^p} .$$

Now choose δ such that $P(\|x_0\| > \delta) < \frac{E[\|x_0\|^p]}{\delta^p}$ and $E\left[\frac{\|x_0\|^p}{\delta^p}\right] = \eta$. It then follows that

$$P(\|x_0\| > \delta) < \eta . \tag{3.2}$$

Now choose t_o such that $(\phi_0, u_0) = a(t_0, E(\|x_0\|^p))$ and $\delta_2 = \delta_2(t_0, \varepsilon, \eta)$ such that $a(t_0, \delta_2) \leq \delta_1$ and $\eta\delta^p < \delta_2$. Then $E\left[\frac{\|x_0\|^p}{\delta^p}\right] = \eta < \frac{\delta_2}{\delta^p} \Rightarrow E[\|x_0\|^p] < \delta_2$. Now

$$\begin{aligned} E[\|x_0\|^p] < \delta_2 &\Rightarrow a(t_0, E[\|x_0\|^p]) = (\phi_0, u_0) < \delta_1 \\ &\Rightarrow (\phi_0, r(t)) < b(\varepsilon^p\eta), \quad t \geq t_0 . \end{aligned}$$

We now claim that the inequality (3.2) implies

$$P\{\|x(t)\| > \varepsilon\} < \eta. \quad t \geq t_0 .$$

Suppose this claim is false, then there would exist a $t_1 > t_0$, such that (3.2) holds and

$$P\{\|x(t_1)\| > \varepsilon\} = \eta . \tag{3.3}$$

Let $x(t)$ be any solution process of (2.1) such that $E[V(t_0, x_0)] \underset{K}{\leq} u_0$ and let t_e be the first exit time of $x(t)$ from $S_\varepsilon = \{x \in R^n : \|x\| < \varepsilon\}$ and let $\tau = \min\{t, t_e\}$. Then by Theorem 2.5 we have

$$(\phi_0, E[V(\tau, x(\tau))]) \leq (\phi_0, r(t))$$

and from (3.1) we have

$$\begin{aligned} E[b(\|x(\tau)\|)] &\leq (\phi_0, E[V(\tau)]) \\ &\leq (\phi_0, r(t)) < b(\varepsilon^p\eta) . \end{aligned}$$

By convexity of b and Jensen's inequality we have

$$b(E[\|x(\tau)\|^p]) \leq E[b(\|x(\tau)\|^p)] < b(\varepsilon^p\eta) . \tag{3.4}$$

From (3.3), (3.4) and Markov's inequality we have

$$b(\eta) = b(P\{\|x(\tau)\| > \varepsilon\}) < b\left(\frac{E[\|x(\tau)\|^p]}{\varepsilon^p}\right) < b\left(\frac{\varepsilon^p\eta}{\varepsilon^p}\right) < b(\eta) .$$

which is absurd. This absurdity justifies our claim.

The proofs for uniform stability in probability, asymptotic stability in probability and uniform asymptotic stability in probability can be given using similar arguments as in the case of stability in probability given above.

Remark 3.7. Theorems giving sufficient conditions for stability with probability one and stability in the mean can be similarly formulated and proved, in a straightforward manner using the arguments in Theorem 3.1.

Theorem 3.2. *Assume that for $(t,x),(t,y) \in D, D \subset [0,\infty) \times S_\rho, \|\sigma(t,x)\| \leq M; M > 0, \|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|, L(t) > 0$ and the solution process $x(t)$ of (2.1) is uniformly asymptotically stable in probability. Then there exists a stochastic cone-valued function V with the following properties:*

(i) *$V \in C[R_+ \times S_\rho, K], V_t, V_x$ and V_{xx} exist and are continuous,*

(ii) *for each $(t,x) \in R_+ \times S_\rho$, $L\,V(t,x) \underset{K}{\leq} g(t, V(t,x))$ where*

$$\begin{aligned} L\,V(t,x) &= V_t(t,x) + V_x(t,x)\, f(t,x) \\ &\quad + \frac{1}{2}\, tr\{V_{xx}(t,x)\, \sigma(t,x)\, \sigma^T(t,x)\}\omega , \\ \omega &= (1,1,\ldots 1)^T \in K \end{aligned}$$

(iii) $E[\|V(t,x)\|] < \infty$,

(iv) *for some $\phi_0 \in K_0^*$ and $(t,x) \in R_+ \times S_\rho$*

$$b(t, \|x\|) \leq (\phi_0, V(t,x)) \leq a(\|x\|) .$$

Proof. (i) Define $V(t,x) = \|x\|^3 e^{-t}\omega$. Then clearly $V \in C[R_+ \times S_\rho, K]$. $V_t = -e^{-t}\|x\|^3\omega, V_x = 6A\|x\|^2 e^{-t}$ where A is an $N \times n$ matrix given by

$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_N \\ x_1 & x_2 & \cdots & x_N \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_N \end{pmatrix}$$

$V_{xx} = 6B\|x\|e^{-t}$, where B is an $n \times n$ matrix given by

$$B = \begin{pmatrix} \|x\| + 4x_1^2 & 4x_1x_2 & \cdots & 4x_1x_n \\ 4x_1x_2 & \|x\| + 4x_2^2 & \cdots & 4x_2x_n \\ \vdots & \vdots & & \vdots \\ 4x_nx_1 & 4x_nx_2 & \cdots & \|x\| + 4x_n^2 \end{pmatrix} .$$

Obviously, V_t, V_x, V_{xx} exist and are continuous.
(ii) Let

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix}$$

then

$$\sigma\sigma^T = \begin{pmatrix} \sum_{j=1}^n \sigma_{1j}^2 & \sum_{j=1}^n \sigma_{1j}\sigma_{2j} & \cdots & \sum_{j=1}^n \sigma_{1j}\sigma_{nj} \\ \sum_{j=1}^n \sigma_{2j}\sigma_{1j} & \sum_{j=1}^n \sigma_{2j}^2 & \cdots & \sum_{j=1}^n \sigma_{2j}\sigma_{nj} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^n \sigma_{nj}\sigma_{1j} & \sum_{j=1}^n \sigma_{nj}\sigma_{2j} & \cdots & \sum_{j=1}^n \sigma_{nj}^2 \end{pmatrix}$$

and $T = V_{xx}\sigma\sigma^T$ is given by

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

and

$$\frac{1}{2} \text{ Trace of } \{V_{xx}\sigma\sigma^T\} = \frac{1}{2}(T_{11} + T_{22} + \ldots + T_{nn})$$

where

$$\begin{aligned} T_{11} &= \{(\|x\| + 4x_1^2)(\sigma_{11}^2 + \sigma_{12}^2 + \ldots + \sigma_{1n}^2) + 4x_1x_2(\sigma_{21}\sigma_{11} + \sigma_{22}\sigma_{12} + \ldots + \sigma_{2n}\sigma_{1n}) \\ &\quad + \ldots + 4x_1x_n(\sigma_{n1}\sigma_{11} + \sigma_{n2}\sigma_{12} + \ldots + \sigma_{nn}\sigma_{1n})\}6e^{-t}\|x\| \\ T_{22} &= \{4x_2x_1(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} + \ldots \sigma_{1n}\sigma_{2n}) + (\|x\| + 4x_2^2)(\sigma_{21}^2 + \sigma_{22}^2 + \ldots + \sigma_{2n}^2) \\ \vdots &\quad + \ldots + 4x_2x_n(\sigma_{n1}\sigma_{21} + \sigma_{n2}\sigma_{22} + \ldots + \sigma_{nn}\sigma_{2n})\}6e^{-t}\|x\| \\ T_{nn} &= \{4x_nx_1(\sigma_{11}\sigma_{n1} + \sigma_{12}\sigma_{n2} + \ldots + \sigma_{1n}\sigma_{nn}) + 4x_nx_2(\sigma_{21}\sigma_{n1} + \sigma_{22}\sigma_{n2} + \ldots + \sigma_{2n}\sigma_{n2}) \\ &\quad + \ldots + (\|x\| + 4x_n^2)(\sigma_{n1}^2 + \sigma_{n2}^2 + \ldots + \sigma_{nn}^2)\}6e^{-t}\|x\| . \end{aligned}$$

Now

$$|T_{ii}| \leq d_i\|x\|^2e^{-t} + m_i\|x\|e^{-t}, \quad i = 1, 2, \ldots n$$

where

$$d_i = \left|\sum_{j=1}^n 6\sigma_{ij}^2\right| \quad \text{and} \quad m_i = \left|\sum_{k=1}^n 24x_ix_k\left(\sum_{j=1}^n \sigma_{ij}\sigma_{kj}\right)\right| .$$

Since $\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|$, then putting $y = 0$ gives $\|f(t,x)\| \leq L(t)\|x\|$ and so $\|V_x(t,x)f(t,x)\| = 6\|A\|L(t)\|x\|^3e^{-t}$ where $\|A\|$ is the matrix norm of A. Therefore

$$\begin{aligned} L\,V(t,x) &= V_t(t,x) + V_x(t,x)f(t,x) + \frac{1}{2}\,Tr\{V_{xx}(t,x)\sigma(t,x)\sigma^T(t,x)\}\omega \\ &= -e^{-t}\|x\|^3\omega + 6A\|x\|^2e^{-t}f(t,x) + \frac{1}{2}(T_{11} + T_{22} + \ldots + T_{nn})\omega \\ &\underset{K}{\leq} 6A\|x\|^2e^{-t}f(t,x) + \frac{1}{2}(T_{11} + T_{22} + \ldots + T_{nn})\omega \\ &\underset{K}{\leq} \left\{6\|A\|\ L(t)\|x\|^3e^{-t} + \sum_{i=1}^n d_i\|x\|^2e^{-t} + \sum_{i=1}^n m_i\|x\|e^{-t}\right\}\omega \end{aligned}$$

choose constants c_1 and c_2 large enough so that $c_1\|x\| \geq 1$ and $c_2\|x\|^2 \geq 1$, then we have

$$\begin{aligned} L\,V(t,x) &\underset{K}{\leq} \left\{6\|A\|L(t) + c_1\sum_{i=1}^n d_i + c_2\sum_{i=1}^n m_i\right\}\|x\|^3\,e^{-t}\omega \\ &= g(t, V(t,x)) . \end{aligned}$$

(iii) For all $(t,x) \in R_+ \times S_\rho$, there exists M, $0 \leq M < \infty$, such that $\|V(t,x)\| \leq M$. Let $p(t,x)$ be any appropriate probability density function for V then

$$E(\|V\|) = \int_\Omega \|V\|\ p(t,x)\ d\Omega = M\int_\Omega p(t,x)\ d\Omega = M < \infty$$

where Ω is an appropriate sample space.
(iv) For some $\phi_0 \in K_0^*$

$$\begin{aligned} (\phi_0, V(t,x)) &= (\phi_0, \|x\|^3e^{-t}\omega) = \|x\|^3e^{-t}(\phi_0, \omega) \\ &\leq (\phi_0, \omega)\|x\|^3 = a(\|x\|, \quad a \in \mathcal{K} . \end{aligned}$$

Since the solution process $x(t)$ of (1.1) is uniformly asymptotically stable in probability then given $\varepsilon > 0, \eta > 0, t_0 \in R_+$, there exist $\delta = \delta(\varepsilon,\eta), T = T(\varepsilon)$ such that

$$P\{\omega : \|x_0\| \geq \delta\} < \eta \Rightarrow P\{\omega : \|x\| > \varepsilon\} < \eta; t \geq T(\varepsilon) + t_0 .$$

It follows that there exist $\psi \in \mathcal{K}$ such that

$$P\{\omega : \|x_0\| \geq \delta\} < \eta \Rightarrow P\{\omega : \|x\| > \psi(\|x\|)\} < \eta .$$

We can take η arbitrarily small such that the above probabilities tend to zero. It then follows that it is fairly certain that the occurrence of the event

$$\|x_0\| < \delta \Rightarrow \|x\| \leq \psi(\|x\|)$$

and so we can find $c \in \mathcal{K}$ such that

$$c\|x\|) \leq c(\psi(\|x\|))$$

and

$$c(\|x\|)e^{-t}\omega \underset{K}{\leq} c(\psi(\|x\|)e^{-t}\omega .$$

For some $\phi_0 \in K_0^*$ we have

$$\begin{aligned}(\phi_0, c(\|x\|)e^{-t}\omega) &\leq (\phi_0, c(\psi(\|x\|))e^{-t}\omega)\\ e^{-t}(\phi_0,\omega)\, c(\|x\|) &\leq (\phi_0, \phi(\|x\|)e^{-t}\omega), \quad \phi \in \mathcal{K}\ .\\ b(t,\|x\|) &\leq (\phi_0, \phi(\|x\|)e^{-t}\omega), \quad b \in \mathcal{K}\ .\end{aligned}$$

Now defining $\phi(\|x\|)$ by $\phi(\|x\|) = \|x\|^3$ we obtain

$$b(t,\|x\|) \leq (\phi_0, V(t,x))\ .$$

And so we obtain

$$b(t,\|x\|) \leq (\phi_0, V(t,x)) \leq a(\|x\|), \quad a, b \in \mathcal{K}\ .$$

Theorem 3.3. Assume that for $(t,x),(t,y) \in D, \|\sigma(t,x)\| \leq M, M > 0, \|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, L(t) > 0$ and the solution process $x(t)$ of (2.1) is uniformly asymptotically stable in the mean. Then there exists a stochastic cone-valued function V with the following properties:

(i) $V \in C[R_+ \times S_\rho, K], V_t, V_x, V_{xx}$ exist and are continuous

(ii) for each $(t,x) \in R_+ \times S_\rho, L\ V(t,x) \underset{K}{\leq} g(t, V(t,x))$ where $L\ V(t,x)$ is as in Theorem 3.2.

(iii) $E[\|V(t,x)\|] < \infty$

(iv) $b(t, E(\|x\|^p)) \leq (\phi_0, V(t,x)) \leq G(\|x\|^p)$ for some $\phi \in K_0^*, (t,x) \in R_+ \times S_\rho, a, b \in \mathcal{K}$ and b is convex.

Proof. (i) Define $V(t,x) = \|x\|^{3p}e^{-t}\omega$. Clearly $V \in C[R_+ \times S_\rho, K]$. $V_t = -e^{-t}\|x\|^{3p}\omega$, $V_x = 6pA\|x\|^{3p-1}e^{-t}$ where A is the $N \times n$ matrix given in Theorem 3.2. $V_{xx} = 6pB_1e^{-t}\|x\|^{3p-2}$, where B_1 is an $n \times n$ matrix given by

$$B_1 = \begin{pmatrix} \|x\| + 2(3p-1)x_1^2 & 2(3p-1)x_1x_2 & \cdots & 2(3p-1)x_1x_n \\ 2(3p-1)x_2x_1 & \|x\| + 2(3p-1)x_2^2 & \cdots & 2(3p-1)x_2x_n \\ \vdots & \vdots & & \vdots \\ 2(3p-1)x_nx_1 & 2(3p-1)x_nx_2 & \cdots & \|x\| + 2(3p-1)x_n^2 \end{pmatrix}.$$

Clearly V_t, V_x, V_{xx} exist and are continuous.

(ii) Following similar computations as in Theorem 3.2 for $\sigma\sigma^T$ and $V_{xx}\sigma\sigma^T = T$ we obtain

$$|T_{ii}| \leq d_i\|x\|^{3p-1}e^{-t} + m_i\|x\|^{3p-2}e^{-t}$$

where

$$d_i = \left|\sum_{j=1}^{n} 6p\ \sigma_{ij}^2\right| \quad \text{and} \quad m_i = \left|\sum_{k=1}^{n} 12p(3p-1)x_ix_k\left(\sum_{j=1}^{n}\sigma_{ij}\sigma_{kj}\right)\right|.$$

Similar arguments as in Theorem 3.2 show that

$$L\ V(t,x) \underset{K}{\leq} g(t, V(t,x))\ .$$

(iii) Since $1 \leq p < \infty$ and $x \in S_\rho$ then $\|x\|^{3p} < \infty$ and so $V = \|x\|^{3p}e^{-t}\omega$ is bounded. Therefore $E(\|V\|) < \infty$.

(iv)

$$\begin{aligned}(\phi_0, V(t,x)) &= (\phi_0, \|x\|^{3p}e^{-t}\omega) = (\phi_0,\omega)e^{-t}\|x\|^{3p}\\ &\leq (\phi_0,\omega)\|x\|^{3p} = a(\|x\|^p), \quad a \in \mathcal{K}\ .\end{aligned}$$

The solution process of (2.1) is uniformly asymptotically stable in the mean implies that given any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon)$ and $T = T(\varepsilon)$ such that the inequality $(E(\|x\|^p))^{1/p} < \delta \Rightarrow (E(\|x\|^p))^{1/p} < \varepsilon, p \geq 1, t \geq T(\varepsilon) + t_0$. Since ε is arbitrary then we can choose $\psi \in \mathcal{K}$ such that

$$E(\|x_0\|^p) < \delta^p \Rightarrow E(\|x\|^p) \leq \psi(\|x\|^p).$$

Then for some $\phi_0 \in K_0^*$ we have $(\phi_0, E(\|x\|^p)e^{-t}\omega) \leq (\phi_0, \psi(\|x\|^p)e^{-t}\omega), (\phi_0,\omega)e^{-t}E(\|x\|^p) \leq (\phi_0, \psi(\|x\|^p)e^{-t}\omega)$. Now for any convex function $\phi \in \mathcal{K}$, Jensen's inequality gives

$$\begin{aligned}(\phi_0, \phi(E(\|x\|^p))e^{-t}\omega) &\leq (\phi_0, E(\phi(\|x\|^p))\ e^{-t}\omega)\\ &\leq (\phi_0, \phi(\psi(\|x\|^p))e^{-t}\omega)\\ (\phi_0,\omega)\ e^{-t}\phi(E(\|x\|^p)) &\leq (\phi_0, c(\|x\|^p)\ e^{-t}\omega), c \in \mathcal{K}\ .\\ b(t, E(\|x\|^p)) &\leq (\phi_0, V(t,x)), \quad b \in \mathcal{K} \text{ and } b \text{ is convex}\end{aligned}$$

where $c(\|x\|^p)$ is defined by $c(\|x\|^p) = \|x\|^{3p}$.

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