Annali di Matematica (2015) 194:673–718
DOI 10.1007/s10231-013-0393-z

Cones of G manifolds and Killing spinors with skew torsion

Ilka Agricola · Jos Höll

Received: 5 June 2013 / Accepted: 20 November 2013 / Published online: 4 December 2013

Abstract This paper is devoted to the systematic investigation of the cone construction for Riemannian G manifolds M, endowed with an invariant metric connection with skew torsion ∇^c, a 'characteristic connection.' We show how to define a $\bar{G}$ structure on the cone $\bar{M} = M \times \mathbb{R}^+$ with a cone metric, and we prove that a Killing spinor with torsion on M induces a spinor on $\bar{M}$ that is parallel for the characteristic connection of the $\bar{G}$ structure. We establish the explicit correspondence between classes of metric almost contact structures on M and almost Hermitian classes on $\bar{M}$, respectively, between classes of G_2 structures on M and $\mathrm{Spin}(7)$ structures on $\bar{M}$. Examples illustrate how this 'cone correspondence with torsion' works in practice.

Keywords Cone of Riemannian manifold · Metric connection with skew torsion · Characteristic connection · G Structure · Killing spinor with torsion · Parallel spinor · G_2 manifold · $\mathrm{Spin}(7)$ manifold · almost contact metric manifold · α-Sasakian manifold · almost Hermitian manifold · Hyper-Kähler manifold with torsion · Tanno deformation

Mathematics Subject Classification (2010) 53 C 25-29 · 58 J 50 · 58 J 60

1 Preliminaries

1.1 Introduction

Given a complete Riemannian spin manifold (M, g), the two most basic equations that a spinor field ψ can fulfill are the parallelism equation and the Killing equation,

$$\nabla^g \psi = 0, \qquad \nabla^g_X \psi = \mu X \cdot \psi \text{ for some } \mu \in \mathbb{R} - \{0\},$$

I. Agricola (✉) · J. Höll
Fachbereich Mathematik und Informatik, Philipps-Universität Marburg,
Hans-Meerwein-Strasse, 35032 Marburg, Germany
e-mail: agricola@mathematik.uni-marburg.de

J. Höll
e-mail: hoellj@mathematik.uni-marburg.de

where ∇^g denotes the Levi-Civita connection. Berger's holonomy theorem yielded that the Ricci-flat manifolds with reduced Riemannian holonomy $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, G_2, or Spin(7) were candidates for manifolds with parallel spinors, and indeed, Wang proved in 1989 that these are the only manifolds admitting parallel spinors, and determined the dimension of the space of parallel spinors [54]. The geometric meaning of the Killing equation stems from the fact that Riemannian Killing spinors realize the equality case in Friedrich's seminal estimate of the first eigenvalue of the Riemannian Dirac operator on compact Riemannian manifolds of positive curvature [24]. Independently, the Killing equation was investigated in theoretical physics for supergravity theories in dimensions 10 and 11 [19] and certain applications in general relativity [45]. The first nontrivial compact examples of Riemannian manifolds with Killing spinors were found in dimensions $5 \leq n \leq 7$ in 1980–1986 ([20,24, 26]). The link to nonintegrable geometry and G structures was established shortly after; for instance, a compact, connected, and simply connected 6-dimensional Hermitian manifold is nearly Kähler if and only if it admits a Riemannian Killing spinor [35]. Similar results hold for Einstein–Sasaki structures in all odd dimension and nearly parallel G_2-manifolds in dimension 7 ([29,30]).

The connection between these two spinorial field equations was recognized by Bryant in 1987, who proved that the cone over the nearly Kähler manifold $\mathrm{SU}(3)/T^2$ was an integrable G_2 manifold, and that the cone over the nearly parallel G_2 manifold $\mathrm{SO}(5)/\mathrm{SO}(3)$ was an integrable Spin(7) manifold [12]. Bär generalized this idea in 1993 and proved that the cone ($\bar{M} = M \times \mathbb{R}^+, \bar{g} = r^2 g + dr^2$) of a (compact) Riemannian spin manifold (M, g) with Riemannian Killing spinors is a (noncompact) Ricci-flat Riemannian spin manifold with $\bar{\nabla}^{\bar{g}}$-parallel spinors. We will loosely call this phenomenon the *cone correspondence*. By combining this cone correspondence with Wang's classification result, one obtains a complete overview about all geometries that can carry Riemannian Killing spinors: together with results by Hijazi [36], the general picture is basically that the nonintegrable geometries listed above are, beside spheres, the *only* possible ones. A great deal of effort has been invested in the actual construction of such nonintegrable geometries. But while there is a rich supply of nonhomogeneous Einstein–Sasaki manifolds (see [13,32], and many others) and nearly parallel G_2 manifolds, compact nearly Kähler manifolds have resisted so far all construction efforts in the nonhomogeneous case, though they are generally believed to exist.

Since then, there has been a lot of progress on G structures on Riemannian manifolds and general holonomy theory. Einstein–Sasaki manifolds, nearly Kähler 6-manifolds, and nearly parallel G_2-manifolds are only special instances of more general Riemannian manifolds with structure group $\mathrm{U}(n)$, $\mathrm{SU}(3)$, or G_2. These can be neatly divided into different classes, first through the study of their characterizing differential equations ([15,16,21,23,33]) and later by more general concepts like intrinsic torsion ([47,48]) and, closely related, characteristic connections—these are, by definition, invariant metric connections with skew torsion ([2, 25]). The integrable geometries covered by Berger's theorem correspond to the 'trivial' class (though, of course, they are highly nontrivial objects). Many examples of different classes were constructed, and their special properties investigated in the last decades. As a common feature, a certain, well-understood subclass of every possible G structure admits a unique G invariant metric connection with skew torsion, the *characteristic connection* ∇^c, and it induces (with a $1/3$ rescaling) a characteristic Dirac operator that generalizes the Dolbeault operator on Hermitian manifolds and Kostant's 'cubic' Dirac operator on naturally reductive homogeneous spaces ([1,3,4,10]).

Again, a big incentive to study G manifolds admitting a characteristic connection came from theoretical physics, more precisely from superstring theory, where the characteristic torsion (by definition, it is a 3-form on the manifold) is interpreted as a higher order flux (see

[34,50] for the first publications on the topic; for more details, we refer to the vast literature on string compactifications). Spinor fields satisfying a generalized kind of Killing/parallelism equation with torsion (the precise equation depends on the model) are identified with supersymmetry transformations. More recently, connections with skew torsion and their Dirac operators are also considered for the spectral action principle and hypothetical applications in cosmic topology [41].

It is well known that the characteristic connection ∇^c can admit a parallel spinor field in more situations than for the Levi-Civita connection ∇^g, and that an analog of Wang's classification result is not possible. For example, any G_2 structure and any $\mathrm{Spin}(7)$ structure admitting a characteristic connection ∇^c has a ∇^c-parallel spinor field, just because G_2 and $\mathrm{Spin}(7)$ are the stabilizers of a generic spinor in dimension 7 and 8, respectively ([27, 39]). More recently, the twistor and Killing equations for the characteristic connection were investigated in [3] and [9]; we will speak of *Killing spinors with torsion* to distinguish them from the Riemannian case. Again, the picture is roughly as follows: there are more G manifolds admitting Killing spinors with torsion than in the Riemannian case, and their geometry is less rigid (for example, they do not have to be Einstein, and the Killing number is not automatically linked to the first eigenvalue of the characteristic Dirac operator). This richness in turn implies that a classification is not possible. One further crucial difference to the Riemannian case is that the families of manifolds admitting parallel spinors resp. Killing spinors with torsion are not disjoint any more; both are described in the language of G structures sketched above; and it is to be discussed in every situation anew what can be said about particular spinor fields.

1.2 Outline

The main purpose of the present paper is to investigate the cone correspondence for G manifolds admitting a characteristic connection. While doing so, several results are obtained that should be of interest in other circumstances as well.

Section 2 is devoted to the general construction. Given a Riemannian manifold (M, g), we denote by $(\bar{M}, \bar{g}) = (M \times \mathbb{R}^+, a^2r^2g + dr^2)$ for some fixed $a > 0$ its cone (we sometimes call a the *cone constant* of $\bar{M}$). Of course, the cone does always exist and carries interesting geometric structures, but if one intends to lift a Killing spinor with torsion from M to $\bar{M}$, one has to choose a suitably, depending on the Killing number α. It is crucial that α is not allowed to vanish, i. e. there is no cone correspondence for parallel spinor fields (but see Corollary 4.16 for an exception). The details of this 'abstract' cone correspondence for most general metric connections with skew torsion are explained in Sect. 2.1. Section 2.2 introduces the G structures that will be of particular interest in this article and their characteristic connections. For metric almost contact structures, we prove a new criterion for the existence of a characteristic connection (Lemma 2.5) and describe the corresponding Chinea-Gonzales classes. For almost Hermitian structures, G_2 structures, and $\mathrm{Spin}(7)$ structures, we quickly recall about their characteristic connections, a few facts that we shall need later.

In Sect. 2.3, we begin to sketch the details of the cone correspondence. Suppose M carries a G structure with characteristic connection ∇^c, and that we can define a $\bar{G}$ structure on the cone $(\bar{M}, \bar{g})$ with characteristic connection $\bar{\nabla}^c$. *Then, an important observation is that the lift of* ∇^c *to* $\bar{M}$ *is not the characteristic connection* $\bar{\nabla}^c$ *of the* $\bar{G}$ *structure on* $\bar{M}$! This happens already in the classical case covered by Bär, where the characteristic connection on M is not ∇^g, while the $\bar{G}$ structure on $\bar{M}$ is integrable, and hence, its characteristic connection is equal to the Levi-Civita connection. Rather, we need as an intermediate step another connection ∇ on M with torsion T such that its lift $\bar{\nabla}$ to $\bar{M}$ with torsion $\bar{T}$ is the characteristic connection on

$\bar{M}$ with respect to the given $\bar{G}$ structure. The torsion T measures in some sense the deviation of the $\bar{G}$ structure from the integrable case, i. e. the classical cone correspondence describes the situations where $T = 0$, hence $\bar{T} = 0$ and $\bar{\nabla} = \bar{\nabla}^{\bar{g}}$. Lemma 2.9 describes the exact correspondence between Killing spinors with torsion on M and $\bar{\nabla}$-parallel spinors on $\bar{M}$.

We then describe in detail the cone correspondence with torsion for two particular situations where M is odd-dimensional. Section 3 treats the case when M is a metric almost contact manifold. We construct an almost Hermitian structure on $\bar{M}$, describe explicitly the intermediate connection ∇ and prove that its lift is the characteristic connection of the almost Hermitian structure. We then establish the correspondence between the different classes of structures on M and $\bar{M}$, first through equations (Theorem 3.10) and then in terms of the different classes (3.11). These results synthesize several approaches to the definition of (some) metric almost contact structures through the almost Hermitian structures that they induce on the cone ([43,44]); for normal structures ($N = 0$), the correspondence was proved independently in the recent preprints [38] and [18] (see Remark 3.7 for details). In Sect. 3.3, the spinor correspondence is described in detail. In [9], it was proved that the Tanno deformation of a $(2n + 1)$-dimensional Einstein–Sasaki manifold and that the 5-dimensional Heisenberg group carry Killing spinors with torsion. As an application, we prove in Sect. 3.4 that these spinors lift to spinors on the cone (it turns out to be conformally Kähler) that are parallel with respect to its characteristic connection. Section 3.5 specializes the previous results to metric almost contact 3-structures.

Section 4 is devoted to the case when M is a G_2 manifold. We construct a $\mathrm{Spin}(7)$ structure on its cone, describe explicitly the intermediate connection ∇ and prove again that its lift is the characteristic connection of the $\mathrm{Spin}(7)$ structure. In 4.2, we establish the explicit correspondence between the different classes on M and $\bar{M}$ (Lemma 4.11 and Theorem 4.13); the results are slightly simpler than in the contact case, because the number of classes is smaller. In 4.3, we establish again the details of the spinor correspondence. In Corollary 4.16, we prove by a clever interpretation of the involved equations that the ∇^c-parallel spinor defining the G_2 structure on M lifts to a parallel spinor for the characteristic connection of the $\mathrm{Spin}(7)$ structure on $\bar{M}$—thus, the spinor correspondence turns out to be as neat as one could expect, and the use of the intermediate connection ∇ is not a draw back at all of the construction.

We end this outline with some words about the cone for even-dimensional manifolds M. The most interesting case would be the lift from an almost Hermitian structure on M to a G_2 structure on $\bar{M}$. As described in several recent publications ([17,37,49]), the construction of a G_2 structure requires the use of Hitchin's flow methods, and it is not very transparent how this could be generalized to cones without having to solve a differential equation in the process. Thus, we reserve such thoughts to a separate, upcoming publication.

2 The general construction

2.1 The cone construction

Consider a Riemannian spin manifold (M, g) equipped with a metric connection ∇ with skew symmetric torsion T and connection form ω, meaning that the tensor $g(T(X, Y), Z) = g(\nabla_X Y - \nabla_Y X - [X, Y], Z)$ is skew symmetric. We are interested in real Killing spinors with respect to the given connection, $\nabla_X \psi = \alpha X\psi$ with $\alpha \in \mathbb{R}\backslash\{0\}$. The aim of this Section is to generalize Bär's cone construction [7] for Riemannian Killing spinors, i. e. the case when $\nabla = \nabla^g$. As an intermediate tool, we define a connection $\tilde{\nabla}$ on the spinor bundle by

$$\tilde{\nabla}_X\psi = \nabla_X\psi + \alpha X\cdot\psi,\ \text{with}\ \alpha\in\mathbb{R}\backslash\{0\}.$$

Denote by $\mathcal{C}(\mathbb{R}^n)$ the Clifford algebra of $\mathbb{R}^n$ with respect to the standard negative definite euclidian scalar product, and by Δ_n the spin module of $\mathrm{Spin}(n)$. We consider the Clifford multiplication for $X\in\mathbb{R}^n\subset\mathcal{C}(\mathbb{R}^n)$ in Δ_n. It is the action of an element of $\mathbb{R}^n\subset\mathfrak{spin}(n)\oplus\mathbb{R}^n=\mathfrak{spin}(n+1)\subset\mathcal{C}(\mathbb{R}^n)$ in Δ_n.

Let $P_{\mathrm{SO}(n)}M$ be the $\mathrm{SO}(n)$-principal bundle of frames, ΣM the spinor bundle and $\rho_n:\mathcal{C}(n)\to GL(\Delta_n)$ the representation of the Clifford algebra, i.e. $\rho_{*|\mathfrak{spin}(n)}$ is the $\mathfrak{spin}(n)$ representation. Let $P_{\mathrm{Spin}(n)}M$ be the $\mathrm{Spin}(n)$-principal bundle. For a local section h in $P_{\mathrm{SO}(n)}M$, we identify TM and $P_{\mathrm{SO}(n)}M\times_{SO(n)}\mathbb{R}^n$ via $X=[h,\eta(dh(X))]$, where η is the solder form. The affine connection $\tilde{\nabla}$ induces a connection in the $\mathrm{Spin}(n+1)$-principal bundle $P_{\mathrm{Spin}(n)}M\times_{\mathrm{Spin}(n)}\mathrm{Spin}(n+1)$ as follows. Let

$$\Phi:\ P_{\mathrm{Spin}(n)}M\to P_{\mathrm{SO}(n)},\quad \theta:\ \mathrm{Spin}(n)\to\mathrm{SO}(n)$$

be the usual projections. We look at $\mathfrak{spin}(n+1)\cong\mathfrak{spin}(n)\oplus\mathbb{R}^n\subset\mathcal{C}(n)$, the restriction of ρ_* to $\mathfrak{spin}(n+1)$ and obtain for a local section k in $P_{\mathrm{Spin}(n)}M$ with $\Phi(k)=h$ and $\Sigma M\ni\psi=[k,\sigma]$,

$$\begin{aligned}\tilde{\nabla}_X[k,\sigma] &= \nabla_X[k,\sigma]+\alpha\cdot[h,\eta(dhX)]\cdot[k,\sigma]\\ &= \left[k, d\sigma(X)+\rho_*(\theta_*^{-1}(\omega(dhX))+\alpha\eta(dhX))\sigma\right].\end{aligned}$$

Thus, we get the $\mathfrak{spin}(n+1)$-valued 1-form $\hat{\omega}:=\Phi^*(\theta_*^{-1}\omega+\alpha\eta)$ on $P_{\mathrm{Spin}(n)}M$. We extend $\hat{\omega}$ to $P_{\mathrm{Spin}(n+1)}M$ as follows: for $b\in P_{\mathrm{Spin}(n)}M$, we have $T_bP_{\mathrm{Spin}(n+1)}M=T_bP_{\mathrm{Spin}(n)}M\oplus dL_b(\mathbb{R}^n)$, where $L_b:\mathrm{Spin}(n+1)\to P_{\mathrm{Spin}(n+1)}M,\ g\mapsto b\cdot g$ and define

$$\hat{\omega}(dL_bY):=Y\in\mathbb{R}^n\subset\mathfrak{spin}(n+1).$$

For any $b\in P_{\mathrm{Spin}(n)}M$, we further extend $\hat{\omega}$ in a $\mathrm{Spin}(n+1)$ equivariant way. One checks that the given form is a connection form. It is the connection form of the connection given by $\tilde{\nabla}$. As in [7], we consider the $\mathrm{SO}(n+1)$-principal bundle

$$P_{\mathrm{SO}(n+1)}M\ :=\ P_{\mathrm{SO}(n)}M\times_{\mathrm{SO}(n)}\mathrm{SO}(n+1)$$

and calculate the corresponding connection form $\tilde{\omega}$ given by $\theta_*^{-1}\Phi^*\tilde{\omega}=\hat{\omega}$ for the projections $\Phi:P_{\mathrm{Spin}(n+1)}M\to P_{\mathrm{SO}(n+1)}M$ and $\theta:\mathrm{Spin}(n+1)\to\mathrm{SO}(n+1)$ and get

$$\tilde{\omega}=\begin{bmatrix}\omega & -2\alpha\eta\\ 2\alpha\eta^t & 0\end{bmatrix}.$$

We now consider the cone $(\bar{M},\bar{g})=(M\times\mathbb{R}^+,a^2r^2g+dr^2)$ for some fixed $a>0$ with principal $\mathrm{SO}(n)$-bundle of frames $P_{\mathrm{SO}(n+1)}\bar{M}$, Levi-Civita connection $\bar{\nabla}^{\bar{g}}$ with connection form $\bar{\omega}^{\bar{g}}$ and projection $\pi:\bar{M}\to M$. For simplicity, we will write $X\in TM$ for a lift to $\bar{M}$ of a vector field on M. We define a tensor $\bar{T}$ on $\bar{M}$ from the torsion tensor T of ∇ via

$$\bar{T}(X,Y)\ :=\ T(X,Y)\ \text{ for } X,Y\perp\partial_r,\quad \partial_r\lrcorner\bar{T}=0.$$

We will notationally not distinguish between the $(2,1)$ torsion tensors and the corresponding skew symmetric $(3,0)$ tensors obtained via $g(T(X,Y),Z)$. For the metrics $g,\bar{g}$ on M and $\bar{M}$, we have $a^2r^2T(X,Y,Z)=\bar{T}(X,Y,Z)$ for $X,Y,Z\perp\partial_r$. From $\bar{T}$, we define on $\bar{M}$ the connection

$$\bar{\nabla}\ :=\ \nabla^{\bar{g}}+\frac{1}{2}\bar{T},$$

whose connection form is $\bar{\omega}$. For $p \in M$ and $s \in \mathbb{R}^+$, the tangent bundle of $\bar{M}$ splits into $T_{(p,s)}\bar{M} = T_pM \oplus \mathbb{R}$, where $d\pi(T\bar{M}) = TM$. Thus, for $X \in TM \subset T\bar{M}$, we will write $''X''$ instead of $''d\pi X''$. With a local orthonormal frame $(X_1, \dots, X_n)$ of M, we have an isomorphism of the last two vector bundles given by ($Y \in \mathbb{R}^{n+1}$)

$$\phi : \pi^* \left(\tilde{P}_{\mathrm{SO}(n+1)}M\right) \times_{\mathrm{SO}(n+1)} \mathbb{R}^{n+1} \to T\bar{M}, \quad [(X_1, .., X_n, \partial_r), Y]$$
$$\mapsto \left[\left(\frac{1}{ar}X_1, .., \frac{1}{ar}X_n, \partial_r\right), Y\right].$$

Thus, we can view the connection $\bar{\omega}$ as a connection of $\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)$, which we again call $\bar{\omega}$. We summarize the different principal bundles with corresponding connections and vector bundles in the following table:

Bundle	Connection form	Vector bundle	Manifold
$P_{\mathrm{SO}(n)}M$	ω	TM	M
$\tilde{P}_{\mathrm{SO}(n+1)}M$	$\tilde{\omega}$		M
$\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)$	$\pi^*\tilde{\omega}$	$\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M) \times_{\mathrm{SO}(n+1)} \mathbb{R}^{n+1}$	$\bar{M}$
$P_{\mathrm{SO}(n+1)}\bar{M}$	$\bar{\omega}$	$T\bar{M}$	$\bar{M}$

To determine $\bar{\omega}$ for a local frame $h := (X_1, .., X_n, \partial_r)$ in $\pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M)$, $X \in T\bar{M}$, we need to compute ($Y \in \pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M) \times_{\mathrm{SO}(n+1)} \mathbb{R}^{n+1}$)

$$\phi^{-1}(\bar{\nabla}_X \phi(Y)) = [h, d(\eta(dhY))(X) + \bar{\omega}(dhX)\eta(dhY)].$$

Let $\tilde{h} := (\frac{1}{ar}X_1, .., \frac{1}{ar}X_n, \partial_r)$ be a local frame in $P_{\mathrm{SO}(n+1)}$. For $Y \in TM \subset \pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M) \times_{\mathrm{SO}(n+1)} \mathbb{R}^{n+1}$, we locally have $Y = [h, (Y_1, .., Y_n, 0)^t]$ for functions $Y_i : M \to \mathbb{R}$ and thus $\phi(Y) = [\tilde{h}, (Y_1, .., Y_n, 0)^t]$. Therefore, $ar\phi(Y)$ is independent of r and thus a lift of a vector field on M. Using the O'Neill formulas [42, p. 206], we compute for lifts X, Y of vector fields in TM and the Levi-Civita connection $\bar{\nabla}^{\bar{g}}$ of $\bar{M}$

$$\bar{\nabla}^{\bar{g}}_{\partial_r}\partial_r = 0, \quad \bar{\nabla}^{\bar{g}}_{\partial_r}X = \bar{\nabla}^{\bar{g}}_X\partial_r = \frac{1}{r}X, \quad \bar{\nabla}^{\bar{g}}_X Y = \nabla^g_X Y - \frac{1}{r}\bar{g}(X, Y)\partial_r.$$

Adding the torsion tensor $\bar{T}$, this implies

$$\bar{\nabla}_{\partial_r}\partial_r = 0, \quad \bar{\nabla}_{\partial_r}X = \bar{\nabla}_X\partial_r = \frac{1}{r}X, \quad \bar{\nabla}_X Y = \nabla_X Y - \frac{1}{r}\bar{g}(X, Y)\partial_r.$$

For $X \in T\bar{M}$ and $Y \in TM \subset \pi^*(\tilde{P}_{\mathrm{SO}(n+1)}M) \times_{\mathrm{SO}(n+1)} \mathbb{R}^{n+1}$ we have

$$\phi^{-1}\left(\bar{\nabla}_{\partial_r}\phi(\partial_r)\right) = \phi^{-1}\left(\bar{\nabla}_{\partial_r}\partial_r\right) = 0 \overset{!}{=} \left[h, d((0..0, 1)^t)(\partial_r) + \bar{\omega}(dh\partial_r)(0..0, 1)^t\right]$$
$$= \left[h, \bar{\omega}(dh\partial_r)(0..0, 1)^t\right]$$

and

$$\phi^{-1}\left(\bar\nabla_{\partial_r}\phi(Y)\right)=\phi^{-1}\left(\bar\nabla_{\partial_r}\frac{1}{ar}ar\phi(Y)\right)=\phi^{-1}\left(\frac{1}{ar}\bar\nabla_{\partial_r}ar\phi(Y)+(\partial_r\frac{1}{ar})ar\phi(Y)\right)$$
$$=\phi^{-1}(\frac{1}{ar}\frac{1}{r}(ar\phi(Y))-\frac{1}{ar^2}ar\phi(Y))=0$$
$$\stackrel{!}{=}\left[h,0+\bar\omega(dh\partial_r)(Y_1,..,Y_n,0)^t\right]$$

and thus $\bar\omega(dh\partial_r)=0$. Furthermore, $X=[\tilde h, ar(X_1,..,X_n,0)^t]=[\tilde h, ar\eta(dhX)]$, and we get

$$\phi^{-1}\left(\bar\nabla_X\phi(\partial_r)\right)=\phi^{-1}\left(\bar\nabla_X\partial_r\right)=\phi^{-1}\left(\frac{1}{r}X\right)=\phi^{-1}\left(\left[\tilde h, a\eta(dhX)\right]\right)=[h,a\eta(dhX)],$$

proving $a\eta=\bar\omega\cdot\partial_r$. Since $\phi(Y)=[\tilde h,(Y_1,..,Y_n,0)^t]$, we have $\bar g(X,ar\phi(Y))=a^2r^2\eta(dhX)^t\cdot(Y_1,..,Y_n)^t$. Furthermore we have

$$\nabla_X ar\phi(Y)=\left[\tilde h, ar(d(Y_1,..Y_n,0)^t(X)+ar(\omega(dhX)(Y_1,..,Y_n)^t,0)^t\right]$$

and obtain

$$\phi^{-1}\left(\bar\nabla_X\phi(Y)\right)=\phi^{-1}\left(\frac{1}{ar}\bar\nabla_X ar\phi(Y)\right)=\phi^{-1}\left(\frac{1}{ar}\nabla_X ar\phi(Y)-\frac{1}{ar}\frac{1}{r}\bar g(X,ar\phi(Y))\partial_r\right)$$
$$=\phi^{-1}\left(\left[\tilde h, d(Y_1,..,Y_n,0)^t(X)+\left(\omega(dhX)(Y_1,..,Y_n)^t,0\right)^t-a\eta(dhX)^t(Y_1,..,Y_n,0)^t\right.\right.$$
$$\left.\left.(0,..,0,1)^t\right]\right).$$

Combining all these results yields

$$\bar\omega=\begin{bmatrix}\omega & a\eta\\ -a\eta^t & 0\end{bmatrix}.$$

If one changes the orientation of $\bar M$ (a local SO($\bar M$) frame is then given by $\left(\frac{1}{ar}X_1,...,\frac{1}{ar}\right.$ $X_2,-\partial_r)$), we obtain the alternative connection form

$$\begin{bmatrix}\omega & -a\eta\\ a\eta^t & 0\end{bmatrix}.$$

For a Killing spinor on M with real Killing number α, we thus choose the cone constant $a=-2\alpha$ for $\alpha<0$ and $a=2\alpha$ for $\alpha>0$. Hence, the cone *depends* on the Killing number, and the construction only makes sense if $\alpha\in\mathbb{R}\backslash\{0\}$, as we had assumed from the beginning. In particular, the results cannot be applied to ∇-parallel spinors ($\alpha=0$). The pullback of the connection $\tilde\omega$ under the projection $\pi:\bar M\to M$ is the same as the connection $\bar\omega$ on $\bar M$, and thus, their holonomy groups $Hol(\tilde\omega)$ and $Hol(\bar\omega)$ are the same. Since the second Stiefel-Whitney class of $\bar M=M\times\mathbb{R}$ is given by [53, p.142]

$$w_2(\bar M)=w_2(M)+w_2(\mathbb{R})+w_1(M)\otimes w_1(\mathbb{R}),$$

we conclude that $\bar M$ is spin, since we assumed M to be spin.

Let us now have a closer look at spinors on M and $\bar M$. A parallel spinor of $(\bar M,\bar\omega)$ is the same as a trivial factor of the action of the holonomy group $Hol(\bar\omega)=Hol(\tilde\omega)$ on Δ_{n+1}. A Killing spinor on (M,ω) corresponds to a trivial factor of the action of the same group on the space Δ_n.

For $n = \dim(M)$ odd, the spin representation splits into $\Delta_{n+1} = \Delta_n^+ \oplus \Delta_n^-$. Changing the orientation of $\bar{M}$ (changing from negative to positive α and vice versa) means interchanging Δ_n^+ and Δ_n^-. Thus, a parallel spinor on $\bar{M}$ is either in Δ_n^+ or in Δ_n^-, giving either a Killing spinor with positive or with negative Killing number α.

For n even, we have $\Delta_n = \Delta_{n+1}$ and, by interchanging the orientation, we obtain for any parallel spinor in $\bar{M}$ one Killing spinor with positive and one with negative Killing number α. We summarize these results in the following lemma:

Lemma 2.1 *For a Riemannian spin manifold (M, g) with metric connection ∇ with skew symmetric torsion T, consider the manifold $(\bar{M}, \bar{g})$ with connection $\bar{\nabla}$ with skew symmetric torsion $\bar{T}$ as constructed above. The following correspondence holds:*

- *If $n = \dim(M)$ is odd, any $\bar{\nabla}$-parallel spinor on $\bar{M}$ corresponds to a ∇-Killing spinor on M, with either positive or negative Killing number $\frac{1}{2}a$ or $-\frac{1}{2}a$.*
- *If n is even, any $\bar{\nabla}$-parallel spinor on $\bar{M}$ corresponds to a pair of ∇-Killing spinors on M with Killing number $\pm\frac{1}{2}a$.*

Remark 2.2 For $\dim M$ even, one can write down the bijection between Killing spinors with torsion with Killing numbers $\pm\alpha$ explicitly: If ψ has Killing number α and decomposes into $\psi = \psi_+ + \psi_-$ in the spin bundle $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, then $\psi_+ - \psi_-$ is a Killing spinor with Killing number $-\alpha$. This is the same argument as in the Riemannian case [8, p.121].

Remark 2.3 The careful reader will have noticed that our cone is slightly more general than in [7], where the computations are done for cone constant $a = 1$. This stems from the fact that in the Riemannian case, the Killing number is determined through $n = \dim M$ and Scal^g (remember that the manifold has to be Einstein), and hence, the cone can be normalized in such a way that $a = 1$. For our applications, this is too restrictive.

2.2 G structures and their characteristic connections

Let (M, g) be an oriented Riemannian manifold with Levi-Civita connection ∇^g. By definition, a G structure on M is a G reduction in the frame bundle of M to some closed subgroup $G \subset \mathrm{SO}(n)$. If M admits a metric connection ∇^c with skew symmetric torsion T^c preserving the G structure, it will be called a *characteristic connection*. The following result proves the uniqueness of the characteristic connection in many geometric situations:

Theorem 2.4 ([6, Thm 2.1.]) *Let $G \subsetneq \mathrm{SO}(n)$ be a connected Lie subgroup acting irreducibly on $\mathbb{R}^n$, and assume that G does not act on $\mathbb{R}^n$ by its adjoint representation. Then, the characteristic connection of a G structure on a Riemannian manifold (M, g) is, if existent, unique.*

This applies, for example, to almost Hermitian structures ($\mathrm{U}(n) \subset \mathrm{SO}(2n)$), G_2 structures in dimension 7 and Spin(7) structures in dimension 8 (but not to metric almost contact structures).

Let us introduce the G structures considered in this article.

2.2.1 Metric almost contact structures

Let M be a $n = 2k + 1$ dimensional manifold. Given a Riemannian metric g, a (1,1)-tensor $\phi : TM \to TM$, a 1-form η with dual vector field ξ of length one, and the $(2, 0)$-tensor F defined by $F(v, w) := g(v, \phi(w))$, we call (M, g, ϕ, η) a metric almost contact structure if

$$\phi^2 = -id + \eta \otimes \xi \qquad \text{and} \qquad g(\phi v, \phi w) = g(v, w) - \eta(v)\eta(w).$$

In [11, Thm 4.1.D], D. Blair shows that $\phi(\xi)=0$ and $\eta\circ\phi=0$. Since

$$g(v,\phi(w))=g(\phi(v),\phi^2(w))+\eta(v)\eta(\phi(w)) \;=\; g(\phi(v),-w+\eta(w)\xi)=-g(\phi(v),w),$$

for all $v, w\in TM$, F is actually a 2-form. In terms of the Levi-Civita connection ∇^g on M, the Nijenhuis tensor of a metric almost contact structure is defined by

$$\begin{aligned} N(X,Y,Z) :=& g((\nabla^g_X\phi)(\phi(Y))-(\nabla^g_Y\phi)(\phi(X))+(\nabla^g_{\phi(X)}\phi)(Y)-(\nabla^g_{\phi(Y)}\phi)(X),Z)\\ &+\eta(X)g(\nabla^g_Y\xi,Z)-\eta(Y)g(\nabla^g_X\xi,Z). \end{aligned}$$

The classification of metric almost contact structures is, alas, relatively involved. For future reference, we recall in the following table the exact definition of the different classes of n-dimensional metric almost contact manifolds given by Chinea and Gonzalez [15].

Class	Defining relation
$\mathcal{C}_1$	$(\nabla^g_X F)(Y,Z)=0,\ \nabla^g\eta=0$
$\mathcal{C}_2$	$dF=\nabla^g\eta=0$
$\mathcal{C}_3$	$(\nabla^g_X F)(Y,Z)-(\nabla^g_{\phi X}F)(\phi Y,Z)=0$
$\mathcal{C}_4$	$(\nabla^g_X F)(Y,Z)=-\frac{1}{n-3}[g(\phi X,\phi Y)\delta F(Z)-g(\phi X,\phi Z)\delta F(Y)$ $-F(X,Y)\delta F(\phi Z)+F(X,Z,\delta F(\phi Y)],\quad \delta F(\xi)=0$
$\mathcal{C}_5$	$(\nabla^g_X F)(Y,Z)=\frac{1}{n-1}[F(X,Z)\eta(Y)-F(X,Y)\eta(Z)]\delta\eta$
$\mathcal{C}_6$	$(\nabla^g_X F)(Y,Z)=\frac{1}{n-1}[g(X,Z)\eta(Y)-g(X,Y)\eta(Z)]\delta F(\xi)$
$\mathcal{C}_7$	$(\nabla^g_X F)(Y,Z)=\eta(Z)(\nabla^g_Y\eta)(\phi X)+\eta(Y)(\nabla^g_{\phi X}\eta)(Z),\quad \delta F=0$
$\mathcal{C}_8$	$(\nabla^g_X F)(Y,Z)=-\eta(Z)(\nabla^g_Y\eta)(\phi X)+\eta(Y)(\nabla^g_{\phi X}\eta)(Z),\quad \delta\eta=0$
$\mathcal{C}_9$	$(\nabla^g_X F)(Y,Z)=\eta(Z)(\nabla^g_Y\eta)(\phi X)-\eta(Y)(\nabla^g_{\phi X}\eta)(Z)$
$\mathcal{C}_{10}$	$(\nabla^g_X F)(Y,Z)=-\eta(Z)(\nabla^g_Y\eta)(\phi X)-\eta(Y)(\nabla^g_{\phi X}\eta)(Z)$
$\mathcal{C}_{11}$	$(\nabla^g_X F)(Y,Z)=-\eta(X)(\nabla^g_\xi F)(\phi Y,\phi Z)$
$\mathcal{C}_{12}$	$(\nabla^g_X F)(Y,Z)=\eta(X)\eta(Z)(\nabla^g_\xi\eta)(\phi Y)-\eta(X)\eta(Y)(\nabla^g_\xi\eta)(\phi Z)$

The most important classes are

- $\mathcal{C}_3\oplus ..\oplus\mathcal{C}_8$, the normal structures characterized by $N=0$,
- $\mathcal{C}_6\oplus C_7$, the quasi Sasaki structures: normal structures satisfying $dF=0$,
- $\mathcal{C}_6$, the α-Sasaki structures: normal structures with $\alpha F=d\eta$ for some constant α,
- Sasaki structures: α-Sasaki structures with $\delta F(\xi)=n-1$.

Other classifications we will not consider here are formulated in terms of the Nijenhuis tensor or by considering the direct (not the twisted) product $M\times\mathbb{R}$ ([16] and [44]). It turns out that the tensor $\alpha(X,Y,Z):=(\nabla^g_X F)(Y,Z)$ will be a useful tool for the investigation of metric almost contact structures. It satisfies the general formula

$$\alpha(X,Y,Z)\;=\;-\alpha(X,Z,Y)\;=\;-\alpha(X,\phi Y,\phi Z)+\eta(Y)\alpha(X,\xi,Z)+\eta(Z)\alpha(X,Y,\xi). \tag{1}$$

This implies

$$\begin{aligned}\alpha(X,Y,\phi Y)&=-\alpha(X,\phi Y,\phi^2 Y)+\eta(Y)\alpha(X,\xi,\phi Y)\\ &=-\alpha(X,Y,\phi Y)+2\eta(Y)\alpha(X,\xi,\phi Y),\end{aligned}$$

so we have

$$\alpha(X, Y, \phi Y) = \eta(Y)\alpha(X, \xi, \phi Y). \tag{2}$$

A metric almost contact structure admits a characteristic connection if and only if its Nijenhuis tensor is skew symmetric and ξ is a Killing vector field, and it is then unique [27, Thm 8.2]. If it exists, its torsion tensor is given by

$$T = \eta \wedge d\eta + dF^{\phi} + N - \eta \wedge (\xi \lrcorner N),$$

where $dF^{\phi} := dF \circ \phi$.

We shall now prove a useful criterion for the existence of a characteristic connection.

Lemma 2.5 *A metric almost contact manifold* (M, g, ϕ, η) *admits a characteristic connection if and only if*

$$(\nabla^g_Y F)(Y, \phi X) + (\nabla^g_{\phi Y} F)(Y, X) = 0.$$

Proof There exists a characteristic connection if and only if the Nijenhuis tensor N is skew symmetric and ξ is a Killing vector field. Since we have

$$g(\nabla^g_Y \xi, Z) = -F(\nabla^g_Y \xi, \phi Z) = (\nabla^g_Y F)(\xi, \phi Z) = (\nabla^g_Y \eta)(Z)$$

and $(\nabla^g_X F)(Z, Y) = g((\nabla^g_X \phi)Y, Z)$, the Nijenhuis tensor on M may be written as

$$\begin{aligned} N(X, Y, Z) = {} & \alpha(X, Z, \phi Y) - \alpha(Y, Z, \phi X) + \alpha(\phi X, Z, Y) - \alpha(\phi Y, Z, X) \\ & + \eta(X)\alpha(Y, \xi, \phi Z) - \eta(Y)\alpha(X, \xi, \phi Z). \end{aligned}$$

Thus, N is skew symmetric if

$$\begin{aligned} 0 = N(X, Y, Y) = {} & \alpha(X, Y, \phi Y) - \alpha(Y, Y, \phi X) - \alpha(\phi Y, Y, X) + \eta(X)\alpha(Y, \xi, \phi Y) \\ & - \eta(Y)\alpha(X, \xi, \phi Y). \end{aligned}$$

With Eq. (2), N is skew symmetric if and only if

$$0 = -\alpha(Y, Y, \phi X) - \alpha(\phi Y, Y, X) + \eta(X)\alpha(Y, \xi, \phi Y). \tag{3}$$

ξ is a Killing vector field if $0 = g(\nabla^g_X \xi, Y) + g(\nabla^g_Y \xi, X) = \alpha(X, \xi, \phi Y) + \alpha(Y, \xi, \phi X)$, and this is satisfied if and only if $\alpha(Y, \xi, \phi Y) = 0$. Together with condition (3), we obtain the condition

$$0 = \alpha(Y, Y, \phi X) + \alpha(\phi Y, Y, X).$$

To see that this is also sufficient, set $X = \xi$. □

In analogy to the almost Hermitian and the G_2 case, we set

Definition 2.6 A metric almost contact manifold admitting a characteristic connection is called a *metric almost contact manifold with torsion*.

With the above lemma, we can easily prove

Theorem 2.7 *Consider a metric almost contact manifold* (M, g, ϕ, η)*. If it is of class*

(1) $\mathcal{C}_1 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7$*, there exists a characteristic connection.*
(2) $\mathcal{C}_2$*,* $\mathcal{C}_5$*,* $\mathcal{C}_9$*,* $\mathcal{C}_{10}$*,* $\mathcal{C}_{11}$ *or* $\mathcal{C}_{12}$ *there is no characteristic connection.*
(3) $\mathcal{C}_8$ *there exists a characteristic connection if and only if* ξ *is a Killing vector field.*

Proof We check the different cases:

In $\mathcal{C}_1$, we have $\alpha(X,X,Y)=\alpha(X,Z,\xi)=0$, and we thus get $\alpha(Y,Y,\phi X)+\alpha(\phi Y,Y,X)=0$.

For a structure given by α in the class $\mathcal{C}_2$, we have

$$\alpha(X,Y,Z)+\alpha(Y,Z,X)+\alpha(Z,X,Y)=\alpha(X,Y,\xi)=0,$$

and Eq. (2) yields

$$\begin{aligned}\alpha(Y,Y,\phi X)+\alpha(\phi Y,Y,X)=&\alpha(Y,Y,\phi X)-\alpha(Y,X,\phi Y)-\alpha(X,\phi Y,Y)\\ &=\alpha(Y,Y,\phi X)+\alpha(Y,\phi Y,X)\overset{(1)}{=}-\alpha(Y,\phi Y,\phi^2 X)\\ &\quad+\alpha(Y,\phi Y,X)\\ &=2\alpha(Y,Y,\phi X).\end{aligned}$$

Thus, the condition $\alpha(Y,Y,\phi X)+\alpha(\phi Y,Y,X)=0$ implies $0=\alpha(Y,Y,\phi^2 X)=-\alpha(Y,Y,X)$ since $\alpha(Y,Y,\xi)=0$. Therefore, α has to be also of class $\mathcal{C}_1$, which implies $\alpha=0$.

In $\mathcal{C}_3$, we have $\alpha(X,Y,Z)=\alpha(\phi X,\phi Y,Z)$ and get

$$\begin{aligned}\alpha(Y,Y,\phi X)+\alpha(\phi Y,Y,X)&=\alpha(Y,Y,\phi X)-\alpha(\phi Y,X,Y)\\ &=\alpha(Y,Y,\phi X)-\alpha(\phi^2 Y,\phi X,Y)=\alpha(Y,Y,\phi X)\\ &\quad+\alpha(Y,\phi X,Y)\\ &=0\end{aligned}$$

since $\alpha(\xi,X,Y)=0$ in $\mathcal{C}_1\oplus\ldots\oplus\mathcal{C}_{10}$.

A structure is of class $\mathcal{C}_3\oplus\ldots\oplus\mathcal{C}_8$ if and only if $N=0$, and thus, we just have to check the condition $\alpha(Y,\xi,\phi Y)=0$, which is satisfied in $\mathcal{C}_4$ and $\mathcal{C}_6$.

$\mathcal{C}_5$ is given by the condition $\alpha(X,Y,Z)=\frac{\delta\eta}{n-1}(F(X,Z)\eta(Y)-F(X,Y)\eta(Z))$ such that the condition $\alpha(Y,\xi,\phi Y)=0$ implies $\delta\eta=0$ and thus $\alpha=0$.

For $(c,b)=(1,-1)$ in $\mathcal{C}_7$, $(c,b)=(-1,-1)$ in $\mathcal{C}_8$, $(c,b)=(1,1)$ in $\mathcal{C}_9$ and $(c,b)=(-1,1)$ in $\mathcal{C}_{10}$, we have

$$\alpha(X,Y,Z)=c\eta(Z)\alpha(Y,X,\xi)+b\eta(Y)\alpha(\phi X,\phi Z,\xi)$$

and get $\alpha(X,Y,\xi)=c\alpha(Y,X,\xi)$ and $\alpha(X,\phi Y,\xi)=b\alpha(X,\phi Y,\xi)$, implying $(1-cb)\alpha(Y,\phi Y,\xi)=0$. Thus, in $\mathcal{C}_7$ and $\mathcal{C}_{10}$, the vector field ξ is Killing. Since in $\mathcal{C}_7$ we have $N=0$, we have a characteristic connection here. In $\mathcal{C}_8$, we have a characteristic connection if and only if ξ is Killing. In $\mathcal{C}_9$ and $\mathcal{C}_{10}$, we have $b=1$ and thus

$$\begin{aligned}\alpha(Y,Y,\phi X)+\alpha(\phi Y,Y,X)&=-\eta(Y)\alpha(\phi Y,X,\xi)+c\eta(X)\alpha(Y,\phi Y,\xi)-\eta(Y)\alpha(Y,\phi X,\xi)\\ &=-2\eta(Y)\alpha(\phi Y,X,\xi)+c\eta(X)\alpha(Y,\phi Y,\xi).\end{aligned}$$

For $X=\xi$, the condition $\alpha(Y,Y,\phi X)+\alpha(\phi Y,Y,X)=0$ implies $\alpha(Y,\phi Y,\xi)=0$, and thus, we have $0=\alpha(\phi Y,X,\xi)$ and also $0=\alpha(\phi^2 Y,X,\xi)=-\alpha(Y,X,\xi)$ since $\alpha(\xi,X,Y)=0$. So, we have already $\alpha=0$.

$\mathcal{C}_{11}$ is given by the condition $\alpha(X,Y,Z)=-\eta(X)\alpha(\xi,\phi Y,\phi Z)$, and thus, with $\alpha(\xi,\xi,X)=0$, we get

$$\alpha(Y,Y,\phi X)+\alpha(\phi Y,Y,X)=\eta(Y)\alpha(\xi,\phi Y,X).$$

Because $\alpha(\xi,\phi Y,X)=0$ already implies $\alpha(\xi,Y,X)=0$, we obtain in this case immediately $\alpha=0$.

In $\mathcal{C}_{12}$ we have $\alpha(X, Y, Z) = \eta(X)\eta(Y)\alpha(\xi, \xi, Z) + \eta(X)\eta(Z)\alpha(\xi, Y, \xi)$ and thus $0 = \alpha(Y, Y, \phi X) + \alpha(\phi Y, Y, X) = \eta(Y)^2\alpha(\xi, \xi, \phi X)$ gives us $\alpha = 0$. □

Remark 2.8 The conditions for a metric almost contact structure to admit a characteristic connection in Theorem 2.7 are sufficient but not necessary. In [46], C. Puhle proves that in the case $n = 5$, there are structures of class $\mathcal{C}_{10} \oplus \mathcal{C}_{11}$ (in his class $\mathcal{W}_4$) carrying a characteristic connection. Thus, a structure with characteristic connection is never of pure class $\mathcal{C}_{10}$ nor of class $\mathcal{C}_{11}$, but it can be of mixed class $\mathcal{C}_{10} \oplus \mathcal{C}_{11}$. But more detailed descriptions are possible in some cases. For example, if we set $Y = \xi$, the equation $0 = \alpha(Y, Y, \phi X) + \alpha(\phi Y, Y, X)$ immediately implies that a structure with characteristic connection is of class $\mathcal{C}_1 \oplus ... \oplus \mathcal{C}_{11}$.

2.2.2 Almost Hermitian structures

Let (M, g) be a $2m$-dimensional Riemannian manifold equipped with a $(1, 1)$-tensor

$$J : TM \to TM \quad \text{with} \quad J^2 = -\mathrm{Id}_{TM}, \quad \text{and} \quad g(JX, JY) = g(X, Y).$$

We define a 2-form $\Omega(X, Y) := g(X, JY)$. Then, (M, g, J, Ω) is called an almost Hermitian manifold. In terms of the Levi-Civita connection ∇^g on M, the Nijenhuis tensor of M is defined to be

$$N(X, Y, Z) = g((\nabla^g_X J)(JY), Z) - g((\nabla^g_Y J)(JX), Z) + g((\nabla^g_{JX} J)(Y), Z) - g((\nabla^g_{JY} J)(X), Z).$$

Almost Hermitian structures were classified by Gray and Hervella in [33] into four classes $\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$. An almost Hermitian manifold admits a characteristic connection if and only if it is of class $\chi_1 \oplus \chi_3 \oplus \chi_4$ [27], and it is always unique (either by explicit computation as in [27] or by the general Theorem 2.4); manifolds of class $\chi_1 \oplus \chi_3 \oplus \chi_4$ are sometimes called *Kähler manifolds with torsion*, although they are evidently not Kählerian. Their characteristic torsion is given by (see for example [2])

$$T = N + d\Omega^J,$$

where $d\Omega^J := d\Omega \circ J$. For a nearly Kähler manifold (class χ_1), this connection was first introduced and investigated by A. Gray; on Hermitian manifolds ($N = 0$, i. e. class $\chi_3 \oplus \chi_4$), it is sometimes called the *Bismut connection* [10]. Almost Hermitian manifolds of class χ_4 are locally conformally Kähler manifolds.

2.2.3 G_2 structures

Let $(M, g,)$ be a 7-dimensional oriented Riemannian manifold. M is said to carry a G_2 structure if it admits a reduction to $G_2 \subset \mathrm{SO}(7)$; alternatively, this amounts to the choice of a generic 3-form ϕ. With respect to a local orthonormal frame $e_1, \ldots, e_7$, such a 3-form can locally be written as

$$\phi = e_{123} + e_{145} + e_{167} + e_{246} - e_{147} - e_{347} - e_{356}.$$

Here, subsequently, we do not distinguish between vectors and covectors and abbreviate the k-form $e_{i_1} \wedge .. \wedge e_{i_k}$ as $e_{i_1..i_k}$. G_2 manifolds were classified by Fernández and Gray in [23] into four classes $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.

Friedrich and Ivanov proved that there is a characteristic connection if and only if the structure is of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$; these manifolds are sometimes called G_2 *manifolds* *with torsion* or *G_2T manifolds* for short. In [27], a concrete description of the torsion can be found (we do not need the explicit formula here). We will often used the skew symmetric endomorphism $P(X, .)$ introduced in [23],

$$\phi(X, Y, Z) = g(X, P(Y, Z)).$$

2.2.4 Spin(7) *structures*

In a similar spirit, an 8-dimensional oriented Riemannian manifold (M, g) is called a Spin(7) manifold if it has a reduction to $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, and this is equivalent to the choice of a 4-form Φ, which, in a local frame $e_1, \ldots, e_8$, can be written as

$$\Phi = \phi + *\phi, \text{ and } \phi = e_{1278} + e_{3478} + e_{5678} + e_{2468} - e_{2358} - e_{1458} - e_{1368}.$$

We define a skew symmetric endomorphism $P(X, Y, .)$ on TM via

$$g(P(X, Y, Z), V) = \Phi(X, Y, Z, V).$$

We extend the metric g to 3-forms on TM in the usual way, i.e. $g(W_1 \wedge W_2 \wedge W_3, V_1 \wedge V_2 \wedge V_3) = \det(g(W_i, V_i))$ for $V_i, W_j \in TM$. For 3-forms $\xi = \sum_{i<j<k} \xi_{ijk} e_{ijk}$ and $\eta = \sum_{i<j<k} \eta_{ijk} e_{ijk}$ let $\eta(\xi)$ be defined as

$$\eta(\xi) := \sum_{i<j<k} \xi_{ijk} \eta(e_i, e_j, e_k) = \sum_{i<j<k} \xi_{ijk} \eta_{ijk} = g(\eta, \xi).$$

We define $p(X)$ via

$$g(X, P(\xi)) = g(p(X), \xi)$$

for $X \in TM$ and a 3-form ξ on M ($P(\xi)$ is well defined, since P is totally skew symmetric). Spin(7) manifolds were classified by Fernández in [21]: they split in the two classes $\mathcal{U}_1$ and $\mathcal{U}_2$. S. Ivanov proves in [39] that such a manifold always carries a characteristic connection.

2.3 The cone correspondence for spinors

Let the cone $(\bar{M}, \bar{g})$ over M with Levi-Civita connection $\bar{\nabla}^{\bar{g}}$ carry a $\bar{G}$ structure and assume that there is a connection ∇ on M such that its lift $\bar{\nabla}$ to $\bar{M}$ with torsion $\bar{T}$ is the characteristic connection on $\bar{M}$ with respect to the given $\bar{G}$ structure.

Given a G structure on M, we shall construct an induced $\bar{G}$ structure on $\bar{M}$ in the following sections. We will see that the characteristic connection ∇^c on M (with torsion T^c) does *not* lift to the characteristic connection $\bar{\nabla}$ on $\bar{M}$ (with torsion $\bar{T}$, introduced by a connection ∇ on M with torsion T). In particular, the lift $\overline{T^c}$ of the characteristic torsion to $\bar{M}$ is not the characteristic torsion on $\bar{M}$. So, the tensor $T^c - T$ is not zero and will play an important role in the following.

We want to study the Killing equation with torsion as discussed in [3]: For the family of connections ($s \in \mathbb{R}$)

$$\nabla^s_X Y = \nabla^g_X Y + 2sT^c(X, Y),$$

a spinor ψ is called a *Killing spinor with torsion* if it satisfies the equation

$$\nabla^s_X \psi = \alpha X \psi$$

for some Killing number $\alpha \in \mathbb{R} - \{0\}$ and some value of s. This definition includes the choice that we do not view a parallel spinor ($\alpha = 0$) as a special case of a Killing spinor. A priori, solutions of this equation with $\alpha \in \mathbb{C} - \mathbb{R}$ are conceivable, but we are not aware of any. In any event, the cone construction would not work for such an α.

The case $s = \frac{1}{4}$ corresponds to the characteristic connection; however, there are many geometric situations in which the Killing equation holds for values $s \neq 1/4$. The connection $\bar{\nabla}^s$ on $\bar{M}$ is then given by $\bar{\nabla}^s = \bar{\nabla}^{\bar{g}} + 2s\bar{T}$. We obtain the following correspondence between connections on $\bar{M}$ and connections on M:

Connections on M	Connections on $\bar{M}$
$\nabla^s = \nabla^g + 2sT^c$	$\bar{\nabla}^{\bar{g}} + 2s\overline{T^c} = \bar{\nabla}^s - 2s(\bar{T} - \overline{T^c})$
$\nabla^g + 2sT = \nabla^s + 2s(T - T^c)$	$\bar{\nabla}^s = \bar{\nabla}^{\bar{g}} + 2s\bar{T}$

A direct application of Lemma 2.1 implies:

Lemma 2.9 *For $\alpha \in \mathbb{R} - \{0\}$, we have the following correspondence between*

Spinors on M	Spinors on $\bar{M}$
$\nabla^s_X \psi = \alpha X \psi$	$\bar{\nabla}^s_X \psi - sX \lrcorner (\bar{T} - \overline{T^c})\psi = 0$
$\nabla^s_X \psi + sX \lrcorner (T - T^c)\psi = \alpha X \psi$	$\bar{\nabla}^s_X \psi = 0$

For $\dim(M)$ *odd, there is one spinor on M with either $\alpha = \frac{1}{2}a$ or $\alpha = -\frac{1}{2}a$.*
If $\dim(M)$ *is even, there is a pair of spinors with Killing numbers $\alpha = \pm\frac{1}{2}a$ on M.*
In particular, for $s = \frac{1}{4}$, we obtain the following correspondence:

Spinors on M	Spinors on $\bar{M}$
$\nabla^c_X \psi = \alpha X \psi$	$\bar{\nabla}_X \psi = \frac{1}{4} X \lrcorner (\bar{T} - \overline{T^c})\psi$
$\nabla^c_X \psi = \alpha X \psi - \frac{1}{4} X \lrcorner (T - T^c)\,\psi$	$\bar{\nabla}_X \psi = 0$

In the following sections, we look at the corresponding structures on $\bar{M}$, their classifications and the correspondences of spinors on M and $\bar{M}$.

3 Metric almost contact structures–almost Hermitian structures on the cone

3.1 Preparations

Let (M, g, ϕ, η) be an n-dimensional metric almost contact structure. As in Sect. 2.1, we construct the twisted cone $\bar{M}$ over M and define an almost Hermitian structure J on $\bar{M}$ via

$$J(ar\partial_r) := \xi, \qquad J(\xi) := -ar\partial_r \qquad \text{and} \qquad J(X) = -\phi(X) \text{ for } X \perp \xi, \partial_r.$$

The identity $\phi^2 = -\mathrm{Id} + \eta \otimes \xi$ immediately implies $J^2 = -\mathrm{Id}$.

Definition 3.1 If M admits a characteristic connection ∇^c with skew symmetric torsion T^c satisfying $\nabla^c\phi = \nabla^c\eta = 0$, we define a connection ∇ with skew symmetric torsion T

$$T \; := \; T^c - 2a\eta \wedge F \text{ and thus } \nabla_X Y \; = \; \nabla^c_X Y - a(\eta \wedge F)(X, Y, .).$$

In particular, if the almost metric contact structure is Sasakian and the Killing number happens to satisfy $|\alpha| = 1/2$ (like in the Riemannian case), the cone is constructed with $a = 1$, and thus $T^c = \eta \wedge d\eta = 2a\eta \wedge F$ and $\nabla = \nabla^g$, the Levi-Civita connection. Thus, ∇ and T measure in some sense the difference to the Riemannian Sasakian case.

Although the role of T is clearly exposed in Sect. 2.3, this is not sufficient to determine T completely. Rather, the formula for T has to be found by trying a suitable Ansatz, the motivation for which comes precisely from the Riemannian case just described. Since T is unique, the definition is justified a posteriori by yielding the desired correspondence.

Theorem 3.2 *If (M, g, ϕ, η) is an almost contact metric structure, $(\bar{M}, \bar{g}, J)$ is an almost Hermitian manifold.*

If furthermore M admits a characteristic connection, consider the connection ∇ defined above. Then, the appendant connection $\bar{\nabla}$ on $\bar{M}$ is almost complex, $\bar{\nabla}J = 0$.

Remark 3.3 This shows in particular that $\bar{\nabla}$ is the unique characteristic connection of $\bar{M}$ with respect to J. Furthermore, the theorem includes the claim that the existence of a characteristic connection for the almost contact metric structure on (M, g, ϕ, η) suffices to imply that the induced almost Hermitian structure on $\bar{M}$ does also admit a characteristic connection.

We first prove

Lemma 3.4 *On M, Definition 3.1 implies*

$$(\nabla_Y\phi)X = ag(Y, X)\xi - a\eta(X)Y, \tag{4}$$

and we have

a) $a\phi(X) = -\nabla_X\xi$,
b) *ξ is a Killing vector field, $g(\nabla_Y\xi, X) = -g(\nabla_X\xi, Y)$, and thus, its integral curves are geodesics,*
c) $d\eta = 2aF + \xi\lrcorner T$.

Proof of Lemma 3.4 Using the definition $\nabla = \nabla^c - a\eta \wedge F$ with the equation $\nabla^c\phi = 0$, we directly compute $(\nabla_Y\phi)X = ag(Y, X)\xi - a\eta(X)Y$. Identity (4) and $\phi(\xi) = 0$ imply for $X \in TM$

$$aX - ag(X, \xi)\xi = -(\nabla_X\phi)\xi \;=\; \nabla_X(\phi(\xi)) - (\nabla_X\phi)\xi = \phi(\nabla_X\xi).$$

Since $\nabla_X\xi \perp \xi$, applying ϕ yields

$$a\phi(X) = \phi(aX - ag(X, \xi)\xi) = -\nabla_X\xi.$$

Since $g(X, \phi(Y)) = -g(\phi(X), Y)$, we can conclude from Eq. (4) the statement b) of the lemma, which is also a consequence of Theorem 8.2 in [27]. For $X, Y \in TM$, we obtain with statement a)

$$\begin{aligned} d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]) \;&=\; Xg(Y, \xi) - Yg(X, \xi) - g([X, Y], \xi) \\ &=\; g(\nabla_X Y, \xi) + g(Y, \nabla_X\xi) - g(\nabla_Y X, \xi) - g(X, \nabla_Y\xi) - g([X, Y], \xi) \\ &= T(X, Y, \xi) - g(Y, a\phi(X)) + g(X, a\phi(Y)) = T(X, Y, \xi) + 2aF(X, Y) \end{aligned}$$

which finishes the proof. □

Proof of Theorem 3.2 One easily checks that $\bar{g}(JX, JY) = \bar{g}(X, Y)$ for $X, Y \in T\bar{M}$, and thus, J is an almost Hermitian structure.
We have to show $\bar{\nabla}J = 0$, meaning $0 = \bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X)$. To do so, we distinguish the following cases:
If $X \in TM$, $X \perp \xi$ and $Y \in TM$, we have

$$\begin{aligned}\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) &= -\bar{\nabla}(\phi(X)) - J(\nabla_Y X - \frac{1}{r}\bar{g}(Y, X)\partial_r)\\ &= -\nabla_Y(\phi(X)) + \frac{1}{r}\bar{g}(Y, \phi(X))\partial_r - J(\nabla_Y X) + \frac{1}{ar^2}\bar{g}(Y, X)\xi\\ &= -(\nabla_Y\phi)(X) - \phi(\nabla_Y X) + a^2 r g(Y, \phi(X))\partial_r\\ &\quad - J(\nabla_Y X) + ag(Y, X)\xi.\end{aligned}$$

With identity (4) and since $\eta(X) = 0$, $\phi(\xi) = 0$, we get

$$\begin{aligned}\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) &= -a\eta(X)Y - \phi(\nabla_Y X) + a^2 r g(Y, \phi(X))\partial_r - J(\nabla_Y X)\\ &= -\phi(\nabla_Y X + ag(Y, \phi(X))\xi) - J(ag(Y, \phi(X))\xi + \nabla_Y X),\end{aligned}$$

which is equal to zero if $\nabla_Y X + ag(Y, \phi(X))\xi$ is perpendicular to ξ and ∂_r. Obviously, it is perpendicular to ∂_r. We have $g(\nabla_Y X + ag(Y, \phi(X))\xi, \xi) = 0$ if

$$\begin{aligned}0 &= g(\nabla_Y X, \xi) + g(Y, a\phi(X)) = -g(X, \nabla_Y \xi) + g(Y, a\phi(X))\\ &= g(X, a\phi(Y)) + g(Y, a\phi(X)) = 0.\end{aligned}$$

If $X \in TM$, $X \perp \xi$ and $Y = \partial_r$, we have $\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) = \frac{1}{r}J(X) - J(\frac{1}{r}X) = 0$.
If $X = \xi$, $Y = \partial_r$ we get

$$\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) = \bar{\nabla}_{\partial_r}(-ar\partial_r) - J(\frac{1}{r}\xi) = -a\partial_r - ar\bar{\nabla}_{\partial_r}\partial_r + a\partial_r = 0.$$

Given $X = \xi$ and $Y = \xi$, we have

$$\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) = -\bar{\nabla}_\xi(ar\partial_r) - J\left(\nabla_\xi\xi - \frac{1}{r}\bar{g}(\xi, \xi)\partial_r\right) = -a\xi + a\xi = 0.$$

If $X = \xi$, $Y \in TM$, $Y \perp \xi$, we have

$$\begin{aligned}\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) &= -\bar{\nabla}_Y(ar\partial_r) - J\left(\nabla_Y\xi - \frac{1}{r}\bar{g}(Y, \xi)\partial_r\right) = -aY + J(a\phi(Y))\\ &= -aY + aY = 0.\end{aligned}$$

Given $X = \partial_r$, $Y \perp \xi$, $Y \in TM$, we get

$$\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) = \bar{\nabla}_Y(\frac{1}{ar}\xi) - J(\frac{1}{r}Y) = -\frac{1}{ar}a\phi(Y) - J(\frac{1}{r}Y) = 0.$$

In the case $X = \partial_r$ and $Y = \xi$, we have

$$\begin{aligned}\bar{\nabla}_Y(J(X)) - J(\bar{\nabla}_Y X) &= \bar{\partial}_\xi(\frac{1}{ar}\xi) - J(\frac{1}{r}\xi) = \frac{1}{ar}\nabla_\xi\xi - \frac{1}{ar^2}\bar{g}(\xi, \xi)\partial_r + a\partial_r\\ &= -a\partial_r + a\partial_r = 0.\end{aligned}$$

The last case is given by $X = Y = \partial_r$. Then, we have $\bar{\nabla}_{\partial_r}(\frac{1}{ar}\xi) = -\frac{1}{ar^2}\xi + \frac{1}{ar}\bar{\nabla}_{\partial_r}\xi = 0$. □

Let (M, g) be a Riemannian manifold such that the above-constructed manifold $(\bar{M}, \bar{g})$ carries an almost Hermitian structure J. We have $J(\partial_r) \perp \partial_r$. We consider the manifold $M = M \times \{1\} \subset \bar{M}$ and define for $X \in TM$: $\xi := aJ(\partial_r)$, $\eta(X) := g(X, \xi)$ and $\phi(X) := -J(X) + \bar{g}(J(X), \partial_r)\partial_r$. We get an almost contact structure on M:

$$\begin{aligned}\phi^2(X) &= -J(-J(X) + \bar{g}(J(X), \partial_r)\partial_r) + \bar{g}(J(-J(X) + \bar{g}(J(X), \partial_r)\partial_r), \partial_r)\partial_r \\ &= -X + \bar{g}(X, J(\partial_r))J(\partial_r) = -X + g(X, \xi)\xi = -X + \eta(X)\xi\end{aligned}$$

and

$$\begin{aligned}g(\phi(X), \phi(Y)) &= \frac{1}{a^2}\bar{g}(-J(X) + \bar{g}(J(X), \partial_r)\partial_r, -J(Y) + \bar{g}(J(Y), \partial_r)\partial_r) \\ &= \frac{1}{a^2}(\bar{g}(J(X), J(Y)) - \bar{g}(X, J\partial_r)\bar{g}(Y, J(\partial_r))) \;=\; g(X, Y) - \eta(X)\eta(Y).\end{aligned}$$

Conversely to Theorem 3.2, one proves:

Theorem 3.5 *Consider the manifold $\bar{M}$ equipped with a connection $\bar{\nabla}$ with skew symmetric torsion $\bar{T}$ being the lift of a connection ∇ with torsion T on M. If the connection $\bar{\nabla}$ is almost complex on $\bar{M}$, we have $(\nabla_X \phi)(Y) = ag(X, Y)\xi - a\eta(Y)X$, and thus, the characteristic connection ∇^c on $M = M \times \{1\}$ has torsion $T^c = T + 2a\eta \wedge F$.*

From now on, we assume that M and $\bar{M}$ admit an almost contact structure and an almost Hermitian structure, respectively, both admitting characteristic connections ∇^c and $\bar{\nabla}$ as introduced above.

3.2 The classification of metric almost contact structures and the corresponding classification of almost Hermitian structures on the cone

We look at the classification of the geometric structures on $\bar{M}$ and M. We first prove the following two lemmata.

Lemma 3.6 *The Nijenhuis tensor $\bar{N}$ of the almost Hermitian structure on $\bar{M}$ restricted to TM and the Nijenhuis tensor N of the almost contact structure on M are related via $a^2r^2N = \bar{N}$. Furthermore, the following conditions are equivalent:*

- $\partial_r \lrcorner \bar{N} = 0$,
- $d\eta(X, \phi Y) + d\eta(\phi X, Y) = 0$ *on* TM,
- $\xi \lrcorner N = 0$.

In particular $N = 0$ if and only if $\bar{N} = 0$.

Remark 3.7 In [38], T. Houri, H. Takeuchi and Y. Yasui considered Hermitian manifolds $\bar{M}$ with a vanishing Nijenhuis tensor $\bar{N}$. They showed that in this case, $N = 0$, and thus, M is a normal almost contact manifold, which also is an immediate consequence of Lemma 3.6. In [18], D. Conti and Th. Madsen investigated ‘Sasaki with torsion’ manifolds, meaning normal ($N = 0$) almost contact metric manifolds with ξ a Killing vector field, and their cones/cylinder; they obtained independently the same result as Houri et al.

Remark 3.8 Since $N = 0$ if and only if $\bar{N} = 0$, the condition $\bar{N} = 0$ is sometimes used for the definition of an almost contact metric manifold to be normal (see for example [15]).

Proof of Lemma 3.6 Since we have

$$\bar{g}((\bar{\nabla}^{\bar{g}}_X J)Y, Z) = \bar{g}((\bar{\nabla}_X J)Y + \frac{1}{2}(\bar{J}T(X, Y) - \bar{T}(X, JY)), Z)$$
$$= -\frac{1}{2}(\bar{T}(X, Y, JZ) + \bar{T}(X, JY, Z)),$$

the Nijenhuis tensor of $\bar{M}$ is given by

$$\bar{N}(X, Y, Z) = \bar{g}((\bar{\nabla}^{\bar{g}}_X J)(JY), Z) - \bar{g}((\bar{\nabla}^{\bar{g}}_Y J)(JX), Z) + \bar{g}((\bar{\nabla}^{\bar{g}}_{JX} J)(Y), Z)$$
$$-\bar{g}((\bar{\nabla}^{\bar{g}}_{JY} J)(X), Z)$$
$$= \bar{T}(X, Y, Z) - \bar{T}(JX, JY, Z) - \bar{T}(JX, Y, JZ) - \bar{T}(X, JY, JZ),$$

whereas the Nijenhuis tensor on M is

$$N(X, Y, Z) = g((\nabla^g_X\phi)(\phi(Y)) - (\nabla^g_Y\phi)(\phi(X)) + (\nabla^g_{\phi(X)}\phi)(Y) - (\nabla^g_{\phi(Y)}\phi)(X), Z)$$
$$+\eta(X)g(\nabla^g_Y\xi, Z) - \eta(Y)g(\nabla^g_X\xi, Z).$$

Identity (4) implies

$$g((\nabla^g_X\phi)(Y), Z) = ag(X, Y)\eta(Z) - ag(X, Z)\eta(Y)$$
$$-\frac{1}{2}\,(T(X, \phi(Y), Z) + T(X, Y, \phi(Z)))$$

and hence, we obtain for

$$N(X, Y, Z) = ag(X, \phi(Y))\eta(Z) - \frac{1}{2}T(X, \phi^2(Y), Z)) - \frac{1}{2}T(X, \phi(Y), \phi(Z))$$
$$- ag(Y, \phi(X))\eta(Z) + \frac{1}{2}T(Y, \phi^2(X), Z)) + \frac{1}{2}T(Y, \phi(X), \phi(Z))$$
$$ag(\phi(X), Y)\eta(Z) - ag(\phi(X), Z)\eta(Y) - \frac{1}{2}T(\phi(X), \phi(Y), Z)) - \frac{1}{2}T(\phi(X), Y, \phi(Z))$$
$$- ag(\phi(Y), X)\eta(Z) + ag(\phi(Y), Z)\eta(X) + \frac{1}{2}T(\phi(Y), \phi(X), Z)) + \frac{1}{2}T(\phi(Y), X, \phi(Z))$$
$$+ \eta(X)g(\nabla^c_Y\xi, Z) - \frac{1}{2}\eta(X)T^c(Y, \xi, Z) - \eta(Y)g(\nabla^c_X\xi, Z) + \frac{1}{2}\eta(Y)T^c(X, \xi, Z),$$

which is the same as

$$= T(X, Y, Z) - \frac{1}{2}\eta(Y)T(X, \xi, Z) - \frac{1}{2}\eta(X)T(\xi, Y, Z) - T(X, \phi(Y), \phi(Z))$$
$$- T(\phi(X), Y, \phi(Z)) - T(\phi(X), \phi(Y), Z) - a\eta(Y)g(\phi(X), Z) + a\eta(X)g(\phi(Y), Z)$$
$$- \frac{1}{2}\eta(X)T(Y, \xi, Z) - \eta(X)aF(Z, Y) + \frac{1}{2}\eta(Y)T(X, \xi, Z) + \eta(Y)aF(Z, X).$$

For $X \in TM$, we have $J(X) + \eta(X)ar\partial_r = J(X - \eta(X)\xi) = -\phi(X - \eta(X)\xi) = -\phi(X)$. Since $\partial_r \lrcorner \bar{T} = 0$ for $X, Y, Z \in TM$, we get

$$\bar{T}(J(X), Y, Z) = -a^2r^2T(\phi(X), Y, Z) \tag{5}$$

and also $\bar{T}(J(X), J(Y), Z) = a^2r^2T(\phi(X), \phi(Y), Z)$ etc. With this result, we have

$$N(X, Y, Z) = \frac{1}{a^2r^2}(\bar{T}(X, Y, Z) - \bar{T}(JX, JY, Z) - \bar{T}(JX, Y, JZ) - \bar{T}(X, JY, JZ))$$

and thus, we get the desired result $\bar{N}(X, Z, Z) = a^2r^2N(X, Y, Z)$ for $X, Y, Z \in TM$.

By definition of the Nijenhuis tensor, we have $\partial_r \lrcorner \bar{N} = 0$ if and only if for $X, Y \in TM$

$$0 = \bar{T}(\xi, JX, Y) + \bar{T}(\xi, X, JY) \Longleftrightarrow 0 = T(\xi, \phi X, Y) + T(\xi, X, \phi Y).$$

The relations $\xi \lrcorner T = d\eta - 2aF$ and

$$F(\phi X, Y) + F(X, \phi Y) = g(\phi X, \phi Y) + g(X, \phi^2 Y) = 0$$

imply that $\partial_r \lrcorner \bar{N} = 0$ holds if and only if $d\eta(\phi X, Y) + d\eta(X, \phi Y) = 0$. In [27], the identity $N(X, Y, \xi) = d\eta(X, Y) - d\eta(\phi X, \phi Y)$ is proved, and we get

$$N(\phi X, Y, \xi) = d\eta(\phi X, Y) + d\eta(X, \phi Y) - \eta(X)d\eta(\xi, \phi Y).$$

The identity $\xi \lrcorner T = d\eta - 2aF$ implies $\xi \lrcorner d\eta = 0$, and thus, $d\eta(\phi X, Y) + d\eta(X, \phi Y) = 0$ if and only if $N(\phi X, Y, \xi) = 0$. Since N is skew symmetric, we have $N(\xi, Y, \xi) = 0$, and thus, $N(\phi X, Y, \xi) = 0$ is equivalent to $\xi \lrcorner N = 0$. □

Lemma 3.9 *For $Z \in T\bar{M}$ let Z_M be the projection of Z onto TM. Then, we have*

$$\delta\Omega(Z) = -(\delta F - a(n-1)\eta)(Z_M).$$

Proof For $X, Y, Z \in T\bar{M}$ we have

$$\begin{aligned}(\bar{\nabla}^{\bar{g}}_X \Omega)(Y, Z) &= (\bar{\nabla}_X \Omega)(Y, Z) - \Omega(-\frac{1}{2}\bar{T}(X, Y), Z) - \Omega(Y, -\frac{1}{2}\bar{T}(X, Z)) \\ &= \frac{1}{2}(\bar{T}(X, JY, Z) + \bar{T}(X, Y, JZ)).\end{aligned}$$

For a local ONB $\{e_1, .., e_n = \xi\}$ of TM, we get the local ONB $\{\bar{e}_1 = \frac{1}{ar}e_1, .., \bar{e}_n = \frac{1}{ar}e_n, \bar{e}_{n+1} = \partial_r\}$ of $T\bar{M}$. In this basis and for $Z \in T\bar{M}$, we compute

$$\begin{aligned}\delta\Omega(Z) = -\sum_{i=1}^{n+1}(\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Omega)(\bar{e}_i, Z) &= -\frac{1}{2}\sum_{i=1}^{n-1}\bar{T}(\frac{1}{ar}e_i, \frac{1}{ar}Je_i, Z) - \frac{1}{2}\bar{T}(\frac{1}{ar}\xi, -\partial_r, Z) \\ &\quad -\frac{1}{2}\bar{T}(\partial_r, J\partial_r, Z).\end{aligned}$$

Since $\partial_r \lrcorner \bar{T} = 0$, with Eq. (5) and the fact that $\phi(e_n) = 0$, we have

$$\begin{aligned}\delta\Omega(Z) &= \frac{1}{2}\sum_{i=1}^{n-1}T(e_i, \phi e_i, Z_M) = \frac{1}{2}\sum_{i=1}^{n-1}(T^c(e_i, \phi e_i, Z_M) - 2a(\eta \wedge F)(e_i, \phi e_i, Z_M)) \\ &= \frac{1}{2}\sum_{i=1}^{n-1}(T^c(e_i, \phi e_i, Z_M) - 2a\eta(Z_M)F(e_i, \phi e_i)) \\ &= \frac{1}{2}\sum_{i=1}^{n}T^c(e_i, \phi e_i, Z_M) + a\eta(Z_M)(n-1) \\ &= -(\delta F - a(n-1)\eta)(Z_M),\end{aligned}$$

finishing the proof. □

We consider the Gray-Hervella classification [33] of almost Hermitian structures, given in Sect. 2.2. Since we want to work with characteristic connections, we will only consider structures of class $\chi_1 \oplus \chi_3 \oplus \chi_4$. We first translate the conditions of this classification for the almost Hermitian structure on $\bar{M}$ to conditions of the almost contact structure on M. For the discussion of the classification of almost contact structures and the correspondences to the classification of almost Hermitian structures, see Theorem 3.11.

Theorem 3.10 *We have the following correspondence between Gray-Hervella classes of almost Hermitian structures on the cone $\bar{M}$ and defining relations of almost contact metric structures on M:*

Class of $\bar{M}$	Defining relation on $\bar{M}$	Corresponding relation on M
Kähler	$\bar{\nabla}^{\bar{g}} J = 0$	$(\nabla^g_X F)(Y, Z) = a\eta(Y)g(X, Z)$ $-a\eta(Z)g(X, Y)$
χ_3	$\delta\Omega = \bar{N} = 0$	$N = 0$, $\delta F = a(n-1)\eta$
χ_4	$(\bar{\nabla}^{\bar{g}}_X \Omega)(Y, Z) = \frac{-1}{n-1}[\bar{g}(X, Y)\delta\Omega(Z)$ $-\bar{g}(X, Z)\delta\Omega(Y) - \bar{g}(X, JY)\delta\Omega(JZ)$ $+\bar{g}(X, JZ)\delta\Omega(JY)]$	$(\nabla^g_X F)(Y, Z) = \frac{\delta F(\xi)}{n-1}(g(X, Z)\eta(Y)$ $-g(X, Y)\eta(Z))$
$\chi_1 \oplus \chi_3$	$\delta\Omega = 0$	$\delta F = a(n-1)\eta$
$\chi_3 \oplus \chi_4$	$\bar{N} = 0$	$N = 0$

Furthermore, a structure on $\bar{M}$ is never nearly Kähler (of class χ_1) nor of mixed class $\chi_1 \oplus \chi_4$.

Proof We have

$$a\eta(Y)g(X, Z) - a\eta(Z)g(X, Y) = a\eta \wedge F(X, Y, \phi Z) + a\eta \wedge F(X, \phi Y, Z).$$

Kähler case: Since the characteristic connection on $\bar{M}$ is unique, we have the following equivalences

$$\bar{\nabla}^{\bar{g}} J = 0 \Leftrightarrow \bar{\nabla}^{\bar{g}} = \bar{\nabla} \Leftrightarrow \bar{T} = 0 \Leftrightarrow T = 0 \Leftrightarrow T^c = 2a\eta \wedge F.$$

For a metric connection $\tilde{\nabla}$ with skew symmetric torsion $\tilde{T}$ on M, one calculates

$$(\tilde{\nabla}_X F)(Y, Z) \;=\; (\nabla^g_X F)(Y, Z) - \frac{1}{2}\tilde{T}(X, \phi Y, Z) - \frac{1}{2}\tilde{T}(X, Y, \phi Z).$$

Thus, $T^c = 2a\eta \wedge F$ implies $(\nabla^g_X F)(Y, Z) = a\eta \wedge F(X, Y, \phi Z) + a\eta \wedge F(X, \phi Y, Z)$ and conversely the condition $(\nabla^g_X F)(Y, Z) = a\eta \wedge F(X, Y, \phi Z) + a\eta \wedge F(X, \phi Y, Z)$ yields

$$(\tilde{\nabla}_X F)(Y, Z) = (a\eta \wedge F - \frac{1}{2}\tilde{T})(X, \phi Y, Z) + (a\eta \wedge F - \frac{1}{2}\tilde{T})(X, Y, \phi Z).$$

The uniqueness of the characteristic connection ∇^c on M thus implies $T^c = 2a\eta \wedge F$.

Case χ_3: Consider an almost Hermitian structure on $\bar{M}$ of class χ_3 defined by $\delta\Omega = \bar{N} = 0$. With Lemma 3.6 and 3.9, we have $\bar{N} = \delta\Omega = 0$ if and only if $N = 0$ and $\delta F - a(n-1)\eta = 0$.

Case χ_4: The defining relation for the class χ_4 of an almost Hermitian manifold $\bar{M}$

$$(\bar{\nabla}^{\bar{g}}_X \Omega)(Y, Z) = \frac{-1}{n-1}[\bar{g}(X, Y)\delta\Omega(Z) - \bar{g}(X, Z)\delta\Omega(Y) - \bar{g}(X, JY)\delta\Omega(JZ) \\ +\bar{g}(X, JZ)\delta\Omega(JY)]$$

translates with Lemma 3.9 for $X, Y, Z \in T\bar{M}$ into

$$\frac{1}{2}\bar{T}(X, Y, JZ) + \frac{1}{2}\bar{T}(X, JY, Z) = \\ \frac{1}{n-1}[\bar{g}(X, Y)(\delta F(Z_M) - a(n-1)\eta(Z_M)) - \bar{g}(X, Z)(\delta F(Y_M) - a(n-1)\eta(Y_M)) \\ - \bar{g}(X, JY)(\delta F((JZ)_M) - a(n-1)\eta((JZ)_M)) + \bar{g}(X, JZ)(\delta F((JY)_M) \\ - a(n-1)\eta((JY)_M))].$$

For $X \in T\bar{M}$, we have $\bar{g}(\partial_r, JX) = -\bar{g}(J\partial_r, X) = -\bar{g}(J\partial_r, X_M) = -a^2r^2 g(\frac{1}{ar}\xi, X_M) = -ar\eta(X_M)$, and for $X \in TM$, we have $(JX)_M = -\phi X$.

In the case where $X = \partial_r$ and $Y, Z \in TM$, the defining relation is equivalent to

$$0 = ar\eta(Y)(\delta F(-\phi Z) - a(n-1)\eta(-\phi Z)) - ar\eta(Z)(\delta F(-\phi Y) - a(n-1)\eta(-\phi Y)).$$

This is satisfied if and only if $0 = (\eta(Y)\delta F(\phi Z) - \eta(Z)\delta F(\phi Y)) = \eta \wedge (\delta F \circ \phi)(Y, Z)$. Taking $Y = \xi$, we receive the condition $F \circ \phi = 0$, which obviously is sufficient too.

If $X = Y = \partial_r$, $Z \in TM$, the defining relation leads to

$$0 = \delta F(Z) - a(n-1)\eta(Z) - ar\eta(Z)(\delta F(\frac{1}{ar}\xi) - \frac{n-1}{r}),$$

which is the same as $0 = \delta F(Z) - \eta(Z)\delta F(\xi) = -\delta F(\phi^2 Z)$, already being satisfied if $\delta F \circ \phi = 0$.

The case $Y = Z = \partial_r$ leads to $0 = 0$.

Given $Y = \partial_r$ and $X, Z \in TM$, we get

$$\frac{1}{2ar}\bar{T}(X, \xi, Z) = \frac{1}{n-1}[ar\eta(X)\delta F(-\phi(Z)) - a^2r^2 F(X, Z)(\delta F(\frac{1}{ar}\xi) - a(n-1)\frac{1}{ar})].$$

Since we already have the condition $\delta F \circ \phi = 0$, this is equivalent to

$$d\eta(X, Z) - 2aF(X, Z) = (\xi \lrcorner T)(X, Z) = \frac{2}{n-1}F(X, Z)(\delta F(\xi) - a(n-1)).$$

This is the same as $d\eta = \frac{2}{n-1}\delta F(\xi)F$. At last ,we look at $X, Y, Z \in TM$. Again, we already have $\delta F \circ \phi = 0$

$$-\frac{1}{2}T(X, Y, \phi Z) - \frac{1}{2}T(X, \phi Y, Z) \\ = \frac{1}{n-1}[g(X, Y)(\delta F(Z) - a(n-1)\eta(Z)) - g(X, Z)(\delta F(Y) - a(n-1)\eta(Y))] \\ = g(X, Y)(\frac{\delta F}{n-1} - a\eta)(Z) - g(X, Z)(\frac{\delta F}{n-1} - a\eta)(Y).$$

Furthermore, we have

$$
\begin{aligned}
&-\frac{1}{2}(T(X,Y,\phi Z)+T(X,\phi Y,Z))\\
&\quad=-\frac{1}{2}\,T^c(X,Y,\phi Z)-\frac{1}{2}T^c(X,\phi Y,Z)+a\eta(X)F(Y,\phi Z)+a\eta(Y)F(\phi Z,X)\\
&\qquad+a\eta(X)F(\phi Y,Z)+a\eta(Z)F(X,\phi Y)\\
&\quad=-\,(\nabla^g_X F)(Y,Z)+a\eta(Y)g(\phi Z,\phi X)+a\eta(Z)g(X,\phi^2 Y)\\
&\quad=-\,(\nabla^g_X F)(Y,Z)+a\eta(Y)g(Z,X)-a\eta(Z)g(X,Y).
\end{aligned}
$$

Thus, we get the equation

$$(\nabla^g_X F)(Y,Z)=g(X,Z)\frac{\delta F}{n-1}(Y)-g(X,Y)\frac{\delta F}{n-1}(Z).$$

Since $\delta F\circ\phi=0$, we have $\delta F=\delta F(\xi)\eta$ and obtain

$$
\begin{aligned}
(\nabla^g_X F)(Y,Z)\eta(Y)-g(X,Y)(\frac{\delta F(\xi)}{n-1}+2a)\eta(Z)\\
=\frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y)-g(X,Y)\eta(Z)).
\end{aligned}
$$

We summarize this result: an almost Hermitian structure on $\bar{M}$, given by an almost contact structure on M, is of class χ_4 if and only if

$$(\nabla^g_X F)(Y,Z)=\frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y)-g(X,Y)\eta(Z)),\quad \delta F\circ\phi=0\ \text{ and }\ d\eta=2\frac{\delta F(\xi)}{n-1}F.$$

The first condition implies the others: for some local orthonormal basis $e_1,\dots,e_n=\xi$ of TM, we have

$$
\begin{aligned}
\delta F(X)&=-\sum_{i=1}^{n}(\nabla^g_{e_i}F)(e_i,X)=-\sum_{i=1}^{n}\frac{\delta F(\xi)}{n-1}(g(e_i,X)\eta(e_i)-\eta(X))\\
&=-\frac{\delta F(\xi)}{n-1}(-n\eta(X)+\eta(X))=\delta F(\xi)\eta(X)
\end{aligned}
$$

and thus, the condition $(\nabla^g_X F)(Y,Z)=\frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y)-g(X,Y)\eta(Z))$ implies $\delta F\circ\phi=0$. Since ξ is a Killing vector field and thus $(\nabla^g_X F)(\xi,\phi Y)=-F(\nabla^g_X\xi,\phi Y)=g(\nabla^g_X\xi,Y)$ is skew symmetric in X and Y, we have

$$d\eta(X,Y)=(\nabla^g_X\eta)(Y)-(\nabla^g_Y\eta)(X)=(\nabla^g_X F)(\xi,\phi Y)-(\nabla^g_Y F)(\xi,\phi X)=2(\nabla^g_X F)(\xi,\phi Y)$$

and with condition $(\nabla^g_X F)(Y,Z)=\frac{\delta F(\xi)}{n-1}(g(X,Z)\eta(Y)-g(X,Y)\eta(Z))$, we already get $d\eta=2\frac{\delta F(\xi)}{n-1}F$.

Case $\chi_1\oplus\chi_3$: The condition for a structure of class $\chi_1\oplus\chi_3$ can be obtained directly from Lemma 3.9.

Case $\chi_3\oplus\chi_4$: An almost Hermitian structure on $\bar{M}$ is of class $\chi_3\oplus\chi_4$ if and only if $\bar{N}=0$. Due to Lemma 3.6, this is equivalent to $N=0$.

Case $\chi_1 \oplus \chi_4$: The condition for an almost Hermitian structure to be of class $\chi_1 \oplus \chi_4$ is the same as for the class χ_4, setting $X = Y$:

$$\begin{aligned}\frac{1}{2}\bar{T}(X, JX, Y) = \frac{1}{n-1}[&\bar{g}(X, X)(\delta F(Y_M) - a(n-1)\eta(Y_M))\\ &- \bar{g}(X, Y)(\delta F(X_M) - a(n-1)\eta(X_M))\\ &+ \bar{g}(X, JY)(\delta F((JX)_M) - a(n-1)\eta((JX)_M))].\end{aligned}$$

The equation is still linear in Y but not in X. We set $X = V + b\partial_r$ for $b \in \mathbb{R}$ and $V \in TM$:

$$\begin{aligned}&\frac{1}{2}\bar{T}(V, JV, Y) + \frac{b}{2ar}\bar{T}(V, \xi, Y)\\ &\quad= \frac{1}{n-1}[(b^2 + a^2r^2 g(V, V))(\delta F(Y_M) - a(n-1)\eta(Y_M))\\ &\qquad - (b\bar{g}(\partial_r, Y) + a^2r^2 g(V, Y_M))(\delta F(V) - a(n-1)\eta(V))\\ &\qquad + (\bar{g}(V, JY) - bar\eta(Y_M))(-\delta F(\phi V) + \frac{b}{ar}\delta F(\xi) - \frac{b(n-1)}{r})].\end{aligned}$$

This is satisfied for any b if and only if

$$\begin{aligned}\frac{1}{2}\bar{T}(V, JV, Y) =&\frac{1}{n-1}[a^2r^2 g(V, V)(\delta F(Y_M) - a(n-1)\eta(Y_M))\\ &- a^2r^2 g(V, Y_M)(\delta F(V) - a(n-1)\eta(V)) + \bar{g}(V, JY)(-\delta F(\phi V))]\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2ar}\bar{T}(V, \xi, Y) = \frac{1}{n-1}[&-\bar{g}(\partial_r, Y)(\delta F(V) - a(n-1)\eta(V))\\ &+\bar{g}(V, JY)(\frac{\delta F(\xi)}{ar} - \frac{(n-1)}{r}) + ar\eta(Y_M)\delta F(\phi V)]\end{aligned} \tag{6}$$

and

$$0 = \delta F(Y_M) - \eta(Y_M)\delta F(\xi) = \delta F(Y_M - \eta(Y_M)\xi) = -\delta F(\phi^2(Y_M)),$$

where the last equation is satisfied if and only if $\delta F \circ \phi = 0$.

For $Y \in TM$ with the condition $\delta F \circ \phi = 0$, equation (6) leads to

$$\frac{1}{2}T(\xi, V, Y) = F(V, Y)(\frac{\delta F(\xi)}{n-1} - a).$$

Since $\xi \lrcorner T = d\eta - 2aF$, we have $d\eta = 2\frac{\delta F(\xi)}{n-1}F$ and thus $dF = 0$.

With Theorem 8.4 in [27], this implies $N = 0$ and the structure is already of class χ_4. Thus, a structure is never of class χ_1 or of mixed class $\chi_1 \oplus \chi_4$. □

We now compare the result of Theorem 3.10 with the 12 classes of almost contact structures given in Sect. 2.2. As in the whole article, we just consider manifolds admitting a characteristic connections (recall that Theorem 2.7 formulates the criterion for its existence).

Theorem 3.11 *If the almost Hermitian structure on $\bar{M}$ is*

- *of class χ_3, then the almost contact structure on M is of class $\mathcal{C}_3 \oplus .. \oplus \mathcal{C}_8$ but not of class $\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ or of class $\mathcal{C}_6$.*
- *of class $\chi_1 \oplus \chi_3$, then the almost contact structure on M is not of class $\mathcal{C}_1 \oplus .. \oplus \mathcal{C}_5 \oplus \mathcal{C}_7 \oplus .. \oplus \mathcal{C}_{12}$ nor of class $\mathcal{C}_6$.*

The almost Hermitian structure on $\bar{M}$ is

- *Kähler if and only if the almost contact structure on M is α-Sasaki (of class $\mathcal{C}_6$) and $\delta F(\xi) = a(n-1)$.*
- *of class χ_4 if and only if the almost contact structure on M is an α-Sasaki structure.*
- *of class $\chi_3 \oplus \chi_4$ if and only if the almost contact structure on M is of class $\mathcal{C}_3 \oplus .. \oplus \mathcal{C}_8$ and there exists a characteristic connection.*

Furthermore, the structure on M is Sasaki if and only if the almost Hermitian structure on $\bar{M}$ is of class χ_4 with $\delta\Omega(\xi) = (a-1)(n-1)$.

Proof If the structure on $\bar{M}$ is of class χ_3, we have $N = 0$, and thus, the structure on M is of class $\mathcal{C}_3 \oplus .. \oplus \mathcal{C}_8$. Furthermore, $\delta F(\xi) = a(n-1)$ holds, but on $\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ we have $\delta F(\xi) = 0$ and a structure on M of class $\mathcal{C}_6$ implies a structure on $\bar{M}$ of class χ_4.

A structure on $\bar{M}$ of class $\chi_1 \oplus \chi_3$ implies on M the relation $\delta F(\xi) \neq 0$, but on $\mathcal{C}_1 \oplus .. \oplus \mathcal{C}_5 \oplus \mathcal{C}_7 \oplus .. \oplus \mathcal{C}_{12}$, we have $\delta F(\xi) = 0$ and again a structure on M of class $\mathcal{C}_6$ implies a structure on $\bar{M}$ of class χ_4.

With Theorem 3.10, a structure on $\bar{M}$ is Kählerian if and only if $(\nabla^g_X F)(Y, Z) = a\eta(Y)g(X, Z) - a\eta(Z)g(X, Y)$ holds on M, which is equivalent for the almost contact structure to be of class $\mathcal{C}_6$ with $\delta F(\xi) = a(n-1)$.

The condition of Theorem 3.10 for a structure of class χ_4 on M is equivalent to the definition of an almost contact structure on $\bar{M}$ to be of class $\mathcal{C}_6$.

In $\mathcal{C}_3 \oplus .. \oplus \mathcal{C}_8$, we have $N = 0$, which together with the existence of a characteristic connection is equivalent to the property that the structure on $\bar{M}$ is of class $\chi_3 \oplus \chi_4$.

A structure on M is Sasaki if and only if it is of class $\mathcal{C}_6$ and $\delta F(\xi) = n - 1$. Due to Theorem 3.10, this is equivalent to the condition for the structure on $\bar{M}$ to be of class χ_4 with $\delta\Omega(\xi) = (a-1)(n-1)$. □

Remark 3.12 If we construct $\bar{M}$ with $a = 1$, we obtain a Kählerian structure, and $(\nabla^g_X F)(Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y)$ defines a Sasakian structure on M. This is the classical case treated by Bär in [7].

3.3 Corresponding spinors on metric almost contact structures and their cones

We shall now work out in detail the abstract spinor correspondence stated in Lemma 2.9 for the case that M carries a metric almost contact structure. The following result serves as a preparation.

Lemma 3.13 *Given a metric almost contact structure with characteristic connection on M, the lift of $\eta \wedge F$ to its cone $\bar{M}$ is given by*

$$\frac{1}{a^3 r^3}(\partial_r \lrcorner \Omega) \wedge \Omega.$$

Proof Since $\partial_r \lrcorner [\frac{1}{a^3 r^3}(\partial_r \lrcorner \Omega) \wedge \Omega] = 0$, we just need to show the equality on TM. For $X, Y \in TM$, we have

$$F(X, Y) = g(X, \phi Y) = -\frac{1}{a^2 r^2}\bar{g}(X, JY + \eta(Y)ar\partial_r) = -\frac{1}{a^2 r^2}\Omega(X, Y)$$

and

$$\eta(X) = g(X, arJ\partial_r) = \frac{1}{ar}\Omega(X, \partial_r)$$

which proves $F = -\frac{1}{a^2 r^2}\Omega$ and $\eta = -\frac{1}{ar}\partial_r \lrcorner \Omega$ on TM. □

We recall the definition of the connections

$$\nabla^s_X Y = \nabla^g_X Y + 2sT^c(X,Y) \text{ and } \bar{\nabla}^s_X Y = \bar{\nabla}^{\bar{g}}_X Y + 2s\bar{T}(X,Y)$$

for $s \in \mathbb{R}$ from Sect. 2.1. Theorem 3.2 yields $T^c = T + 2a\eta \wedge F$, and since $\bar{T} = a^2r^2T$ and $\overline{T^c} = a^2r^2T^c$, we get $\overline{T^c} - \bar{T}$ as the lift of $2a^3r^2\eta \wedge F$ to $\bar{M}$. With Lemma 3.13, we obtain $\overline{T^c} - \bar{T} = \frac{2}{r}(\partial_r \lrcorner \Omega) \wedge \Omega$.

Theorem 3.14 *Assume that the almost contact metric manifold (M, g, ϕ, η) admits a characteristic connection and is spin. Then, there is for $\alpha = \frac{1}{2}a$ or $\alpha = -\frac{1}{2}a$:*

(1) A one to one correspondence between Killing spinors with torsion

$$\nabla^s_X \psi = \alpha X \psi$$

on M and parallel spinors of the connection $\bar{\nabla}^s + \frac{4s}{r}(\partial_r \lrcorner \Omega) \wedge \Omega$ on $\bar{M}$ with cone constant a

$$\bar{\nabla}^s_X \psi + \frac{2s}{r}(X \lrcorner (\partial_r \lrcorner \Omega) \wedge \Omega)\psi = 0,$$

(2) A one to one correspondence between $\bar{\nabla}^s$-parallel spinors on $\bar{M}$ with cone constant a and spinors on M satisfying

$$\nabla^s_X \psi - 2asX \lrcorner (\eta \wedge F)\psi = \alpha X \psi.$$

In particular, for $s = \frac{1}{4}$, we get the correspondence

Spinors on M	Spinors on $\bar{M}$
$\nabla^c_X \psi = \alpha X \psi$	$\bar{\nabla}_X \psi = -\frac{1}{2r} X \lrcorner ((\partial_r \lrcorner \Omega) \wedge \Omega)\psi$
$\nabla^c_X \psi = \alpha X \psi + \frac{a}{2} X \lrcorner (\eta \wedge F)\psi$	$\bar{\nabla}_X \psi = 0$

Remark 3.15 Since $\bar{\nabla} = \bar{\nabla}^{\bar{g}} + \frac{1}{2}\bar{T}$ is the characteristic connection of the almost Hermitian structure on $\bar{M}$, we can write

$$\bar{T} = \bar{N} + d\Omega^J,$$

where $d\Omega^J = d\Omega \circ J$. Thus, one can rewrite all equations above. For example, the correspondence (1) of Theorem 3.14 is given with spinors on $\bar{M}$ satisfying

$$\bar{\nabla}^{\bar{g}}_X \psi + sX \lrcorner [\bar{N} + d\Omega^J + \frac{2}{r}(\partial_r \lrcorner \Omega) \wedge \Omega]\psi = 0.$$

Equivalently, one can use the description of T^c on M given by $T^c = \eta \wedge d\eta + dF^\phi + N - \eta \wedge (\xi \lrcorner N)$ ([27]) to rewrite the second correspondence. Note that this also implies that $\bar{T} = \bar{N} + d\Omega^J$ is the lift of

$$a^2r^2T = a^2r^2(T^c - 2a\eta \wedge F) = a^2r^2(\eta \wedge (d\eta - 2aF) + dF^\phi + N - \eta \wedge (\xi \lrcorner N))$$

to $\bar{M}$, in particular we have $\partial_r \lrcorner (\bar{N} + d\Omega^J) = 0$.

3.4 Examples

In this Section, we shall discuss several examples of metric almost contact structures and the special spinor fields that exist on them and on their cones. In particular, we shall describe several situations where the cone carries a parallel spinor field for the characteristic connection $\bar{\nabla}$ of its almost Hermitian structure.

Example 3.16 For a metric almost contact manifold (M, g, ϕ, η), the deformation

$$g_t := tg + (t^2 - t)\eta \otimes \eta, \quad \xi_t := \frac{1}{t}\xi, \quad \eta_t := t\eta, \quad t > 0$$

is often used for different purposes and constructions. It was introduced by Tanno [52], which explains why it is either called *Tanno deformation* or *D-homothetic deformation*. It has the property that if the original manifold is K-contact or Sasaki, then the deformed manifold $(M, g_t, \xi_t, \eta_t, \phi)$ has again this property.

In [9, Cor.2.18], it was proved that any Sasakian η-Einstein manifold (with certain weak relations between the curvature parameters) carries Killing spinors with torsion, while Einstein–Sasaki manifolds can never admit Killing spinors with nontrivial torsion [3]. Since any η-Einstein manifold can be Tanno deformed into an Einstein manifold ([51,52]), it is thus sufficient to restrict our attention to Tanno deformations of Einstein–Sasaki manifolds. It is well known that these carry Riemannian Killing spinors [29].

In [9], the Killing spinors with torsion on the Tanno deformation of an Einstein–Sasaki manifold (M, g, ϕ, η) of dimension $n = 2k + 1 \geq 5$ are constructed as follows. Consider the one-dimensional subbundles of the spinor bundle Σ_t of (M, g_t) defined by

$$L_1(\Sigma_t) := \{\psi \in \Sigma_t \mid \phi(X)\psi = -iX\psi \ \forall X \perp \xi\},$$
$$L_2(\Sigma_t) := \{\psi \in \Sigma_t \mid \phi(X)\psi = iX\psi \ \forall X \perp \xi\}.$$

Define $\epsilon = \pm 1$ to be the number satisfying $e_1\phi(e_1)...e_k\phi(e_k)\xi\psi = \epsilon i^{k+1}\psi$ for a local orthonormal frame $e_1, \phi(e_1), .., e_k, \phi(e_k), \xi$ on M. Theorem 2.22 from [9] then states that the spinors $\psi_1 \in L_1(\Sigma_t)$ and $\psi_2 \in L_2(\Sigma_t)$ are Killing spinors with torsion for $s_t = \frac{k+1}{4(k-1)}(\frac{1}{t} - 1)$ with Killing numbers

$$\beta_{1,t} = \frac{\epsilon}{2}\frac{2kt - (k+1)}{t(k-1)} = \frac{\epsilon}{2}(1 - 4s_t) \quad \text{and} \quad \beta_{2,t} = (-1)^{k+1}\beta_{1,t} \quad (*)$$

respectively. For $t = 1$, there is no deformation, and indeed, the parameter s_t is then zero, and the two spinors are just classical Riemannian Killing spinors. Since $(M, g_t, \xi_t, \eta_t, \phi)$ with fundamental 2-form F_t is Sasakian, the characteristic torsion of ∇^c is given by $T^c = \eta_t \wedge d\eta_t = 2\eta_t \wedge F_t$. Thus, the Killing equation

$$\nabla^{g_t}_X \psi_i + s_t(X \lrcorner T^c)\psi_i = \beta_{i,t} X\psi_i, \quad i = 1, 2$$

can equivalently be reformulated as

$$\nabla^{g_t}_X \psi_i + \frac{1}{4}(X \lrcorner T^c)\psi_i - (1 - 4s_t)\frac{1}{4}(X \lrcorner T^c)\psi_i = \beta_{i,t} X\psi_i.$$

If $1 - 4s_t = 0$, both Killing numbers $\beta_{i,t}$ vanish by equation $(*)$ and the Killing equation is reduced to $\nabla^c\psi_i = 0$—the spinor fields ψ_i are ∇^c-parallel, and, as observed before, the cone construction is not possible. The condition $1 - 4s_t > 0$ is equivalent to $t > \frac{k+1}{2k}$, and we observe that in this case, the last equation is exactly of the form treated in Theorem 3.14, case

(2) for $s = 1/4$ and $a = 2|\beta_{i,t}| = 1 - 4s_t > 0$. Recall that we know from Theorem 3.11 that the cone $(\bar{M}, \bar{g}_t)$ of the Tanno deformation is a locally conformally Kähler manifold (class χ_4). Hence, we can conclude from Theorem 3.14, case (2):

Theorem 3.17 *Let (M, g, ϕ, η) be an Einstein–Sasaki manifold of dimension $2k + 1 \geq 5$. Consider its Tanno deformation $(M, g_t, \xi_t, \eta_t, \phi)$ for $t > \frac{k+1}{2k}$ and the cone $(\bar{M}, \bar{g}_t, J_t)$ constructed with cone constant $a = 1 - 4s_t > 0$ and endowed with the conformally Kähler structure described before. Then, the two Killing spinors with torsion on $(M, g_t, \xi_t, \eta_t, \phi)$ induce each a spinor on the cone $(\bar{M}, \bar{g}_t, J_t)$ that is parallel with respect to its characteristic connection $\bar{\nabla}$.*

Although Killing spinors with torsion do exist on $(M, g_t, \xi_t, \eta_t, \phi)$ for $0 < t < \frac{k+1}{2k}$, Theorem 3.14, case (2) cannot be applied because the signs do not match. Of course, case (1) does still hold, and therefore, we obtain a spinor field satisfying a more complicated equation on $\bar{M}$. For $t = 1$ (meaning $s_t = 0$), Theorem 3.17 is the classical cone correspondence between Riemannian Killing spinors on Einstein–Sasaki manifolds and Riemannian parallel spinors on their cone [7].

Example 3.18 We shall now prove the existence of parallel spinors on the cone for a manifold that is not Sasaki and that cannot be deformed into a manifold carrying Riemannian Killing spinors.

The Heisenberg group H is defined to be the following Lie subgroup of $\mathrm{Gl}(4, \mathbb{R})$:

$$H := \left\{ \begin{bmatrix} 1 & u & v & z \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix} \; : \; u, v, x, y, z \in \mathbb{R} \right\}.$$

The vector fields $u_1 = \partial_u$, $u_2 = \partial_x + u\partial_z$, $u_3 = \partial_v$, $u_4 = \partial_y + v\partial_z$ and $u_5 = \partial_z$ form a basis of the left-invariant vector fields. For $\rho > 0$, we consider the metric ([40])

$$g_\rho = \frac{1}{\rho}(\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}v^2 + \mathrm{d}y^2) + (\mathrm{d}z - u\mathrm{d}x - v\mathrm{d}y)^2$$

and get an orthonormal frame $e_1 = \sqrt{\rho}u_1$, $e_2 = \sqrt{\rho}u_2$, $e_3 = \sqrt{\rho}u_3$, $e_4 = \sqrt{\rho}u_4$ and $e_5 = u_5$. On H, there exists a left-invariant spin structure such that $e_1e_2e_3e_4e_5\psi = i\psi$ for all spinor fields ψ, which is the one we choose. We consider the almost contact structures given by $\xi := e_5$ and the fundamental 2-forms

$$F_1 := e_1 \wedge e_2 - e_3 \wedge e_4 \text{ and } F_2 := -(e_1 \wedge e_2 + e_3 \wedge e_4).$$

It is a lengthy but routine calculation to determine the class of these metric almost contact structures. Together with Theorem 3.11, the final result is:

Lemma 3.19 *(1) (H, g_ρ) is never an Einstein manifold $\forall \rho > 0$, and its Tanno deformation is again a metric in the same family of metrics.*
(2) The structure F_1 is of class $\mathcal{C}_7$, and the structure F_2 is of class $\mathcal{C}_6$ (for $\rho = 2$, F_2 is Sasakian).
(3) The almost Hermitian structure on $\bar{M}$ induced by F_1 is Hermitian (mixed class $\chi_3 \oplus \chi_4$), and the almost Hermitian structure on $\bar{M}$ induced by F_2 is locally conformally Kähler (class χ_4). With respect to the orthonormal frame $X_i := \frac{1}{ar}e_i$ for $i = 1, \ldots, 5$ and $X_6 := \partial_r$, they are given by

$$\Omega_1 = -X_1 \wedge X_2 + X_3 \wedge X_4 + X_5 \wedge X_6 \text{ and } \Omega_2 = X_1 \wedge X_2 + X_3 \wedge X_4 + X_5 \wedge X_6.$$

In particular, $N_i = \bar{N}_i = 0$ and $dF_i = 0$ for $i = 1, 2$. Becker-Bender calculates in [9] that the characteristic connection for both structures is given by $T^c = -\rho(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5$. One checks that $d\eta = -\rho(e_1 \wedge e_2 + e_3 \wedge e_4)$, hence $d\eta = \rho F_2$, whereas F_1 is not proportional to $d\eta$. She also proves that ψ_1 and ψ_2, defined via the equations

$$\phi_2(X)\psi_1 = -iX\psi_1 \ \forall X \perp \xi \text{ and } \phi_2(X)\psi_2 = iX\psi_2 \ \forall X \perp \xi,$$

where ϕ_j is the $(1, 1)$ tensor to the 2-form F_j for $j = 1, 2$, are Killing spinors with torsion for $s = -\frac{3}{4}$ with Killing number ρ and $-\rho$, respectively:

$$\nabla_X^{-\frac{3}{4}}\psi_1 = \rho X\psi_1 \text{ and } \nabla_X^{-\frac{3}{4}}\psi_2 = -\rho X\psi_2.$$

If we set $\rho_1 = \rho$, $\rho_2 = -\rho$, we can rewrite these equations as

$$\nabla_X^c\psi_i - (X\lrcorner T^c)\psi_i = \rho_i X\psi_i.$$

On the other hand, let us consider again the equation from Theorem 3.14, case (2), for $s = 1/4$:

$$\nabla_X^c\psi - \frac{a}{2}X\lrcorner(\eta \wedge F)\psi = \alpha X\psi.$$

Since a has to be chosen as $a = 2|\alpha| = 2|\rho_i| = 2\rho$, we conclude that both Killing spinors ψ_1, ψ_2 with torsion on the Heisenberg group satisfy this equation for the structure $F = F_2$. Therefore, their lifts to the cone are parallel for the characteristic connection of the conformally Kähler structure Ω_2. We see at once that the argument can be generalized as follows:

Lemma 3.20 *Let (M, g, ϕ, η) be an α-Sasaki structure (class C_6) satisfying $d\eta = \lambda F$ for some $\lambda > 0$ and admitting a Killing spinor with torsion with Killing number $\alpha = \lambda$ or $\alpha = -\lambda$ for $s = -3/4$. Then, its cone is a locally conformally Kähler manifold (class χ_4), and the spinor lifts to a parallel spinor on $\bar{M}$ with respect to its characteristic connection.*

Let us have a closer look at the characteristic connections $\bar{\nabla}^i$, induced by the connections ∇^i with torsions $T^i = T^c - 2a\eta \wedge F_i$ on M, and the s-dependent connections $\bar{\nabla}^{s,i} := \bar{\nabla}^{\bar{g}} + 2s\bar{T}^i$ $(i = 1, 2)$. Since $F_1 \neq F_2$, we see that the characteristic connections $\bar{\nabla}^i$ (of the almost Hermitian structures Ω_i) on $\bar{M}$ do not coincide, despite the fact that the characteristic connections (of the metric almost structures F_i) coincide on M, $i = 1, 2$. This illustrates neatly the subtle dependence of the construction on the underlying geometric structure, not only its characteristic connection.

The equivalence of the characteristic connections for F_1 and F_2 on M implies that the connections $\bar{\nabla}^{s,i} + \frac{4s}{r}(\partial_r\lrcorner\Omega_i) \wedge \Omega_i$ are the same for $i = 1, 2$, $s = -3/4$. As discussed above, this connection is in turn just the characteristic connection of the locally conformally Kähler structure Ω_2, and hence, we have the following relation between the Kähler forms:

$$d\Omega_2^{J_2} = -3\,[d\Omega_i^{J_i} + \frac{2}{r}(\partial_r\lrcorner\Omega_i) \wedge \Omega_i] \qquad i = 1, 2.$$

In particular, we can apply Theorem 3.14 case (1) for $i = 1$ and can state that the differential equation for the two $\bar{\nabla}^2$-parallel spinors on $\bar{M}$ can equally be written

$$0 = \bar{\nabla}_X^{-\frac{3}{4},1}\psi - \frac{3}{2r}X\lrcorner((\partial_r\lrcorner\Omega_1) \wedge \Omega_1)\psi \ = \ \bar{\nabla}_X^{\bar{g}}\psi - \frac{3}{4}X\lrcorner[d\Omega_1 + \frac{2}{r}(\partial_r\lrcorner\Omega_1) \wedge \Omega_1]\psi.$$

Example 3.21 Another example (see [9]) is given by the homogeneous space $M := \mathrm{SO}(3) \times \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ with the embedding

$$\mathrm{SO}(2) \ni A(t) := \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mapsto \left[A(t), A\left(\frac{t}{2}\right)^{-1} \right].$$

As an orthonormal basis of a reductive complement of $\mathfrak{so}(2)$ in $\mathfrak{so}(3) \times \mathfrak{sl}(2, \mathbb{R})$, we choose

$$e_1 := D_1 \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, 0 \right), \quad e_2 := D_1 \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, 0 \right),$$
$$e_5 := \left(c_1 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, c_2 \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right), \quad e_3 := \frac{1}{2} D_2 \left(0, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right),$$
$$e_4 := \frac{1}{2} D_2 \left(0, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right),$$

such that $c_1 + c_2 \neq 0$, $D_1^2 = c_1(c_1 + c_2)$, $D_2^2 = -c_2(c_1 + c_2)$ and the numbers $c_1, -c_2$ and $(c_1 + c_2)$ have the same signature. We consider the almost contact structure (M, ξ, F) defined via

$$\xi := e_5 \text{ and } F = e_1 \wedge e_2 + e_3 \wedge e_4.$$

Then, the characteristic connection ∇^c has torsion $T^c = -c_1 e_1 \wedge e_2 \wedge e_5 - c_2 e_3 \wedge e_4 \wedge e_5$.

Lemma 3.22 *The almost contact structure (M, ξ, F) is normal. Furthermore, the almost Hermitian structure on $\bar{M}$, constructed with $a = \frac{-c_1 - c_2}{4}$, induced by the almost contact structure (M, ξ, F) is of class χ_3, and thus, the structure (M, ξ, F) is of mixed class $\mathcal{C}_3 \oplus .. \oplus \mathcal{C}_8$.*

Proof One uses Theorem 3.10 and proves that the almost contact structure (M, ξ, F) is normal, satisfies $\delta F = (-c_1 - c_2)\eta$ and never satisfies $(\nabla^g_X F)(Y, Z) = a\eta(Y)g(X, Z) - a\eta(Z)g(X, Y)$. Thus, the structure on $\bar{M}$ is never Kähler, and for $a = \frac{-c_1 - c_2}{4}$, it really is of class χ_3. □

In this example, we only have Killing spinors with torsion satisfying $\nabla^s_X \psi = \alpha X \psi$ for $\alpha = 0$. But since the construction of $\bar{M}$ explicitly depends on $2\alpha = a \neq 0$, we cannot lift these spinors to $\bar{M}$.

3.5 Metric almost contact 3-structures

Let M be a manifold of dimension $n = 4m - 1$ with 3 metric almost contact structures given by ξ_i, η_i and ϕ_i for $i = 1, 2, 3$ (see [14] for a more detailed description of metric almost contact 3-structures). Looking at the cone $\bar{M}$, we define the three almost Hermitian structures

$$\begin{aligned} &J_1(ar\partial_r) := \xi_1, && J_1(\xi_1) = -ar\partial_r, && J_1(V) = -\phi_1(V) \text{ for } V \perp \xi_1, \partial_r, \\ &J_2(ar\partial_r) := \xi_2, && J_2(\xi_2) = -ar\partial_r, && J_2(V) = -\phi_2(V) \text{ for } V \perp \xi_2, \partial_r, \\ &J_3(ar\partial_r) := -\xi_3, && J_3(\xi_3) = ar\partial_r, && J_3(V) = -\phi_3(V) \text{ for } V \perp \xi_3, \partial_r. \end{aligned}$$

Conversely, let $\bar{M}$ be a $4m$ dimensional manifold with three almost Hermitian structures J_1, J_2 and J_3. We can define three almost contact structures

$$\begin{aligned}
\xi_1 &:= +aJ_1(\partial_r), \qquad \phi_1(X) := -J_1(X) + \bar{g}(J_1(X), \partial_r)\partial_r.\\
\xi_2 &:= +aJ_2(\partial_r), \qquad \phi_2(X) := -J_2(X) + \bar{g}(J_2(X), \partial_r)\partial_r.\\
\xi_3 &:= -aJ_3(\partial_r), \qquad \phi_3(X) := +J_3(X) - \bar{g}(J_3(X), \partial_r)\partial_r.
\end{aligned}$$

on $M = M \times \{1\} \subset \bar{M}$.
A $4m$-dimensional manifold with three almost Hermitian structures $\Omega_i = \bar{g}(., J_i.)$ is called *hyper-Kähler with torsion (HKT)* if the almost Hermitian structures are integrable ($\bar{N}_i = 0$) and

$$J_1 \circ d\Omega_1 = J_2 \circ d\Omega_2 = J_3 \circ d\Omega_3.$$

We can apply Theorem 3.2 to each of these structures and prove.

Theorem 3.23 *The three almost Hermitian structures on $\bar{M}$ satisfy the relation $J_1J_2 = -J_2J_1 = J_3$ if and only if ξ_1, ξ_2 and ξ_3 are orthonormal and the almost contact structures on M satisfy the following*

$$\phi_3\phi_2 = -\phi_1 + \eta_2 \otimes \xi_3, \quad \phi_2\phi_3 = +\phi_1 + \eta_3 \otimes \xi_2, \quad \phi_1\phi_3 = -\phi_2 + \eta_3 \otimes \xi_1, \tag{7}$$

$$\phi_3\phi_1 = +\phi_2 + \eta_1 \otimes \xi_3, \quad \phi_2\phi_1 = -\phi_3 + \eta_1 \otimes \xi_2, \quad \phi_1\phi_2 = +\phi_3 + \eta_2 \otimes \xi_1, \tag{8}$$

where η_i is the dual to ξ_i for $i = 1, 2, 3$. The appendant connection $\bar{\nabla}$ satisfies $\bar{\nabla} J_2 = \bar{\nabla} J_3 = \bar{\nabla} J_1 = 0$ if and only if the characteristic connections $\nabla^{c,i}$ on M of the three almost Hermitian structures (η_i, ϕ_i) are such that the corresponding connections ∇^i constructed in Definition 3.1 coincide $\nabla^1 = \nabla^2 = \nabla^3 =: \nabla$.

In this case, we get the additional commutator relations

$$[\xi_1, \xi_2] = 2a\xi_3 - T(\xi_1, \xi_2), \quad [\xi_2, \xi_3] = 2a\xi_1 - T(\xi_2, \xi_3), \quad [\xi_3, \xi_1] = 2a\xi_2 - T(\xi_3, \xi_1).$$

If furthermore the almost contact structures are normal, the three almost Hermitian structures on $\bar{M}$ form an HKT structure.

Proof Given three almost Hermitian structures satisfying the relation $J_1J_2 = -J_2J_1 = J_3$, we compute

$$\begin{aligned}
\phi_3(\phi_2(X)) &= -J_3(J_2(X)) + \bar{g}(J_3(J_2(X)), \partial_r)\partial_r + \bar{g}(J_2(X), \partial_r)J_3(\partial_r)\\
&\quad - \bar{g}(J_3(X), \partial_r)\bar{g}(J_3(\partial_r), \partial_r)\partial_r\\
&= -\phi_1(X) - \bar{g}(X, J_2\partial_r)J_3(\partial_r) = -\phi_1(X) - a^2 g(X, J_2\partial_r)J_3(\partial_r)\\
&= -\phi_1(X) + g(X, \xi_2)\xi_3,
\end{aligned}$$

and similarly for the other relations. Conversely, given three almost Hermitian structures satisfying Eqs. (7) and (8), we plug in ξ_1, ξ_2, and ξ_3 and, with $\phi_i(\xi_i) = 0$ for $i = 1, 2, 3$, we obtain immediately

$$\phi_1(\xi_2) = \xi_3, \quad \phi_1(\xi_3) = -\xi_2, \quad \phi_2(\xi_1) = -\xi_3, \quad \phi_2(\xi_3) = \xi_1, \quad \phi_3(\xi_1) = \xi_2,$$
$$\phi_3(\xi_2) = -\xi_1.$$

Since all ϕ_i, leave the vector space $V := \operatorname{span}(\xi_1, \xi_2, \xi_3)$ invariant, and since they are orthonormal, they also leave $V^\perp$ invariant. For $X \perp \xi_1, \xi_2, \xi_3, \partial_r$, we have

$$J_1(J_2(X)) = \phi_1(\phi_2(X)) = \phi_3(X) = J_3(X) = -\phi_2(\phi_1(X)) = -J_2(J_1(X)).$$

For ξ_1 we obtain

$$J_1(J_2(\xi_1)) = -J_1(\phi_2(\xi_1)) = J_1(\xi_3) = -\phi_1(\xi_3) = \xi_2 \quad = J_2(ar\partial_r) = -J_2(J_1(\xi_1)) \\ = \phi_3(\xi_1) = J_3(\xi_1)$$

and similarly for ξ_2, ξ_3 and ∂_r. For a connection as in Theorem (3.2), we have that all almost Hermitian structures are parallel under $\bar{\nabla}$ and for $X, Y \in TM$

$$[X, Y] = \bar{\nabla}^{\bar{g}}_X Y - \bar{\nabla}^{\bar{g}}_Y X = \bar{\nabla}_X Y - \bar{\nabla}_Y X - \bar{T}(X, Y).$$

Thus, the commutator relations are given by

$$\begin{aligned}[\xi_1, \xi_2] &= a^2[J_1(\partial_r), J_2(\partial_r)] = a^2(\bar{\nabla}_{J_1(\partial_r)} J_2(\partial_r) - \bar{\nabla}_{J_2(\partial_r)} J_1(\partial_r)) - \bar{T}(\xi_1, \xi_2) \\ &= a^2(J_2(\bar{\nabla}_{J_1(\partial_r)}\partial_r) - J_1(\bar{\nabla}_{J_2(\partial_r)}\partial_r)) - \bar{T}(\xi_1, \xi_2) \\ &= a^2(J_2(J_1(\partial_r)) - J_1(J_2(\partial_r))) - \bar{T}(\xi_1, \xi_2) \\ &= -2a^2 J_3(\partial_r) - \bar{T}(\xi_1, \xi_2) = 2a\xi_3 - T(\xi_1, \xi_2).\end{aligned}$$

The other relations are to be calculated similarly.

If the almost contact structures are normal, then the almost Hermitian structures are normal, and with the formula for the torsion given in Remark 3.15, we have

$$\bar{g}(\bar{\nabla}_X, Y, Z) = \bar{g}(\bar{\nabla}^{\bar{g}}_X Y, Z) + \frac{1}{2}(J_i \circ d\Omega_i)(X, Y, Z)$$

for any $i = 1, 2, 3$. This implies $J_1 \circ d\Omega_1 = J_2 \circ d\Omega_2 = J_3 \circ d\Omega_3$. □

Remark 3.24 In [22, Section 7], the authors obtain a similar result (but without a description of the characteristic connections). Furthermore, they investigate more closely the conditions for the HKT structure to be *strong* ($J_i \circ d\Omega_i$ is closed).

Remark 3.25 If we rescale the metric such that $a = 1$ and if $T = 0$, we have 3 Kählerian structures on $\bar{M}$ and thus 3 Sasakian structures on M. Then, the commutator relations in Theorem 3.23 ensure that the structures on M form a 3-Sasakian structure. This is Lemma 5 of [7]: a one to one correspondence between hyper-Kähler structures on $\bar{M}$ and 3-Sasaki structures on M.

Remark 3.26 We emphasize that it is not necessary that the three characteristic connections $\nabla^{c,i}$, $i = 1, 2, 3$ coincide in order to apply Theorem 3.23, only the connections ∇^i with torsion $T^i = T^{c,i} - 2a\eta_i \wedge F_i$ have to be equal. If M is a 3-Sasakian manifold, $T^i = 0$ for $i = 1, 2, 3$ and thus $\nabla^1 = \nabla^2 = \nabla^3 = \nabla^g$. In this case, there exists a special G_2 structure on M which will be discussed in Example 4.20.

4 G_2 structures – Spin(7) structures on the cone

4.1 Preparations

Let (M, g, ϕ, P) be a G_2 manifold (see Sect. 2.2). We cite a classical but for us crucial result by Fernandez and Gray:

Lemma 4.1 ([23, Lemma 2.7])

$$\begin{aligned}*\phi(V, W, X, Y) &= g(P(V, W), P(X, Y)) - g(V, X)g(W, Y) + g(V, Y)g(W, X) \\ &= \phi(V, W, P(X, Y)) - g(V, X)g(W, Y) + g(V, Y)g(W, X).\end{aligned}$$

Remark 4.2 In [23], this formula is stated differently,

$$*\phi(V, W, X, Y) = -g(P(V, W), P(X, Y)) + g(V, X)g(W, Y) - g(V, Y)g(W, X).$$

This is due to the standard 3-form ϕ used by Fernández and Gray, which corresponds to the orientation opposite to ours. This changes the sign of the Hodge operator.

Now, we are able to prove

Lemma 4.3 *For any metric connection ∇ with skew torsion on M, the G_2 form ϕ satisfies*

$$(\nabla_Z * \phi)(V, W, X, Y) \;=\; (\nabla_Z\phi)(V, W, P(X, Y)) + (\nabla_Z\phi)(X, Y, P(V, W)).$$

If ∇ satisfies $\nabla\phi = a * \phi$ for some $a > 0$, we have the simplified relation

$$\begin{aligned}(\nabla_Z * \phi)(V, W, X, Y) = a[&\phi(X, Y, V)g(Z, W) - \phi(X, Y, W)g(Z, V)\\ &+\phi(V, W, X)g(Z, Y) - \phi(V, W, Y)g(Z, X)].\end{aligned}$$

Proof For any metric connection with skew torsion, we have

$$\begin{aligned}(\nabla_Z * \phi)(V, W, X, Y) =&Z * \phi(V, W, X, Y) - *\phi(\nabla_Z V, W, X, Y) - *\phi(V, \nabla_Z W, X, Y)\\ &- *\phi(V, W, \nabla_Z X, Y) - *\phi(V, W, X, \nabla_Z Y).\end{aligned}$$

Since ∇ is metric, g is parallel and with Lemma 4.1, we get

$$\begin{aligned}&= Z\phi(V, W, P(X, Y)) - \phi(\nabla_Z V, W, P(X, Y)) - \phi(V, \nabla_Z W, P(X, Y))\\ &\quad - \phi(V, W, P(\nabla_Z X, Y)) - \phi(V, W, P(X, \nabla_Z Y)) - \phi(V, W, \nabla_Z P(X, Y))\\ &\quad + \phi(V, W, \nabla_Z P(X, Y)).\end{aligned}$$

We have $\phi(V, W, (\nabla_Z P)(X, Y)) = g(P(V, W), (\nabla_Z P)(X, Y)) = (\nabla_Z\phi)(X, Y, P(V, W))$ and thus we get

$$(\nabla_Z * \phi)(V, W, X, Y) \;=\; (\nabla_Z\phi)(V, W, P(X, Y)) + (\nabla_Z\phi)(X, Y, P(V, W)).$$

The condition $\nabla\phi = a * \phi$ implies

$$(\nabla_Z * \phi)(V, W, X, Y) \;=\; -a * \phi(P(X, Y), Z, V, W) - a * \phi(P(V, W), Z, X, Y)$$

and aplying once again Lemma 4.1 yields

$$\begin{aligned}(\nabla_Z * \phi)(V, W, X, Y) &= -a\phi(P(X, Y), Z, P(V, W)) - a\phi(P(V, W), Z, P(X, Y))\\ &\quad +ag(P(X, Y), V)g(Z, W)\\ &\quad -ag(P(X, Y), W)g(Z, V) + ag(P(V, W), X)g(Z, Y)\\ &\quad -ag(P(V, W), Y)g(Z, X)\\ &= a[\phi(X, Y, V)g(Z, W) - \phi(X, Y, W)g(Z, V)\\ &\quad +\phi(V, W, X)g(Z, Y) - \phi(V, W, Y)g(Z, X)],\end{aligned}$$

which finishes the proof. □

We define a 4-form on the cone $\bar{M}$ via

$$\Phi(\partial_r, X, Y, Z) \;:=\; a^3r^3\phi(X, Y, Z), \qquad \Phi(X, Y, Z, W) \;:=\; a^4r^4 * \phi(X, Y, Z, W)$$

for $X, Y, Z, W \in TM$. Since $\partial_r \lrcorner \Phi$ locally is a G_2-structure on $\partial_r^\perp$, Φ is a Spin(7)-structure on $\bar{M}$. As in Sect. 3, given a characteristic connection on M with respect to ϕ, we construct a connection ∇ with skew symmetric torsion T on M such that its lift $\bar{\nabla}$ to $\bar{M}$ with torsion $\bar{T}$ is the characteristic connection on $\bar{M}$ with respect to Φ. Since we have $T = \bar{T}_{|TM}$ and $\partial_r \lrcorner \bar{T} = 0$, we have $\bar{T} = T = 0$ in case of a parallel $\mathrm{Spin}(7)$ structure with respect to the Levi-Civita connection on $\bar{M}$, and thus, ∇ is the Levi-Civita connection on M.

Definition 4.4 Let (M, g, ϕ) be a G_2T manifold with characteristic connection ∇^c. We define a metric connection ∇ with skew symmetric torsion T via

$$T := T^c - \frac{2a}{3}\phi.$$

As in the metric almost contact case (see the comments in Definition 3.1), T cannot be computed abstractly, but it is found through an educated guess and justified a posteriori from its properties.

Theorem 4.5 *The connection* ∇ *satisfies*

$$\nabla\phi = a * \phi,$$

and Φ *is parallel with respect to* $\bar{\nabla}$*, the appendant connection on* $\bar{M}$.

Proof We have for the Riemannian connection ∇^g on M

$$\begin{aligned}\nabla_X\phi(Y, Z, W) &= X\phi(Y, Z, W) - \phi(\nabla^g_X Y, Z, W) - \phi(Y, \nabla^g_X Z, W) - \phi(Y, Z, \nabla^g_X W)\\ &\quad -\frac{1}{2}\phi(T(X, Y), Z, W) - \frac{1}{2}\phi(Y, T(X, Z), W) - \frac{1}{2}\phi(Y, Z, T(X, W))\\ &= (\nabla^c_X\phi)(Y, Z, W) + \frac{1}{2}\phi((T^c - T)(X, Y), Z, W)\\ &\quad +\frac{1}{2}\phi(Y, (T^c - T)(X, Z), W) + \frac{1}{2}\phi(Y, Z, (T^c - T)(X, W))\end{aligned}$$

and because $\nabla^c\phi = 0$, we have

$$\begin{aligned}\nabla_X\phi(Y, Z, W) = &= \frac{1}{2}[(T^c - T)(X, Y, P(Z, W)) + (T^c - T)(X, Z, P(W, Y))\\ &\quad +(T^c - T)(X, W, P(Y, Z))]\\ &= \frac{a}{3}[\phi(X, Y, P(Z, W)) + \phi(X, Z, P(W, Y)) + \phi(X, W, P(Y, Z))].\end{aligned}$$

With Lemma 4.1, we obtain

$$\begin{aligned}a * \phi(X, Y, Z, W) &= \frac{a}{3}[*\phi(X, Y, Z, W) + *\phi(X, Z, W, Y) + *\phi(X, W, Y, Z)]\\ &= \frac{a}{3}[\phi(X, Y, P(Z, W)) + \phi(X, Z, P(W, Y)) + \phi(X, W, P(Y, Z))\\ &\quad -g(X, Z)g(Y, W) + g(X, W)g(Y, Z) - g(X, W)g(Z, Y)\\ &\quad +g(X, Y)g(Z, W) - g(X, Y)g(W, Z) + g(X, Z)g(W, Y)]\\ &= \nabla_X\phi(Y, Z, W),\end{aligned}$$

which proves the first statement. To show $\bar{\nabla}\Phi = 0$ on $\bar{M}$, we look at several cases. Let always be $V, W, X, Y, Z \in TM$.

Case 1: If ∂_r is one of the arguments, we compute

$$\begin{aligned}(\bar{\nabla}_W \Phi)(\partial_r, X, Y, Z) &= Wa^3r^3\phi(X, Y, Z) - \frac{1}{r}\Phi(W, X, Y, Z) - r^3a^3\phi(\nabla_W X, Y, Z)\\ &\quad -r^3a^3\phi(X, \nabla_W Y, Z) - r^3a^3\phi(X, Y, \nabla_W Z)\\ &= a^3r^3(\nabla_W\phi)(X, Y, Z) - \frac{1}{r}\Phi(W, X, Y, Z)\\ &= a^4r^3 * \phi(W, X, Y, Z) - \frac{1}{r}\Phi(W, X, Y, Z) = 0.\end{aligned}$$

Case 2: If the direction of the derivative is equal to ∂_r, we obtain

$$\begin{aligned}(\bar{\nabla}_{\partial_r}\Phi)(X, Y, Z, W) &= \partial_r(a^4r^4 * \phi(X, Y, Z, W)) - 4\frac{1}{r}\Phi(X, Y, Z, W)\\ &= 4r^3a^4 * \phi(X, Y, Z, W) - 4\frac{1}{r}\Phi(X, Y, Z, W) = 0.\end{aligned}$$

Case 3: If the direction of the derivative and one argument are equal to ∂_r, we compute

$$(\bar{\nabla}_{\partial_r}\Phi)(\partial_r, X, Y, Z) = \partial_r(a^3r^3\phi(X, Y, Z)) - 3a^3r^3\frac{1}{r}\phi(X, Y, Z) = 0.$$

Case 4: On TM, we have:

$$\begin{aligned}(\bar{\nabla}_V\Phi)(W, X, Y, Z) &= a^4r^4V * \phi(W, X, Y, Z) - \Phi(\bar{\nabla}_V W, X, Y, Z) - \Phi(W, \bar{\nabla}_V X, Y, Z)\\ &\quad -\Phi(W, X, \bar{\nabla}_V Y, Z) - \Phi(W, X, Y, \bar{\nabla}_V Z)\\ &= a^4r^4V * \phi(W, X, Y, Z) - \Phi(\nabla_V W - \frac{1}{r}\bar{g}(V, W)\partial_r, X, Y, Z)\\ &\quad -\Phi(W, \nabla_V X - \frac{1}{r}\bar{g}(V, X)\partial_r, Y, Z)\\ &\quad -\Phi(W, X, \nabla_V Y - \frac{1}{r}\bar{g}(V, Y)\partial_r, Z) - \Phi(W, X, Y, \nabla_V Z\\ &\quad -\frac{1}{r}\bar{g}(V, Z)\partial_r)\\ &= a^4r^4(\nabla_V * \phi)(W, X, Y, Z) + r^4a^5[g(V, W)\phi(X, Y, Z)\\ &\quad -g(V, X)\phi(W, Y, Z) + g(V, Y)\phi(W, X, Z) - g(V, Z)\phi(W, X, Y)],\end{aligned}$$

which is equal to zero due to Lemma 4.3. □

Conversely, given a Spin(7) structure $(\bar{M}, \bar{g}, \Phi, \bar{P}, \bar{p})$ on $\bar{M}$ (see Sect. 2.2 for the definitions), $\partial_r \lrcorner \Phi$ is a G_2 structure with respect to the metric a^2g on $M = M \times \{1\} \subset \bar{M}$ and thus

$$\phi := \frac{1}{a^3}\partial_r \lrcorner \Phi$$

defines a G_2 structure on M with respect to the metric g. To prove the following theorem, we need

Lemma 4.6 *If $*$ is the Hodge operator on M with respect to g and $*_{a^2g}$ is the Hodge operator on M with respect to the metric a^2g, we have for any 3-form ω*

$$*_{a^2g}\omega = a * \omega.$$

Proof Let e_i for $i = 1..7$ be an orthonormal basis with dual basis e^i on M with respect to g. Then, $\frac{1}{a}e_i$ with dual ae^i is a orthonormal basis with respect to a^2g. We define $e^{\{i,j,k\}} := e^i \wedge e^j \wedge e^k$ and $e^{\{i,j,k,j\}} := e^i \wedge e^j \wedge e^k \wedge e^l$ as well as $(se)^{\{i,j,k\}} := se^i \wedge se^j \wedge se^k$ for $s \in \mathbb{R}$ and $(se)^{\{i,j,k,j\}}$, respectively. Then, we have

$$*_{a^2g}e^{\{i,j,k\}} = \frac{1}{a^3} *_{a^2g} (ae)^{\{i,j,k\}} = \frac{1}{a^3}(ae)^{\{1,..,7\}\setminus\{i,j,k\}} = \frac{1}{a^3}a^4 e^{\{1,..,7\}\setminus\{i,j,k\}} = a * e^{\{i,j,k\}},$$

which proves the lemma. □

Theorem 4.7 *Given a* Spin(7) *structure on* $\bar{M}$ *with characteristic connection* $\bar{\nabla}$ *being the lift of a connection* ∇ *on* M*, we have for the* G_2 *structure* ϕ *induced by* Φ

$$\nabla\phi = a * \phi$$

and the characteristic connection on (M, g, ϕ) *is given by* $T^c = T + \frac{2a}{3}\phi$*.*

Proof We have for $W, X, Y, Z \in TM$

$$\begin{aligned}(\nabla_W\phi)(X, Y, Z) &= \frac{1}{a^3}[W\Phi(\partial_r, X, Y, Z)\\ &\quad - \Phi(\partial_r, \nabla_W X, Y, Z) - \Phi(\partial_r, X, \nabla_W Y, Z) - \Phi(\partial_r, X, Y, \nabla_W Z)]\\ &= \frac{1}{a^3}[(\bar{\nabla}_W\Phi)(\partial_r, X, Y, Z) + \Phi(\bar{\nabla}_W\partial_r, X, Y, Z)] = \frac{1}{a^3}\Phi(W, X, Y, Z).\end{aligned}$$

With Lemma 8 of [7] and the definition of ϕ, we conclude $\Phi|_{TM} = *_{a^2g}(\partial_r \lrcorner \Phi) = *_{a^2g}(a^3\phi) = a^4 * \phi$, where $*_{a^2g}$ is the Hodge operator on $M \subset \bar{M}$ with respect to the metric a^2g. The last equality follows from Lemma 4.6. Thus, we get

$$\nabla\phi = a * \phi.$$

For the connection ∇^c with torsion $T^c = T + \frac{2a}{3}\phi$, we calculate as in the proof of Theorem 4.5

$$\begin{aligned}(\nabla^c_X\phi)(Y, Z, W) &= (\nabla_X\phi)(Y, Z, W) + \frac{1}{2}[(T - T^c)(X, Y, P(Z, W))\\ &\quad + (T - T^c)(X, Z, P(W, Y)) + (T - T^c)(X, W, P(Y, Z))]\\ &= a * \phi(X, Y, Z, W) - \frac{a}{3}[\phi(X, Y, P(Z, W)) + \phi(X, Z, P(W, Y))\\ &\quad + \phi(X, W, P(Y, Z))]\end{aligned}$$

which is equal to zero due to Lemma 4.1. Since the characteristic connection of a G_2 manifold is unique, this proves the Theorem. □

Remark 4.8 As in the metric almost contact case, $T = T^c - \frac{2a}{3}\phi$ measures the 'deviation' of the G_2 structure from a nearly parallel G_2 structure; for then, $T^c = \frac{2a}{3}\phi$, i.e. $T = 0$ and thus $\nabla = \nabla^g$ lifts to the Levi-Civita connection on $\bar{M}$, reflecting the fact that the Spin(7) structure on the cone is then integrable. That ∇ plays indeed a geometric role beyond being an auxiliary tool, and that this role is that of a the Levi-Civita connection for a nearly parallel G_2 manifold and is confirmed by Theorem 4.5, since it states that the equation $\nabla^g\phi = a * \phi$ for the nearly parallel case generalizes to $\nabla\phi = a * \phi$ for any G_2T manifold.

4.2 The classification of G_2 structures and the corresponding classification of Spin(7) structures on the cone

We will now discuss the classification of Fernández [21] of Spin(7) structures on $\bar{M}$ given in Sect. 2.2 and compute the correspondence to the classification of G_2 structures [23]. Again, we are only interested in structures carrying a characteristic connection (G_2 structures of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$). We write X_M for the projection on TM of a vector field X in $T\bar{M}$. We summarize some useful identities:

Lemma 4.9 *(1) P can be expressed through ϕ on TM: $P(Y, Z) = \sum_l \phi(e_l, Y, Z)e_l$.*
(2) For any metric connection $\tilde{\nabla}$ with skew torsion on M, we have:

$$(\tilde{\nabla}_X \phi)(Y, Z, V) = g((\tilde{\nabla}_X P)(Y, Z), V),$$
$$(\tilde{\nabla}_X P)(Y, Z) = \sum_l g(e_l, (\tilde{\nabla}_X P)(Y, Z))e_l = \sum_l (\tilde{\nabla}_X \phi)(e_l, Y, Z)e_l.$$

(3) For ∇, this can be simplified to $(\nabla_X P)(Y, Z) = a \sum_l *\phi(X, e_l, Y, Z)e_l$.
(4) P, ϕ, and $\bar{P}$ are related by $(X, Y, Z \in TM)$

$$\bar{g}(\bar{P}(X, Y, Z), \partial_r) = -a^3 r^3 \phi(X, Y, Z), \quad \bar{P}(\partial_r, X, Y) = ar P(X, Y),$$
$$\bar{P}(Y, Z, V)_M = ar^2 (\nabla_Y P)(Z, V).$$

(5) The derivative of Φ on $\bar{M}$ can be expressed in terms of ϕ on M $(X, Y, Z, V, W \in TM)$:

$$(\bar{\nabla}^{\bar{g}}_X \Phi)(\partial_r, Z, V, W) = a^3 r^3 [(\nabla^g - \nabla)_X \phi](Z, V, W), (\bar{\nabla}^{\bar{g}}_X \Phi)(Y, Z, V, W)$$
$$= a^4 r^4 [(\nabla^g - \nabla)_X * \phi](Y, Z, V, W).$$

Proof Statements (1)–(3) are easily checked. To prove statement (4) for $X, Y, Z \in TM$, we have

$$\bar{g}(\bar{P}(\partial_r, X, Y), Z) = \Phi(\partial_r, X, Y, Z) = a^3 r^3 \phi(X, Y, Z) = ar \bar{g}(P(X, Y), Z),$$

thus $\bar{P}(\partial_r, X, Y) = ar P(X, Y)$. Furthermore,

$$\bar{g}(X, \bar{P}(Y, Z, V)) = \Phi(Y, Z, V, X) = a^3 r^4 (\nabla_Y \phi)(Z, V, X) = a^3 r^4 g(X, (\nabla_Y P)(Z, V))$$
$$= ar^2 \bar{g}(X, (\nabla_Y P)(Z, V)),$$

and thus $\bar{P}(Y, Y, V)_M = ar^2 (\nabla_Y P)(Z, V)$. For (5) and vector fields $X, Y, Z, V, W \in TM$, we calculate

$$\begin{aligned}
&2(\bar{\nabla}^{\bar{g}}_X \Phi)(\partial_r, Z, V, W) \\
&\quad = 2(\bar{\nabla}_X \Phi)(\partial_r, Z, V, W) + \Phi(\partial_r, \bar{T}(X, Z), V, W) \\
&\quad\quad + \Phi(\partial_r, Z, \bar{T}(X, V), W) + \Phi(\partial_r, Z, V, \bar{T}(X, W)) \\
&\quad = a^3 r^3 [\phi(T(X, Z), V, W) + \phi(Z, T(X, V), W) + \phi(Z, V, T(X, W))] \\
&\quad = 2a^3 r^3 [\phi((\nabla_X - \nabla^g_X)Z, V, W) + \phi(Z, (\nabla_X - \nabla^g_X)V, W) + \phi(Z, V, (\nabla_X - \nabla^g_X)W)] \\
&\quad = -2a^3 r^3 [(\nabla - \nabla^g)_X \phi](Z, V, W),
\end{aligned}$$

and similarly

$$
\begin{aligned}
&(\bar{\nabla}^{\bar{g}}_X \Phi)(Y, Z, V, W)\\
&= \frac{1}{2}[\Phi(\bar{T}(X, Y), Z, V, W) + \Phi(Y, \bar{T}(X, Z), V, W) + \Phi(Y, Z, \bar{T}(X, V), W)\\
&\quad + \Phi(Y, Z, V, \bar{T}(X, W))]\\
&= \frac{a^4 r^4}{2}[*\phi(T(X, Y), Z, V, W) + *\phi(Y, T(X, Z), V, W) + *\phi(Y, Z, T(X, V), W)\\
&\quad + *\phi(Y, Z, V, T(X, W))]\\
&= -a^4 r^4[(\nabla - \nabla^g)_X *\phi](Y, Z, V, W) = a^4 r^4[(\nabla^g - \nabla)_X *\phi](Y, Z, V, W),
\end{aligned}
$$

which finishes the proof. □

Remark 4.10 Since the characteristic connection of the Spin(7) structure on $\bar{M}$ is unique (see Sect. 2.2), we can conclude for any such structure satisfying $\bar{\nabla}^{\bar{g}}\Phi = 0$ that $\nabla = \nabla^g$ and thus $\nabla^g \phi = a*\phi$, and the G_2 structure is of class $\mathcal{W}_1$. Conversely, given a connection ∇ with skew symmetric torsion and $\nabla\phi = a*\phi$, we construct ∇^c via $T^c := T - \frac{2a}{3}\phi$, which satisfies $\nabla^c\phi = 0$ and thus is unique. Hence, a metric connection with skew symmetric torsion and the property $\nabla\phi = *\phi$ is unique.

For any tensor R on M, we introduce the notation $R \llcorner X$ to denote $R(-, X)$.
We extend the metric g to arbitrary k-tensors R, S via an orthonormal frame $e_1, \ldots, e_n$

$$g(R, S) := \sum_{i_1,..,i_k=1}^{n} R(e_{i_1}, .., e_{i_k}) S(e_{i_1}, .., e_{i_k}).$$

Lemma 4.11 *A* Spin(7) *structure on $\bar{M}$ is of class $\mathcal{U}_1$ if and only if on M*

- *$g(\nabla^g\phi, *\phi) = a g(*\phi, *\phi)$, and*
- *for every $X \in TM$ we have $g(*\phi, [(\nabla - \nabla^g) * \phi]\llcorner X) = 3g(\phi, [(\nabla - \nabla^g)\phi]\llcorner X)$.*

The structure on $\bar{M}$ is of class $\mathcal{U}_2$ if and only if the following conditions are satisfied for $X, Y, Z, X_1, .., X_4 \in TM$ and a local orthonormal frame $e_1, .., e_7$ of TM:

- *$\delta\Phi|_{TM} = 0$ on TM, which is equivalent to $0 = \sum_{i=1}^{7}[(\nabla^g - \nabla)_{e_i} * \phi](e_i, X, Y, Z)$*
- *$0 = \sum_{i=1}^{4}\sum_{l<j<8}(-1)^i \delta\phi(e_l, e_j)\phi(e_l, e_j, X_i)\phi(X_1, .., \hat{X}_i, .., X_4)$*
- *$28[(\nabla^g - \nabla)_W * \phi](X_1, X_2, X_3, X_4) = \sum_{i=1}^{4}\sum_{l<j<8}(-1)^{i+1}\delta\phi(e_l, e_j)\phi(e_l, e_j, X_i) * \phi(W, X_1, .., \hat{X}_i, .., X_4)$.*

Proof We consider a local $\bar{g}$-orthonormal frame $\bar{e}_1 = \frac{1}{ar}e_1, .., \bar{e}_7 = \frac{1}{ar}e_7, e_8 = \partial_r$ of $T\bar{M}$ such that $e_1, .., e_7$ is a local orthonormal frame of TM. With Lemma 4.2 of [21], a Spin(7) structure is defined to be of class $\mathcal{U}_1$ if and only if

$$0 = -6\delta\Phi(\bar{p}(X)) = \sum_{i,k,j=1}^{8}(\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \bar{e}_k, \bar{P}(\bar{e}_i, \bar{e}_j, \bar{e}_k), X).$$

For $X \in TM$, we have

$$
\begin{aligned}
0 &= -6\delta\Phi(\bar{p}(X)) = \sum_{i,k,j=1}^{8} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \bar{e}_k, \bar{P}(\bar{e}_i, \bar{e}_j, \bar{e}_k), X)\\
&= \sum_{i,k,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \bar{e}_k, \bar{P}(\bar{e}_i, \bar{e}_j, \bar{e}_k), X) + 2\sum_{i,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_j, \partial_r, \bar{P}(\bar{e}_i, \bar{e}_j, \partial_r), X)\\
&= \frac{1}{a^6 r^6}\sum_{i,k,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, ar^2(\nabla_{e_i}P)(e_j, e_k) + \bar{g}(\bar{P}(e_i, e_j, e_k), \partial_r)\partial_r, X)\\
&\quad +2\frac{1}{a^4 r^4}\sum_{i,j=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, \partial_r, arP(e_i, e_j), X)\\
&= \frac{1}{a^5 r^4}\sum_{i,k,j=1}^{7} a^4 r^4[(\nabla^g - \nabla)_{e_i} * \phi](e_j, e_k, (\nabla_{e_i}P)(e_j, e_k), X)\\
&\quad -\frac{1}{a^3 r^3}\sum_{i,k,j=1}^{7} \phi(e_i, e_j, e_k)(\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, \partial_r, X)\\
&\quad -2\frac{a^3 r^3}{a^3 r^3}\sum_{i,j=1}^{7} ([\nabla^g - \nabla]_{e_i}\phi)(e_j, P(e_i, e_j), X)\\
&= \sum_{i,k,j,l=1}^{7} [(\nabla^g - \nabla)_{e_i} * \phi](e_j, e_k, *\phi(e_i, e_l, e_j, e_k)e_l, X)\\
&\quad -3\sum_{i,k,j=1}^{7} \phi(e_i, e_j, e_k)([\nabla^g - \nabla]_{e_i}\phi)(e_j, e_k, X)\\
&= g(*\phi, (\nabla^g - \nabla) * \phi \llcorner X) - 3g(\phi, (\nabla^g - \nabla)\phi \llcorner X).
\end{aligned}
$$

In case $X = \partial_r$, we deduce from Lemma 4.9:

$$
\begin{aligned}
0 &= \sum_{i,j,k=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, \bar{P}(e_i, e_j, e_k), \partial_r) = ar^2\sum_{i,j,k=1}^{7} (\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_j, e_k, (\nabla_{e_i}P)(e_j, e_k), \partial_r)\\
&= -a^4 r^5\sum_{i,j,k=1}^{7} [(\nabla^g - \nabla)_{e_i}\phi](e_j, e_k, (\nabla_{e_i}P)(e_j, e_k))\\
&= -a^4 r^5\left[\sum_{i,j,k,l=1}^{7} (\nabla^g_{e_i}\phi)(e_j, e_k, e_l)(\nabla_{e_i}\phi)(e_j, e_k, e_l)\right.\\
&\quad \left. - \sum_{i,j,k,l=1}^{7} (\nabla_{e_i}\phi)(e_j, e_k, e_l)(\nabla_{e_i}\phi)(e_j, e_k, e_l)\right]\\
&= -a^4 r^5[g(\nabla^g\phi, \nabla\phi) - g(\nabla\phi, \nabla\phi)] = -a^5 r^5[g(\nabla^g\phi, *\phi) - ag(*\phi, *\phi)],
\end{aligned}
$$

and thus, we have $g(\nabla^g\phi, *\phi) = ag(*\phi, *\phi)$. A Spin(7) structure is of class $\mathcal{U}_2$ if it satisfies

$$28(\bar{\nabla}^{\bar{g}}_W\Phi)(X_1, X_2, X_3, X_4) = -\sum_{i=1}^{4}(-1)^{i+1}[\delta\Phi(\bar{p}(X_i))\Phi(W, X_1, .., \hat{X}_i, .., X_4) \\ +7\bar{g}(W, X_i)\delta\Phi(X_1, .., \hat{X}_i, .., X_4)]. \quad (9)$$

Suppose $W = X_1 = \partial_r$ and $X_2, X_3, X_4 \in TM$. For a 3-form ξ on TM, we have

$$\bar{g}(\bar{p}(\partial_r), \xi) = \bar{g}(\partial_r, \bar{P}(\xi)) = -\Phi(\partial_r, \xi) = -a^3r^3\phi(\xi) = \bar{g}(-a^3r^3\phi, \xi)$$

and thus $\bar{p}(\partial_r) = -a^3r^3\phi$. Since $\partial_r \lrcorner \bar{T} = 0$, we have $\bar{\nabla}^{\bar{g}}_{\partial_r}\Phi = 0$ and the defining relation of the class $\mathcal{U}_2$ reduces to

$$\begin{aligned} 0 &= \delta\Phi(p(\partial_r))\Phi(\partial_r, X_2, X_3, X_4) + 7\delta\Phi(X_2, X_3, X_4) \\ &= \delta\Phi(-a^6r^6\phi(X_2, X_3, X_4)\phi + 7X_2 \wedge X_3 \wedge X_4). \end{aligned}$$

Since $a^6r^6\phi(X_2, X_3, X_4)\phi - 7X_2 \wedge X_3 \wedge X_4$ spans $\Lambda^3(TM)$, we have $\delta\Phi = 0$ on TM.

For $X, Y, Z \in TM$, we have

$$\begin{aligned} 0 = \delta\Phi(X, Y, Z) &= -\sum_{i=1}^{8}(\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_i, X, Y, Z) = -\frac{1}{a^2r^2}\sum_{i=1}^{7}(\bar{\nabla}^{\bar{g}}_{e_i}\Phi)(e_i, X, Y, Z) \\ &= -a^2r^2\sum_{i=1}^{7}[(\nabla^g - \nabla)_{e_i} * \phi](e_i, X, Y, Z). \end{aligned}$$

For $X \in TM$, we have

$$\begin{aligned} \delta\Phi(\bar{p}(X)) &= \delta\Phi(\sum_{i<j<k=1}^{8}\bar{g}(\bar{p}(X), \bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_k)\bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_k) \\ &= \delta\Phi(\sum_{i<j<8}\bar{g}(\bar{p}(X), \bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_8)\bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_8) \\ &= \sum_{i<j<8}\bar{g}(\bar{p}(X), \bar{e}_i \wedge \bar{e}_j \wedge \bar{e}_8)\delta\Phi(\bar{e}_i, \bar{e}_j, \partial_r) \\ &= -\sum_{k=1}^{7}\sum_{i<j<8}(\bar{\nabla}^{\bar{g}}_{\bar{e}_k}\Phi)(\bar{e}_k, \bar{e}_i, \bar{e}_j, \partial_r)\bar{g}(X, \bar{P}(\bar{e}_i, \bar{e}_j, \partial_r)) \\ &= \sum_{k=1}^{7}\sum_{i<j<8}a^3r^3(\nabla^g_{\bar{e}_k}\phi)(\bar{e}_k, \bar{e}_i, \bar{e}_j)\Phi(\bar{e}_i, \bar{e}_j, \partial_r, X) \\ &= a^6r^6\sum_{k=1}^{7}\sum_{i<j<8}(\nabla^g_{\bar{e}_k}\phi)(\bar{e}_k, \bar{e}_i, \bar{e}_j)\phi(\bar{e}_i, \bar{e}_j, X) \\ &= -\sum_{i<j<8}\delta\phi(e_i, e_j)\phi(e_i, e_j, X). \end{aligned}$$

Suppose $W = \partial_r$ and $X_1, \dots, X_4 \in TM$. Then, Eq. (9) gives us

$$0 = \sum_{i=1}^{4} (-1)^{i+1} \delta\Phi(\bar{p}(X_i)) a^3 r^3 \phi(X_1, .., \hat{X}_i, .., X_4)$$

$$= a^3 r^3 \sum_{i=1}^{4} \sum_{l<j<8} (-1)^i \delta\phi(e_l, e_j)\phi(e_l, e_j, X_i)\phi(X_1, .., \hat{X}_i, .., X_4).$$

For $W, X_i \in TM$, Eq. (9) reduces to

$$28(\bar{\nabla}^{\bar{g}}_W \Phi)(X_1, X_2, X_3, X_4) = 28a^4 r^4 [(\nabla^g - \nabla)_W * \phi](X_1, X_2, X_3, X_4),$$

which is equal to

$$-\sum_{i=1}^{4} (-1)^{i+1} [\delta\Phi(\bar{p}(X_i))\Phi(W, X_1, .., \hat{X}_i, .., X_4) + 7\bar{g}(W, X_i)\delta\Phi(X_1, .., \hat{X}_i, .., X_4)]$$

$$= a^4 r^4 \sum_{i=1}^{4} \sum_{l<j<8} (-1)^{i+1} \delta\phi(e_l, e_j)\phi(e_l, e_j, X_i) * \phi(W, X_1, .., \hat{X}_i, .., X_4).$$

This proves the statement. □

Remark 4.12 One can use Lemma 4.3 and Lemma 4.9 to simplify these equations in rather lengthy calculations. The property

$$0 = \sum_{i=1}^{7} [(\nabla^g - \nabla)_{e_i} * \phi](e_i, X, Y, Z)$$

can for example be simplified to

$$0 = g((\phi \llcorner Y) \llcorner Z, \delta\phi \llcorner X) + g(\phi \llcorner X, (\nabla^g \phi \llcorner Y) \llcorner Z) - g(\phi \llcorner X, (*\phi \llcorner Y) \llcorner Z).$$

Another simplification (see Lemma 4.19) will be used in the example.

Theorem 4.13 *If the* Spin(7) *structure on the cone* $\bar{M}$ *is of class* $\mathcal{U}_1$*, then:*

- *The* G_2 *structure* ϕ *on* M *cannot be of class* $\mathcal{W}_3 \oplus \mathcal{W}_4$*.*
- *The* G_2 *structure is of class* $\mathcal{W}_1$ *if and only if the* Spin(7) *structure is integrable.*

If the structure on $\bar{M}$ *is of class* $\mathcal{U}_2$*, then the structure on* M *is never of class* $\chi_1 \oplus \chi_3$*.*

Proof Since the relation $g(\nabla^g \phi, *\phi) = 0$ defines the class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, we conclude the first result directly from Lemma 4.11. Now, assume the G_2 structure ϕ is of class $\mathcal{W}_1$, i. e. nearly parallel G_2 (see [23]):

$$\nabla^g \phi = \frac{1}{168} g(\nabla^g \phi, *\phi) * \phi.$$

Taking the scalar product with $*\phi$ on both sides leads to

$$g(\nabla^g \phi, *\phi) = \frac{1}{168} g(\nabla^g \phi, *\phi) g(*\phi, *\phi).$$

With the Spin(7) structure being of class $\mathcal{U}_1$ and the calculation above, we get $g(*\phi, *\phi) = \frac{1}{168} g(*\phi, *\phi) g(*\phi, *\phi)$ and thus $g(*\phi, *\phi) = 168$. Therefore,

$$\nabla^g \phi = \frac{1}{168} g(\nabla^g \phi, *\phi) * \phi = a \frac{1}{168} g(*\phi, *\phi) * \phi = a * \phi.$$

Thus, $\nabla^g\phi = \nabla\phi = a * \phi$, and with Remark 4.10, we get $\nabla = \nabla^g$ and $\bar{\nabla}^{\bar{g}} = \bar{\nabla}$. Since $\bar{\nabla}\Phi = 0$, the Spin(7) structure on $\bar{M}$ is integrable.

Consider a structure on $\bar{M}$ of class $\mathcal{U}_2$. With Lemma 4.11, we get $\delta\Phi = 0$ on TM. To see that this structure is integrable, it is sufficient to show $\partial_r \lrcorner \delta\Phi = 0$, see [21]. We have for $X, Y \in TM$

$$(\partial_r \lrcorner \delta\Phi)(X,Y) = -\sum_{i=1}^{8}(\bar{\nabla}^{\bar{g}}_{\bar{e}_i}\Phi)(\bar{e}_i, \partial_r, X, Y) = ar\sum_{i=1}^{7}((\nabla^g - \nabla)_{e_i}\phi)(e_i, X, Y)$$
$$= -ar\delta\phi(X,Y).$$

This is equal to zero if the structure on M is cocalibrated (of class $\chi_1 \oplus \chi_3$, defined by $\delta\phi = 0$). □

4.3 Corresponding spinors on G_2 manifolds and their cones

Since we have $T - T^c = -\frac{2a}{3}\phi$, the difference $\bar{T} - \overline{T^c}$ is the lift of $a^2r^2T - a^2r^2T^c = -\frac{2a}{3}a^2r^2\phi$. Furthermore, $\frac{1}{a^3r^3}\partial_r \lrcorner \Phi$ is the lift of ϕ to $\bar{M}$, and hence, we have

$$\bar{T} - \overline{T^c} = -\frac{2}{3r}\partial_r \lrcorner \Phi.$$

Now Lemma 2.9 implies:

Theorem 4.14 *For a G_2T manifold with characteristic connection ∇^c and for $\alpha = \frac{1}{2}a$ or $\alpha = -\frac{1}{2}a$, there is*

(1) A one to one correspondence between Killing spinors with torsion

$$\nabla^s_X\psi = \alpha X\psi$$

on M, and parallel spinors of the connection $\bar{\nabla}^s + \frac{4s}{3r}\partial_r \lrcorner \Phi$ on $\bar{M}$ with cone constant a

$$\bar{\nabla}^s_X\psi + \frac{2s}{3r}(X \lrcorner (\partial_r \lrcorner \Phi))\psi = 0.$$

(2) A one to one correspondence between $\bar{\nabla}^s$-parallel spinors on $\bar{M}$ with cone constant a and spinors on M satisfying

$$\nabla^s_X\psi = \alpha X\psi + \frac{2as}{3}(X \lrcorner \phi)\psi.$$

In particular for $s = \frac{1}{4}$, we get the correspondence

Spinors on M	Spinors on $\bar{M}$
$\nabla^c_X\psi = \alpha X\psi$	$\bar{\nabla}_X\psi = -\frac{1}{6r}(X \lrcorner (\partial_r \lrcorner \Phi))\psi$
$\nabla^c_X\psi = \alpha X\psi + \frac{a}{6}(X \lrcorner \phi)\ \psi$	$\bar{\nabla}_X\psi = 0$

Remark 4.15 As for metric almost contact structures (see Remark 3.15), one can use the characterization $\bar{T} = -\delta\Phi - \frac{7}{6} * (\theta \wedge \Phi)$ with $\theta = \frac{1}{7} * (\delta\Phi \wedge \Phi)$ (see [39]) and the description of T^c given in Theorem 4.8 of [27] to rewrite these equations in terms of the geometric data of the Spin(7) structure.

Theorem 4.14 states, as before, the general correspondence between spinors on the base and spinors on the cone. However, G_2T manifolds, i. e. carrying a characteristic connection ∇^c, enjoy a further, very special property: The G_2 structure ϕ induces a unique spinor field ψ of length one, and this spinor field is ∇^c-parallel, $\nabla^c\psi = 0$. This is due to the fact that G_2 is the stabilizer of a generic spinor in Δ_7, the spin representation in dimension 7. For a nearly parallel G_2 manifold, it is well known that ψ is just the Riemannian Killing spinor (see [27,28,30,31] for all these results). Thus, ψ induces in this case the ∇^g-parallel spinor of the integrable Spin(7) structure on the cone. We prove that this result carries over to all admissible G_2 manifolds.

Corollary 4.16 *Let (M, g, ϕ) be a G_2T manifold with characteristic connection ∇^c, ψ the ∇^c-parallel spinor field defined by ϕ. Then, ψ satisfies*

$$\nabla^c_X\psi = -\frac{a}{2}X\psi + \frac{a}{6}(X\lrcorner\phi)\psi$$

for every $a > 0$ and induces a $\bar{\nabla}$-parallel spinor on the cone $\bar{M}$, constructed with cone constant a and endowed with its induced Spin(7) *structure.*

Proof The crucial observation is the algebraic identity

$$(X\lrcorner\phi)\cdot\psi = 3\,X\cdot\psi$$

that holds for all vector fields X. Since The 7-dimensional standard representation $\mathbb{R}^7$ of G_2 is isomorphic to the G_2 representation

$$\Lambda^2_7 = \{X\lrcorner\phi \mid X \in \mathbb{R}^7\} \subset \Lambda^2(\mathbb{R}^7) = \mathfrak{so}(7) = \Lambda^2_7 \oplus \mathfrak{g}_2,$$

it is clear that there exists a constant c s. t. $(X\lrcorner\phi)\cdot\psi = cX\cdot\psi$; one then computes its explicit value in any realization of the spin representation. Thus, the equation for ψ follows, and we can apply Theorem 4.14. □

Be cautious that ∇^c may have more parallel spinor fields than just ψ; for these, we cannot define a suitable 'lifted' spinor on the cone, unless one finds a similar trick to write the spinor field equation in a form covered by Theorem 4.14.

Remark 4.17 In Theorem 1.1 of [39], S. Ivanov proves that any Spin(7) manifold admits a spinor field that is parallel with respect to the characteristic connection. Corollary 4.16 gives an explicit construction of this spinor in case the Spin(7) manifold is the cone of an admissible G_2 manifold.

Remark 4.18 Since Corollary 4.16 holds for any G_2T manifold, one could also carry out the whole study without using the 3-form ϕ and the 4-form Φ: the spinor field ψ describes the G_2 structure completely, and then, one considers the induced $\bar{\nabla}$-parallel spinor φ on the cone described in Corollary 4.16 and establishes the correspondence between the G_2 classes and the Spin(7) classes by studying the equations satisfied by ψ and φ.

4.4 Examples

To simplify the calculations in the example, we reformulate the second condition for a G_2 structure on M to imply a Spin(7) structure of class $\mathcal{U}_1$ on $\bar{M}$ of Lemma 4.11. So we only have to calculate ϕ, $*\phi$ and $\nabla^g\phi$ to check the conditions. We omit the proof of the following result, and it is a lengthy, but straight forward continuation of the calculations in the proof of Lemmas 4.11 and 4.3.

Lemma 4.19 *The second condition of Lemma 4.11*

$$g(*\phi,[(\nabla-\nabla^g)*\phi]\llcorner X) = 3g(\phi,[(\nabla-\nabla^g)\phi]\llcorner X)$$

is equivalent to

$$\begin{aligned}0=\sum_{i,k,j,l,m=1}^{7}\Big[&*\phi(e_i,e_j,e_k,e_l)(\nabla^g_{e_i}\phi)(e_j,e_k,e_m)\phi(e_m,e_l,X)\\&+*\phi(e_i,e_j,e_k,e_l)(\nabla^g_{e_i}\phi)(e_l,X,e_m)\phi(e_m,e_j,e_k)\\&-*\phi(e_i,e_j,e_k,e_l)*\phi(e_i,e_j,e_k,e_m)\phi(e_m,e_l,X)\\&-*\phi(e_i,e_l,e_j,e_k)*\phi(e_i,e_l,X,e_m)\phi(e_m,e_j,e_k)\Big]\\&+3\sum_{i,k,j=1}^{7}\Big[-\phi(e_i,e_j,e_k)(\nabla^g_{e_i}\phi)(e_j,e_k,X)+a\,\phi(e_i,e_j,e_k)*\phi(e_i,e_j,e_k,X)\Big].\end{aligned}$$

Example 4.20 Let $(M,\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3)$ be a 7 dimensional 3-Sasaki manifold with corresponding 2-forms F_i, $i=1,2,3$. Let η_i for $i=1,..,7$ be the dual of a local basis $\{e_1=\xi_1,e_2=\xi_2,e_3=\xi_3,e_4,..,e_7\}$, such that

$$F_1=-\eta_{23}-\eta_{45}-\eta_{67},\quad F_2=\eta_{13}-\eta_{46}+\eta_{57},\quad F_3=-\eta_{13}-\eta_{47}-\eta_{56}.$$

Here, for $\eta_i\wedge..\wedge\eta_j$, we write $\eta_{i,..,j}$. In [5], it is explained that there is no characteristic connection as such, but one can construct a cocalibrated G_2 structure

$$\begin{aligned}\phi=\eta_1\wedge F_1+\eta_2\wedge F_2+\eta_3\wedge F_3+4\eta_1\wedge\eta_2\wedge\eta_3&=\eta_{123}-\eta_{145}-\eta_{167}-\eta_{246}+\eta_{257}\\&-\eta_{347}-\eta_{356}\end{aligned}$$

with characteristic connection ∇^c and torsion $T^c=\eta_1\wedge d\eta_1+\eta_2\wedge d\eta_2+\eta_3\wedge d\eta_3$ that is very well adapted to the 3-Sasakian structure. It is therefore called the *canonical G_2 structure* of the underlying 3-Sasakian structure. Corollary 4.16 ensures then the existence of a $\bar\nabla$-parallel spinor field on $\bar M$.

We calculate the class of the Spin(7) structure on $\bar M$ of the canonical G_2 structure using Lemma 4.11.

Theorem 4.21 *The* Spin(7) *structure on the cone constructed from the canonical G_2 structure of a* 3*-Sasakian manifold is of class $\mathcal{U}_1$ if and only if the cone constant is $a=\frac{15}{14}$.*

Proof Due to the formulation of the second condition of Lemma 4.11 given in Lemma 4.19, we just need to calculate $*\phi$ and $\nabla^g\phi$. Obviously, $*\phi$ is given by

$$*\phi=\eta_{4567}-\eta_{2367}-\eta_{2345}-\eta_{1357}+\eta_{1346}-\eta_{1256}-\eta_{1247}.$$

To get $\nabla^g\phi$, we observe

$$\begin{aligned}\nabla^g_{e_j}\phi=&(\nabla^g_{e_j}\eta_1)\wedge F_1+(\nabla^g_{e_j}\eta_2)\wedge F_2+(\nabla^g_{e_j}\eta_3)\wedge F_3\\&+\eta_1\wedge(\nabla^g_{e_j}F_1)+\eta_2\wedge(\nabla^g_{e_j}F_2)+\eta_3\wedge(\nabla^g_{e_j}F_3)\\&+4(\nabla^g_{e_j}\eta_1)\wedge\eta_2\wedge\eta_3+4\eta_1\wedge(\nabla^g_{e_j}\eta_2)\wedge\eta_3+4\eta_1\wedge\eta_2\wedge(\nabla^g_{e_j}\eta_3)\end{aligned}$$

and since (η_i,F_i) are Sasakian structures, we have $(\nabla^g_{e_j}F_i)(Y,Z)=g(e_j,Z)\eta_i(Y)-g(e_j,Y)\eta_i(Z)$. Thus, $\nabla^g_{e_j}F_i=\eta_j\wedge\eta_i$ for $i=1,2,3$ and $j=1,..,7$ implying $\eta_i\wedge(\nabla^g_{e_j}F_i)=0$. Since

$$(\nabla^g_X \eta_i)Y = g(Y, \nabla^g_X \xi_i) = g(Y, -\phi_i X) = F_i(X, Y)$$

we have $\nabla^g_X \eta_i = X \lrcorner F_i$ and get

$$\begin{aligned}\nabla^g_{e_j}\phi =& (e_j \lrcorner F_1) \wedge F_1 + (e_j \lrcorner F_2) \wedge F_2 + (e_j \lrcorner F_3) \wedge F_3 \\ &+ 4(e_j \lrcorner F_1) \wedge \eta_2 \wedge \eta_3 + 4\eta_1 \wedge (e_j \lrcorner F_2) \wedge \eta_3 + 4\eta_1 \wedge \eta_2 \wedge (e_j \lrcorner F_3).\end{aligned}$$

This gives us

$$\begin{aligned}&\nabla^g_{e_1}\phi = -\eta_{346} + \eta_{357} + \eta_{247} + \eta_{256}, \quad \nabla^g_{e_2}\phi = \eta_{345} + \eta_{367} - \eta_{147} - \eta_{156},\\ &\nabla^g_{e_3}\phi = -\eta_{245} - \eta_{267} + \eta_{146} - \eta_{157}, \quad \nabla^g_{e_4}\phi = 3\,(-\eta_{235} + \eta_{567} + \eta_{136} - \eta_{127}),\\ &\nabla^g_{e_5}\phi = 3\,(\eta_{234} - \eta_{467} - \eta_{137} - \eta_{126}), \quad \nabla^g_{e_6}\phi = 3\,(-\eta_{237} + \eta_{457} - \eta_{134} + \eta_{125}),\\ &\nabla^g_{e_7}\phi = 3\,(\eta_{236} - \eta_{456} + \eta_{135} + \eta_{124}).\end{aligned}$$

Using an appropriate computer algebra system, we easily calculate

$$g(\nabla^g\phi, *\phi) = 180, \quad g(*\phi, *\phi) = 168,$$

thus the first condition of Lemma 4.11 is satisfied if $a = \frac{15}{14}$. Using the formulation given in Lemma 4.19 of the second condition, one easily checks that the this condition is satisfied for any a. □

We expect that for all other values of the cone constant a, the structure is of generic class $\mathcal{U}_1 \oplus U_2$, but the system of equations that one obtains is extremely involved.

Acknowledgments Both authors thank Thomas Friedrich (Berlin) for his steady mathematical interaction. Ilka Agricola acknowledges financial support by the DFG within the priority program 1388 "Representation theory". Jos Höll thanks Philipps-Universität Marburg for a Ph.D. grant. He is member of the 'Graduate Center for Life and Natural Sciences' of Philipps-Universität Marburg and of the special graduate program 'Lie theory and complex geometry' of the Department of Mathematics and Computer Science. Finally, we thank the referee for his careful reading of the manuscript.

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