

CONES OF EFFECTIVE DIVISORS ON THE BLOWN-UP \mathbb{P}^3 IN GENERAL LINES

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ABSTRACT. We compute the facets of the effective cones of divisors on the blow-up of \mathbb{P}^3 in up to five lines in general position. We prove that up to six lines these threefolds are weak Fano and hence Mori Dream Spaces.

1. INTRODUCTION

In classical algebraic geometry, the study of linear systems in \mathbb{P}^n of hypersurfaces of degree d with prescribed multiplicities at a collection of s points in general position was investigated by many authors for over a century (see [10] for an overview).

We will briefly recall the most important results in this area. The well-known Alexander-Hirschowitz theorem [1] classifies completely the dimensionality problem for linear systems with double points (see [5, 7, 8, 19] for more recent and simplified proofs). Besides this theorem, general results about linear systems are rare and few things are known. For arbitrary number of points in the projective space \mathbb{P}^n , the dimensionality problem for linear systems was analysed in the articles [3, 4, 12]. On the one hand, linear systems can be studied via techniques of commutative algebra. In fact, the dimensionality problem is related via apolarity to the Fröberg-Iarrobino Weak and Strong conjectures [15, 16]. These conjectures give a predicted value for the Hilbert series of an ideal generated by s general powers of linear forms in the polynomial ring with $n + 1$ variables. On the other hand, linear systems correspond in birational geometry to the space of global sections of line bundles on the blown-up space \mathbb{P}^n in points. From this perspective, the dimensionality problem of such linear systems reduces to vanishing theorems of divisors on blown-up spaces in points and along higher dimensional cycles of the base locus. Vanishing theorems are important in algebraic geometry because they are related to positivity properties of divisors. In this direction, there are positivity conjectures in birational geometry that involve complete understanding of cohomology groups of divisors (see [13] for an introduction).

For a small number of points, namely whenever $s \leq n + 3$, understanding the cohomology of divisors on the blown-up \mathbb{P}^n in s points is easier since this space is a Mori dream space, see [6, Theorem 1.3]. A projective variety X is called a Mori dream space if its Cox ring is finitely generated. The Cox ring of an algebraic variety naturally generalizes the homogeneous coordinate ring of the projective space. Mori dream spaces are generalizations of toric varieties that have a polynomial Cox ring and their geometry can be encoded by combinatorial data. For example, Mori dream spaces have rational polyhedral effective cone. In other words, the cone of

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effective divisors can be described as intersection of half-spaces in $N^1(X)_{\mathbb{R}} \cong \mathbb{R}^m$, the Néron-Severi group of X tensored with the real numbers.

Motivated by the study of the classical interpolation problems in \mathbb{P}^n started by the Italian school of algebraic geometry, in this article we study the blown-up projective space \mathbb{P}^3 in a collection of s lines in general position l_1, \dots, l_s . We give a description of the cone of effective divisors whenever s is bounded above by five. The technique for finding the facets of the effective cone was first developed for blown-up \mathbb{P}^n in $n + 3$ points in [4].

Moreover, in this article we prove that these threefolds are weak Fano, hence Mori dream spaces, for a number of lines bounded above by six. We also prove that for $s \geq 7$ these threefolds are not weak Fano.

These notes are organized as follows. In Section 2 we introduce the notation that will be used throughout. In Section 3 we describe cycles of the base locus of linear systems of divisors on the blown-up space in a collection of lines in general position. In section 4 we prove the first main result of these notes, i.e. we bound the number of blown-up general lines to obtain a weak Fano threefolds. Section 5 contains the second main result of these notes, a complete description of the effective cone of the blown-up projective space at up to five general lines.

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2. PRELIMINARIES AND NOTATION

Let $X_s := \text{Bl}_s \mathbb{P}^3$ be the blow-up of \mathbb{P}^3 along s lines l_1, \dots, l_s in general position and let E_1, \dots, E_s denote the corresponding exceptional divisors. We use H to denote the class of the pull-back of a general hyperplane in $\text{Bl}_s \mathbb{P}^3$. The Picard group $\text{Pic}(\text{Bl}_s \mathbb{P}^3)$ is spanned by the hyperplane class H and the exceptional divisors E_i .

Remark 2.1. Notice that the hyperplane class H is a blown-up projective plane at s points, with exceptional divisors $e_i = E_i|_H$, $i = 1, \dots, s$, and line class $h := H|_H$. Each exceptional divisor E_i is a Hirzebruch surface \mathbb{F}_0 , isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and the restriction of the hyperplane class $H|_{E_i}$ is the class of a fiber.

Let

$$(2.1) \quad D = dH - \sum_{i=1}^s m_i E_i,$$

with $d, m_1, \dots, m_s \in \mathbb{Z}$, be any line bundle in $\text{Pic}(X_s)$.

Remark 2.2. If $d, m_i \geq 0$, $|D|$ corresponds to the linear system $\mathcal{L}_d(m_1, \dots, m_s)$ of degree- d surfaces of \mathbb{P}^3 that have multiplicity at least m_i along l_i , for $i \in \{1, \dots, s\}$.

Remark 2.3. The canonical divisor of the blown-up projective space \mathbb{P}^3 in s lines is

$$(2.2) \quad K_{\text{Bl}_s \mathbb{P}^3} = -4H + \sum_{i=1}^s E_i.$$

3. BASE LOCUS LEMMAS

Lemma 3.1. *Let $\mathcal{L}_d(m_1, \dots, m_s)$ be the linear system interpolating s lines with multiplicities m_i . If $l_i, l_j, l_k \subset \mathbb{P}^3$, with $i, j, k \in \{1, \dots, s\}$, are three lines in general position, then there exists a (unique) smooth quadric surfaces Q_{ijk} in \mathbb{P}^3 containing the lines l_i, l_j, l_k . Moreover such a quadric splits off $\mathcal{L}_d(m_1, \dots, m_s)$ at least $k_{ijk} := \max(m_i + m_j + m_p - d, 0)$ times.*

Proof. It is enough to prove the claim for $s = 3$, and we will denote the lines by l_1, l_2, l_3 . One can easily compute that $h^0(\mathbb{P}^3, \mathcal{L}_2(1, 1, 1)) = 1$; the quadric is the unique element of this linear system and it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. By generality, $l_1, l_2, l_3 \subset \mathbb{P}^3$ belong to the same ruling of Q_{123} . Notice that each element of $\mathcal{L}_d(m_1, m_2, m_3)$ intersects each line of the other ruling of the quadric at least $m_1 + m_2 + m_3$ times. Therefore, if $m_1 + m_2 + m_3 > d$, the quadric Q_{123} is in the base locus of $\mathcal{L}_d(m_1, m_2, m_3)$. To compute the multiplicity of containment one can check that in the residual linear system, $\mathcal{L}_d(m_1, m_2, m_3) - Q_{123}$, the integer k_{123} drops by 1 and this concludes the proof. \square

Lemma 3.2. *Let $\mathcal{L}_d(m_1, \dots, m_s)$ as in Lemma 3.1. Any collection of four general lines l_i, l_j, l_k, l_l in \mathbb{P}^3 determines two transversal lines t, t' . For any $s \geq 4$ these transversals are contained in the base locus of the linear system $\mathcal{L}_d(m_1, \dots, m_s)$ with multiplicity at least $k_t = k_{t'} := \max(m_i + m_j + m_k + m_l - d, 0)$.*

Proof. It is enough to prove the claim for $s = 4$, and for simplicity we will denote the lines by l_1, l_2, l_3, l_4 . The existence of two distinct transversals lines is straightforward. We will denote by t, t' such lines, see Figure 1.

Notice that all elements of $\mathcal{L}_d(m_1, m_2, m_3, m_4)$ intersect t (resp. t') at least $m_1 + m_2 + m_3 + m_4$ times, therefore if $m_1 + m_2 + m_3 + m_4 > d$ the two transversals are part of the base locus of $\mathcal{L}_d(m_1, m_2, m_3, m_4)$. In fact, the multiplicity of containment of these transversals is at least k_t . \square

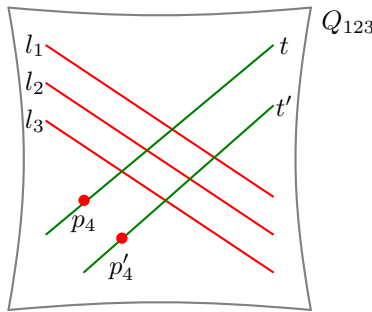


FIGURE 1. Q_{123}

4. WEAK FANO CASE

In this section we prove that $X_s = \text{Bl}_s \mathbb{P}^3$, the blow-up of \mathbb{P}^3 in s lines in general position, is *weak Fano* if and only if $s \leq 6$. In particular this implies that for $s \leq 6$ it is a Mori dream space. The notion of weak Fano varieties relates to positivity properties of linear series.

Notice that knowing the base locus of the linear series associated to a given divisor D makes it simpler to determine when D is nef, since the curves that intersect the divisor negatively can only be contained in the diminished base locus of the divisor $\mathbb{B}_-(D)$ (see [14]).

Let us first recall some definitions and properties connected to the Minimal Model Program that give a precise way of detecting Mori dream spaces.

Definition 4.1. Let X be a smooth projective variety. We say that

- X is *weak Fano* if $-K_X$ is big and nef.
- X is *log Fano* if there exists an effective divisor D such that $-(K_X + D)$ is ample and the pair (X, D) has *Kawamata log terminal* singularities (see [2] for the definition).

Remark 4.2. We have

$$\text{weak Fano} \Rightarrow \text{log Fano} \Rightarrow \text{Mori Dream Space.}$$

The first implication follows trivially from Definition 4.1, while the second one follows from the results contained in [2].

If $s \leq 2$, the threefold X_s is a normal toric variety. For $s = 3$, X_s is isomorphic to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along the small diagonal. This variety is one of the Fano 3-folds of Picard number four in Fujita's classification, see [18]. Therefore X_3 is log Fano, hence a Mori dream space. For $s \geq 4$ it was not known, at the best of our knowledge, whether X_s is a Mori dream space.

Remark 4.3. The intersection table on X_s is given by the following:

$$H^3 = 1, \quad H^2 E_i = 0, \quad H E_i^2 = -1, \quad E_i^2 E_j = 0, \quad E_i^3 = 2,$$

for every distinct indices $i, j \in \{1, \dots, s\}$. Therefore we can compute the top self-intersection of the anti-canonical divisor: $(-K_{X_s})^3 = 64 - 10s$. If $-K_{X_s}$ is nef, it fails to be big whenever $s \geq 7$, see e.g. [17, Theorem 2.2.16].

Theorem 4.4. *The threefold X_s is weak Fano if and only if $s \leq 6$.*

Proof. By Remark 4.3, it is enough to prove that $-K_s$ is nef for $s \leq 6$. For $s \leq 4$, $-K_s$ is clearly nef, because it can be written as a sum of nef divisors given by pencils $H - E_i$.

We will consider the cases $s = 5, 6$. If $s = 5$, notice that for every permutation $\sigma \in \mathfrak{S}_5$, the union of divisors

$$(2H - E_{\sigma(1)} - E_{\sigma(2)} - E_{\sigma(3)}) + (H - E_{\sigma(4)}) + (H - E_{\sigma(5)})$$

belongs to the anti-canonical system. Similarly, if $s = 6$, for every permutation $\sigma \in \mathfrak{S}_6$, the union

$$(2H - E_{\sigma(1)} - E_{\sigma(2)} - E_{\sigma(3)}) + (2H - E_{\sigma(4)} - E_{\sigma(5)} - E_{\sigma(6)})$$

belongs to the anti-canonical system. Therefore the base locus of $-K_{X_s}$, for $s = 5, 6$, is contained in the intersection of all these unions, which is $\sum_{i=1}^s E_i$. We also know that any curve C intersecting $-K_X$ negatively is contained in $\mathbb{B}_-(-K_X) \subseteq \bigcup_{i=1}^s E_i$. Now, since for any $i \in \{1, \dots, s\}$ the exceptional divisor E_i is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, any curve C contained in E_i is such that $C \cdot E_i \leq 0$ and $C \cdot E_j = 0$ for any $j \in \{1, \dots, s\} \setminus \{i\}$. In particular this implies that $-K_{X_s}$ is nef. \square

The following is an immediate consequence of Theorem 4.2 and Remark 4.2

Corollary 4.5. *If $s \leq 6$, the threefold X_s is a Mori dream space.*

5. CONES OF EFFECTIVE DIVISORS

In this section we will consider \mathbb{Q} -Cartier divisors on $X_s = \text{Bl}_s \mathbb{P}^3$, the blow-up of \mathbb{P}^3 in s general lines. Any divisor here will have the form (2.1), where the assumption on the coefficients has been relaxed to $d, m_i \in \mathbb{Q}$. The purpose is in fact to describe the boundary of the effective cone. These cones can be extremely complicated for a general variety. In the specific case of Mori dream spaces they are polyhedral but there is no reason to expect that the hyperplanes cutting the facets of such cone are described by integral equations in the degree and the multiplicities.

For $s \leq 2$, since X_s is a normal toric variety, the cone of effective divisors can be computed by means of techniques from toric geometry, see for instance [11]. In this section we extend the description of the effective cone from the toric case to the case $s = 3, 4, 5$, in particular showing that it is a rational and polyhedral cone.

We will let (d, m_1, \dots, m_s) be the coordinates of the Néron-Severi group of X_s . In the next sections we will give equations describing the cones of effective divisors for $s \leq 5$.

5.1. The case of three lines.

Theorem 5.1. *If $s \leq 3$, the effective cone of X_3 is the closed rational polyhedral cone given by the following inequalities:*

$$(5.1) \quad 0 \leq d,$$

$$(5.2) \quad m_i \leq d, \quad \forall i \in \{1, \dots, s\},$$

$$(5.3) \quad m_i + m_j \leq d, \quad \forall i, j \in \{1, \dots, s\}, i \neq j \text{ (if } s = 2, 3\text{)}.$$

Proof. Notice that if $m_i < 0$ for some $i \in \{1, \dots, s\}$, then D is effective if and only if $D + m_i E_i$ is. Hence we may assume that $m_i \geq 0$ for all $i = 1, \dots, s$. In this case (5.3) implies (5.2).

Assume first of all that D is effective. Clearly $d \geq 0$. Assume $m_i + m_j > d$, for some $i \neq j$. Each point p of $\mathbb{P}^3 \setminus \{l_1, \dots, l_s\}$ lies on the line spanned by points $p_i \in l_i$ and $p_j \in l_j$. By Bézout's Theorem such a line is contained in the base locus of $\mathcal{L}_d(m_1, \dots, m_s)$. Hence by Remark 2.2, the strict transform on X_s of such a line is contained in the base locus of $|D|$, and in particular so does the inverse image of p . Therefore each point of $X_s \setminus \{E_1, \dots, E_s\}$ is in the base locus of $|D|$, hence $|D| = \emptyset$.

We now prove that a divisor D that satisfies the inequalities is effective.

If $s = 2$, we can write $D = (d - m_1 - m_2)H + m_1(H - E_1) + m_2(H - E_2)$ and conclude, as D is sum of effective divisors with non-negative coefficients.

Assume $s = 3$ and $m_1 + m_2 + m_3 - d \leq 0$. We can for instance write $D = (d - m_1 - m_2 - m_3)E_1 + (d - m_2 - m_3)(H - E_1) + m_2(H - E_2) + m_3(H - E_3)$ and conclude. If, instead, $k_{123} = m_1 + m_2 + m_3 - d > 0$, we can write $D = k_{123}(2H - E_1 - E_2 - E_3) + (d - m_2 - m_3)(H - E_1) + (d - m_1 - m_3)(H - E_2) + (d - m_1 - m_2)(H - E_3)$. \square

5.2. The case of four lines.

Theorem 5.2. *The effective cone of X_4 is the closed rational polyhedral cone given by the following inequalities:*

$$(5.4) \quad 0 \leq d,$$

$$(5.5) \quad m_i \leq d, \quad \forall i \in \{1, \dots, 4\},$$

$$(5.6) \quad m_i + m_j \leq d, \quad \forall i, j \in \{1, \dots, 4\}, i \neq j,$$

$$(5.7) \quad (m_1 + \dots + m_4) + m_i \leq 2d, \quad \forall i \in \{1, \dots, 4\},$$

$$(5.8) \quad 2(m_1 + \dots + m_4) \leq 3d.$$

Proof. As in the proof of Theorem 5.1, we may assume that $m_i \geq 0$, for all $i = 1, 2, 3, 4$. In this case (5.5) is redundant.

If D is effective, then (5.4) and (5.6) are satisfied by Theorem 5.1.

We now prove that $D \geq 0$ implies (5.7). Notice that, without loss of generality, we may prove that the inequality is satisfied for a choice of index $i \in \{1, 2, 3, 4\}$. Set $i = 1$ and consider the pencil of planes containing l_1 . Notice that it covers the whole space \mathbb{P}^3 . Therefore in order to conclude it is enough to prove that if the inequality is violated, then each point of the general plane of the pencil is contained in the base locus of $|D|$, making it empty. Let Δ be a general plane containing l_1 . Each line l_j , $j = 2, 3, 4$, intersects Δ in a point: write $p_j := l_j \cap \Delta$ (Figure 2). By the generality assumption, the points p_2, p_3, p_4 are not collinear. We will consider the net of conics in Δ passing through these three points.

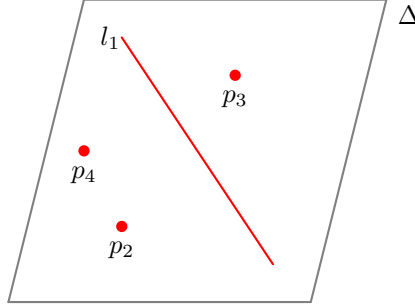


FIGURE 2. Δ

Let us denote by $\tilde{\Delta}$ the strict transform of Δ in X_4 , that is a projective plane blown-up in three non-collinear points. The exceptional divisor E_1 intersects the blown-up plane $\tilde{\Delta}$ in the class of a general line. Precisely, we have the following intersection table:

$$h := H|_{\tilde{\Delta}} = E_1|_{\tilde{\Delta}},$$

$$e_j := E_j|_{\tilde{\Delta}}, \quad j = 2, 3, 4.$$

The classes h and e_j for $j = 2, 3, 4$ generate the Picard group of $\tilde{\Delta}$. Consider the restriction

$$D|_{\tilde{\Delta}} = (d - m_1)h - \sum_{j=2}^4 m_j e_j.$$

The intersection number between $D|_{\tilde{\Delta}}$ and the strict transform on $\tilde{\Delta}$ of a conic through p_2, p_3, p_4 is $2(d - m_1) - m_2 - m_3 - m_4$. If this number is negative, namely if (5.7) is violated with $i = 1$, then each conic through p_2, p_3, p_4 is contained in

the base locus of $|D|_{\tilde{\Delta}}$. Since these conics cover Δ , we conclude $\tilde{\Delta}$ is contained in the base locus of $|D|$. This is a contradiction since a pencil of planes can not be contained in the fixed part of an effective divisor.

We now prove the last inequality. Notice first of all that if $m_1 + m_2 + m_3 + m_4 \leq d$, then (5.8) follows obviously from (5.6). Hence we may assume that $m_1 + m_2 + m_3 + m_4 > d$. Observe that each point of $X_4 \setminus \{E_1, \dots, E_4\}$ sits on the strict transform of a line in \mathbb{P}^3 that is transversal to t and t' , because t and t' are skew. Using Lemma 3.2 and arguing as for the proof of Theorem 5.1, we conclude that if $2(m_1 + \dots + m_4) > 3d$ then $|D|$ is empty.

We now prove that a divisor D satisfying (5.4), (5.6), (5.7) and (5.8) is effective.

Consider first the case $m_{i_1} + m_{i_2} + m_{i_3} \leq d$ for all $\{i_1, i_2, i_3\} \subset \{1, 2, 3, 4\}$. Write $D = D' + m_4(H - E_4)$. Then

$$D' = (d - m_4)H - m_1E_1 - m_2E_2 - m_3E_3$$

is effective by Theorem 5.1. Hence we conclude in this case.

After reordering the indices $\{1, 2, 3, 4\}$ if necessary, we assume that $k_{123} = m_1 + m_2 + m_3 - d > 0$. It is enough to prove that

$$D'' := D - k_{123}(2H - E_1 - E_2 - E_3)$$

satisfies (5.4), (5.6), (5.7) and (5.8) to conclude. The coefficients of the expanded expression

$$D'' =: d'H - \sum_{i=1}^3 m'_i E_i - m_4 E_4$$

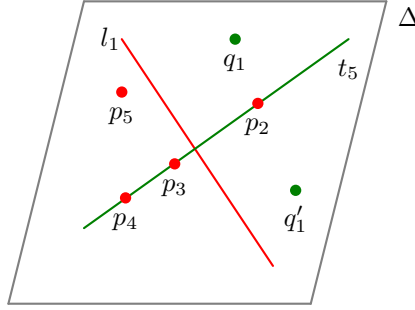
are $d' = 3d - 2(m_1 + m_2 + m_3)$, $m'_1 = d - m_2 - m_3$, $m'_2 = d - m_1 - m_3$, $m'_3 = d - m_1 - m_2$. Since all of these coefficients are positive, it is enough to check that (5.6), (5.7) and (5.8) are satisfied. The divisor D'' satisfies condition (5.6) because D does. Indeed $m'_i + m'_j \leq d'$ is equivalent to $m_i + m_j \leq d$, for $1 \leq i < j \leq 3$; moreover $m'_i + m_4 \leq d'$ is equivalent to $(m_1 + m_2 + m_3 + m_4) + m_i \leq 2d$, for all $1 \leq i \leq 3$, i.e. condition (5.7). Furthermore D'' satisfies (5.7) with $i \in \{1, 2, 3\}$ if and only if D does; D'' satisfies (5.7) with $i = 4$ if and only if D satisfies (5.8). Finally, D'' satisfies (5.8) because D does; we leave the details to the reader. \square

5.3. The case of five lines.

Lemma 5.3. *There does not exist any cubic surface containing five general lines of \mathbb{P}^3 .*

Proof. Assume that there exists a cubic surface S containing 5 general lines of \mathbb{P}^3 . It has to be irreducible. In fact there is no quadric surface containing four lines and no plane containing two lines. Therefore $S|_{\Delta}$ is a plane cubic, for each plane $\Delta \subset \mathbb{P}^3$.

Now, for every $i = 1, \dots, 5$, denote by t_i, t'_i the two transversal lines to the four lines $\{l_1, \dots, \tilde{l}_i, \dots, l_5\} \subset \mathbb{P}^3$. By Lemma 3.2, t_i, t'_i are contained in S , for every $i = 1, \dots, 5$. Let now $\Delta \subset \mathbb{P}^3$ be the plane spanned by l_1 and t_5 (see Figure 3).

FIGURE 3. Δ_1^5

The restricted linear series $|S|_{\Delta}$ is the linear series of plane cubics containing l_1 and t_5 and passing through the three points $l_5 \cap \Delta$, $t_1 \cap \Delta$, $t'_1 \cap \Delta$. By the generality assumption, these three points are not collinear, hence the restricted series is empty. This concludes the proof. \square

Remark 5.4. Assuming that a collection of lines is in general position, is much stronger than assuming that the lines are skew.

Recall that any smooth cubic surface S may be realized as the blow-up of the projective plane \mathbb{P}^2 in six points p_1, \dots, p_6 in general position. There are 27 lines in S : the 6 exceptional divisors, the strict transforms of the lines through two of the p_i 's, and the strict transforms of the conics through five of the p_i 's.

Therefore, for any choice of five among the exceptional divisors on the blown-up \mathbb{P}^2 , there is a line intersecting them, whereas for general lines on the cubic surface, this can happen for at most four lines, see Lemma 3.2.

Theorem 5.5. *The effective cone of X_5 is the closed rational polyhedral cone given by the following inequalities:*

$$(5.9) \quad 0 \leq d,$$

$$(5.10) \quad m_i \leq d, \quad \forall i \in \{1, \dots, 5\},$$

$$(5.11) \quad m_i + m_j \leq d, \quad \forall i, j \in \{1, \dots, 5\}, i \neq j,$$

$$(5.12) \quad (m_1 + \dots + m_5) + m_i - m_j \leq 2d, \quad \forall i, j \in \{1, \dots, 5\}, i \neq j,$$

$$(5.13) \quad (m_1 + \dots + m_5) + m_i \leq 2d, \quad \forall i \in \{1, \dots, 5\},$$

$$(5.14) \quad 2(m_1 + \dots + m_5) \leq 3d.$$

Proof. We may assume that $m_i \geq 0$, for all $i = 1, 2, 3, 4, 5$. In this case (5.10) and (5.12) are redundant.

We prove that the conditions (5.9), ..., (5.14) are necessary for a divisor D on $\text{Bl}_5 \mathbb{P}^3$ to be effective.

If D is effective, then (5.9) and (5.11) are satisfied by Theorem 5.1.

The proof that $D \geq 0$ implies (5.13) is the same as the one for the inequality (5.7) in Theorem 5.2, where we consider the pencil of conics in Δ passing to four points p_2, p_3, p_4 and $p_5 := l_5 \cap \Delta$.

We now prove that $D \geq 0$ implies (5.14). If $m_i = 0$ for some $i \in \{1, \dots, 5\}$, we conclude by Theorem 5.2; therefore we may assume $m_i > 0$. If $(m_1 + \dots + m_5) - m_i \leq d$ for some $i \in \{1, \dots, 5\}$, then (5.14) follows from (5.13). Therefore we shall assume

$(m_1 + \cdots + m_5) - m_i > d$ for every i . Assume by contradiction that (5.14) is not satisfied. Let us consider the plane Δ containing l_1 (Figure 4).

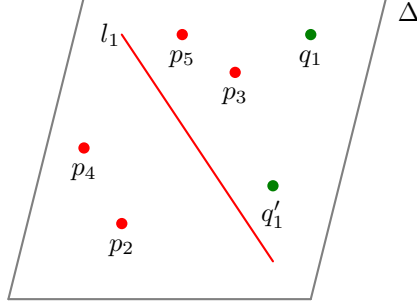


FIGURE 4. Δ'_1

Let Δ , t_i and t'_i be as above and set $q_1 := t_1 \cap \Delta$ and $q'_1 := t'_1 \cap \Delta$. Assume Δ is general. Note that the set of points $\{q_1, p_2, \dots, p_5\}$ is in general position in Δ . If (5.14) is violated, then the strict transform in $\hat{\Delta}$ of the unique conic in Δ passing through these five points is contained in the base locus of $|D|_{\hat{\Delta}}$ with multiplicity $(m_2 + m_3 + m_4 + m_5 - d) + m_2 + m_3 + m_4 + m_5 - 2(d - m_1) = 2(m_1 + \cdots + m_5) - 3d > 0$. Assume Δ contains one of the transversal lines to l_1 , say t_2 . We interpret this as a degeneration of the conic to the union of the lines spanned by $\{q_1, p_2\}$ and $\{q'_1, p_2\}$ while the plane Δ specializes. Each of these two lines is contained in the base locus with multiplicity $2(m_1 + \cdots + m_5) - 3d$. This shows that the conic bundle over \mathbb{P}^1 , having fibers these conics, is contained in the base locus of D with multiplicity $2(m_1 + \cdots + m_5) - 3d$. It is a surface of degree $\delta \geq 3$, that is linearly equivalent to the following divisor

$$S = \delta H - \sum_{j=2}^5 E_j.$$

Consider the divisor

$$\begin{aligned} D' &= D - S = (d - \delta)H - m_1 E_1 - \sum_{j=2}^5 (m_j - 1)E_j \\ &:= d'H - \sum_{j=1}^5 m'_j E_j. \end{aligned}$$

Since $S \subseteq \text{Bs}(|D|)$, one obtains that D is effective if and only if D' is effective. Moreover $d \geq \delta$. Now, we distinguish two cases.

Case 1. If $m'_j = 0$ for some $j \in \{2, \dots, 5\}$, then we can think of D' as a divisor living on the space blown-up along up to four lines and conclude by Theorem 5.2.

Case 2. Assume $m'_j > 0$ for all $j \in 1, \dots, 5$. We claim that $(m'_1 + \cdots + m'_5) - m'_j > d'$ for every j . This is very easy to check if $j \in \{2, \dots, 5\}$. If $j = 1$, we compute

$$\begin{aligned} 0 &< (2(m_1 + \cdots + m_5) - 3d) + (3\delta - 8) = 2(m'_1 + \cdots + m'_5) - 3d' \\ &= ((m'_1 + \cdots + m'_5) + m_1 - 2d') + ((m'_1 + \cdots + m'_5) - m_1 - d') \\ &\leq ((m'_1 + \cdots + m'_5) - m_1 - d'), \end{aligned}$$

where the last inequality follows from the fact that D' is effective, as we already proved that (5.13) is necessary condition for this. Moreover the first inequality, that just follows from the assumption that (5.14) is violated and the fact that $\delta \geq 3$, shows that S is contained in the base locus of $|D'|$. Hence we reduced D to a divisor D' which satisfies the same assumptions and that has smaller degree and multiplicities along the lines. We can proceed as above until we reduce to either a divisor with negative degree or to Case 1. This concludes this part of the proof.

We now prove that a divisor D satisfying all inequalities is effective.

Consider first of all the case $m_{i_1} + m_{i_2} + m_{i_3} \leq d$, for all triples of pairwise distinct indices $\{i_1, i_2, i_3\} \subset \{1, \dots, 5\}$. Modulo reordering the indices, if necessary, we may assume $m_5 \geq m_i$ for $i = 1, \dots, 4$. Set

$$T := 4H - \sum_{i=1}^5 E_i - E_5.$$

This divisor is effective, because we can write it as a sum of effective divisors $T = (2H - E_1 - E_2 - E_5) + (2H - E_3 - E_4 - E_5)$. Consider the following difference:

$$D'' := 2D - m_5 T.$$

If D'' is effective we conclude, in fact when describing the effective cone we are only interested in effectivity up to numerical equivalence. Notice that D'' is a divisors that lives in $\text{Bl}_4 \mathbb{P}^3$, the blow-up of \mathbb{P}^3 along four lines, in fact we can write

$$D'' = (2d - 4m_5)H - \sum_{i=1}^4 (2m_i - m_5)E_i := d''H - \sum_{i=1}^4 m_i'' E_i.$$

Notice that by assumption, $d'' \geq 0$, $m_i'' \leq d''$ and $m_i'' + m_j'' \leq d''$. To show that D'' is effective, it is enough to verify that it satisfies the inequality (5.8) of Theorem 5.2; we conclude by noticing that

$$2 \sum_{i=1}^4 m_i'' - 3d'' = 2(2(m_1 + \dots + m_5) - 3d),$$

so that the inequality $2 \sum_{i=1}^4 m_i'' \leq 3d''$ is equivalent to (5.14).

Modulo reordering the indices if necessary, assume that $m_1 + m_2 + m_3 > d$, namely that the quadric through the lines l_1, l_2, l_3 is contained in the base locus of D with multiplicity $k_{123} = m_1 + m_2 + m_3 - d$, see Lemma 3.1. It is enough to prove that

$$D - k_{123}(2H - E_1 - E_2 - E_3)$$

is effective. To conclude it is enough to check that it satisfies (5.9), ..., (5.14) and then to use the preceding case. We leave the details to the reader. \square

5.4. Extremal rays of the effective cones.

Corollary 5.6. *The extremal rays of $\text{Eff}(X_2)$ are*

$$\begin{aligned} &\text{cone}(E_i), \quad 1 \leq i \leq 2, \\ &\text{cone}(H - E_i), \quad 1 \leq i \leq 2. \end{aligned}$$

Proof. The proof of Theorem 5.1 implies that every effective divisor on X_2 can be written as linear combination with positive coefficients of the effective divisors E_i and $H - E_i$, for $i = 1, 2$. Hence these divisors form a set of generators for the effective cone.

We now show that each of these generators spans an extremal rays of the effective cone in $N^1(X_2)_{\mathbb{R}} \cong \mathbb{R}^3$. In order to do this we use the equations describing the facets of the effective cone in Theorem 5.1 and we obtain:

$$\begin{aligned} \text{cone}(E_1) &= \{d = 0\} \cap \{m_2 = 0\}, \\ \text{cone}(E_2) &= \{d = 0\} \cap \{m_1 = 0\}, \\ \text{cone}(H - E_1) &= \{m_1 = d\} \cap \{m_1 + m_2 = d\}, \\ \text{cone}(H - E_2) &= \{m_2 = d\} \cap \{m_1 + m_2 = d\}. \end{aligned}$$

□

Corollary 5.7. *The extremal rays of $\text{Eff}(X_s)$, $s = 3, 4, 5$ are*

$$\begin{aligned} \text{cone}(E_i), \quad 1 \leq i \leq s, \\ \text{cone}(H - E_i), \quad 1 \leq i \leq s, \\ \text{cone}(2H - E_{i_1} - E_{i_2} - E_{i_3}), \quad 1 \leq i_1 < i_2 < i_3 \leq s. \end{aligned}$$

Proof. The proofs of Theorem 5.1, Theorem 5.2 and Theorem 5.5 respectively provide a recipe for writing D as sum of positive multiples of the generators listed.

To see that these generators span the extremal rays of the effective cone, we argue as in the proof of Corollary 5.6. For $s = 3$, using Theorem 5.1, we obtain the following description:

$$\begin{aligned} \text{cone}(E_{i_1}) &= \{d = 0\} \cap \bigcap_{i \neq i_1} \{m_i = d\} \cap \{m_1 + m_2 + m_3 - m_{i_1} = d\}, \\ \text{cone}(H - E_{i_1}) &= \{m_{i_1} = d\} \cap \bigcap_{i \neq i_1} \{m_{i_1} + m_i = d\}. \end{aligned}$$

Each ray of the form $\text{cone}(E_{i_1})$ (resp. $\text{cone}(H - E_{i_1})$) is intersection of four (resp. three) hyperplanes of $N^1(X_3)_{\mathbb{R}} \cong \mathbb{R}^4$.

For $s = 4$, using Theorem 5.2, we obtain

$$\begin{aligned} \text{cone}(E_{i_1}) &= \{d = 0\} \cap \bigcap_{i \neq i_1} \{m_i = d\} \cap \bigcap_{i, j \neq i_1} \{m_i + m_j = d\}, \\ \text{cone}(H - E_{i_1}) &= \{m_{i_1} = d\} \cap \bigcap_{i \neq i_1} \{m_{i_1} + m_i = d\} \\ &\quad \cap \{(m_1 + \cdots + m_4) + m_{i_1} = 2d\}, \\ \text{cone}(2H - E_{i_1} - E_{i_2} - E_{i_3}) &= \bigcap_{i, j \in \{i_1, i_2, i_3\}} \{m_i + m_j = d\} \\ &\quad \cap \bigcap_{i \in \{i_1, i_2, i_3\}} \{(m_1 + \cdots + m_4) + m_i = 2d\} \\ &\quad \cap \{2(m_1 + \cdots + m_4) = 3d\}. \end{aligned}$$

We leave it to the reader to verify the statement for $s = 5$, using Theorem 5.5. □

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