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CONEAT SUBMODULES AND CONEAT-FLAT MODULES

Engin Büyükaşık and Yilmaz Durğun

Abstract. A submodule N of a right R-module M is called *coneat* if for every simple right R-module S, any homomorphism $N \to S$ can be extended to a homomorphism $M \to S$. M is called *coneat-flat* if the kernel of any epimorphism $Y \to M \to 0$ is coneat in Y. It is proven that (1) coneat submodules of any right R-module are coclosed if and only if R is right K-ring; (2) every right R-module is coneat-flat if and only if R is right V-ring; (3) coneat submodules of right injective modules are exactly the modules which have no maximal submodules if and only if R is right small ring. If R is commutative, then a module M is coneat-flat if and only if M^+ is m-injective. Every maximal left ideal of R is finitely generated if and only if every absolutely pure left R-module is m-injective. A commutative ring R is perfect if and only if every coneat-flat module is projective. We also study the rings over which coneat-flat and flat modules coincide.

1. Introduction

A subgroup A of an abelian group B is said to be *neat* in B if $pA = A \cap pB$ for every prime integer p. The notion of neat subgroup was generalized to modules by Renault (see, [12]). Namely, a submodule N of a right R-module M is called *neat* in M, if for every simple right R-module S, $\mathrm{Hom}(S, M) \to \mathrm{Hom}(S, M/N) \to 0$ is epic. Dually, in [8], a submodule N of a right R-module M is called *coneat* in M if $\mathrm{Hom}(M, S) \to \mathrm{Hom}(N, S) \to 0$ is epic for every simple right R-module S. The notions of neat and coneat are coincide over the ring of integers. By [8, Theorem], the commutative domains over which neat and coneat submodules coincide are exactly the domains with finitely generated maximal ideals (i.e., N-domains). This result was extended to certain commutative rings in [5]. Recently, modules related to neat and coneat submodules are considered by several authors. In [5], a right R-module M is called absolutely neat (resp. coneat) if M is a neat (resp. coneat) submodule of any module containing it. According to [16], a right R-module M is m-injective

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if for any maximal right ideal I of R, any homomorphism $I \to M$ can be extended to a homomorphism $R \to M$. By Theorem 3.4, a right R-module M is absolutely neat if and only if M is m-injective.

A ring R is called right C-ring if $\mathrm{Soc}(R/I) \neq 0$ for each proper essential right ideal I of R. Left perfect rings, right semiartinian rings and almost perfect domains are right C-rings.

A dual notion of m-injective modules has been studied in [1] and [2]. A module M is called neat-flat if the kernel of any epimorphism $F \to M \to 0$ is a neat submodule of F. Closed submodules of any right R-module are neat, and neat submodules of any right R-module are closed if and only if R is a right C-ring (see, [9, Theorem 5]). In [21], a module M is called *weak-flat* if the kernel of any epimorphism $F \to M \to 0$ is a closed submodule of F. Hence, summing up we get, R is a right C-ring if and only if every neat-flat right R-module is weak-flat.

We call M *coneat-flat* if the kernel of any epimorphism $Y \to M \to 0$ is coneat in Y. In this paper, several characterizations of coneat submodules and coneat-flat modules are given. Some known results are generalized, and relations between coneat-flat modules and flat, m-injective, absolutely pure and projective modules are studied.

In Section 2, it is shown that a submodule N of a right R-module M is coneat if and only if for every maximal submodule K of N, N/K is a direct summand of M/K. A ring R is a right V-ring if and only if submodules of right R-modules are coneat. R is right small if and only if its absolutely coneat right modules are precisely those modules M such that $M = \mathrm{Rad}(M)$.

In Section 3, we prove that, a module M is coneat-flat if and only if $M \cong P/N$ where P is a projective R-module and N is a coneat submodule of P. An R-module M is coneat-flat if and only if and only if M^+ is m-injective, over commutative rings. R is a right V-ring if and only if every right R-module is coneat-flat.

In Section 4, we prove that, if R is a left C-ring, then a right R-module M is flat if and only if $\mathrm{Tor}_1^R(M, S) = 0$ for each simple left R-module S. If R is a commutative C-ring, then coneat-flat modules are only the flat modules, and the converse holds when R is noetherian. R is a left N-ring (i.e., maximal left ideals are finitely generated) if and only if every absolutely pure module is m-injective. A ring R is left artinian if and only if m-injective left R-modules are precisely those modules M with M^+ is projective.

In Section 5, we consider the projectivity of coneat-flat modules. We show that, if R is right perfect, then every coneat-flat R-module is projective, the converse hold if R is commutative. Finitely presented coneat-flat modules are projective, over semiperfect rings and over commutative rings.

Throughout, R is a ring with an identity element and all modules are unital right R-modules, unless otherwise stated. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. We use the notation $E(M)$,

$\operatorname{Soc}(M)$, $\operatorname{Rad}(M)$, for the injective hull, socle, radical of M respectively. By $N \leq M$, we mean that N is a submodule of M.

2. Characterization and closure properties of coneat submodules

In this section, several characterizations and some properties of coneat submodules are given. Recall that a submodule K of M is called *small* in M (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule T of M. A submodule $L \leq M$ is called *coclosed in M* if $L/N \ll M/N$ implies $L = N$ for every $N \leq L$.

Proposition 2.1. *For a submodule $N \leq M$ the following are equivalent.*

(1) *N is coneat in M.*
(2) *If $K \leq N$ with N/K finitely generated and $N/K \ll M/K$, then $K = N$.*
(3) *For any maximal submodule K of N, N/K is a direct summand of M/K.*
(4) *If K is a maximal submodule of N, then there exists a maximal submodule L of M such that $K = N \cap L$.*

Proof. (1) ⇒ (4) Let K be a maximal submodule of N and $\pi : N \to N/K$ be the canonical epimorphism. By the hypothesis, there exists a homomorphism $f : M \to N/K$ such that $f|_N = \pi$. Then $\operatorname{Ker} f$ is a maximal submodule of M and $N + \operatorname{Ker} f = M$. So that $N \cap \operatorname{Ker} f$ is a maximal submodule of N. Then $\pi(N \cap \operatorname{Ker} f) = f(N \cap \operatorname{Ker} f) = 0$. Therefore $K = N \cap \operatorname{Ker} f$.

(3) ⇒ (1) Let S be a simple right R-module and $f : N \to S$ a nonzero homomorphism. Since f is an epimorphism, without loss of generality we may assume that $S = N/K$ for some maximal submodule K of N. So that $\operatorname{Ker} f$ is a maximal submodule of N. Then, by (3), $M/\operatorname{Ker} f = (N/\operatorname{Ker} f) \oplus (L/\operatorname{Ker} f)$ for some $L \leq M$. Let $\tilde{f} : N/\operatorname{Ker} f \to N/K$ be the isomorphism induced by f. Consider the canonical epimorphisms $\pi : M \to M/\operatorname{Ker} f$ and $\pi' : M/\operatorname{Ker} f \to N/\operatorname{Ker} f$. Then the homomorphism $g = \tilde{f}\pi'\pi$ is the extension of f.

(2) ⇒ (3) is clear.

(3) ⇒ (2) Suppose N/K is finitely generated and $N/K \ll M/K$ for some proper submodule $K \leq N$. Then there is a maximal submodule T of N such that $K \leq T$ and $N/T \ll M/T$, because N/T is the image of N/K under the canonical epimorphism $f : M/K \to M/T$, a contradiction.

(3) ⇔ (4) is straight forward. □

Properties of coclosed modules in [4, 3.7] are adapted to coneat submodules as follows. The proof is omitted.

Proposition 2.2. *Let $K \leq L \leq M$ be submodules. Then the following hold.*

(1) *If L is coneat in M, then L/K is coneat in M/K.*
(2) *If $K \leq \operatorname{Rad}(L)$ and L/K is coneat in M/K, then L is coneat in M.*

(3) *If* $L \leq M$ *is coneat, then* $K \leq \operatorname{Rad}(M)$ *implies* $K \leq \operatorname{Rad}(L)$; *hence* $\operatorname{Rad}(L) = L \cap \operatorname{Rad}(M)$.
(4) *If* $f : M \to N$ *is a small epimorphism and* L *is coneat in* M, *then* $f(L)$ *is coneat in* N.
(6) *If* K *is coneat in* M, *then* K *is coneat in* L *and the converse is true if* L *is coneat in* M.

The proof of [20, Lemma A.4] can be adapted to prove the following.

Proposition 2.3. *Let* $K \leq L \leq M$ *be submodules of* M. *If* K *is coneat in* M *and* L/K *is coneat in* M/K, *then* L *is coneat in* M.

Proof. Suppose X is a submodule of L such that L/X finitely generated and L/X is small in M/X. Firstly we will prove that $K/K\cap X$ is small in $M/K\cap X$.
Assume the contrary. Then there is an R-module W such that

$$(*) \qquad K \cap X \leq W \text{ and } W + K = M.$$

Suppose $L/[K + (W \cap X)]$ is not small in $M/[K + (W \cap X)]$. Then there is an R-module Z such that $K + (W \cap X) \leq Z$ and $Z + L = M$. Since $K \leq Z$, $Z = Z \cap W + K$ by $(*)$, and so $M = Z \cap W + L$. By smallness of L/X is small in M/X, $Z \cap W + X = M$. Now $W = Z \cap W + X \cap W$, $W \leq Z$. Finally, since $Z + W = M$, $Z = M$. Recall that L/K is coneat in M/K and $L/[K + (W \cap X)]$ is epimorphic image of the finitely generated module L/X. Hence, $L = K + W \cap X$ by Proposition 2.1(2). By modular law, $X = K \cap X + W \cap X$, and $X \leq W$. Then $K + X = L$. Since L/X is small in M/X, $W = M$ by $(*)$. By our assumption K is coneat in M, hence $K = K \cap X$ and $K \leq X$. Since L/X is an epimorphic image of L/K and L/K is coneat in M/K, $L = X$ by Proposition 2.1(2), again. □

Proposition 2.4 ([15, Lemma 6.1])**.** *Let* A *be a submodule of an* R*-module* B *and* $i_A : A \hookrightarrow B$ *be the inclusion map. For a* right *ideal* I *of* R, $A \cap IB = IA$ *if and only if* $R/I \otimes A \overset{1_{R/I} \otimes i_A}{\to} R/I \otimes B$ *is injective.*

An exact sequence $0 \to A \overset{f}{\to} B \to C$ is said to be *coneat exact* if $f(A)$ is a coneat submodule of B. A monomorphism $f : A \to B$ is said to be a coneat monomorphism, if the short exact sequence $0 \to A \overset{f}{\to} B \to B/f(A) \to 0$ is coneat exact. Neat-exact sequences are defined in the same manner.

Theorem 2.5. *Let* R *be a commutative ring and* $f : N \to M$ *be a monomorphism. The following are equivalent.*
(1) $f(N)$ *is a coneat submodule of* M.
(2) $S \otimes_R N \overset{1_S \otimes f}{\to} S \otimes_R M$ *is a monomorphism for each simple* R*-module* S.
(3) $mf(N) = f(N) \cap mM$ *for each maximal ideal* m *of* R.

Proof. (1) ⇔ (2) By [8, Proposition 3.1].
(2) ⇔ (3) Follows by Proposition 2.4. □

Remark 2.6. If N is a pure submodule of M, then $NI = N \cap MI$ for every left ideal of R (see, [10, Corollary 4.92]). Therefore, over commutative rings, every pure submodule is coneat by Theorem 2.5(3). This fact will be used in the sequel.

Corollary 2.7. *Let R be a commutative ring. The following are equivalent.*

(1) $0 \to A \to B \to C \to 0$ *is coneat exact.*
(2) $0 \to C^+ \to B^+ \to A^+ \to 0$ *is neat exact.*

Proof. By Theorem 2.5(2) and the adjoint isomorphism

$$(M \otimes N)^+ \cong \mathrm{Hom}(M, N^+).$$ □

Let M be an R-module with $\mathrm{Rad}\, M = M$. It is easy to see that $\mathrm{Hom}(M, S) = 0$ for each simple module. Hence,

Corollary 2.8. *Let M be a right R-module with* $\mathrm{Rad}(M) = M$. *Then M is absolutely coneat.*

A ring R is said to be right *small* if $R_R \ll E(R_R)$. A ring R is small if and only if $E = \mathrm{Rad}(E)$ for every injective R-module E (see, [11, Proposition 3.3]).

Proposition 2.9. *The following statements are equivalent for a ring R.*

(1) *R is a right small ring.*
(2) *Absolutely coneat right R-modules are precisely those modules N such that* $\mathrm{Rad}(N) = N$.

Proof. (1) ⇒ (2) Let E be the injective hull of N. Then $\mathrm{Rad}(E) = E$ as R is a small ring. Suppose N is coneat in E. So that $\mathrm{Rad}(N) = N \cap \mathrm{Rad}(E) = N$ by Proposition 2.2(3). The rest of (2) by Corollary 2.8.

(2) ⇒ (1) Every injective right R-module E is absolutely coneat. Then (2) implies $\mathrm{Rad}(E) = E$, and so R is a small ring by [11, Proposition 3.3]. □

Let R be a ring and M be a nonzero R-module. M is called *coatomic* if every proper submodule N of M is contained in a maximal submodule of M, i.e., $\mathrm{Rad}(M/N) \neq 0$.

Proposition 2.10. *Let M be a module and N be a coatomic submodule of M. Then N is coneat in M if and only if it is coclosed in M.*

Proof. Suppose N is coneat and $N/X \ll M/X$ for some proper submodule $X \leq N$. Since N is coatomic, X is contained in a maximal submodule, say K, of N. Then $N/K \ll M/K$, and this contradicts with the fact that N is coneat. Hence N is coclosed. The converse implication is obvious. □

In [19], a ring R is called right K-ring if every non-zero small right R-module is coatomic. Dedekind domains and right max rings (i.e., every nonzero right R-module has a maximal submodule) are right K-rings.

Theorem 2.11. *R is a right K-ring if and only if coneat submodules of any right R-module are coclosed.*

Proof. For the necessity, let M be a non-zero small module and suppose M/K has no maximal submodules, i.e., $\mathrm{Rad}(M/K) = M/K$ for some proper submodule K of M. Then M/K is small and coneat submodule in $E(M/K)$. Hence M/K is coclosed in $E(M/K)$ by (1). This gives a contradiction, since coclosed submodules are not small. Consequently, K is contained in a maximal submodule of M, and so M is coatomic.

For the sufficiency, suppose the contrary that, there is a module M and a submodule N of M which is coneat but not coclosed. Then there is a proper submodule K of N such that $N/K \ll M/K$. By Proposition 2.2(1), N/K is a coneat submodule of M/K. Then N/K is coatomic by the hypothesis, and so N/K is coclosed by Proposition 2.10, a contradiction. □

3. Coneat-flat modules

It is well known that, a right R-module M is flat if and only if any short exact sequence of the form $0 \to K \xrightarrow{f} N \to M \to 0$ is pure exact, i.e., $f(K)$ is a pure submodule of N. It is natural to ask for which right R-modules P any short exact sequence ending with P is coneat exact? In this section several characterizations of such modules are given.

A right R-module M is called coneat-flat if the kernel of any epimorphism $Y \to M \to 0$ is a coneat submodule of Y. Clearly, projective modules are coneat-flat but the converse need not be true in general (see, Theorem 5.1).

Theorem 3.1. *The following are equivalent for an R-module M:*

(1) *M is coneat-flat.*
(2) *$\mathrm{Ext}^1_R(M, S) = 0$ for each simple R-module S.*
(3) *There is a coneat exact sequence $0 \to K \to L \to M \to 0$ with L projective.*
(4) *There is a coneat exact sequence $0 \to K \to L \to M \to 0$ with L coneat-flat.*

Proof. (1) ⇒ (2) Let $\mathbb{E} : 0 \to S \xrightarrow{\alpha} L \to M \to 0$ be a short exact sequence with S simple right R-module. Since M is coneat-flat, S is coneat in L, and there is a homomorphism $\beta : L \to S$ such that the following diagram is commutative.

$$\begin{array}{rccccccc} \mathbb{E}: & 0 \longrightarrow & S & \xrightarrow{\alpha} & L & \longrightarrow & P & \longrightarrow 0 \\ & & {\scriptstyle 1_S}\downarrow & \swarrow_{\beta} & & & & \\ & & S & & & & & \end{array} \tag{3.1}$$

Then $1_S = \beta\alpha$, and so the sequence $\mathbb{E}$ splits. Hence $\mathrm{Ext}^1_R(M, S) = 0$.

(2) ⇒ (3) Assuming (2). There is a short exact sequence $\mathbb{E} : 0 \to C \to F \to M \to 0$ with F free R-module. Applying $\mathrm{Hom}_R(-, S)$, we obtain the exact

sequence $0 \to \mathrm{Hom}_R(M,S) \to \mathrm{Hom}_R(F,S) \to \mathrm{Hom}_R(C,S) \to \mathrm{Ext}^1_R(M,S) = 0$.
That is, $\mathrm{Hom}_R(\mathbb{E},S)$ is exact for every simple R-module S, and so $\mathbb{E}$ is coneat exact.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) Let $s : B \to M$ be any epimorphism. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Ker}\, s & \to & X & \xrightarrow{\alpha} & L & \to & 0\\
 & & \| & & \downarrow t & & \downarrow \beta & & \\
0 & \to & \mathrm{Ker}\, s & \to & B & \xrightarrow{s} & M & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

$\beta\alpha = st$ is coneat epimorphism, i.e., $\mathrm{Ker}(st)$ is a coneat submodule of X, by Proposition 2.3. Then s is coneat epimorphism by Proposition 2.2(1). This completes the proof. □

By Theorem 3.1, we get the following.

Corollary 3.2. *The class of coneat-flat modules is closed under extensions, direct sums, direct summands and coneat quotients. In particular, coneat-flat modules are closed under pure quotients over commutative rings.*

Proof. Coneat-flat modules are closed under extensions, direct sums, direct summands and coneat quotients by Theorem 3.1, and under pure quotients by Remark 2.6 and Theorem 3.1. □

Proposition 3.3. *Let R be a commutative ring and M be an R-module. Then M is coneat-flat if and only if* $\mathrm{Tor}_R(M,S) = 0$ *for each simple R-module S.*

Proof. Let $0 \to K \xrightarrow{i} F \to M \to 0$ be a short exact sequence with F projective. Applying $- \otimes S$, we get

$$0 = \mathrm{Tor}(F,S) \to \mathrm{Tor}(M,S) \to K \otimes S \overset{i\otimes 1_S}{\to} F \otimes S \to M \otimes S \to 0.$$

Then $i \otimes 1_S$ is a monomorphism if and only if $\mathrm{Tor}(M,S) = 0$. Now the proof is clear by Theorem 2.5 and Theorem 3.1. □

Proposition 3.4. *The following are equivalent for a right R-module M.*

(1) *M is m-injective.*
(2) *M is a neat submodule of an m-injective module.*
(3) *M is a neat submodule of every module containing it.*
(4) $\mathrm{Ext}^1_R(S, M) = 0$ *for every simple right R-module S .*

Proof. (1) $\Leftrightarrow$ (4) Let I be a right ideal of R. Then applying $\mathrm{Hom}(-, M)$ to the short exact sequence $0 \to I \xrightarrow{i} R \to R/I \to 0$, we get $0 \to \mathrm{Hom}(R/I, M) \to \mathrm{Hom}(R, M) \xrightarrow{i^*} \mathrm{Hom}(I, M) \to \mathrm{Ext}^1(R/I, M) \to \mathrm{Ext}^1_R(R, M) = 0$. Then i^* is epic if and only if $\mathrm{Ext}^1_R(R/I, M) = 0$.

(2) $\Leftrightarrow$ (3) By [5, Theorem 3.3].

(3) $\Leftrightarrow$ (4) By [5, Theorem 3.4. (i)$\Leftrightarrow$(ii)]. □

Proposition 3.5. *Let R be a commutative ring. An R-module M is coneat-flat if and only if M^+ is m-injective.*

Proof. Let S be a simple R-module. We have the standard isomorphism

$$\mathrm{Ext}^1_R(S, M^+) \cong \mathrm{Tor}^R_1(M, S)^+.$$

Now, the proof is immediate by Proposition 3.3 and Proposition 3.4. □

Corollary 3.6. *Let R be a commutative ring. The class of coneat-flat modules is closed under pure submodules.*

Proof. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of R-modules with B coneat-flat. Then the short exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ splits. By Proposition 3.5 the module B^+ is m-injective, and so A^+ is m-injective. Then A is coneat-flat by Proposition 3.5, again. □

Proposition 3.7. *The following statements are equivalent for a ring R.*

(1) *R is a right V-ring.*
(2) *for every right R-module M every submodule of M is coneat in M.*
(3) *every right R-module is coneat-flat.*

Proof. (1) $\Rightarrow$ (2) is clear, since every simple right R-module is injective by (1).

(2) $\Rightarrow$ (3) Let M be a right R-module. Consider an epimorphism $f : F \to M$ with F free right R-module. Then $\mathrm{Ker}\, f$ is a coneat submodule of F by (2). Therefore M is coneat-flat by Theorem 3.1.

(3) $\Rightarrow$ (1) Let S be a simple R-module and E be an injective module containing S. By the hypothesis E/S is coneat-flat. Hence the sequence $0 \to S \to E \to E/S \to 0$ splits by Theorem 3.1, and so S is injective. □

4. When coneat-flat modules are flat

In this section, we study the flatness of coneat-flat modules, and the character of coneat-flat modules. We begin with the following. A module right R-module M is called *cotorsion* if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat R-module F.

Example 4.1. (1) Let R be a valuation domain with a non finitely generated maximal ideal P. Then $\mathrm{Rad}(P) = P^2 = P$, and so P is a coneat submodule of R by Corollary 2.8. Hence R/P is coneat-flat by Theorem 3.1. On the other hand, R/P is a not a flat R-module, since R/P is a torsion R-module.

(2) Let R be a regular ring that is not a right V-ring. Then there exists a flat module which is not coneat-flat by Proposition 3.7.

In light of Example 4.1, it is natural to consider the rings over which coneat-flat and flat modules coincide. We begin with the following lemma.

Lemma 4.2. *Let R be a ring and S be a simple R-module. If R is commutative or semilocal, then S is cotorsion.*

Proof. First suppose R is commutative and let $I = Ann_R(S)$. Then clearly S is an R/I-module. Since R/I is simple, S is cotorsion as an R/I-module. So that S is a cotorsion R-module by [18, Proposition 3.3.3]. If R is semilocal, then $J(R).S = 0$ and so S is an $R/J(R)$-module. As R is semilocal, $R/J(R)$ is semisimple and so S is a cotorsion $R/J(R)$-module. Now, S is a cotorsion R-module by [18, Proposition 3.3.3], again. □

Corollary 4.3. *Suppose R is commutative or a semilocal ring. Then every flat module is coneat-flat.*

Proof. Let S be a simple R-module. Then S is a cotorsion module by Lemma 4.2. Therefore $\mathrm{Ext}^1_R(M, S) = 0$, and so M is coneat-flat by Theorem 3.1. □

Remark 4.4. A commutative domain R is called almost perfect if R/I is a perfect ring for each nonzero ideal I of R. It is clear that almost perfect domains are C-rings. In [14], the authors prove that, if R is an almost perfect domain, then an R-module M is injective if and only if $\mathrm{Ext}^1_R(S, M) = 0$ (i.e., M is m-injective) for each simple module S. Actually, one of the characterization of right C-rings is the following: R is a right C-ring if and only if every m-injective right R-module is injective (see, [16, Lemma 4]).

Proposition 4.5. *Let R be a left C-ring. A right R-module M is flat if and only if $\mathrm{Tor}^R_1(M, S) = 0$ for each simple left R-modules S.*

Proof. Necessity is clear. For the sufficiency assume that $\mathrm{Tor}^R_1(M, S) = 0$ for each simple left R-modules S. Then $0 = \mathrm{Tor}^R_1(M, S)^+ \cong \mathrm{Ext}^1_R(S, M^+)$ implies M^+ is m-injective by Theorem 3.4. Therefore M^+ is injective, because R is a left C-ring. Hence M is flat by [7, Theorem 3.2.10]. □

Proposition 4.6. *Let R be a commutative ring. Consider the following statements.*

(1) *R is a C-ring.*
(2) *Coneat-flat R-modules are flat.*

Then (1) $\Rightarrow$ (2). *If R is a noetherian, then* (2) $\Rightarrow$ (1).

Proof. (1) ⇒ (2) By Corollary 3.3 and Proposition 4.5.

(2) ⇒ (1) Let M be an m-injective R-module. Then M^+ is flat by the hypothesis and Theorem 4.10. As R is noetherian, M is injective by [3, Theorem 2]. Hence R is a C-ring. □

Theorem 4.7. *The following are equivalent for a commutative ring R.*

1. *Every coneat-flat module is flat.*
2. *Flat modules are precisely those modules M satisfying*

$$\mathrm{Ext}^1(M, \prod_{i\in I} S_i) = 0,$$

where the S_i's are all the non-isomorphic simple modules.

Proof. (1) ⇒ (2) By Lemma 4.2, simple modules are cotorsion. Then $\prod_{i\in I} S_i$ is cotorsion, since cotorsion modules are closed under direct products. Hence, if M is flat, then $\mathrm{Ext}^1_R(M, \prod_{i\in I} S_i) = 0$. Conversely, suppose $\mathrm{Ext}^1_R(M, \prod_{i\in I} S_i) = 0$. Then $\mathrm{Ext}^1_R(M, S_i) = 0$ for each $i \in I$. So that M is coneat-flat by Theorem 3.1. Hence M is flat by (1).

(2) ⇒ (1) Suppose M is coneat-flat. Then $\mathrm{Ext}^1_R(M, S) = 0$ for each simple R-module S. So that $\mathrm{Ext}^1_R(M, \prod_{i\in I} S_i) = 0$ for any index set I and simple R-modules S_i. Hence M is flat by (2). □

Proposition 4.8. *Let R be a commutative N-ring and M be an arbitrary R-module. Then the following hold.*

1. *M is m-injective if and only if M^+ is coneat-flat.*
2. *M is m-injective if and only if M^{++} is m-injective.*
3. *M is coneat-flat if and only if M^{++} is coneat-flat.*
4. *Any direct product of coneat-flat modules is coneat-flat.*
5. *Any direct product of copies of R is coneat-flat.*
6. *The class of m-injective modules is closed under pure quotients.*

Proof. (1) An R-module M is m-injective module if and only if M^+ is coneat-flat by [13, Theorem 9.51], since R is an N-ring

(2) M is m-injective if and only if M^+ is coneat-flat by (1), and M^+ is coneat-flat if and only if M^{++} is m-injective by Proposition 3.5.

(3) If M is coneat-flat, then M^+ is m-injective by Proposition 3.5. So M^{+++} is m-injective by (2), and hence M^{++} is coneat-flat. Conversely, if M^{++} is coneat-flat, then M is coneat-flat by Corollary 3.6, since M is a pure submodule of M^{++}.

(4) Let $(M_i)_{i\in J}$ be a family of coneat-flat R-modules. Since the class of coneat-flat modules is closed under direct sums, $\bigoplus_{i\in J} M_i$ is coneat-flat. So $(\bigoplus M_i)^{++} \cong (\prod M_i^+)^+$ is coneat-flat by (3). Since $\oplus_{i\in J} M_i^+$ is a pure submodule of $\prod_{i\in J} M_i^+$, $(\oplus_{i\in J} M_i^+)^+$ is a direct summand of $(\prod_{i\in J} M_i^+)^+$, and so $(\oplus_{i\in J} M_i^+)^+ \cong \prod_{i\in J} M_i^{++}$ is coneat-flat. Since coneat-flat modules are closed

under pure submodules and $\prod_{i\in J} M_i$ is a pure submodule of $\prod_{i\in J} M_i^{++}$, the module $\prod_{i\in J} M_i$ is coneat-flat.

(5) By (4).

(6) Take any pure exact sequence $0 \to A \to B \to C \to 0$ with B m-injective. Then we have a split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By (1), B^+ is coneat-flat, and so C^+ is coneat-flat. Then C is m-injective by (1), again. □

An R-module M is called *absolutely pure* if it is pure in every module containing it as a submodule. It is well known that, a ring R is left noetherian if and only if every absolutely pure left R-module is injective.

Proposition 4.9. *R is a left N-ring if and only if every absolutely pure left R-module is m-injective.*

Proof. (⇒) Let M be an absolutely pure left R-module. Since R is a left N-ring, $\mathrm{Ext}^1_R(S, M) = 0$ for each simple left R-module S. That is, M is m-injective.

(⇐) Let S be a simple left R-module. Then $\mathrm{Ext}^1_R(S, M) = 0$ for each absolutely pure left R-module M by the assumption. Then S is finitely presented by [6, Proposition]. □

Theorem 4.10. *Let R be a ring. The following statements are equivalent.*

(1) (a) *M is a flat right R-module if and only if $\mathrm{Tor}^R_1(M, S) = 0$ for each simple left R-module S,*
 (b) *R is a left N-ring.*

(2) *M is an m-injective left R-module if and only if M^+ is flat.*

(3) *M is an m-injective left R-module if and only if M is an absolutely pure left R-module.*

Proof. (1) ⇒ (2) Let M be a left R-module and S be a simple left R-module. Suppose M is m-injective. Then $0 = \mathrm{Ext}^1_R(S, M)^+ \cong \mathrm{Tor}^R_1(M^+, S)$ by [13, Theorem 9.51], and so M^+ is flat by (1). Conversely suppose M^+ is flat. Then M^{++} is injective by [13, Theorem 3.52], and so M is absolutely pure, since M is pure in M^{++}. Therefore M is m-injective by Proposition 4.9.

(2) ⇒ (3) Firstly, we shall prove that a right R-module M is flat if and only if M^{++} is flat. Then R is left coherent by [3, Theorem 1]. Suppose M is a flat right R-module. Then M^+ is (m-)injective, and so M^{++} is flat by (2). Now, conversely suppose M^{++} is a flat right R-module. Then M is flat, since M is pure submodule of M^{++} and flat modules closed under pure submodules.

Let M be a left R-module. Then M^+ is flat if and only if M is absolutely pure by [3, Theorem 1], since R is left coherent. Hence the rest of (3) follows by (2).

(3) ⇒ (1) Suppose $\mathrm{Tor}^R_1(M, S) = 0$ for each simple left R-module S. Then $\mathrm{Ext}^1_R(S, M^+) = 0$, and so M^+ is m-injective. Then M^+ is absolutely pure by (3). Therefore M^+ is injective, since it is pure-injective. Thus M is flat. This proves (a), and (b) follows by Proposition 4.9. □

Proposition 4.11. *Let R be a commutative ring. Consider the following statements.*

(1) *R is a C-ring.*
(2) *Coneat-flat R-modules are flat.*

Then (1) $\Rightarrow$ (2). *If R is a noetherian, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) By Proposition 3.3 and Proposition 4.5.

(2) $\Rightarrow$ (1) Let M be an m-injective R-module. Then M^+ is flat by the hypothesis and Theorem 4.10. As R is noetherian, M is injective by [3, Theorem 2]. Hence R is a C-ring. □

It is easy to see that, a left N-ring and left semiartinian ring is left noetherian. The following is a slight generalization of this fact.

Corollary 4.12. *If R is a left N-ring and a left C-ring, then R is left noetherian.*

Proof. By Proposition 4.5 and Theorem 4.10, a left R-module M is m-injective if and only if it is absolutely pure. So that every absolutely pure left module is injective. Hence R is left noetherian. □

Note that, Corollary 4.12, generalizes [5, Theorem 4.1 (ii)$\Rightarrow$(i)].

In [3, Theorem 4], the authors proves that, R is left artinian if and only if a left module M is injective exactly when M^+ is projective. We show that, this result still holds if we replace m-injective by injective.

Theorem 4.13. *Let R be a ring. The following are equivalent.*

(1) *R is left artinian.*
(2) *A left R-module M is m-injective if and only if M^+ is projective.*

Proof. (1) $\Rightarrow$ (2) R is a left C-ring by (1), and so m-injective modules are injective. Now, (2) follows by [3, Theorem 4].

(2) $\Rightarrow$ (1) Firstly, we show that a left R-module M is m-injective if and only if M is absolutely pure.

Let M be an absolutely pure left R-module. Consider the pure exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Then the short exact sequence $0 \to (E(M)/M)^+ \to E(M)^+ \to M^+ \to 0$ splits. Then $E(M)^+$ is projective, and hence M^+ is projective. By (2), M is m-injective. Conversely, let M be an m-injective left R-module. Since M is pure in M^{++} and M^{++} is injective, M is absolutely pure.

Then a left R-module M is m-injective if and only if M is absolutely pure if and only if M^+ is projective. By [3, Theorem 3], R is right perfect, and so it is a left C-ring, i.e., m-injective left R-modules are injective. Hence R is left artinian by [3, Theorem 4] and (2). □

5. When coneat-flat modules are projective

In this section, we shall consider when coneat-flat modules are projective. We begin with the following result.

Theorem 5.1. *Consider the following statements.*

(1) *R is a right perfect ring.*

(2) *Every coneat-flat right R-module is projective.*

Then (1) $\Rightarrow$ (2). *If R is either commutative or semilocal, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Let P be a coneat-flat module. Consider a short exact sequence $0 \to K \to F \to P \to 0$ with F free module. Since R is perfect, F is supplemented by [17, 43.9]. So K has a supplement in F, that is, $K + N = F$ and $A \cap N \ll N$ for some submodule N of F. On the other hand, K is coatomic, as R is a perfect ring. Then K is a coclosed submodule of F by Proposition 2.10. So that $K \cap N \ll K$. Hence K and N are mutual supplements, and so $K \oplus N = F$ by [17, 41.15]. Therefore $N \cong F/K \cong P$ is projective.

(2) $\Rightarrow$ (1) Let M be a flat module. By Corollary 4.3, M is coneat-flat, and so M is projective by (2). Hence R is a perfect ring. □

The following is an immediate consequence of Theorem 5.1.

Corollary 5.2. *Let R be a perfect ring. Then an R-module P is projective if and only if* $\mathrm{Ext}^1_R(P, S) = 0$ *for every simple R-module S.*

An epimorphism $f : N \to M$ is said to be a *small cover* of M if $\mathrm{Ker}\, f \ll N$. Moreover, if N is projective, then f is called a *projective cover.*

Proposition 5.3. *Let R be a ring and M be a right R-module with a projective cover $f : P \to M$. Set $K = \mathrm{Ker}\, f$. Then M is a coneat-flat module if and only if* $\mathrm{Rad}(K) = K$.

Proof. ($\Rightarrow$) Assume $\mathrm{Rad}(K) \neq K$. Then K has a maximal submodule, say A. By Proposition 2.1, there exists a maximal submodule L of P such that $A = K \cap L$. Then $K \leq \mathrm{Rad}\, P$ implies $K = K \cap \mathrm{Rad}(P) \leq K \cap L = A$. Contradiction. Hence (2) holds.

($\Leftarrow$) By Corollary 2.8 and Theorem 3.1. □

Corollary 5.4. *Let R be a semiperfect ring. Then finitely presented coneat-flat modules are projective.*

Lemma 5.5. *Let R be a commutative ring and M be a coneat-flat R-module. Then, for all maximal ideals m of R, M_m is a coneat-flat R_m-module.*

Proof. Since M is a coneat-flat R-module, there is a short exact sequence $0 \to K \to F \to M \to 0$ where K is coneat submodule of F with F is a projective R-module by Theorem 3.1. By exactness of localization, for all maximal ideals m of R, the sequence $0 \to K_m \to F_m \to M_m \to 0$ is exact. Since $mK = K \cap mF$

for all maximal ideals m of R, we have $m_m K_m = K_m \cap m_m F_m$. Therefore M_m is a coneat-flat R_m-module by Theorem 2.5.

□

Corollary 5.6. *Let R be a commutative ring. Then a finitely presented R-module M is coneat-flat if and only if it is projective.*

Proof. Sufficiency is clear. For the necessity, suppose M is coneat-flat. Let m be a maximal ideal of R. Then M_m is a coneat-flat R_m-module by Lemma 5.5. So that M_m is projective (and so flat) over R_m by Corollary 5.4. Then M is flat by [10, page 160, Exercise 14]. Therefore M is projective by [10, Theorem 4.30]. □

References

[1] E. Büyükaşık and Y. Durğun, *Neat-flat modules*, http://arxiv.org/abs/1306.2860.

[2] ———, *Absolutely s-pure modules and neat-flat modules*, Comm. Alg. **43** (2015), no. 2, 384–399.

[3] T. J. Cheatham and D. R. Stone, *Flat and projective character modules*, Proc. Amer. Math. Soc. **81** (1981), no. 2, 175–177.

[4] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting Modules*, Frontiers in Mathematics, Basel, 2006.

[5] S. Crivei, *Neat and coneat submodules of modules over commutative rings*, Bull. Aust. Math. Soc. **89** (2014), no. 2, 343–352.

[6] E. Enochs, *A note on absolutely pure modules*, Canad. Math. Bull. **19** (1976), no. 3, 361–362.

[7] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter & Co., Berlin, 2000.

[8] L. Fuchs, *Neat submodules over integral domains*, Period. Math. Hungar. **64** (2012), no. 2, 131–143.

[9] A. I. Generalov, *Weak and ω-high purities in the category of modules*, Mat. Sb. (N.S.) **105(147)** (1978), no. 3, 389–402, 463.

[10] T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York, 1999.

[11] C. Lomp, *On the splitting of the dual Goldie torsion theory*, Algebra and its applications (Athens, OH, 1999), 377–386, Contemp. Math., 259, Amer. Math. Soc., Providence, RI, 2000.

[12] G. Renault, *Étude de certains anneaux liés aux sous-modules compléments d'un A-module*, C. R. Acad. Sci. Paris **258** (1964), 4888–4890.

[13] J. Rotman, *An Introduction to Homological Algebra*, Academic Press Inc., New York, 1979.

[14] L. Salce, *Almost perfect domains and their modules*, Commutative algebra-Noetherian and non-Noetherian perspectives, 363–386, Springer, New York, 2011.

[15] E. G. Skljarenko, *Relative homological algebra in the category of modules*, Uspehi Mat. Nauk **33** (1978), no. 3(201), 85–120.

[16] P. F. Smith, *Injective modules and prime ideals*, Comm. Algebra **9** (1981), no. 9, 989–999.

[17] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

[18] J. Xu, *Flat Covers of Modules*, Springer-Verlag, Berlin, 1996.

[19] H. Zöschinger, *Koatomare Moduln*, Math. Z. **170** (1980), no. 3, 221–232.

[20] ———, *Schwach-injektive Moduln*, Period. Math. Hungar. **52** (2006), no. 2, 105–128.

[21] ———, *Schwach-Flache Moduln*, Comm. Algebra **41** (2013), no. 12, 4393–4407.

Engin Büyükaşık
İZMİR Institute of Technology
Department of Mathematics
35430, İZMİR, Turkey
E-mail address: enginbuyukasik@iyte.edu.tr

Yilmaz Durğun
BİTLİS Eren University
Department of Mathematics
13000, BİTLİS, Turkey
E-mail address: ydurgun@beu.edu.tr