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CONES OF MATRICES AND SET-FUNCTIONS AND 0–1 OPTIMIZATION*

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Abstract. It has been recognized recently that to represent a polyhedron as the projection of a higher-dimensional, but simpler, polyhedron, is a powerful tool in polyhedral combinatorics. A general method is developed to construct higher-dimensional polyhedra (or, in some cases, convex sets) whose projection approximates the convex hull of 0–1 valued solutions of a system of linear inequalities. An important feature of these approximations is that one can optimize any linear objective function over them in polynomial time.

In the special case of the vertex packing polytope, a sequence of systems of inequalities is obtained such that the first system already includes clique, odd hole, odd antihole, wheel, and orthogonality constraints. In particular, for perfect (and many other) graphs, this first system gives the vertex packing polytope. For various classes of graphs, including t-perfect graphs, it follows that the stable set polytope is the projection of a polytope with a polynomial number of facets.

An extension of the method is also discussed which establishes a connection with certain submodular functions and the Möbius function of a lattice.

Key words. polyhedron, cone, vertex packing polytope, perfect graph, Möbius function

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0. Introduction. One of the most important methods in combinatorial optimization is that which represents each feasible solution of the problem by a 0–1 vector (usually the incidence vector of the appropriate set), and then describes the convex hull K of the solutions by a system of linear inequalities. In the nicest cases (e.g., in the case of the bipartite matching problem) we obtain a system that has polynomial size (measured in the natural "size" n of the problem). In such a case, we can compute the maximum of any linear objective function in polynomial time by solving a linear program. In other cases, however, the convex hull of feasible solutions has exponentially many facets and so can only be described by a linear program of exponential size. For many combinatorial optimization problems (including those solvable in polynomial time), this exponentially large set of linear inequalities is still "nice" in one sense or another. We mention two possible notions of "niceness":

—Given an inequality in the system, there is a polynomial size certificate of the fact that it is valid for K. If this is the case, the problem of determining whether a given vector is in K is in the complexity class co-NP.

—There is a polynomial time separation algorithm for the system; that is, given a vector, we can check in polynomial time whether it satisfies the system, and if not, we can find an inequality in the system that is violated. It follows, then, from general results on the ellipsoid method (see Grötschel, Lovász, and Schrijver [14]) that every linear objective function can be optimized over K in polynomial time.

Many important theorems in combinatorial optimization provide such "nice" descriptions of polyhedra. Important examples of polyhedra with "nice" descriptions are matching polyhedra, matroid polyhedra, stable set polyhedra for perfect graphs, etc. On the other hand, stable set polyhedra, in general, or travelling salesman polyhedra, are not known to have "nice" descriptions (and probably do not have any). Typically, to find such a "nice" description and to prove its correctness, one needs ad

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hoc methods depending on the combinatorial structure. However, one can mention two general ideas that can help in obtaining such linear descriptions:

—*Gomory-Chvátal cuts.* Let P be a polytope with integral vertices. Assume that we have already found a system of linear inequalities valid for P whose *integral* solutions are precisely the integral vectors in P. The solution set of this system is a polytope K containing P which will in general be larger than P. We can generate further linear inequalities valid for P (but not necessarily for K) as follows. Given a linear inequality

$$\sum_i a_i x_i \leqq \alpha$$

valid for K, where the a_i are integers, the inequality

$$\sum_i a_i x_i \leqq \lfloor \alpha \rfloor$$

is still valid for P but may eliminate some part of K. Gomory [11] used a special version of this construction in his integer programming algorithm. If we take all inequalities obtainable in this way, they define a polytope K' with $P \subseteq K' \subset K$. Repeating this with K' in place of K we obtain K'', etc. Chvátal [8] proved that in a finite number of steps, we obtain the polytope P itself.

Unfortunately, the number of steps needed may be very large; it depends not only on the dimension but also on the coefficients of the system with which we start. Another problem with this procedure is that there is no efficient way known to implement it algorithmically. In particular, even if we know how to optimize a linear objective function over K in polynomial time (say, K is given by an explicit, polynomial size linear program), and $K' = P$, we know of no general method to optimize a linear objective function over P in polynomial time.

—*Projection representation* (*new variables*). This method has received much attention lately. The idea is that a projection of a polytope may have more facets than the polytope itself. This remark suggests that even if P has exponentially many facets, we may be able to represent it as the projection of a polytope Q in higher (but still polynomial) dimension, having only a polynomial number of facets. Among others, Barahona [4]; Liu [16]; Ball, Liu, and Pulleyblank [3]; Maculan [19]; Balas and Pulleyblank [1], [2]; Barahona and Mahjoub [5]; and Cameron and Edmonds [6] have provided nontrivial examples of such a representation. It is easy to see that such a representation can be used to optimize linear objective functions over P in polynomial time. In the negative direction, Yannakakis [26] proved that the travelling salesman polytope and the matching polytope of complete graphs cannot be represented this way, assuming that the representation is "canonical." (Let $P \subseteq \mathbb{R}^n$ and $P' \subseteq \mathbb{R}^m$ be two polytopes. We say that a projection representation $\pi: P' \to P$ is *canonical* if the group Γ of isometries of $\mathbb{R}^n$ preserving P has an action as isometries of $\mathbb{R}^m$ preserving P' so that the projection commutes with these actions. Such a representation is obtained, e.g., when new variables are introduced in a "canonical" way—in the case of the travelling salesman polytope, this could mean variables assigned to edges or certain other subgraphs, and constraints on these new variables are derived from local properties. If we have to start with a reference orientation, or with specifying a root, then the representation obtained will not be canonical.) No negative results seem to be known without this symmetry assumption.

One way to view our results is to provide a general procedure to create such liftings. The idea is to extend the method of Grötschel, Lovász, and Schrijver [12] for finding maximum stable sets in perfect graphs to general 0-1 programs. We represent a feasible subset not by its incidence vector v but by the matrix vv^T. This squares the number of variables, but in return we obtain two new powerful ways to write down linear constraints. Projecting back to the "usual" space, we obtain a procedure somewhat similar to the Gomory-Chvátal procedure: it "cuts down" a convex set K to a new convex set K' so that all 0–1 solutions are preserved. In contrast to the Gomory-Chvátal cuts, however, any subroutine to optimize a linear objective function over K can be used to optimize a linear objective function over K'. Moreover, repeating the procedure at most n times, we obtain the convex hull P of 0–1 vectors in K.

Our method is closely related to recent work of Sherali and Adams [22]. They introduce new variables for products of the original ones and characterize the convex hull, in this high-dimensional space, of vectors associated with 0–1 solutions of the original problem. In this way they obtain a sequence of relaxations of the 0–1 optimization problem, the first of which is essentially the N operator introduced in § 1 below. Further, members of the two sequences of relaxations are different but closely related; some of our results in § 3, in particular, formula (6) and Theorem 3.3, follow directly from their work.

This method is also related to (but different from) the recent work of Pemantle, Propp, and Ullman [20] on the tensor powers of linear programs.

In § 1, we describe the method in general, and prove its basic properties. Section 2 contains applications to the vertex packing problem, one of the best studied combinatorial optimization problems. It will turn out that our method gives in one step almost all of the known classes of facets of the vertex packing polytope. It will follow, in particular, that if a graph has the property that its stable set polytope is described by the clique, odd hole, and odd antihole constraints, then its maximum stable set can be found in polynomial time.

In § 3 we put these results in a wider context by raising the dimension even higher. We introduce exponentially many new variables; in this high-dimensional space, rather simple and elegant polyhedral results can be obtained. The main part of the work is to "push down" the inequalities to a low dimension and to carry out the algorithms using only a polynomial number of variables and constraints. It will turn out that the methods in § 1, as well as other constructions like TH (G), as described in Grötschel, Lovász, and Schrijver [13], [14], follow in a natural way.

1. Matrix cuts. In this section we describe a general construction for "lifting" a 0–1 programming problem in n variables to n^2 variables, and then projecting it back to the n-space so that cuts, i.e., tighter inequalities still valid for all 0–1 solutions, are introduced. It will be convenient to deal with homogeneous systems of inequalities, i.e., with convex cones rather than polytopes. Therefore we embed the n-dimensional space in $\mathbb{R}^{n+1}$ as the hyperplane $x_0 = 1$. (The 0th variable will play a special role throughout.)

One way to view our constructions is to generate *quadratic* inequalities valid for all 0–1 solutions. These may be viewed as homogeneous linear inequalities in the $\binom{n}{2}+n+1$-dimensional space, and they define a cone there. (This space can be identified with the space of symmetric $(n+1)\times(n+1)$ matrices.) We then combine these quadratic inequalities to eliminate all quadratic terms in order to obtain linear inequalities not derivable directly. This corresponds to projecting the cone down the $n+1$-dimensional space.

1.a. The construction of matrix cones and their projections. Let K be a convex cone in $\mathbb{R}^{n+1}$. Let K^* be its polar cone, i.e., the cone defined by

$$K^* = \{u \in \mathbb{R}^{n+1}: u^T x \geqq 0 \text{ for all } x \in K\}.$$

We denote by K^o the cone spanned by all 0–1 vectors in K. Let Q denote the cone spanned by all 0–1 vectors $x \in \mathbb{R}^{n+1}$ with $x_0 = 1$. We are interested in determining K^o, and generally we may restrict ourselves to subcones of Q. We denote by e_i the ith unit vector, and set $f_i = e_0 - e_i$. Note that the cone Q^* is spanned by the vectors e_i and f_i. For any $(n+1)\times(n+1)$ matrix Y, we denote by $\bar{Y}$ the vector composed of the diagonal entries of Y.

Let $K_1 \subseteq Q$ and $K_2 \subseteq Q$ be convex cones. We define the cone $M(K_1, K_2) \subseteq \mathbb{R}^{(n+1)\times(n+1)}$ consisting of all $(n+1)\times(n+1)$ matrices $Y = (y_{ij})$ satisfying (i), (ii), and (iii) below (for motivation, the reader may think of Y as a matrix of the form xx^T, where x is a 0–1 vector in $K_1 \cap K_2$).

(i) Y is symmetric;

(ii) $\bar{Y} = Ye_0$, i.e., $y_{ii} = y_{0i}$ for all $1 \leqq i \leqq n$;

(iii) $u^T Y v \geqq 0$ holds for every $u \in K_1^*$ and $v \in K_2^*$.

Note that (iii) can be rewritten as

(iii′) $YK_2^* \subseteq K_1$.

We shall also consider a slightly more complicated cone $M_+(K_1, K_2)$, consisting of matrices Y satisfying the following condition, in addition to (i), (ii), and (iii):

(iv) Y is positive semidefinite.

From the assumption that K_1 and K_2 are contained in Q it follows that every $Y = (y_{ij}) \in M(K_1, K_2)$ satisfies $y_{ij} \geqq 0$, $y_{ij} \leqq y_{ii} = y_{0i} \leqq y_{00}$, and $y_{ij} \geqq y_{ii} + y_{jj} - y_{00}$.

These cones of matrices are defined by linear constraints and so their polars can also be expressed quite nicely. Let U_{psd} denote the cone of positive semidefinite $(n+1)\times(n+1)$ matrices (which is self-dual in the space U_{sym} of symmetric matrices), and U_{skew} the linear space of skew symmetric $(n+1)\times(n+1)$ matrices (which is the orthogonal complement of U_{sym}). Let U_1 denote the linear space of $(n+1)\times(n+1)$ matrices (w_{ij}), where $w_{0j} = -w_{jj}$ for $1 \leqq j \leqq n$, $w_{00} = 0$ and $w_{ij} = 0$ if $i \neq 0$ and $i \neq j$. Note that U_1 is generated by the matrices $f_i e_i^T$ $(i = 1, \cdots, n)$.

With this notation, we have, by definition,

$$M(K_1, K_2)^* = U_1 + U_{\text{skew}} + \text{cone}\,\{uv^T\colon u \in K_1^*,\ v \in K_2^*\},$$

and

$$M_+(K_1, K_2)^* = U_1 + U_{\text{skew}} + U_{\text{psd}} + \text{cone}\,\{uv^T\colon u \in K_1^*,\ v \in K_2^*\}.$$

Note that only the last term depends on the cones K_1 and K_2. In this term, it would be enough to let u and v run over extreme rays of K_1^* and K_2^*, respectively. So if K_1 and K_2 are polyhedral, then so is $M(K_1, K_2)$, and the number of its facets is at most the product of the numbers of facets of K_1 and K_2.

Note that U_{psd} and hence $M_+(K_1, K_2)$ will generally be nonpolyhedral.

We project down these cones from the $(n+1)\times(n+1)$-dimensional space to the $(n+1)$-dimensional space by letting

$$N(K_1, K_2) = \{Ye_0\colon Y \in M(K_1, K_2)\} = \{\bar{Y}\colon Y \in M(K_1, K_2)\}$$

and

$$N_+(K_1, K_2) = \{Ye_0\colon Y \in M_+(K_1, K_2)\} = \{\bar{Y}\colon Y \in M_+(K_1, K_2)\}.$$

Clearly, $M(K_1, K_2) = M(K_2, K_1)$ and so $N(K_1, K_2) = N(K_2, K_1)$ (and similarly for the "+" subscripts).

If $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a linear transformation mapping the cone Q onto itself, then clearly $M(AK_1, AK_2) = AM(K_1, K_2)A^T$. If $n \geqq 2$, then from $AQ = Q$ it easily follows that $A^T e_0$ is parallel to e_0, and hence $N(AK_1, AK_2) = AN(K_1, K_2)$. In particular, we can "flip" coordinates, replacing x_i by $x_0 - x_i$ for some $i \neq 0$.

If K_1 and K_2 are polyhedral cones, then so too are $M(K_1, K_2)$ and $N(K_1, K_2)$. The cones $M_+(K_1, K_2)$ and $N_+(K_1, K_2)$ are also convex (but generally not polyhedral), since (iv) is equivalent to an infinite number of linear inequalities.

LEMMA 1.1. $(K_1 \cap K_2)^o \subseteq N_+(K_1, K_2) \subseteq N(K_1, K_2) \subseteq K_1 \cap K_2$.

Proof. (1) Let x be any nonzero 0-1 vector in $K_1 \cap K_2$. Since $K_1 \subseteq Q$, we must have $x_0 = 1$. Using this it is easy to check that the matrix $Y = xx^T$ satisfies (i)-(iv). Hence $x = Ye_0 \in N_+(K_1, K_2)$.

(2) $N_+(K_1, K_2) \subseteq (K_1, K_2)$ trivially.

(3) Let $x \in N(K_1, K_2)$. Then there exists a matrix Y satisfying (i)-(iv) such that $x = Ye_0$. Now, by our hypothesis that $K_1 \subseteq Q$, it follows that $e_0 \in K_1^*$, and hence by (iii'), $x = Ye_0$ is in K_2. Similarly, $x \in K_1$. □

We will see that, in general, $N(K_1, K_2)$ will be much smaller than $K_1 \cap K_2$.

The reason why we consider two convex cones instead of one is technical. We shall need only two special choices: either $K_1 = K_2 = K$ or $K_1 = K$, $K_2 = Q$. It is easy to see that

$$N(K_1 \cap K_2, K_1 \cap K_2) \subseteq N(K_1, K_2) \subseteq N(K_1 \cap K_2, Q).$$

This suggests that it would suffice to consider $N(K, K)$; but, as we shall see, $N(K, Q)$ behaves algorithmically better (see Theorem 1.6 and the remark following it), and this is why we allow two different cones. To simplify notation, we set $N(K) = N(K, Q)$ and $M(K) = M(K, Q)$. In this case, $K_2^* = Q^*$ is generated by the vectors e_i and f_i, and hence (iii') has the following convenient form:

(iii'') Every column of Y is in K; the difference of the first column and any other column is in K.

1.b. Properties of the cut operators. We give a lemma that yields a more explicit representation of constraints valid for $N(K)$ and $N_+(K)$. Unfortunately, the geometric meaning of $N(K)$ and $N_+(K)$ is not immediate; Lemmas 1.3 and 1.5 may be of some help in visualizing these constructions.

LEMMA 1.2. *Let $K \subseteq Q$ be a convex cone in $\mathbb{R}^{n+1}$ and $w \in \mathbb{R}^{n+1}$.*

(a) *$w \in N(K)^*$ if and only if there exist vectors $a_1, \cdots, a_n \in K^*$, a real number λ, and a skew symmetric matrix A such that $a_i + \lambda e_i + Ae_i \in K^*$ for $i = 1, \cdots, n$, and $w = \sum_{i=1}^n a_i + A\mathbf{1}$ (where $\mathbf{1}$ denotes the all-1 vector).*

(b) *$w \in N_+(K)^*$ if and only if there exist vectors $a_1, \cdots, a_n \in K^*$, a real number λ, a positive semidefinite symmetric matrix B, and a skew symmetric matrix A such that $a_i + \lambda e_i + Ae_i + Be_i \in K^*$ for $i = 1, \cdots, n$, and $w = \sum_{i=1}^n a_i + A\mathbf{1} + B\mathbf{1}$.*

Proof. Assume that $w \in N(K)^*$. Then $we_0^T \in M(K)^*$, and so we can write

$$we_0^T = \sum_i a_t b_t^T + \sum_{i=1}^n \lambda_i e_i f_i^T + A,$$

where $a_t \in K^*$, $b_t \in Q^*$, $\lambda_i \in \mathbb{R}$, and A is a skew symmetric matrix. Since Q^* is spanned by the vectors e_i and f_i, we may express the vectors b_i in terms of them and obtain a representation of the form

$$we_0^T = \sum_{i=1}^n a_i e_i^T + \sum_{i=1}^n \bar{a}_i f_i^T + \sum_{i=1}^n \lambda_i e_i f_i^T + A, \tag{1}$$

where $a_i, \bar{a}_i \in K^*$. Multiplying (1) by e_j from the right we get

$$0 = a_j - \bar{a}_j - \lambda_j e_j + Ae_j. \tag{2}$$

Multiplying (1) by e_0 and using (2) we get

$$w=\sum_{i=1}^{n}\bar{a}_i+\sum_{i=1}^{n}\lambda_ie_i+Ae_0=\sum_{i=1}^{n}a_i+\sum_{i=1}^{n}Ae_i+Ae_0=\sum_{i=1}^{n}a_i+A\mathbf{1}.$$

Here $a_j-\lambda_je_j+Ae_j=\bar{a}_j\in K^*$. Since, trivially, $e_j\in K^*$, this condition remains valid if we decrease λ_j. Hence we can choose all the $\lambda_j=-\lambda$ equal. This proves the necessity of the condition given in (a).

The sufficiency of the condition, as well as of assertion (b), are proved by similar arguments. □

Our next lemma gives a geometric property of $N(K)$, which is easier to apply than the algebraic properties discussed before. Let $H_i=\{x\in\mathbb{R}^{n+1}: x_i=0\}$ and $G_i=\{x\in\mathbb{R}^{n+1}: x_i=x_0\}$. Clearly, H_i and G_i are hyperplanes supporting Q at a facet, and all facets of Q are determined this way.

LEMMA 1.3. *For every convex cone* $K\subseteq Q$ *and every* $1\leqq i\leqq n$,

$$N(K)\subseteq(K\cap H_i)+(K\cap G_i).$$

Proof. Consider any $x\in N(K)$ and let $Y\in M(K)$ be a matrix such that $Ye_0=x$. Let y_i denote the ith column of Y. Then by (ii), $y_i\in G_i$ and by (iii″), $y_i\in K$, so $y_i\in K\cap G_i$. Similarly, $y_0-y_i\in K\cap H_i$, and so $Ye_0=y_0=(y_0-y_i)+y_i\in(K\cap H_i)+(K\cap G_i)$. □

Let us point out the following consequence of this lemma: if $K\cap G_i=\{0\}$, then $N(K)\subseteq H_i$. If, in particular, K meets both opposite facets of Q only in the 0 vector, then $N(K)=\{0\}$. This may be viewed as a very degenerate case of Gomory-Chvátal cuts (see below for more on the connection with Gomory-Chvátal cuts).

One could define a purely geometric cutting procedure based on this lemma: for each cone $K\subseteq Q$, consider the cone

$$N_0(K)=\bigcap\nolimits_i((K\cap G_i)+(K\cap H_i)). \tag{3}$$

This cone is similar to $N(K)$ but is generally bigger. We remark that this cone could also be obtained from a rather natural matrix cone by projection: this arises by imposing (ii), (iii), and the following restricted form of (i): $y_{0i}=y_{i0}$ for $i=1,\cdots,n$.

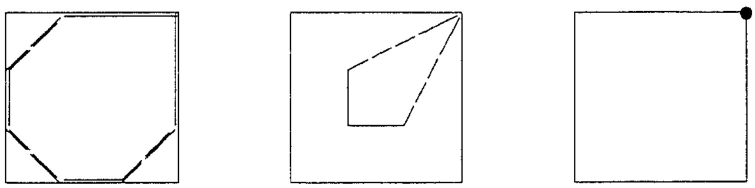
Figure 1 shows the intersection of three cones in $\mathbb{R}^3$ with the hyperplane $x_3=1$: the cones K, $N(K)$, and $N(N(K))$, and the constraints implied by Lemma 1.3. We see that the cone in Lemma 1.3 gets close to $N(K)$ but does not coincide with it.

We remark that $N(K\cap H_i)=N(K)\cap H_i$ for $i=1,\cdots,n$; it should be noted that $N(K\cap H_i)$ does not depend on whether it is computed as a cone in $\mathbb{R}^{n+1}$ or in H_i.

We can get a better approximation of K^o by iterating the operator N. Define $N^t(K)$ recursively by $N^0(K)=K$ and $N^t(K)=N(N^{t-1}(K))$ for $t\geqq 1$.

THEOREM 1.4. $N^n(K)=K^o$.

Proof. Consider the unit cube Q' in the hyperplane $x_0=0$ and let $1\leqq t\leqq n$. Consider any face F of Q' of dimension $n-t$ and let $\bar{F}$ be the union of faces of Q' parallel to



K $\quad N(K)$ $\quad N^2(K)$

FIG. 1

F. We prove, by induction on t, that

$$N^t(K) \subseteq \text{cone}\,(K \cap \bar{F}). \tag{4}$$

For $t=n$, this is just the statement of the theorem. For $t=1$, this is equivalent to Lemma 1.3.

We may assume that F contains the vector e_0. Let F' be an $(n-t+1)$-dimensional face of Q' containing F and let i be an index such that $F' \cap H_i = F$. Then, by the induction hypothesis,

$$N^{t-1}(K) \subseteq \text{cone}\,(K \cap \overline{F'}).$$

Hence by Lemma 1.3,

$$\begin{aligned} N^t(K) = N(N^{t-1}(K)) &\subseteq \text{cone}\,(N^{t-1}(K) \cap (H_i \cup G_i)) \\ &\subseteq \text{cone}\,([\text{cone}\,(K \cap \overline{F'}) \cap H_i] \cup [\text{cone}\,(K \cap \overline{F'}) \cap G_i]). \end{aligned}$$

Now H_i is a supporting plane of cone $(K \cap \overline{F'})$ and hence its intersection with the cone is spanned by its intersection with the generating set of the cone:

$$\text{cone}\,(K \cap \overline{F'}) \cap H_i = \text{cone}\,(K \cap \overline{F'} \cap H_i) \subseteq \text{cone}\,(K \cap \bar{F}).$$

Similarly,

$$\text{cone}\,(K \cap \overline{F'}) \cap G_i \subseteq \text{cone}\,(K \cap \bar{F}).$$

Hence (4) follows. □

Next we show that if we use positive semidefiniteness, i.e., we consider $N_+(K)$, then an analogue of Lemma 1.3 can be obtained that is more complicated but important in the applications to combinatorial polyhedra.

LEMMA 1.5. *Let $K \subseteq Q$ be a convex cone and let $a \in \mathbb{R}^{n+1}$ be a vector such that $a_i \leqq 0$ for $i=1, \cdots, n$ and $a_0 \geqq 0$. Assume that $a^Tx \geqq 0$ is valid for $K \cap G_i$ for all i such that $a_i < 0$. Then $a^Tx \geqq 0$ is valid for $N_+(K)$.*

(The condition that $a_0 \geqq 0$ excludes only trivial cases. The condition that $a_i \leqq 0$ is a normalization, which can be achieved by flipping coordinates.)

Proof. First, assume that $a_0=0$. Consider a subscript i such that $a_i<0$. (If no such i exists, we have nothing to prove.) Then for every $x \in G_i \setminus \{0\}$, we have $a^Tx \leqq a_ix_i < 0$, and so, $x \notin K$. Hence $K \cap G_i = \{0\}$, and so by Lemma 1.3, $N_+(K) \subseteq N(K) \subseteq K \cap H_i$. As this is true for all i with $a_i<0$, we know that $a^Tx=0$ for all $x \in N_+(K)$.

Second, assume that $a_0>0$. Let $x \in N_+(K)$ and let $Y \in M_+(K)$ be a matrix with $Ye_0=x$. For any $1 \leqq i \leqq n$, the vector Ye_i is in K by (iii″) and in G_i by (ii); so by the assumption on a, $a^TYe_i \geqq 0$ whenever $a_i<0$. Hence $a^TY(a_0e_0-a)=a^TY(-a_1e_1-\cdots-a_ne_n) \geqq 0$ (since those terms with $a_i=0$ do not contribute to the sum anyway), and hence $a^TY(a_0e_0) \geqq a^TYa \geqq 0$ by positive semidefiniteness. Thus $a^TYe_0=a^Tx \geqq 0$. □

1.c. Algorithmic aspects. Next we turn to some algorithmic aspects of these constructions. We have to start by sketching the framework we are using; for a detailed discussion, see Grötschel, Lovász, and Schrijver [14].

Let K be a convex cone. A *strong separation oracle* for the cone K is a subroutine that, given a vector $x \in \mathbb{Q}^{n+1}$, either returns that $x \in K$ or returns a vector $w \in K^*$ such that $x^Tw<0$. A *weak separation oracle* is a version of this which allows for numerical errors: its input is a vector $x \in \mathbb{Q}^n$ and a rational number $\varepsilon>0$, and it either returns the assertion that the euclidean distance of x from K is at most ε, or returns a vector w such that $|w| \geqq 1$, $w^Tx \leqq \varepsilon$, and the euclidean distance of w from K^* is at most ε. If the cone K is spanned by 0–1 vectors, then we can strengthen a weak separation oracle to a strong one in polynomial time.

Let us also recall the following consequence of the ellipsoid method: Given a weak separation oracle for a convex body, together with some technical information (say, the knowledge of a ball contained in the body and of another one containing the body), we can optimize any linear objective function over the body in polynomial time (again, allowing an arbitrarily small error). If we have a weak separation oracle for a convex cone $K \subseteq Q$, then we can consider its intersection with the halfspace $x_0 \leqq 1$; using the above result, we can solve various important algorithmic questions concerning K in polynomial time. We mention here the weak separation problem for the polar cone K^*.

THEOREM 1.6. *Suppose that we have a weak separation oracle for K. Then the weak separation problem for $N(K)$ as well as for $N_+(K)$ can be solved in polynomial time.*

Proof. Suppose that we have a (weak) separation oracle for the cone K. Then we have a polynomial time algorithm to solve the (weak) separation problem for the cone $M(K)$. In fact, let Y be any matrix. If it violates (i) or (ii), then this is trivially recognized and a separating hyperplane is also trivially given. (iii) can be checked as follows: we have to know if $Yu \in K$ holds for each $u \in Q^*$. Clearly it suffices to check this for the extreme rays of Q^*, i.e., for the vectors e_i and f_i. But this can be done using the separation oracle for K.

Since $N(K)$ is a projection of K, the weak separation problem for $N(K)$ can also be solved in polynomial time (by the general results from [14]).

In the case of $N_+(K)$, all we have to add is that the positive semidefiniteness of the matrix Y can be checked by Gaussian elimination, pivoting always on diagonal entries. If we always pivot positive elements, the matrix is positive semidefinite. If the test fails, it is easy to construct a vector v with $v^TYv < 0$; this gives, then, a hyperplane separating Y from the cone. $\square$

We remark that this proof does not remain valid for $N(K, K)$. In fact, let K be the cone induced by the incidence vectors of perfect matchings of a graph G with m nodes (with "1" appended as a 0th entry). Then the separation problem for K can be solved in polynomial time. On the other hand, consider the matrix $Y = (Y_{ij})$, where

$$y_{ij} = \begin{cases} 1, & \text{if } i = j \quad \text{or} \quad i = 0 \quad \text{or} \quad j = 0, \\ -4(m+2)/m^2, & \text{otherwise.} \end{cases}$$

Then $Y \in M(K, K)$ if and only if G is 3-edge-colorable, which is NP-complete to decide. We do not know if Theorem 1.6 extends to $N(K, K)$, but suspect that it does not.

Note, however, that if K is given by an explicit system of linear inequalities, then $M(K, K)$ is described by a system of linear inequalities of polynomial size and so the separation problem for $N(K, K)$ and $N_+(K, K)$ can be solved in polynomial time. In this case, we get a projection representation of $N(K)$ and of $N(K, K)$ from polyhedra with a polynomial number of facets. It should be remarked that this representation is canonical.

1.d. Stronger cut operators. We could use stronger versions of this procedure to get convex sets smaller than $N(K)$.

One possibility is to consider $N(K, K)$ instead of $N(K) = N(K, Q)$. It is clear that $N(K, K) \subseteq N(K)$. Trivially, Theorem 1.4 and Lemma 1.3 remain valid if we replace $N(K)$ by $N(K, K)$. Unfortunately, it is not clear whether Theorem 1.6 also remains valid. The problem is that now we have to check whether $YK^* \subseteq K$, and unfortunately K^* may have exponentially many, or even infinitely many, extreme rays. If K is given by a system of linear inequalities, then this is not a problem. So in this case we could consider the sequence $N(K, K)$, $N(N(K, K), K)$, etc. This shrinks down faster to K^o than $N^t(K)$, as we shall see in the next section.

The following strengthening of the projection step in the construction seems quite interesting. For $v \in \mathbb{R}^{n+1}$, let $M(K)v = \{Yv: Y \in M(K)\}$. So $N(K) = M(K)e_0$. Now define

$$\hat{N}(K) = \bigcap_{v \in \operatorname{int}(Q^*)} M(K)v.$$

Note that the intersection can be written in the form

$$\hat{N}(K) = \bigcap_{u \in Q^*} M(K)(e_0 + u).$$

It is easy to see that

$$K^o \subseteq \hat{N}(K) \subseteq N(K).$$

The following lemma gives a different characterization of $\hat{N}(K)$.

LEMMA 1.7. *$x \in \hat{N}(K)$ if and only if for every $w \in \mathbb{R}^{n+1}$ and every $u \in Q^*$ such that $(e_0 + u)w^T \in M(K)^*$, we have $w^T x \geqq 0$.*

In other words, $\hat{N}(K)^*$ is generated by those vectors w for which there exists a $v \in \operatorname{int}(Q^*)$ such that $vw^T \in M(K)^*$.

Proof. (Necessity) Let $x \in \hat{N}(K)$, $w \in \mathbb{R}^{n+1}$, and $v \in \operatorname{int}(Q^*)$ such that $vw^T \in M(K)^*$. Then in particular x can be written as $x = Yv$, where $Y \in M(K)$. So $w^T x = w^T Y v = Y \cdot (vw^T) \geqq 0$.

(Sufficiency) Assume that $x \notin \hat{N}(K)$. Then there exists a $v \in \operatorname{int}(K^*)$ such that $x \notin M(K)v$. Now $M(K)v$ is a convex cone, and hence it can be separated from x by a hyperplane, i.e., there exists a vector $w \in \mathbb{R}^{n+1}$ such that $w^T x < 0$ but $w^T Y v \geqq 0$ for all $Y \in M(K)$. This latter condition means that $vw^T \in M(K)^*$, i.e., the condition given in the lemma is violated. □

The cone $\hat{N}(K)$ satisfies important constraints that the cones $N(K)$ and $N_+(K)$ do not. Let $b \in \mathbb{R}^{n+1}$, and define $F_b = \{x \in \mathbb{R}^{n+1}: b^T x \geqq 0\}$.

LEMMA 1.8. *Assume that $N(K \cap F_b) = \{0\}$. Then $-b \in \hat{N}(K)^*$.*

Proof. If $N(K \cap F_b) = \{0\}$, then for every matrix $Y \in M(K \cap F_b)$ we have $Ye_0 = 0$. In particular, $Y_{00} = 0$ and hence $Y = 0$. So $M(K \cap F_b) = \{0\}$. Since clearly

$$M(K \cap F_b)^* = M(K)^* + \operatorname{cone}\{bu^T: u \in Q^*\},$$

this implies that $M(K)^* + \{bu^T: u \in Q^*\} = \mathbb{R}^{(n+1)\times(n+1)}$. So, in particular, we can write $-be_0^T = Z + bu^T$ with $Z \in M(K)^*$ and $u \in Q^*$. Hence $-b(e_0 + u)^T \in M(K)^*$. By the previous lemma, this implies that $-b \in \hat{N}(K)^*$. □

We can use this lemma to derive a geometric condition on $\hat{N}(K)$ similar to Lemma 1.5.

LEMMA 1.9. *Let $K \subseteq Q$ be a convex cone and assume that $e_0 \notin K$. Then*

$$\hat{N}(K) \subseteq (K \cap G_1) + \cdots + (K \cap G_n).$$

In other words, if $a^T x \geqq 0$ is valid for all of the faces $K \cap G_i$, then it is also valid for $\hat{N}(K)$.

Proof. Let $b = -a + te_0$, where $t > 0$. Consider the cone $K \cap F_b$. By the definition of b, this cone does not meet any facet G_i of Q in any nonzero vector. Hence by Lemma 1.3, $N(K \cap F_b)$ is contained in every facet H_i of Q, and hence $N(K \cap F_b) \subseteq \operatorname{cone}(e_0)$. But $N(K \cap F_b) \subseteq K$ and so $N(K \cap F_b) = \{0\}$.

Hence by Lemma 1.7, we get that $-b = a - te_0 \in \hat{N}(K)^*$. Since this holds for every $t < \alpha$ and $\hat{N}(K)^*$ is closed, the lemma follows. □

Applying this lemma to the cone in Fig. 1, we can see that we obtain K^o in a single step. The next corollary of Lemma 1.9 implies that at least some of the Gomory-Chvátal cuts for K are satisfied by $\hat{N}(K)$.

COROLLARY 1.10. *Let $1 \leqq k \leqq n$ and assume that $\sum_{i=1}^{k} x_i > 0$ holds for every $x \in K$. Then $\sum_{i=1}^{k} x_i \geqq x_0$ holds for every $x \in \hat{N}(K)$.*

The proof consists of applying Lemma 1.9 to the projection of K on the first $k+1$ coordinates.

Unfortunately, we do not know if Theorem 1.6 remains valid for $\hat{N}(K)$. Of course, the same type of projection can be defined starting with $M_+(K)$ or with $M(K, K)$ instead of $M(K)$, and properties analogous to those in Lemmas 1.8, 1.9 can be derived.

2. Stable set polyhedra. We apply the results in the previous section to the stable set problem. To this end, we first survey some known methods and results on the facets of stable set polytopes.

2.a. Facets of stable set polyhedra and perfect graphs. Let $G = (V, E)$ be a graph with no isolated nodes. Let $\alpha(G)$ denote the maximum size of any stable set of nodes in G. For each $A \subseteq V$, let $\chi^A \in \mathbb{R}^V$ denote its incidence vector. The *stable set polytope* of G is defined as

$$\mathrm{STAB}\,(G) = \mathrm{conv}\,\{\chi^A\colon A \text{ is stable}\}.$$

So the vertices of STAB (G) are just the 0–1 solutions of the system of linear inequalities

$$x_i \geqq 0 \quad \text{for each } i \in V, \tag{1}$$

and

$$x_i + x_j \leqq 1 \quad \text{for each } ij \in E. \tag{2}$$

In general, STAB (G) is much smaller than the solution set of (1), (2), which we denote by FRAC (G) ("fractional stable sets"). In fact, they are equal if and only if the graph is bipartite. The polytope FRAC (G) has many nice properties; what we will need is that its vertices are half-integral vectors.

There are several classes of inequalities that are satisfied by STAB (G) but not necessarily by FRAC (G). Let us mention some of the most important classes. The *clique constraints* strengthen the class (2): for each clique B, we have

$$\sum_{i \in B} x_i \leqq 1. \tag{3}$$

Graphs for which (1) and (3) are sufficient to describe STAB (G) are called *perfect.* It was shown by Grötschel, Lovász, and Schrijver [12] that the weighted stable set problem can be solved in polynomial time for these graphs.

The *odd hole constraints* express the nonbipartiteness of the graph: if C induces a chordless odd cycle in G, then

$$\sum_{i \in C} x_i \leqq \tfrac{1}{2}(|C| - 1). \tag{4}$$

Of course, the same inequality holds if C has chords; but in this case it easily follows from other odd hole constraints and edge constraints. Nevertheless, it will be convenient that, if we apply an odd hole constraint, we do not have to check whether the circuit in question is chordless.

Graphs for which (1), (2), and (4) are sufficient to describe STAB (G) are called *t-perfect.* Graphs for which (1), (3), and (4) are sufficient are called *h-perfect.* It was shown by Grötschel, Lovász, and Schrijver [13] that the weighted stable set problem can be solved in polynomial time for h-perfect (and hence also for t-perfect) graphs.

The *odd antihole constraints* are defined by sets D that induce a chordless odd cycle in the complement of G:

$$\sum_{i\in D} x_i \leqq 2. \tag{5}$$

We shall see that the weighted stable set problem can be solved in polynomial time for all graphs for which (1)-(5) are enough to describe STAB (G) (and for many more graphs).

All constraints (2)-(5) are special cases of the *rank constraints*: let $U \subseteq V$ induce a subgraph G_U, then

$$\sum_{i\in U} x_i \leqq \alpha(G_U). \tag{6}$$

Of course, many of these constraints are inessential. To specify some that are essential, let us call a graph G *α-critical* if it has no isolated nodes and $\alpha(G-e) > \alpha(G)$ for every edge e. Chvátal [9] showed that if G is a connected α-critical graph then the rank constraint

$$\sum_{i\in V(G)} x_i \leqq \alpha(G)$$

defines a facet of STAB (G).

(Of course, in this generality, rank constraints are ill behaved: given any one of them, we have no polynomial time procedure to verify that it is indeed a rank constraint, since we have no polynomial time algorithm to compute the stability number of the graph on the right-hand side. For the special classes of rank constraints introduced above, however, it is easy to verify that a given inequality belongs to them.)

Finally, we remark that not all facets of the stable set polytope are determined by rank constraints. For example, let U induce an odd wheel in G, with center $u_0 \in U$. Then the constraint

$$\sum_{i\in U\setminus\{u_0\}} x_i + \frac{|U|-2}{2} x_{u_0} \leqq \frac{|U|-2}{2}$$

is called a *wheel constraint*. If, e.g., $V(G) = U$, then the wheel constraint induces a facet of the stable set polytope.

Another class of nonrank constraints of a rather different character are *orthogonality constraints*, introduced by Grötschel, Lovász, and Schrijver [12]. Let us associate with each vertex $i \in V$, a vector $v_i \in \mathbb{R}^n$, so that $|v_i| = 1$ and nonadjacent vertices correspond to orthogonal vectors. Let $c \in \mathbb{R}^n$ with $|c| = 1$. Then

$$\sum_{i\in V} (c^T v_i)^2 x_i \leqq 1$$

is valid for STAB (G). The solution set of these constraints (together with the nonnegativity constraints) is denoted by TH (G). It is easy to show that

$$\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{FRAC}(G).$$

In fact, STAB (G) satisfies all the clique constraints. Note that there are infinitely many orthogonality constraints for a given graph, and TH (G) is in general nonpolyhedral (it is polyhedral if and only if the graph is perfect). The advantage of TH (G) is that every linear objective function can be optimized over it in polynomial time. The algorithm involves convex optimization in the space of matrices, and was the main motivation for our studies in the previous section. We shall see that these techniques give substantially better approximations of STAB (G) over which one can still optimize in polynomial time.

2.b. The "N" operator. To apply the results in the previous chapter, we homogenize the problem by introducing a new variable x_0 and consider STAB (G) as a subset of the hyperplane H_0 defined by $x_0 = 1$. We denote by St (G) the cone spanned by the vectors

$$\begin{pmatrix} 1 \\ \chi^A \end{pmatrix} \in \mathbb{R}^{V \cup \{0\}},$$

where A is a stable set. We get STAB (G) by intersecting ST (G) with the hyperplane $x_0 = 1$. Similarly, let FR (G) denote the cone spanned by the vectors $\binom{1}{x}$, where $x \in$ FRAC (G). Then FR(G) is determined by the constraints

$$x_i \geqq 0 \quad \text{for each } i \in V,$$

and

$$x_i + x_j \leqq x_0 \quad \text{for each } ij \in E.$$

Since it is often easier to work in the original n-dimensional space (without homogenization), we shall use the notation $N(\text{FRAC}(G)) = N(\text{FR}(G)) \cap H_0$, and similarly for N_+, $\hat{N}$, etc. We shall also abbreviate $N(\text{FRAC}(G))$ by $N(G)$, etc. Since FRAC (G) is defined by an explicit linear program, one can solve the separation problem for it in polynomial time. We shall say briefly that the polytope is *polynomial time separable.* By Theorem 1.6, we obtain the following.

THEOREM 2.1. *For each fixed $r \geqq 0$, $N_+^r(G)$, as well as $N^r(G)$, are polynomial time separable.*

It should be remarked that, in most cases, if we use $N^r(G)$ as a relaxation of STAB (G), then it does not really matter whether the separation subroutine returns hyperplanes separating the given $x \notin N^r(G)$ from $N^r(G)$ or only from STAB (G). Hence it is seldom relevant to have a separation subroutine for a given relaxation, say, $N^r(G)$; one could use just as well a separation subroutine for any other convex body containing STAB (G) and contained in $N^r(G)$ (such as, e.g., $N_+^r(G)$). Hence the polynomial time separability of $N_+^r(G)$ is substantially deeper than the polynomial time separability of $N^r(G)$ (even though it does not imply it directly).

In the rest of this section we study the question of how much this theorem gives us: which graphs satisfy $N_+^r(G) = \text{STAB}(G)$ for small values of r, and more generally, which of the known constraints are satisfied by $N(G)$, $N_+(G)$, etc. With a little abuse of terminology, we shall not distinguish between the original and homogenized versions of clique, odd hole, etc., constraints.

It is a useful observation that if $Y = (y_{ij}) \in M(\text{FR}(G))$, then $y_{ij} = 0$ whenever $ij \in E(G)$. In fact, the constraint $x_i + x_j \leqq 1$ must be satisfied by Ye_i, and so $y_{ii} + y_{ji} \leqq y_{0i} = y_{ii}$ by nonnegativity. This implies $y_{ij} = 0$.

Let $a^T x \leqq b$ be any inequality valid for STAB (G). Let $W \subseteq V$ and let $a_W \in \mathbb{R}^W$ be the restriction of a to W. For every $v \in V$, if $a^T x \leqq b$ is valid for STAB (G), then $a_{V-v}^T x \leqq b$ is valid for STAB $(G - v)$ and $a_{V-\Gamma(v)-v}^T x \leqq b - a_v$ is valid for STAB $(G - \Gamma(v) - v)$. Let us say that these inequalities arise from $a^T x \leqq b$ by the *deletion* and *contraction* of node v, respectively. Note that if $a^T x \leqq b$ is an inequality such that for some v, both the deletion and contraction of v yield inequalities valid for the corresponding graphs, then $a^T x \leqq b$ is valid for G.

Let K be any convex body containing STAB (G) and contained in FRAC (G). Now Lemma 1.3 implies the following lemma.

LEMMA 2.2. *If $a^T x \leqq b$ is an inequality such that for some $v \in V$, both the deletion and contraction of v give an inequality valid for K, then $a^T x \leqq b$ is valid for $N(K)$.*

This lemma enables us to characterize completely the constraints obtained in one step (not using positive semidefiniteness).

THEOREM 2.3. *The polytope $N(G)$ is exactly the solution set of the nonnegativity, edge, and odd hole constraints.*

Proof. (1) It is obvious that $N(G)$ satisfies the nonnegativity and edge constraints. Consider an odd hole constraint $\sum_{i \in C} x_i \leqq \frac{1}{2}(|C|-1)$. Then for any $i \in C$, both the contraction and deletion of i result in an inequality trivially valid for FRAC(G). Hence the odd hole constraint is valid for $N(G)$ by Lemma 2.2.

(2) Conversely, assume that $x \in \mathbb{R}^V$ satisfies the nonnegativity, edge, and odd hole constraints. We want to show that there exists a nonnegative symmetric matrix $Y = (y_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)}$ such that $y_{i0} = y_{ii} = x_i$ for all $1 \leqq i \leqq n$, $y_{00} = 1$, and

$$x_i + x_j + x_k - 1 \leqq y_{ik} + y_{jk} \leqq x_k$$

for all $i, j, k \in V$ such that $ij \in E$ (the lower bound comes from the condition that $Yf_k \in \mathrm{FR}(G)$; the upper, from the condition that $Ye_k \in \mathrm{FR}(G)$). Note that the constraint has to hold in particular when $i = k$; then the upper bound implies that $y_{ij} = 0$, while the lower bound is automatically satisfied.

The constraints on the y's are of a special form: they involve only two variables. We can therefore use the following (folklore) lemma, which gives a criterion for the solvability of such a system, more combinatorial than the Farkas lemma.

LEMMA 2.4. *Let $H = (W, F)$ be a graph and let two values $0 \leqq a(ij) \leqq b(ij)$ be associated with each edge of H. Let $U \subseteq W$ also be given. Then the linear system*

$$a(ij) \leqq y_i + y_j \leqq b(ij) \qquad (ij \in F),$$

$$y_i \geqq 0 \qquad (i \in W),$$

$$y_i = 0 \qquad (i \in U)$$

has no solution if and only if there exists a sequence of (not necessarily distinct) vertices $v_0, v_1, \cdots, v_p$ such that v_i and v_{i+1} are adjacent (the sequence is a walk), and one of the following holds:

(a) *p is odd and $b(v_0v_1) - a(v_1v_2) + b(v_2v_3) - \cdots + b(v_{p-1}v_p) < 0$;*

(b) *p is even, $v_0 = v_p$, and $b(v_0v_1) - a(v_1v_2) + b(v_2v_3) - \cdots - a(v_{p-1}v_p) < 0$;*

(c) *p is even, $v_p \in U$, and $b(v_0v_1) - a(v_1v_2) + b(v_2v_3) - \cdots - a(v_{p-1}v_p) < 0$;*

(d) *p is odd, $v_0, v_p \in U$, and $-a(v_0v_1) + b(v_1v_2) - a(v_2v_3) - \cdots - a(v_{p-1}v_p) < 0$.*

In our case, we have as W the set of all pairs $\{i, j\}$ $(i \neq j)$, U is the subset consisting of the edges of G, two pairs, $\{i, j\}$ and $\{k, l\}$, are adjacent in H if and only if $i = k$ and $je \in E(G)$, and $a(ij, jk) = x_i + x_j + x_k - 1$, $b(ij, jk) = x_j$. We want to verify that if x satisfies all the odd hole constraints, then none of the walks of types (a)-(d) in the lemma above can occur. Let us ignore, for a while, how the walk ends. The vertices of the walk in H correspond to pairs ij; the edges in the walk correspond to triples (ijk) such that $ik \in E$. Let us call this edge the *bracing edge* of the triple. We have to add up alternately x_j and $1 - x_i - x_j - x_k$; call the triple *positive* and *negative* accordingly.

Let w be a vertex of G that is not an element of the first and last pair v_0 and v_p. Then following the walk, w may become an element of a v_i, stay an element for a while, and then cease to be; this may be repeated, say, $f(w)$ times. It is then easy to see that the total contribution of the variable x_w to the sum is $-f(w)x_w$.

It is easy to settle case (b) now. Then any v_i can be considered first, and so the above counting applies to each vertex (unless all pairs v_i share a vertex of G, which is a trivial case). So the sum

$$b(v_0v_1)-a(v_1v_2)+b(v_2v_3)-\cdots-a(v_{p-1}v_p)=\frac{p}{2}-\sum_w f(w)x_w.$$

But note that every vertex w occurs in exactly $2f(w)$ bracing edges. If we add up the edge constraints for all bracing edges, we get $p-\sum_w 2f(w)x_w \geqq 0$, which shows that (b) cannot occur.

Cases (a) and (c) take only a little care around the end of the walk, and are left to the reader. Let us show how case (d) can be settled, which is the only case in which the odd hole constraints are needed.

Consider again the bracing edges of the triples, but now, count the pairs v_0 and v_p (which are edges of G) as bracing edges. Again, it is easy to see that the total sum in question is $(p+1)/2-\sum f(w)x_w$, where each w is contained in exactly $2f(w)$ bracing edges. Unfortunately, we now have $p+2$ bracing edges, so adding up the edge constraints for them would not yield the nonnegativity of the sum. But observe that the multiset of bracing edges (we count an edge that is bracing in more than one triple with multiplicity) forms an Eulerian graph, and is, therefore, the union of circuits. Since the total number of bracing edges, $p+2$, is odd, at least one of these circuits is odd. Add up the odd hole constraint for this circuit and the edge constraint, divided by two, for each of the remaining bracing edges. We get that $\sum_w f(w)x_w \leqq (p+1)/2$, which shows that (d) cannot occur. □

COROLLARY 2.5. *If G is t-perfect, then* STAB (G) *is the projection of a polytope whose number of facets is polynomial in n. Moreover, this representation is canonical.* □

This corollary generalizes a result of Barahona and Mahjoub [5] that constructs a projection representation for series-parallel graphs. It could also be derived in an alternative way. The separation problem for the odd cycle inequalities can be reduced to n shortest path problems (see [13]). Following this construction, one can see that a vector x is in the stable set polytope of a t-perfect graph if and only if n potential functions exist in an auxiliary graph. This yields a representation of STAB (G) as the projection of a polytope with $O(n^2)$ facets. (We are grateful to the referee for this remark.)

2.c. The repeated "N" operator. Next, we prove a theorem which describes a large class of inequalities valid for $N^r(G)$ for a given r. The result is not as complete as in the case $r=1$, but it does show that the number of constraints obtainable grows very quickly with r.

Let $a^Tx\leqq b$ be any inequality valid for STAB (G). By Theorem 1.4, there exists an $r\geqq 0$ such that $a^Tx\leqq b$ is valid for $N^r(G)$. Let the *N-index* of the inequality be defined as the least r for which this is true. We can define (and will study later) the N_+-index analogously. Note that in each version, the index of an inequality depends only on the subgraph induced by those nodes having a nonzero coefficient. In particular, if these nodes induce a bipartite graph, then the inequality has N-index 0. We can define the N-index of a graph as the largest N-index of the facets of STAB (G). The N-index of G is 0 if and only if G is bipartite; the N-index of G is 1 if and only if G is t-perfect. Lemma 2.2 implies the following corollary (using the obvious fact that the N-index of an induced subgraph is never larger than the N-index of the whole graph).

COROLLARY 2.6. *If for some node v, $G-v$ has N-index k, then G has N-index at most $k+1$.* □

The following lemma about the iteration of the operator N will be useful in estimating the N-index of a constraint.

LEMMA 2.7. $1/(k+2)\mathbf{1} \in N^k(G)$ $(k \geqq 0)$.

Proof. We use induction on k. The case $k=0$ is trivial. Consider the matrix $Y=(y_{ij}) \in \mathbb{R}^{(V\cup\{0\})\times(V\cup\{0\})}$ defined by

$$y_{ij} = \begin{cases} 1 & \text{if } i=j=0, \\ 1/(k+1), & \text{if } i=0 \text{ and } j>0 \text{ or } i>0 \text{ and } j=0 \text{ or } i=j>0, \\ 0, & \text{otherwise.} \end{cases}$$

Then $Y \in M(N^{k-1}(\mathrm{FR}(G)))$, since

$$Ye_i = \frac{1}{k+2}(e_0+e_i) \in \mathrm{ST}(G) \subseteq N^{k-1}(\mathrm{FR}(G))$$

and

$$Yf_i = \frac{k+1}{k+2}e_0 + \sum_{j\neq 0,i}\frac{1}{k+2}e_j \leqq \frac{k+1}{k+2}\left(e_0 + \frac{1}{k+1}\sum_{j\in V}e_j\right) \in N^{k-1}(\mathrm{FR}(G)),$$

and so by the monotonicity of $N^{k-1}(\mathrm{FR}(G))$, $Yf_i \in N^{k-1}(\mathrm{FR}(G))$. Hence the first column of Y is in $N^k(\mathrm{FR}(G))$, and thus $1/(k+2)\mathbf{1} \in N^k(G)$. □

From these two facts, we can derive some useful bounds on the N-index of a graph.

COROLLARY 2.8. *Let G be a graph with n nodes and at least one edge. Assume that G has stability number $\alpha(G)=\alpha$ and N-index k. Then*

$$\frac{n}{\alpha} - 2 \leqq k \leqq n - \alpha - 1.$$

Proof. The upper bound follows from Corollary 2.6, applying it repeatedly to all but one nodes outside a maximum stable set. To show the lower bound, assume that $k<(n/\alpha)-2$. Then the vector $(1/(k+2))\mathbf{1}$ does not satisfy the constraint $\sum_i x_i \leqq \alpha$ and so it does not belong to STAB (G). Since it belongs to $N^k(G)$ by Lemma 2.7, it follows that $N^k(G) \neq \mathrm{STAB}(G)$—a contradiction. □

It follows in particular that the N-index of a complete graph on t vertices is $t-2$. The N-index of an odd hole is 1, as an odd whole is a t-perfect graph. The N-index of an odd antihole with $2k+1$ nodes is k; more generally, we have the following corollary.

COROLLARY 2.9. *The N-index of a perfect graph G is $w(G)-2$. The N-index of a critically imperfect graph G is $w(G)-1$.*

Next we study the index of a single inequality. Let $a^Tx \leqq b$ be any constraint valid for STAB (G) $(a \in \mathbb{Z}_+^V, b \in \mathbb{Z}_+)$. Define the *defect* of this inequality as $2\times \max\{a^T - b : x \in \mathrm{FRAC}(G)\}$. The factor 2 in front guarantees that this is an integer. In the special case when we consider the constraint $\sum_i x_i \leqq \alpha(G)$ for an α-critical graph G, the defect is just the *Gallai class number* of the graph (see Lovász and Plummer [18] for a discussion of α-critical graphs, in particular of the Gallai class number).

Given a constraint, its defect can be computed in polynomial time, since optimizing over FRAC (G) is an explicit linear program. The defect of a constraint is particularly easy to compute if the constraint defines a facet of STAB (G). This is shown by the following lemma, which states a property of facets of STAB (G) of independent interest.

LEMMA 2.10. *Let $\sum_i a_i x_i \leqq b$ define a facet of* STAB (G), *different from those determined by the nonnegativity and edge constraints. Then every vector v maximizing*

a^Tx over FRAC (G) *has $v_i=\frac{1}{2}$ whenever $a_i>0$. In particular,*

$$\max\{a^Tx\colon x\in \text{FRAC}(G)\}=\tfrac{1}{2}\sum_i a_i$$

and the defect of the inequality is $\sum_i a_i-2b$.

Proof. Let v be any vertex of FRAC (G) maximizing a^Tx. It suffices to prove that $v_i\neq 1$ whenever $a_i>0$; this will imply that the vector $(\frac{1}{2},\cdots,\frac{1}{2})^T$ also maximizes a^Tx, and to achieve the same objective value, v must have $v_i=\frac{1}{2}$ whenever $a_i>0$.

Let $U=\{i\in V\colon v_i=1\}$ and assume, by way of contradiction, that $a(U)>0$. Clearly U is a stable set. If we choose v so that U is minimal (but of course nonempty), then $a_i>0$ for every $i\in U$. Let $\Gamma(U)$ denote the set of neighbors of U. Let X be any stable set in G whose incidence vector χ^X is a vertex on the facet of STAB (G) determined by $a^Tx=b$.

Consider the set $Y=U\cup(X\backslash\Gamma(U))$. Clearly, Y is stable and $a(Y)=a(X)+a(U\backslash X)-a(\Gamma(U)\cap X)$. So, by the optimality of X, we have

$$a(U\backslash X)\leqq a(\Gamma(U)\cap X).$$

On the other hand, consider the vector $w\in\mathbb{R}^V$ defined by

$$w_i=\begin{cases}1, & \text{if } i\in U\cap X,\\ 0, & \text{if } i\in\Gamma(U)\backslash X,\\ \frac{1}{2}, & \text{otherwise.}\end{cases}$$

Then $w\in$ FRAC (G) and $a^Tw\geqq a^Tv+\frac{1}{2}a(\Gamma(U)\cap X)-\frac{1}{2}a(U\backslash X)\geqq a^Tv$. By the optimality of v, we must have equality, and so $a(U\backslash X)=a(\Gamma(U)\cap X)$. But this means that χ^X satisfies the linear equation

$$\sum_{i\in U\cup\Gamma(U)} a_ix_i=a(U).$$

So this linear equation is satisfied by every vertex of the facet determined by $a^Tx=b$. The only way this can happen is that it is the equation $a^Tx=b$ itself. But then $a^Tv=b$ and so $a^Tv\leqq b$ also defines a facet of FRAC (G), which was excluded. □

We need some further, related lemmas about stable set polytopes. These may be viewed as weighted versions of results on graphs with the so-called König property; see [18, § 6.3].

LEMMA 2.11. *Let $a\in\mathbb{R}^V_+$ and assume that*

$$\max\{a^Tx\colon x\in\text{STAB}(G)\}<\max\{a^Tx\colon x\in\text{FRAC}(G)\}.$$

Let E' be the set of those edges ij for which $y_i+y_j=1$ holds for every vector $y\in$ FRAC (G) *maximizing a^Tx. Then (V,E') is nonbipartite.*

Proof. Suppose that (V,E') is bipartite. Let z be a vector in the relative interior of the face F of FRAC (G) maximizing a^Tx. Then clearly

$$E'=\{ij\in E\colon z_i+z_j=1\}$$

and

$$F=\{x\in\text{FRAC}(G)\colon x_i+x_j=1 \text{ for all } ij\in E\}.$$

Let (U,W) be a bipartition of (V,E'). In every connected component of (V,E'), $z_i\geqq\frac{1}{2}$ on at least one color class and hence we may choose (U,W) so that $z_i\geqq\frac{1}{2}$ for all $i\in W$. Then, W is a stable set in the whole graph G. Hence it follows that $\chi^W\in F$. This implies that $\max\{a^Tx\colon x\in\text{STAB}(G)\}=\max\{a^Tx\colon x\in\text{FRAC}(G)\}$—a contradiction. □

LEMMA 2.12. *As in the previous lemma, let $a \in \mathbb{R}_+^V$ and assume that*

$$\max\{a^Tx: x \in \mathrm{STAB}(G)\} < \max\{a^Tx: x \in \mathrm{FRAC}(G)\}.$$

Then there exists an $i \in V$ such that every vector $y \in \mathrm{FRAC}(G)$ maximizing a^Tx has $y_i = \frac{1}{2}$.

Proof. Let E' be as before. Then by Lemma 2.11, there exists an odd circuit C in G such that $E(C) \subseteq E'$. If y is any vector in FRAC (G) maximizing a^Tx, then by the definition of E', $y_i + y_j = 1$ for every edge $ij \in E(C)$, and hence $y_i = \frac{1}{2}$ for every $i \in V(C)$. □

Now we can state and prove our theorem, which shows the connection between defect and the N-index.

THEOREM 2.13. *Let $a^Tx \leqq b$ be an inequality with integer coefficients valid for* STAB (G) *with defect r and N-index k. Then*

$$\frac{r}{b} \leqq k \leqq r.$$

Proof. (Upper bound) We use induction on r. If $r = 0$ we have nothing to prove, so suppose that $r > 0$. Then Lemma 2.12 can be applied and we get that there is a vertex i such that every vector y optimizing a^Tx over FRAC (G) has $y_i = \frac{1}{2}$. Note that trivially $a_i > 0$.

We claim that both the contraction and deletion of i result in constraints with smaller defect. In fact, let y be a vertex of FRAC(G) maximizing a_{V-i}^Tx. If y also maximizes a^Tx, then $y_i = \frac{1}{2}$ and hence

$$2(a_{V-i}^Ty - b) = 2(a^Ty - b) - a_i < 2(a^Ty - b) = r.$$

On the other hand, if y does not maximize a^Tx, then

$$2(a_{V-i}^Ty - b) \leqq 2(a^Ty - b) < 2 \cdot \max\{a^Tx - b: x \in \mathrm{FRAC}(G)\} = r.$$

The assertion follows similarly for the contraction. Hence by the induction hypothesis, the contraction and deletion of i yield constraints valid for $N^{r-1}(G)$. It follows by Lemma 2.2 that $a^Tx \leqq b$ is valid for $N^r(G)$.

(Lower bound) By Lemma 2.7, $(1/(k+2))\mathbf{1} \in N^k(G)$, and so $a^Tx \leqq b$ must be valid for $(1/(k+2))\mathbf{1}$. So $(1/(k+2))a^T\mathbf{1} \leqq b$ and hence

$$k \geqq \frac{a^T\mathbf{1}}{b} - 2 = \frac{r}{b}. \qquad □$$

It follows from our discussions that for an odd antihole constraint, the lower bound is tight. On the other hand, it is not difficult to check that for a rank constraint defined by an α-critical subgraph that arises from K_p by subdividing an edge by an even number of nodes, the upper bound is tight.

We would like to mention that Ceria [7] proved that $N(\mathrm{FRAC}(G), \mathrm{FRAC}(G))$ also satisfies, among others, the K_4-constraints. We do not study the operator $K \mapsto N(K, K)$ here in detail, but a thorough comparison of its strength with N and N_+ would be very interesting.

A class of graphs interesting from the point of view of stable sets is the class of line-graphs: the stable set problem for these graphs is equivalent to the matching problem. In particular, it is polynomial time solvable and Edmonds's description of the matching polytope [10] provides a "nice" system of linear inequalities describing the stable set polytope of such graphs. The N-index of line-graphs is unbounded; this follows, e.g., by Corollary 2.8. This also follows from Yannakakis's result [26] mentioned in the Introduction, since bounded N-index would yield a representation of the matching polytope as a projection of a polytope with a polynomial number of facets. We do not know whether or not the N_+-index of line-graphs remains bounded.

2.d. The "N_+" operator. Now we turn to the study of the operator N_+ for stable set polytopes. We do not have as general results for the operator N_+ as for the operator N, but we will be able to show that many constraints are satisfied even for very small r.

Lemma 1.5 implies the following lemma.

LEMMA 2.14. *If $a^Tx \leqq b$ is an inequality valid for* STAB (G) *such that for all $v \in V$ with a positive coefficient the contraction of v gives an inequality with N_+-index at most r, then $a^Tx \leqq b$ has N_+-index at most $r+1$.*

The clique, odd hole, odd wheel, and odd antihole constraints have the property that, contracting any node with a positive coefficient, we get an inequality in which the nodes with positive coefficients induce a bipartite subgraph. Hence, we have the following corollary.

COROLLARY 2.15. *Clique, odd hole, odd wheel, and odd antihole constraints have N_+-index* 1.

Hence all h-perfect (in particular all perfect and t-perfect) graphs have N_+-index at most 1. We can also formulate the following recursive upper bound on the N_+-index of a graph.

COROLLARY 2.16. *If $G-\Gamma(v)-v$ has N_+-index at most r for every $v \in V$, then G has N_+-index at most $r+1$.*

Next, we consider the orthogonality constraints. To this end, consider the cone M_{TH} of $(V\cup\{0\})\times(V\cup\{0\})$ matrices $Y=(y_{ij})$ satisfying the following constraints:

(i) Y is symmetric;

(ii) $y_{ii}=y_{i0}$ for every $i \in V$;

(iii′) $y_{ij}=0$ for every $ij \in E$;

(iv) Y is positive semidefinite.

As remarked, (iii′) is a relaxation of (iii) in the definition of $M_+(\mathrm{FR}\,(G))$. Hence $M_+(\mathrm{FR}\,(G)) \subseteq M_{\mathrm{TH}}$.

LEMMA 2.17. $\mathrm{TH}\,(G)=\{Ye_0\colon Y\in M_{\mathrm{TH}},\ e_0^TYe_0=1\}$.

Proof. Let $x \in \mathrm{TH}\,(G)$. Then, by the results of Grötschel, Lovász, and Schrijver [13], x can be written in the form $x_i=(v_0^Tv_i)^2$, where the v_i $(i\in V)$ form an orthonormal representation of the complement of G and v_0 is some vector of unit length. Set $x_0=1$ and define $Y_{ij}=v_i^Tv_j\sqrt{x_ix_j}$. Then it is easy to verify that $Y\in M_{\mathrm{TH}}$ and $Ye_0=x$.

The converse inclusion follows by a similar direct construction. □

This representation of TH (G) is not a special case of the matrix cuts introduced in § 1 (though it is clearly related). In § 3 we will see that, in fact, TH (G) is in a sense more fundamental than the relaxations of STAB (G) constructed in § 1. Right now we can infer the following.

COROLLARY 2.18. *Orthogonality constraints have N_+-index* 1.

We conclude with an upper bound on the N_+-index of a single inequality. Since $\alpha(G-\Gamma(v)-v)<\alpha(G)$, Lemma 2.14 gives, by induction, Corollary 2.19.

COROLLARY 2.19. *If $a^Tx\leqq b$ is an inequality valid for* STAB (G) *such that the nodes with positive coefficient induce a graph with independence number r, then $a^Tx \leqq b$ has N_+-index at most r. In particular, $a^Tx\leqq b$ has index at most b.*

Let us turn to the algorithm aspects of these results. Theorem 2.1 implies the following corollary.

COROLLARY 2.20. *The maximum weight stable set problem is polynomial time solvable for graphs with bounded N_+-index.*

Note that even for small values of r, quite a few graphs have N_+-index at most r. Collecting previous results, we obtain Corollary 2.21.

COROLLARY 2.21. *For any fixed $r\geqq 0$, if* STAB (G) *can be defined by constraints $a^Tx\leqq b$ such that either the defect of the constraint is at most r or the support contains* *no stable set larger than r, then the maximum weight stable set problem is polynomial time solvable for G.*

3. Cones of set-functions. Vectors in $\mathbb{R}^S$ are just functions defined on the one-element subsets of a set S; the symmetric matrices in the previous sections can be considered as functions defined on unordered pairs. We show that if we consider set-functions, i.e., functions defined on all subsets of S, then some of the previous considerations become more general and sometimes even simpler.

In fact, most of the results extend to a general finite lattice in the place of the boolean algebra, and we present them in this generality for the sake of possible other applications.

3.a. Preliminaries: Vectors on lattices. Let us start with some general facts about functions defined on lattices. Given a lattice L, we associate with it the matrix $Z=(\zeta_{ij})$, called the *zeta-matrix* of the lattice, defined by

$$\zeta_{ij}=\begin{cases}1, & \text{if } i\leqq j,\\ 0, & \text{otherwise.}\end{cases}$$

For $j\in L$, let ζ^j denote the jth column of the zeta matrix, i.e., let

$$\zeta^j(i)=\zeta_{ij}.$$

If we order the rows and columns of Z compatibly with the partial ordering defined by the lattice, it will be upper triangular with 1's in its main diagonal. Hence it is invertible, and its inverse $M=Z^{-1}$ is an integral matrix of the same shape. This inverse is a very important matrix, called the *Möbius matrix* of the lattice. Let

$$M=(\mu(i,j))_{i,j\in\mathscr{L}}.$$

The function μ is called the *Möbius function* of the lattice. From the discussion above, we see that $\mu(i,i)=1$ for all $i\in\mathscr{L}$, and $\mu(i,j)=0$ for all $i,j\in\mathscr{L}$ such that $i\not\leqq j$. Moreover, the definition of M implies that for every pair of elements $a\leqq b$ of the lattice,

$$\sum_{a\leqq i\leqq b}\mu(a,i)=\begin{cases}1, & \text{if } a=b,\\ 0, & \text{otherwise;}\end{cases}$$

and

$$\sum_{a\leqq i\leqq b}\mu(i,b)=\begin{cases}1, & \text{if } a=b,\\ 0, & \text{otherwise.}\end{cases}$$

Either one of these identities provides a recursive procedure to compute the Möbius function. It is easy to see from this procedure that the value of the Möbius function $\mu(i,j)$, where $i\leqq j$, depends only on the internal structure of the interval $[i,j]$. Also note the symmetry in these two identities. This implies that if μ^* denotes the Möbius function of the lattice turned upside down, then

$$\mu^*(i,j)=\mu(j,i).$$

For $j\in L$, let μ^j denote the jth column of the Möbius matrix, i.e., let

$$\mu^j(i)=\mu_{ij}.$$

We denote by μ_j the jth row of the Möbius matrix, and by $\mu_{[i,j]}$ the restriction of μ_i to the interval $[i,j]$, i.e., the vector defined by

$$\mu_{[i,j]}(k)=\begin{cases}\mu(i,k), & \text{if } k\leqq j,\\ 0, & \text{otherwise.}\end{cases}$$

The Möbius function of a lattice generalizes the Möbius function in number theory, and it can be used to formulate an inversion formula extending the Möbius inversion in number theory. Let $g \in \mathbb{R}^L$ be a function defined on the lattice. The zeta matrix can be used to express its *lower* and *upper summation function*:

$$(Z^T g)(i) = \sum_{j \leqq i} g(j),$$

and

$$(Zg)(i) = \sum_{j \geqq i} g(j).$$

Given (say) $f = Zg$, we can recover g uniquely by

$$g(i) = (Mf)(i) = \sum_{j \geqq i} \mu(i, j) f(j).$$

The function g is called the *upper Möbius inverse* of f. The *lower Möbius inverse* is defined analogously.

There is a further simple but important formula relating a function to its inverse. Given a function $f \in \mathbb{R}^L$, we associate with it the matrix $W^f = (w_{ij})$, where

$$w_{ij} = f(i \vee j).$$

We also consider the diagonal matrix D^f with $(D^f)_{ii} = f(i)$. Then it is not difficult to prove the following identity (Lindström [15], Wilf [24]).

LEMMA 3.1. *If g is the upper Möbius inverse of f, then $W^f = ZD^g Z^T$.*

For more on Möbius functions, see Rota [21], Lovász [17, Chap. 2], or Stanley [23, Chap. 3].

A function $f \in \mathbb{R}^L$ will be called *strongly decreasing* if $Mf \geqq 0$. Since $f = Z(Mf)$, this is equivalent to saying that f is a nonnegative linear combination of the columns of Z, i.e., of the vectors ζ_j. So strongly decreasing functions form a convex cone $H = H(L)$, which is generated by the vectors ζ^j, $j \in L$. Also by definition, the polar cone H^* is generated by the rows of M, i.e., by the vectors μ_j.

Let us mention that the vector $\mu_{[i,j]}$ is also in H^* for every $i \leqq j$. This is straightforward to check by calculating the inner product of $\mu_{[i,j]}$ with the generators ζ_j of H. It is easy to see that strongly decreasing functions are nonnegative, monotone decreasing, and *supermodular*, i.e., they satisfy

$$f(i \vee j) + f(i \wedge j) \geqq f(i) + f(j).$$

Lemma 3.1 implies Corollary 3.2.

COROLLARY 3.2. *A function f is strongly decreasing if and only if W^f is positive semidefinite.*

It follows, in particular, that f is strongly decreasing if and only if for every $x \in \mathbb{R}^L$,

$$x^T W^f x = \sum_{x_i x_j} f(i \vee j) \geqq 0.$$

It is, in fact, worthwhile to mention the following identity, following immediately from Lemma 3.1. Let f, $x \in \mathbb{R}^L$ and let $g = Mf$ and $y = Zx$. Then

$$x^T W^f x = \sum_{i \in L} g(i) y(i)^2.$$

In particular, if f is strongly decreasing, then

$$x^T W^f x \geqq g(0) x(0)^2. \tag{1}$$

Remark. Let $L=2^S$, and let $f\in\mathbb{R}^L$ such that $f(\emptyset)=1$. Then f is strongly decreasing if and only if there exist random events A_s ($s\in S$) such that for every $X\subseteq S$,

$$\operatorname{Prob}\left(\prod_{s\in X} A_s\right)=f(X).$$

(If this is the case, $(Mf)(X)$ is the probability of the atom $\prod_{s\in X}A_s\prod_{s\in S-X}\bar{A}_s$.) In particular, we obtain from (1) that for any $\lambda\in\mathbb{R}^L$ with $\lambda(0)=1$,

$$\sum_{X,Y}\lambda_X\lambda_Y\operatorname{Prob}\left(\prod_{s\in X\cup Y} A_s\right)\geqq\operatorname{Prob}\left(\prod_{i\in S}\bar{A}_i\right).$$

This is a combinatorial version of the Selberg sieve in number theory (see [17, Chap. 2]). Inequality (1) can be viewed as Selberg's sieve for general lattices; see Wilson [25].

The lattice structure also induces a "multiplication," which leads to the *semigroup algebra* of the semigroup $(L,\vee)$. Given a, $b\in\mathbb{R}^L$, we define the vector $a\vee b\in\mathbb{R}^L$ by

$$(a\vee b)(k)=\sum_{i\vee j=k}a(i)b(j).$$

In particular,

$$e_i\vee e_j=e_{i\vee j}$$

(and the rest of the definition is obtained by distributivity). It is straightforward to see that this operation is commutative, associative, and distributive with respect to the vector addition, and has unit element e_0 (where 0 is the zero element of the lattice). This semigroup algebra has a very simple structure: elementary calculations show that

$$Z^T(a\vee b)(k)=(Z^Ta)(k)\cdot(Z^Tb)(k), \tag{2}$$

and hence the semigroup algebra is isomorphic to the direct product of $|L|$ copies of $\mathbb{R}$. It also follows from (2) that a vector a has an inverse in this algebra if and only if $(Z^Ta)(k)\neq 0$ for all k.

Another identity which will be useful is the following:

$$(a\vee b)^Tc=a^TW^cb. \tag{3}$$

Using this, we can express the fact that a vector c is strongly decreasing as follows:

$$(a\vee a)^Tc\geqq 0\quad\text{for every } a\in\mathbb{R}^L.$$

In particular it follows that H^* *is generated by the vectors* $a\vee a$, $a\in\mathbb{R}^L$. Comparing this with our previous characterization, it follows that the vectors μ_j must be of the form $a\vee a$. In fact, $\mu_j\vee\mu_j=\mu_j$; more generally, the vectors $\mu_{[i,j]}$ are also idempotent. Using (2) it is easy to see that the idempotents are exactly the vectors of the form $\sum_{i\in I}\mu_i$, where $I\subseteq L$. Moreover, the "$\vee$" product of any two vectors μ_i is zero.

3.b. Optimization in lattices. Given a subset $F\subseteq L$, we denote by cone (F) the convex cone spanned by the vectors ζ^i, $i\in F$. Since these vectors are extreme rays of H, and all extreme rays of H are linearly independent, it is, in principle, trivial to describe F by linear inequalities. It is determined by the system

$$\mu_j^Tx\begin{cases}=0, & \text{if } i\notin F,\\ \geqq 0, & \text{if } i\in F.\end{cases} \tag{4}$$

But since cone (F) is generally not full-dimensional, it may have many other minimal descriptions. For example, in the case when F is an order ideal (i.e., $x \in F$, $y \leqq x$ imply $y \in F$), cone (F) could be described by

(5) $$x \in H, \qquad x(i)=0 \quad \text{for all } i \notin F.$$

Hence

(6) $$\text{cone}\,(F)^* = \{a \in \mathbb{R}^L\colon (Z^T a)(k) \geqq 0 \text{ for all } k \in F\}.$$

Our main concern will be to describe the projection of cone (F) on the subspace spanned by a few "small" elements in the lattice. Let I be the set of these "interesting" lattice elements. We consider $\mathbb{R}^I$ as the subspace of $\mathbb{R}^L$ spanned by the elements of I. For any convex cone $k \subseteq H$, let K_I denote the intersection of K with $\mathbb{R}^I$ and let K/I denote the projection of K onto $\mathbb{R}^I$. Then $(K^*)_I \subseteq K^*$ is the set of linear inequalities valid for K involving only variables corresponding to elements of I. Also, $(K^*)_I$ is the polar of K/I with respect to the linear space $\mathbb{R}^I$.

For example, in the case when $L=2^S$, where S is an n-element set, we can take I as the set of all singletons and $\emptyset$. If we project cone (F) on this subspace, and intersect the projection with the hyperplane $x_\emptyset = 1$, then we recover the polyhedron usually associated with F (namely, the convex hull of incidence vectors of members of F). Note that the projection itself is just the homogenization introduced in § 1. The cone Q considered in § 1 is just H/I.

From these considerations we can infer the following theorem, due (in a slightly different form) to Sherali and Adams [22].

THEOREM 3.3. *If* $\mathcal{F} \subseteq 2^S$ *then* conv $\{\chi^A\colon A \in \mathcal{F}\}$ *is the projection of the following cone to singleton sets*:

$$x_\emptyset = 0, \qquad \mu_j^T x \geqq 0 \qquad (j \in \mathcal{F}), \qquad \mu_j^T x = 0 \qquad (j \notin \mathcal{F}).$$

The $(n \geqq 1) \times (n+1)$ matrices Y used in § 1 can be viewed in this framework in two different ways. First, they can be viewed as portions of the vector $x \in \mathbb{R}^{2^S}$ determined by the entries indexed by $\emptyset$, singletons, and pairs; the linear constraints on $M(K)$ used in § 1 are only the constraints we can derive in a natural way from the constraints involving just the first $n+1$ variables.

Second, the matrices Y also occur as principal minors of the corresponding (huge) matrix W^x. So the positive semidefiniteness constraint for $M_+(K)$ is just a relaxation of the condition that for $x \in H$, W^x is positive semidefinite. (It is interesting to observe that while by Corollary 3.2, the positive semidefiniteness of W^x is a polyhedral condition, this relaxation of it is not.)

Let us discuss the case of the stable set polytope. We have a graph $G=(V, E)$ and we take $S=V$, $L=2^S$. Let F consist of the stable sets of G. Then cone $(F) \subseteq \mathbb{R}^L$ is defined by the constraints

$$x \in H, \qquad x_{ij}=0 \quad \text{for every } ij \in E.$$

We can relax the first constraint by stipulating that the upper left $(n+1)\times(n+1)$ submatrix W_0^x of W^x is positive semidefinite. Then these submatrices form exactly the cone M_{TH} as introduced in § 2. As we have seen, the projection of this cone to $\mathbb{R}^I$, intersected with the hyperplane $x_0 = 1$, gives the body TH (G).

Note that the "supermodularity" constraints $x_{ij} - x_i - x_j + x_0 \geqq 0$ are linear constraints valid for H, and involve only the variables indexed by sets with cardinality at most 2, but they do not follow from the positive semidefiniteness of W_0^x. Using these inequalities we obtain from $x_{ij}=0$ the constraint $x_i \geqq x_j \leqq x_0$ for every edge $ij \in E$.

Returning to our general setting, we are going to interpret the operators N, N_+, and $\hat{N}$ in this general setting, using the group algebra. In order to describe the projection of cone (F) on $\mathbb{R}^I$, we want to generate linear constraints valid for cone (F) such that only the coefficients corresponding to elements of I are nonzero. To this end, we use the semigroup algebra to combine constraints to yield new constraints for cone (F). (This may temporarily yield constraints having some further nonzero coefficients, which we can eliminate afterwards.)

We have already seen that $a \vee a \in \text{cone}\,(F)^*$ for every a. From (2) and (6) we can read off the following further rules:

(a) If a, $b \in \text{cone}\,(F)^*$, then $a \vee b \in \text{cone}\,(F)^*$.

(b) If $a \in \text{int}\,(\text{cone}\,(F)^*)$ and $a \vee b \in \text{cone}\,(F)^*$, then $b \in \text{cone}\,(F)^*$.

In rule (b), we can replace the condition that $a \in \text{int}\,(\text{cone}\,(F)^*)$ by the perhaps more manageable condition that $a = e_0 + c$ with $c \in \text{cone}\,(F)^*$. In fact, $e_0 \in \text{int}\,(\text{cone}\,(F)^*)$ and hence for every $c \in \text{cone}\,(F)^*$, $e_0 + c \in \text{int}\,(\text{cone}\,(F)^*)$. Conversely, if $a \in \text{int}\,(\text{cone}\,(F)^*)$, then for a sufficiently small $t > 0$, $a - te_0 \in \text{cone}\,(F)^*$. Set $c = (a - e_0)/t$, then $c + e_0 \in \text{cone}\,(F)^*$ and $(c + e_0) \vee b = (a \vee b)/t \in \text{cone}\,(F)^*$, and hence $b \in \text{cone}\,(F)^*$.

If $Z^T a > 0$, then rule (b) follows from rule (a). In fact, let $c(k) = 1/(Z^T a)(k)$, and $d = M^T c$. Then d is the inverse of a, that is, $d \vee a = e_0$, and $(Z^T d)(k) = c(k) > 0$ for all k, so $d \in \text{cone}\,(F)^*$. Hence

$$b = (a \vee b) \vee d \in \text{cone}\,(F)^*,$$

by rule (a).

For two cones K_1, $K_2 \subseteq \mathbb{R}^L$, we denote by $K_1 \vee K_2$ the cone spanned by all vectors $u_1 \vee u_2$, where $u_i \in K_i$. (The set of all vectors arising in this way is not convex in general.) This operation generalizes the construction of $N(K_1, K_2)$, $N_+(K_1, K_2)$, and $\hat{N}(K)$ in the following sense.

PROPOSITION 3.4. *Let $L = 2^S$, I, the set consisting of $\emptyset$, and the singleton subsets of S, and let K_1, $K_2 \subseteq H/I$ be two convex cones. Then*

(i) $N(K_1, K_2)^* = ((K_1^*)_I \vee (K_2^*)_I)_I$;

(ii) $N_+(K_1, K_2)^* = ((K_1^*) \vee (K_2^*)_I + \mathbb{R}^I \vee \mathbb{R}^I)_I$.

Proof of (i). First, we assume that $w \in ((K_1^*)_I \vee (K_2^*)_I)_I$. Then we can write $w = \sum_t a_t \vee b_t$, where $a_t \in (K_1^*)_I$ and $b_t \in (K_2^*)_I$. Let $x \in N(K_1, K_2)$; then we can write $x = Ye_0$ with $Y = (y_{ij} \in M(K_1, K_2))$. Define the vector $y \in \mathbb{R}^L$ by

$$y(k) = \begin{cases} x_k, & \text{if } k \in I, \\ y_{ij}, & \text{if } k = \{y, j\}, \\ 0, & \text{else.} \end{cases}$$

Then we have

$$w^T x = w^T y = \sum_t (a_t \vee b_t)^T y = \sum_t a_t^T Y b_t \geqq 0.$$

This proves that $w \in N(K_1, K_2)^*$.

Second, assume that $w \in N(K_1, K_2)^*$. Then we can write

$$we_0^T = \sum_t a_t b_t^T + \sum_{i=1}^n \lambda_i e_i f_i^T + A,$$

where $a_t \in K_1^*$, $b_t \in K_2^*$, $\lambda_i \in \mathbb{R}$, and A is a skew symmetric matrix. Now it is easy to check that

$$w = \sum_t (a_t \vee b_t),$$

and so $w \in ((K_1^*)_I \vee (K_2^*)_I)_I$.

The proof of part (ii) is analogous. □

Next we show that the construction of $\hat{N}$ is, in fact, a special case of the application of rule (b).

LEMMA 3.5. *Let $L = 2^S$, I, the set consisting of $\emptyset$, and the singleton subsets of S, and let $K \subseteq H/I$, a convex cone. Then*

$$\hat{N}(K)^* = \{a \in \mathbb{R}^I : \exists b \in \text{int}\,(K^*)_I \text{ such that } a \vee b \in (K^*)_I \vee (Q^*)_I\}.$$

The proof is analogous to that of Proposition 3.3, and is omitted.

We can use the formula in Proposition 3.4 to formulate a stronger version of the repetition of the operator N. Note that

$$N^2(K)^* = [[(K^*)_I \vee (Q^*)_I]_I \vee (Q^*)_I]_I \subseteq [(K^*)_I \vee (Q^*)_I \vee (Q^*)_I]_I,$$

and similarly, if we denote $(Q^*)_I \vee \cdots \vee (Q^*)_I$ (r factors) by Q_r, then

$$N^r(K)^* \subseteq [(K^*)_I \vee Q_r]_I.$$

Now it is easy to see that the cone Q_r is spanned by the vectors $\mu_{[i,j]}$ where $i \subseteq j$ and $|j| \leqq r$. For fixed r, this is a polynomial number of vectors. Let $\bar{N}^r(K)$ denote the polar cone of $[(K^*)_I \vee Q_r]_I$ in the linear space $\mathbb{R}^I$. Then $\bar{N}^r(K) \subseteq N^r(K)$.

For the case of boolean algebras (and in a quite different form), the sequence $\bar{N}^r(K)$ of relaxations of K^o was introduced by Sherali and Adams [22], who also showed that $\bar{N}^n(K) = K^o$.

It is easy to see that if K is polynomial time separable, then so is $\bar{N}^r(K)$ for every fixed r: to check whether $x \in \bar{N}^r(K)$, it suffices to check whether there exist vectors $a^{[i,j]} \in (K^*)_I$ for every i and j with $i \subseteq j$ and $|n| \leqq r$ such that $a = \sum_{i,j} a^{[i,j]} \vee \mu_{[i,j]} \in \mathbb{R}^I$ and $a^T x < 0$. This is easily done in polynomial time using the ellipsoid method.

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