SIAM J. CONTROL OPTIM.
Vol. 45, No. 5, pp. 1769–1800

CONEWISE LINEAR SYSTEMS: NON-ZENONESS AND OBSERVABILITY*

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Abstract. Conewise linear systems are dynamical systems in which the state space is partitioned into a finite number of nonoverlapping polyhedral cones on each of which the dynamics of the system is described by a linear differential equation. This class of dynamical systems represents a large number of piecewise linear systems, most notably, linear complementarity systems with the P-property and their generalizations to affine variational systems, which have many applications in engineering systems and dynamic optimization. The challenges of dealing with this type of hybrid system are due to two major characteristics: mode switchings are triggered by state evolution, and states are constrained in each mode. In this paper, we first establish the absence of Zeno states in such a system. Based on this fundamental result, we then investigate and relate several state observability notions: short-time and T-time (or finite-time) local/global observability. For the short-time observability notions, constructive, finitely verifiable algebraic (both sufficient and necessary) conditions are derived. Due to their long-time mode-transitional behavior, which is very difficult to predict, only partial results are obtained for the T-time observable states. Nevertheless, we completely resolve the T-time local observability for the bimodal conewise linear system, for finite T, and provide numerical examples to illustrate the difficulty associated with the long-time observability.

Key words. piecewise linear systems, conewise linear systems, hybrid systems, Zeno behavior, observability

AMS subject classifications. 34C99, 34A40, 90C33, 93B07, 93C10

DOI. 10.1137/050645166

1. Introduction.

A conewise linear system (CLS) is a hybrid dynamical system consisting of a finite number of linear ordinary differential equations (ODEs) that are active on certain polyhedral cones which partition the whole Euclidean state space. Each of these cones is called a *mode* of the system; transitions between modes occur along a state trajectory. Many piecewise linear systems can be formulated as CLSs; among these, linear complementarity systems (LCSs) [15, 6] are perhaps the most prominent. Specifically, an LCS is defined by a linear time-invariant ODE containing an algebraic variable that is required to be a solution to a finite-dimensional linear complementarity problem (LCP). Collectively, these piecewise linear systems and their generalizations, such as the differential variational inequalities (DVIs) [25], have found a wide range of applications in nonsmooth mechanical systems, switched electrical

*Received by the editors November 14, 2005; accepted for publication (in revised form) May 21, 2006; published electronically December 11, 2006.
http://www.siam.org/journals/sicon/45-5/64516.html

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networks and control systems, and dynamic optimization in operations research and economics. See the two surveys [4, 31] and the recent papers [7, 8, 16, 17] as well as the references therein.

As with all switched dynamical systems, a critical issue associated with a CLS is whether infinitely many mode transitions exist in any finite time along a state trajectory, i.e., the *Zeno behavior* of the CLS. Such an issue was studied in a different setting for piecewise analytic systems in [5, 35] two decades ago. It has regained considerable attention and received extensive treatment in the hybrid system literature in the past few years, e.g., [19, 32, 37], due to its fundamental role in the study of numerical simulations and basic system and control properties of hybrid systems. Adding to the recent study of the Zeno issue [7] for complementarity systems, the paper [33] introduces several important notions of non-Zenoness and non-Zeno states of an LCS and establishes the "strong non-Zenoness" for an LCS with the P-property and the "weak non-Zenoness" for a broader class of LCSs. The paper [24] further extends the Zeno study to a nonlinear complementarity system (NCS) and to the DVI; it shows the strong non-Zenoness for an NCS satisfying the strong regularity condition and investigates certain system properties using these non-Zeno results.

Having its roots in the very early stages of modern control theory [20], observability is a fundamental concept in systems and control. Roughly speaking, observability refers to the ability of reconstructing the initial state from given output observations. This notion is well understood for linear systems [10]. However, characterization of observability of nonlinear systems (with control inputs) becomes a very hard problem. For instance, one has to analyze many different observability concepts of nonlinear systems, and only local sufficient conditions are available for small-time observability [23]; see the algebraic approach for analytic systems [2]. Moreover, checking these conditions can become a computationally untractable task [3].

The observability of hybrid systems has attracted growing attention in recent years. Mode and state observability of discrete-time switched linear systems are studied in [1], under the assumption that mode sequences are arbitrary; linear algebraic tests are provided, and the decidability is discussed. The paper [36] analyzes the observability of jump-linear systems and linear hybrid systems; necessary and sufficient conditions in terms of algebraic tests are given. Several observability notions are proposed for piecewise affine hybrid systems in [12]; sufficient conditions are obtained for the observability test and are used for observer design. Other related results include observability of Turing machines and its connection to hybrid systems [11]. For more discussions on observability analysis and observer design, see the references cited in the above-mentioned papers.

The present paper deals with the non-Zenoness and state observability of CLSs, assuming a linear system output. Compared with the observability results in the literature, there are two unique characteristics of the CLS that make the observability analysis challenging and different from the prior results: (i) mode switchings are triggered by state evolution instead of being arbitrarily chosen; (ii) the state is restricted to a cone in each mode. Due to the first property, the issues of well-posedness and Zenoness of system solutions become nontrivial. The second property implies that classical matrix rank conditions are insufficient to characterize observability properties. Moreover, a state trajectory is at best only once differentiable with respect to time and is not differentiable with respect to the initial condition. These properties necessitate the development of new tools to handle observability issues for this class of nonsmooth systems.

The organization of the paper is as follows. In section 2, we begin with the introduction of the main object of the study, CLSs, and discuss certain fundamental properties of the solutions. We then prove the non-Zenoness and piecewise analyticity of such solutions in section 3. Various kinds of observability notions are investigated, and the relation between them is discussed in section 4. It is shown that the linear dynamics, together with the conic state space partition, is instrumental in the derivation of constructive necessary and sufficient conditions for certain observability notions, particularly for short-time observability. Detailed investigation of T-time local observability of bimodal CLSs is given in section 5. For the long-time observability, we present examples to demonstrate several interesting properties that make this observability notion challenging to characterize, even for bimodal CLSs.

2. Conewise linear systems. Consider the ODE

$$\dot{x} \;=\; f(x), \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise affine (PA) function; i.e.,

(a) f is continuous, and

(b) a positive integer m and a family of affine functions $\{f_i\}_{i=1}^m$ exist such that $f(x) \in \{f_i(x)\}_{i=1}^m$ for all $x \in \mathbb{R}^n$.

We call systems of the form (2.1) *piecewise affine* systems. The representation (2.1) describes the system at hand in an implicit way via the component functions $\{f_i\}_{i=1}^m$. Alternatively, a geometric representation of (2.1) can be obtained by invoking well-known properties of PA functions (see, e.g., [14]). To elaborate on this, we recall that a finite collection of polyhedra in $\mathbb{R}^n$, denoted Ξ, is a *polyhedral subdivision* of $\mathbb{R}^n$ if

(a) the union of all polyhedra in Ξ is equal to $\mathbb{R}^n$,

(b) each polyhedron in Ξ is of dimension n, and

(c) the intersection of any two polyhedra in Ξ is either empty or a common proper face of both polyhedra.

For every PA function f, one can find a polyhedral subdivision of $\mathbb{R}^n$ and a finite family of affine functions $\{g_i\}$ such that f coincides with one of the functions $\{g_i\}$ on each polyhedron in Ξ [14, Proposition 4.2.1]. Let such a polyhedral subdivision be given by $\Xi = \{\mathcal{X}_i\}_{i=1}^m$, where each polyhedron $\mathcal{X}_i$, called a *piece* of the system, is described by a finite system of linear inequalities:

$$\mathcal{X}_i \;=\; \{x \mid \; C_i x + d_i \;\geqslant\; 0\,\} \tag{2.2}$$

for a certain matrix $C_i \in \mathbb{R}^{m_i \times n}$ and vector $d_i \in \mathbb{R}^{m_i}$; also write $g_i(x) = A_i x + b_i$ for some matrix A_i and vector b_i. With these definitions, we can write the system (2.1) in the equivalent form

$$\dot{x} \;=\; A_i x + b_i \quad \text{ if } x \;\in\; \mathcal{X}_i. \tag{2.3}$$

In this case, continuity of the function f is equivalent to the following implication:

$$x \;\in\; \mathcal{X}_i \cap \mathcal{X}_j \;\Rightarrow\; A_i x + b_i = A_j x + b_j. \tag{2.4}$$

Since a PA function must be globally Lipschitz continuous (see, e.g., [14]), it follows from well-known ODE theory that the PA system (2.1) must admit a unique solution, which is denoted by $x(t,\xi)$, that is continuously differentiable (i.e., C^1) in time for any initial state $x(0) = \xi$. Moreover, it was recently proved in [26] that for each t, $x(t,\cdot)$

is a "semismooth" function on $\mathbb{R}^n$, meaning that it is "B(ouligand)-differentiable" (i.e., locally Lipschitz continuous and directionally differentiable) everywhere with the directional derivative along a prescribed direction given as the unique solution of a "first-order variational equation."

Throughout this paper, we focus on a particular type of PA system obtained by taking $b_i = d_i = 0$ for all i in (2.2) and (2.3). In this case, the system takes the form

$$\dot{x} \,=\, A_i x \quad \text{if } \; x \in \mathcal{X}_i \equiv \{ x \mid \; C_i x \geqslant 0 \}. \tag{2.5}$$

The continuity requirement of the right-hand side of (2.5) reduces to

$$x \in \mathcal{X}_i \cap \mathcal{X}_j \;\Rightarrow\; A_i x \,=\, A_j x; \tag{2.6}$$

i.e., $\mathcal{X}_i \cap \mathcal{X}_j \subseteq \ker(A_i - A_j)$, where ker denotes the kernel of a matrix. Since the pieces $\mathcal{X}_i$ are cones in this case, we call the system (2.5) a *conewise linear system* (CLS). Without loss of generality, we assume throughout that each matrix C_i contains no zero rows. Under this assumption, and by the fact that $\mathcal{X}_i$ is full dimensional, it follows that for each index $\ell = 1, \ldots, m_i$, there exists a vector $\widehat{x}^\ell \in \mathcal{X}_i$ such that $(C_i \widehat{x}^\ell)_\ell > 0$. Therefore, we must have

$$\varnothing \neq \operatorname{int} \mathcal{X}_i \,=\, \{ x \mid \; C_i x \,>\, 0 \}, \tag{2.7}$$

where int denotes the interior of a set. By property (c) of a polyhedral subdivision, it follows that $\mathcal{X}_j \cap \operatorname{int} \mathcal{X}_i = \varnothing$ for all $i \neq j$.

Associated with the "forward-time" system (2.5) is a *backward-time* system that allows us to obtain reverse-time results easily from a forward-time analysis. Specifically, for any given $T > 0$, define $x^{\mathrm{r}}(t) \equiv x(T - t)$. We have $x^{\mathrm{r}}(0) = x(T)$ and

$$\dot{x}^{\mathrm{r}} \,=\, \widetilde{A}_i \, x^{\mathrm{r}} \quad \text{if } x^{\mathrm{r}} \in \mathcal{X}_i, \tag{2.8}$$

where $\widetilde{A}_i \equiv -A_i$. Obviously, the latter system remains a CLS. The reverse-time system can be used to derive backward-time results pertaining to the forward-time trajectory. For instance suppose that $x(t_0, \xi) = x(t_0, \eta) = z^0$ for some $t_0 > 0$ and some ξ and η in $\mathbb{R}^n$. By considering the reverse-time trajectory starting at time t_0 and going backwards in time until the initial time $t = 0$ and by using the uniqueness of the solution to the reverse-time system given an initial condition, it follows that $\xi = x^{\mathrm{r}}(t_0, z^0) = \eta$. In words, this observation says that if two forward-time trajectories starting at two initial conditions ξ and η ever intersect at some common future time, then these two trajectories must in fact be identical at all times.

CLSs form a special class of linear hybrid systems (see, for instance, [22]). In fact, they can be cast as hybrid automata for which

(a) the vector fields in each location are linear,
(b) the invariant sets are solid polyhedral cones,
(c) the guard sets are the boundaries of these cones, and
(d) the reset maps are all identity.

In what follows, we look at two specific examples of CLSs.

Example 2.1. Bimodal CLSs are the simplest CLSs with only 2 pieces; i.e., $m = 2$ and $\Xi = \{\mathcal{X}_1, \mathcal{X}_2\}$. We claim that any such system can be described by the ODE

$$\dot{x} \,=\, Ax + b \, \max(0, c^T x) \tag{2.9}$$

for some $n \times n$ matrix A and n-vectors b and c. This is a nontrivial assertion; indeed, we need to show that given (2.5) and (2.6), we can identify the matrix A and the two vectors b and c such that (2.9) is equivalent to the given bimodal CLS. The proof is as follows. Since $\Xi = \{\mathcal{X}_1, \mathcal{X}_2\}$ is a polyhedral division of $\mathbb{R}^n$, it follows that

$$\operatorname{int} \mathcal{X}_1 \cap \operatorname{int} \mathcal{X}_2 = \varnothing, \tag{2.10}$$

$$\mathcal{X}_1 \cup \mathcal{X}_2 = \mathbb{R}^n. \tag{2.11}$$

In view of (2.10), it follows from [34, Theorem 3.3.4] that there exists a hyperplane, say $\mathcal{H} = \{x \in \mathbb{R}^n \mid c^T x = 0\}$, such that $\mathcal{X}_1 \subseteq \mathcal{H}_+ \equiv \{x \in \mathbb{R}^n \mid c^T x \geqslant 0\}$ and $\mathcal{X}_2 \subseteq \mathcal{H}_- \equiv \{x \in \mathbb{R}^n \mid c^T x \leqslant 0\}$. We claim that $\mathcal{X}_1 = \mathcal{H}_+$. To see this, note that

$$(\operatorname{int} \mathcal{H}_+ \setminus \mathcal{X}_1) \cap \mathcal{X}_1 = \varnothing, \tag{2.12}$$

$$(\operatorname{int} \mathcal{H}_+ \setminus \mathcal{X}_1) \cap \mathcal{X}_2 \subseteq \operatorname{int} \mathcal{H}_+ \cap \mathcal{H}_- = \varnothing. \tag{2.13}$$

Then, (2.11) together with (2.12)–(2.13) implies that $\operatorname{int} \mathcal{H}_+ \setminus \mathcal{X}_1 = \varnothing$, i.e., $\operatorname{int} \mathcal{H}_+ \subseteq \mathcal{X}_1$. Since $\mathcal{X}_1$ is contained in $\mathcal{H}_+$ and it is a closed set, we get $\mathcal{X}_1 = \mathcal{H}_+$. In a similar fashion, one can show that $\mathcal{X}_2 = \mathcal{H}_-$. Then, we get $\mathcal{X}_1 \cap \mathcal{X}_2 = \mathcal{H}_+ \cap \mathcal{H}_- = \mathcal{H} = \ker(c^T)$. Hence, we can write (2.5)–(2.6) as

$$\dot{x} = \begin{cases} A_1 x & \text{if} \quad c^T x \geqslant 0, \\ A_2 x & \text{if} \quad c^T x \leqslant 0, \end{cases} \tag{2.14}$$

where A_1 and A_2 satisfy $c^T x = 0 \Rightarrow A_1 x = A_2 x$. Equivalently, $A_1 - A_2 = bc^T$ for some n-vector b. Thus, (2.14) becomes (2.9) with $A \equiv A_2$.

Example 2.2. A broad class of CLSs consists of the following *linear cone complementarity system* (LCCS):

$$\begin{aligned} \dot{x} &= Ax + Bz, \\ \mathcal{C} \ni z &\perp Cx + Dz \in \mathcal{C}^*, \end{aligned} \tag{2.15}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $\mathcal{C}$ is a polyhedral cone and $\mathcal{C}^*$ is its dual (as in convex analysis [27]), and $a \perp b$ means $a^T b = 0$. A wealth of examples, from various areas of engineering as well as operations research, of these piecewise linear (hybrid) systems can be found in [31, 30, 16]. For references on the analysis of the general LCCS, we refer the reader to [8, 17, 7, 28, 29, 18]. A special case of interest emerges when $\mathcal{C} = \mathbb{R}^p_+$; the resulting LCCS is called simply a *linear complementarity system* (LCS). A fundamental subclass of the LCSs arises when all the principal minors of the matrix D are positive. Such matrices are called P-matrices in the literature of mathematical programming. The class of P-matrices is very broad (see [13]); in particular, it includes the class of positive definite (not necessarily symmetric) matrices. Most importantly, P-matrices play a fundamental role in the LCP, i.e., the problem of finding a p-vector z satisfying

$$0 \leqslant z \perp q + Dz \geqslant 0 \tag{2.16}$$

for a given p-vector q and a $p \times p$ matrix D. We denote the latter problem by LCP(q, D). It is well known that the LCP(q, D) admits a unique solution for all $q \in \mathbb{R}^p$ if and only if D is a P-matrix; see [13, Theorem 3.3.7]. Moreover, for each q there exists an index set $\alpha \subseteq \{1, 2, \ldots, p\}$ with complement $\bar{\alpha}$ such that

(a) $-(D_{\alpha\alpha})^{-1} q_\alpha \geqslant 0$ and $q_{\bar{\alpha}} - D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1} q_\alpha \geqslant 0$,

(b) the unique solution z of the LCP (q, D) is given by $z_\alpha = -(D_{\alpha\alpha})^{-1} q_\alpha$ and $z_{\bar{\alpha}} = 0$.

This shows that the solution mapping $q \mapsto z$ is a piecewise linear function on $\mathbb{R}^p$. Based on the above facts, we can rewrite the LCS (2.15) in the form of the CLS (2.5) as follows:

(2.17)
$$\dot{x} \,=\, (A - B_{\bullet\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\bullet})x \quad \text{if} \quad \begin{bmatrix} -D_{\alpha\alpha}^{-1} & 0 \\ -D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1} & I_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} C_{\alpha\bullet} \\ C_{\bar{\alpha}\bullet} \end{bmatrix} x \,\geqslant\, 0.$$

There are generalizations of the LCP results to the linear cone complementarity problem (LCCP), which can then be applied to the LCCS. In what follows, we discuss one such generalization that does not require the LCCP to have a unique solution. Let us denote by $\mathrm{SOL}(\mathcal{C}, q, D)$ the solution set of the LCCP

$$\mathcal{C} \,\ni\, z \,\perp\, q + Dz \,\in\, \mathcal{C}^*.$$

It has been observed [9, 25] that the LCCS (2.15) has a unique C^1 trajectory $x(t)$ for all initial conditions $x(0) = x^0 \in \mathbb{R}^n$ if and only if $B\mathrm{SOL}(\mathcal{C}, Cx, D)$ is a singleton for all $x \in \mathbb{R}^n$. If the latter singleton condition holds, then it follows that (2.15) is equivalent to

$$\dot{x} \,=\, Ax + B\mathrm{SOL}(\mathcal{C}, Cx, D), \tag{2.18}$$

where the right-hand side is a piecewise linear function of x on $\mathbb{R}^n$. Thus (2.18) is a CLS. A special case where $B\mathrm{SOL}(\mathcal{C}, Cx, D)$ is a singleton for all $x \in \mathbb{R}^n$ occurs when D is positive semidefinite (albeit not necessarily symmetric), $C\mathbb{R}^n \subseteq -D\mathcal{C} + \mathcal{C}^*$, and $(\mathcal{C} - \mathcal{C}) \cap \ker(D + D^T) \subseteq \ker B$. The first two conditions imply that $\mathrm{SOL}(\mathcal{C}, Cx, D)$ is a nonempty polyhedron for all $x \in \mathbb{R}^n$, and the last assumption ensures that $B\mathrm{SOL}(\mathcal{C}, Cx, D)$ is a singleton; see [14] for a proof of these facts.

Unlike the LCS with a P-matrix D, it is not straightforward to write down the pieces of the system (2.18); nevertheless, this can be achieved by introducing multipliers to the constraints defining the cone $\mathcal{C}$, which we write as

$$\mathcal{C} \,=\, \{\, z \,|\, Gz \,\geqslant\, 0 \,\} \tag{2.19}$$

for some matrix $G \in \mathbb{R}^{r \times p}$. Letting $\lambda \in \mathbb{R}^r$ be the vector of multipliers corresponding to the constraint $Gz \geqslant 0$, the complementarity condition $\mathcal{C} \ni z \perp Cx + Dz \in \mathcal{C}^*$ is equivalent to

$$0 \,=\, Cx + Dz - G^T\lambda \quad \text{and} \quad 0 \,\leqslant\, \lambda \,\perp\, Gz \,\geqslant\, 0.$$

The pieces of the CLS (2.18) can be identified as follows. Define for each index subset α of $\{1, \ldots, r\}$, with complement $\bar{\alpha}$, the polyhedral cone $\widehat{\mathcal{X}}_\alpha \subset \mathbb{R}^n$ consisting of all vectors x for which there exist (z, λ_α) such that

$$\begin{aligned} 0 \quad &= Cx + Dz - (\,G_{\alpha\bullet}\,)^T\lambda_\alpha, \\ \lambda_\alpha \,\geqslant\, 0 \,&=\, G_{\alpha\bullet}z, \quad \text{and} \quad \lambda_{\bar{\alpha}} = 0 \,\leqslant\, G_{\bar{\alpha}\bullet}z. \end{aligned}$$

While there may be multiple pairs (z, λ_α) satisfying the above linear inequality system for a given x (which explains why $\widehat{\mathcal{X}}_\alpha$ is a polyhedral cone), the vector Bz is a constant among all such pairs, as long as $B\mathrm{SOL}(\mathcal{C}, Cx, D)$ is a singleton. Moreover, with some linear algebraic manipulations, it can be deduced that $Ax + B\mathrm{SOL}(\mathcal{C}, Cx, D) = \widehat{A}_\alpha x$ for some matrix $\widehat{A}_\alpha$ for all $x \in \widehat{\mathcal{X}}_\alpha$. Notice that the family $\{\widehat{\mathcal{X}}_\alpha\}$ for α ranging over all subsets of $\{1, \ldots, r\}$ may not form a polyhedral subdivision of $\mathbb{R}^r$ (for one thing, some of them could overlap); nevertheless, they are enough to show that the right-hand side of (2.18) is a piecewise linear function of x.

2.1. Some structural properties. In this subsection, we establish some basic structural properties of the CLS (2.5). First we review some well-known concepts. An ordered tuple $a \equiv (a_1, \ldots, a_k)$ of real numbers is said to be *lexicographically nonnegative* if either $a = 0$ or its first nonzero component is positive. In this case, we write $a \succcurlyeq 0$. If a is nonzero and lexicographically nonnegative, we say that a is lexicographically positive. In this case, we write $a \succ 0$. Sometimes, we also use the signs "$\preccurlyeq$" and "$\prec$" with the obvious meanings. The set of lexicographically nonnegative k-tuples forms a convex, albeit not closed, cone in $\mathbb{R}^k$. A finite collection of n-dimensional vectors $(y^1, y^2, \ldots, y^k)$ for some positive integer k is said to be lexicographically nonnegative (positive), denoted $(y^1, y^2, \ldots, y^k) \succcurlyeq (\succ)0$, if for each $j = 1, \ldots, n$, the k-dimensional tuple $(y^1_j, \ldots, y^k_j)$ is lexicographically nonnegative (positive).

For a given $x^0 \in \mathbb{R}^n$, the solution trajectory $x(t, x^0)$ of the CLS (2.5) does not necessarily stay in the same piece as the initial state x^0 for all sufficiently small $t > 0$. To ensure the latter persistence property, we define the following sets: for all $i = 1, \ldots, m$,

$$\mathcal{Y}_i \equiv \left\{ x \in \mathbb{R}^n \,\middle|\, \left(C_i x, C_i A_i x, \ldots, C_i A_i^{n-1} x \right) \succcurlyeq 0 \right\}. \tag{2.20}$$

Obviously, $\mathcal{Y}_i$ is a convex, albeit not closed, cone in $\mathbb{R}^n$; it bears a close connection with the set $\mathcal{X}_i$ as described in the following result, whose proof is elementary and thus omitted. In the result, we let cl denote the closure of a set.

LEMMA 2.3. *Assume that every matrix C_i has no zero rows. The following statements hold for all $i = 1, \ldots, m$:* (a) $\mathcal{Y}_i \subseteq \mathcal{X}_i$; (b) $\operatorname{int} \mathcal{X}_i \subseteq \mathcal{Y}_i$; (c) $\operatorname{cl} \mathcal{Y}_i = \mathcal{X}_i$; *and* (d) $\mathcal{Y}_i - \mathcal{Y}_i = \mathcal{X}_i - \mathcal{X}_i = \mathbb{R}^n$.

The next lemma characterizes the elements of the set $\mathcal{Y}_i$; it shows in particular that the solution trajectory $x(t, x^0)$ stays in one piece, say $\mathcal{X}_i$, for all $t > 0$ sufficiently small if and only if $x^0 \in \mathcal{Y}_i$.

LEMMA 2.4. *The three statements below are equivalent for any vector $x^0 \in \mathbb{R}^n$:*

(a) $x^0 \in \mathcal{Y}_i$;

(b) *there exists a positive number ε such that $x(t, x^0) \in \mathcal{X}_i$ for all $t \in [0, \varepsilon]$;*

(c) *for some (equivalently, any) positive n-vector $c = (c_1, \ldots, c_n)$, there exists a number $\mu_0 > 0$ such that $\sum_{k=0}^{n-1} c_{k+1}\, \mu^k A_i^k x^0 \in \mathcal{X}_i$ for all $\mu \in [0, \mu_0]$.*

Proof. The equivalence of (a) and (b) is easy. We prove only the equivalence of (a) and (c). Observe that (c) is equivalent to $\sum_{k=0}^{n-1} c_{k+1}\, \mu^k (C_i A_i^k x^0)_\ell \geqslant 0$ for each index $\ell = 1, \ldots, m_i$ and all $\mu \in [0, \mu_0]$. Therefore, the positivity of $c_1, \ldots, c_n$ and μ implies $((C_i x^0)_\ell, \ldots, (C_i A_i^{n-1} x^0)_\ell) \succcurlyeq 0$ for all indices $\ell = 1, \ldots, m_i$. Hence $x^0 \in \mathcal{Y}_i$. Conversely, suppose (c) holds, but (a) does not. Then there exist an index $\ell \in \{1, \ldots, m_i\}$ and an integer $0 \leq k \leq n-1$ such that $(C_i A_i^k x^0)_\ell < 0$ and $(C_i A_i^j x^0)_\ell = 0$ for all $j = 0, \ldots, k-1$. Therefore, for sufficiently small $\mu > 0$, $\sum_{k=0}^{n-1} c_{k+1} \mu^k (C_i A_i^k x^0)_\ell < 0$. This contradicts (c). ∎

In general, a given initial state x^0 may be contained in multiple cones $\mathcal{X}_i$ and $\mathcal{Y}_i$. This motivates the definition of the following index sets. Given $\xi \in \mathbb{R}^n$, define

$$\mathcal{I}(\xi) \equiv \{ i \mid \ \xi \in \mathcal{X}_i \} \quad \text{and} \quad \mathcal{J}(\xi) \equiv \{ i \mid \ \xi \in \mathcal{Y}_i \}. \tag{2.21}$$

Basic relations between these sets are summarized in the following lemma.

LEMMA 2.5. *The following statements hold for any $\xi \in \mathbb{R}^n$:*

(a) *a neighborhood $\mathcal{N}$ of ξ exists such that $\mathcal{N} \subseteq \bigcup_{i \in \mathcal{I}(\xi)} \mathcal{X}_i$;*

(b) $\mathcal{J}(\xi) \subseteq \mathcal{I}(\xi)$;

(c) $A_i \xi = A_j \xi$ *if* $i, j \in \mathcal{I}(\xi)$;

(d) $A_i^k \xi = A_j^k \xi$ *for all positive integers k if* $i, j \in \mathcal{J}(\xi)$.

Proof. The proof is easy and omitted. ∎

With respect to the reverse-time system (2.8) where $\widetilde{A}_i = -A_i$, both sets

$$\mathcal{Y}_i^{\mathrm{r}} \equiv \left\{ x \in \mathbb{R}^n \,|\, (C_i x, C_i \widetilde{A}_i x, \ldots, C_i \widetilde{A}_i^{n-1} x) \succcurlyeq 0 \right\} \text{ and } \mathcal{J}^{\mathrm{r}}(\xi) \equiv \{ i \,|\, \xi \in \mathcal{Y}_i^{\mathrm{r}} \}$$

are not necessarily equal to the respective sets $\mathcal{Y}_i$ and $\mathcal{J}(\xi)$ that are defined with respect to the original forward-time system (2.5). Nevertheless, the forward-time trajectory and the reverse-time trajectory are equal in any interval. In particular, if $i \in \mathcal{J}^{\mathrm{r}}(x(t_*, \xi))$, where $t_* > 0$, then there exists $\varepsilon > 0$ such that $C_i x(t, \xi) \geqslant 0$ for all $t \in [t_* - \varepsilon, t_*]$.

3. Non-Zeno property of CLSs. In the hybrid systems literature, the occurrence of an infinite number of mode transitions within a finite time interval is called the *Zeno behavior* with reference to the ancient Greek philosopher Zeno's paradoxes.[1] Our goal in this section is to show that the CLS does not exhibit such behavior. At the end of the section, we will compare the main result specialized to the LCS with those obtained in [33].

DEFINITION 3.1. *The PA system* (2.1) *is said to satisfy the* (forward and backward) non-Zeno property *if for any $x^0 \in \mathbb{R}^n$ and any $t' \geq 0$, there exist $\varepsilon_\pm > 0$ and $\mathcal{X}_{i\pm} \in \Xi$ such that $x(t, x^0) \in \mathcal{X}_{i+}$ for all $t \in [t', t' + \varepsilon_+]$ (forward-time non-Zeno) and, for $t' > 0$, $x(t, x^0) \in \mathcal{X}_{i-}$ for all $t \in [t' - \varepsilon_-, t']$ (backward-time non-Zeno).*

(Note: the backward-time non-Zeno property is not defined at the initial time $t' = 0$; since the trajectory $x(t, x^0)$ is in principle defined only for $t \geqslant 0$, in the backward-time non-Zeno property at the time $t' > 0$, the scalar $\varepsilon_- > 0$ is taken to be less than t'.)

The following result shows that the backward non-Zeno property of the forward-time CLS (2.5) is equivalent to the forward non-Zeno property of the reverse-time CLS (2.8). It allows us to focus our attention on the forward non-Zeno property subsequently.

PROPOSITION 3.2. *The system* (2.5) *has the backward non-Zeno property if and only if the system* (2.8) *has the forward non-Zeno property.*

Proof. Suppose that (2.5) has the backward non-Zeno property. Let $x^0 \in \mathbb{R}^n$ and $t' \geqslant 0$ be given. Consider the reverse-time trajectory $x^{\mathrm{r}}(t, x^0)$ beginning at x^0 and terminating at a state $\xi^0 \equiv x^{\mathrm{r}}(T, x^0)$ at time $T > t'$. We then have $x^{\mathrm{r}}(t', x^0) = x(T - t', \xi^0)$. It follows by the backward non-Zeno property of the forward CLS (2.5) that an $\varepsilon > 0$ and an $\mathcal{X}_{i-}$ exist such that $x(t, x^0) \in \mathcal{X}_{i-}$ for all $t \in [T - t' - \varepsilon, T - t']$. Hence $x^{\mathrm{r}}(\widetilde{t}, x^0) = x(T - \widetilde{t}, \xi^0) \in \mathcal{X}_{i-}$ for all $\widetilde{t} \in [t', t' + \varepsilon]$. Therefore, the system (2.8) has the forward non-Zeno property. The converse can be proved similarly. ∎

The next lemma is the first step in showing that the CLSs do not have the Zeno behavior.

LEMMA 3.3. *The following three statements are equivalent.*

(a) *The CLS* (2.5) *satisfies the forward non-Zeno property.*

(b) $\cup_{i=1}^m \mathcal{Y}_i = \mathbb{R}^n$.

(c) *For every $\xi \in \mathbb{R}^n$, $\mathcal{J}(\xi) \neq \varnothing$.*

Proof. Suppose (a) holds. Since for an arbitrary $x^0 \in \mathbb{R}^n$, there exist an $\varepsilon > 0$ and a piece $\mathcal{X}_i$ such that $x(t, x^0) \in \mathcal{X}_i$ for all $t \in [0, \varepsilon]$, we have $x^0 \in \mathcal{Y}_i$ by Lemma 2.4. Thus $\mathbb{R}^n \subseteq \cup_{i=1}^m \mathcal{Y}_i$, which clearly yields (b). For the converse, let $x^0 \in \mathbb{R}^n$ and $t' \geqslant 0$ be given. By (b), the vector $\xi \equiv x(t', x^0)$ belongs to some $\mathcal{Y}_{i+}$. Hence Lemma 2.4 implies the existence of $\varepsilon_+ > 0$ such that $x(t, \xi) = x(t' + t, x^0) \in \mathcal{X}_{i+}$ for all $t \in [0, \varepsilon_+]$. Thus (a) and (b) are equivalent. The equivalence of (b) and (c) is clear. ∎

[1]The most well known of these four paradoxes dealing with counterintuitive aspects of space and time is Achilles and the turtle.

We need another technical lemma in order to prove the main result of this section.

LEMMA 3.4. *Let $\xi \in \mathbb{R}^n$ be arbitrary. For any polynomial $p : \mathbb{R} \to \mathbb{R}^n$ with $p(0) = \xi$, there exists $i \in \mathcal{I}(\xi)$ such that $p(\mu) \in \mathcal{X}_i$ for all sufficiently small $\mu > 0$.*

Proof. Let $p(\mu)$ be a polynomial satisfying $p(0) = \xi$. For each index $i \in \mathcal{I}(\xi)$, there are only three possible cases:

(i) $p(\mu) \in \mathcal{X}_i$ for all sufficiently small $\mu > 0$;

(ii) $p(\mu) \notin \mathcal{X}_i$ for all sufficiently small $\mu > 0$;

(iii) there exists an infinite sequence of positive scalars $\{\mu_k\}$ all distinct and converging to zero as $k \uparrow \infty$ such that, for all $k = 1, 2, \ldots$, $p(\mu_{2k-1}) \in \mathcal{X}_i$ and $p(\mu_{2k}) \notin \mathcal{X}_i$.

If the claim of the lemma does not hold, then for each index $i \in \mathcal{I}(\xi)$ either (ii) or (iii) must hold. We claim that (iii) must hold at least for one $i \in \mathcal{I}(\xi)$ in this case. To show this, it is enough to prove that (ii) cannot hold for all $i \in \mathcal{I}(\xi)$. Suppose, on the contrary, that (ii) holds for all $i \in \mathcal{I}(\xi)$. Then, one gets $p(\mu) \notin \cup_{i \in \mathcal{I}(\xi)} \mathcal{X}_i$ for all sufficiently small $\mu > 0$. This, however, contradicts part (a) of Lemma 2.5. Therefore, there exists $i \in \mathcal{I}(\xi)$ such that (iii) holds. Without loss of generality, we may assume that the sequence $\{\mu_k\}$ is strictly decreasing. For each k, since $p(\mu_{2k}) \notin \mathcal{X}_i$, there exists an index ℓ_k such that $(C_i p(\mu_{2k}))_{\ell_k} < 0$. Since there are only finitely many such indices ℓ_k, there exists an index ℓ_0 such that $(C_i p(\mu_{2k}))_{\ell_0} < 0$ for infinitely many k's. Without loss of generality, we may assume that $(C_i p(\mu_{2k}))_{\ell_0} < 0$ for all k. Since $(C_i p(\mu_{2k-1}))_{\ell_0} \geqslant 0$, it follows that for all k, there exists $\bar{\mu}_k \in [\mu_{2k-1}, \mu_{2k})$ such that $(C_i p(\bar{\mu}_k))_{\ell_0} = 0$. Since the $\bar{\mu}_k$'s are all distinct (because the sequence $\{\mu_k\}$ is strictly decreasing) and $(C_i p(\mu))_{\ell_0}$ is a polynomial in μ with finitely many roots, we have $(C_i p(\mu))_{\ell_0} \equiv 0$ for all μ. This is a contradiction. ∎

With the help of the last two lemmas, we can now formally state and prove the absence of Zeno behavior in the CLS.

THEOREM 3.5. *The CLS* (2.5) *has the non-Zeno property; i.e., $\cup_{i=1}^m \mathcal{Y}_i = \mathbb{R}^n = \cup_{i=1}^m \mathcal{Y}_i^{\mathrm{r}}$.*

Proof. In view of Proposition 3.2, it suffices to show that any CLS has the forward non-Zeno property. In turn, by Lemma 3.3, it is enough to show that $\cup_{i=1}^m \mathcal{Y}_i = \mathbb{R}^n$. Take any $\xi \in \mathbb{R}^n$. Set $\eta_0 = \xi$ and $\mathcal{I}_0 = \mathcal{I}(\xi)$. The continuity implies that $A_i \eta_0 = A_j \eta_0$ for all $i, j \in \mathcal{I}_0$. Define

$$\mathcal{I}_1 \equiv \{ i \in \mathcal{I}_0 \mid \eta_0 + \mu \eta_1 \in \mathcal{X}_i \ \forall \text{ sufficiently small } \mu > 0 \},$$

where $\eta_1 \equiv A_i \eta_0$ for any $i \in \mathcal{I}_0$. Lemma 3.4 guarantees that $\mathcal{I}_1 \neq \varnothing$. Note that

$$A_i(\eta_0 + \mu \eta_1) = A_j(\eta_0 + \mu \eta_1) \tag{3.1}$$

for all $i, j \in \mathcal{I}_1$ and for all sufficiently small $\mu > 0$. Since $\mathcal{I}_1 \subseteq \mathcal{I}_0$, (3.1) implies that $A_i \eta_1 = A_j \eta_1$ for all $i, j \in \mathcal{I}_1$. Define

$$\mathcal{I}_2 \equiv \{ i \in \mathcal{I}_0 \mid \eta_0 + \mu \eta_1 + \mu^2 \eta_2 \in \mathcal{X}_i \ \forall \text{ sufficiently small } \mu > 0 \},$$

where $\eta_2 \equiv A_i \eta_1$ for any $i \in \mathcal{I}_1$. Again, Lemma 3.4 guarantees that this set is nonempty. We claim that $\mathcal{I}_2 \subseteq \mathcal{I}_1$. To see this, let $i \in \mathcal{I}_2$. We need to show that $C_i(\eta_0 + \mu \eta_1) \geqslant 0$ for all $\mu > 0$ sufficiently small. Since $i \in \mathcal{I}_2$, we must have $C_i \eta_0 \geqslant 0$. If ℓ is an index such that $(C_i \eta_0)_\ell = 0$, then we must have $(C_i \eta_1)_\ell \geqslant 0$. Hence the claim holds. Therefore, for all $i, j \in \mathcal{I}_2$, we have $A_i(\eta_0 + \mu \eta_1 + \mu^2 \eta_2) = A_j(\eta_0 + \mu \eta_1 + \mu^2 \eta_2)$ for all $\mu > 0$ sufficiently small. Since $\mathcal{I}_2 \subseteq \mathcal{I}_1$, we deduce $A_i \eta_2 = A_j \eta_2$. Next, define

$$\mathcal{I}_3 \equiv \{ i \in \mathcal{I}_0 \mid \eta_0 + \mu \eta_1 + \mu^2 \eta_2 + \mu^3 \eta_3 \in \mathcal{X}_i \ \forall \text{ sufficiently small } \mu > 0 \},$$

where $\eta_3 \equiv A_i \eta_2$ for any $i \in \mathcal{I}_2$. In a similar fashion, we can show that $\mathcal{I}_3 \subseteq \mathcal{I}_2$ and $A_i \eta_3 = A_j \eta_3$ for all $i, j \in \mathcal{I}_3$. Continuing this process, we can eventually define $\mathcal{I}_{n-1}$, which is nonempty and is contained in $\mathcal{I}_{n-2} \subseteq \cdots \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_0$. We claim that $i \in \mathcal{I}_{n-1}$ implies that $\xi \in \mathcal{Y}_i$. To see this, note that

$$\eta_0 + \mu\eta_1 + \cdots + \mu^{n-1}\eta_{n-1} = \xi + \mu A_i \xi + \cdots + \mu^{n-1} A_i^{n-1} \xi$$

for any $i \in \mathcal{I}_{n-1}$ because of the nested inclusions of the index sets $\mathcal{I}_j$ for $j = 0, 1, \ldots, n-1$. Hence, $i \in \mathcal{I}_{n-1}$ implies that $\sum_{k=0}^{n-1} \mu^k A_i^k \xi \in \mathcal{X}_i$ for all $\mu > 0$ sufficiently small; thus condition (c) of Lemma 2.4 is satisfied with c being the vector of all ones. Consequently, $\xi \in \mathcal{Y}_i$. ∎

The non-Zeno property is closely related to the boundedness of the number of "mode transitions" defined as follows.

DEFINITION 3.6. *Let $x(t, x^0)$ be a solution trajectory of the CLS* (2.5) *over a time interval $[0, T]$, $T > 0$, and let $t_* \in (0, T)$. We say that t_* is* not *a switching time if there exist $i \in \{1, \ldots, m\}$ and $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_i$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$; otherwise, we say that t_* is a* switching time*, and that the CLS has a mode transition or* mode switching *at t_*.*

With this definition, we easily obtain the following result from the non-Zeno property of the CLS. The proof is by a compactness argument and resembles that of Proposition 8 in [33].

THEOREM 3.7. *Let $x(t, x^0)$ be a solution trajectory of the CLS* (2.5) *on an open time interval containing $[0, T]$. Then there is a finite number of switching times in $[0, T]$. Hence, any such trajectory $x(\bullet, x^0)$ is a continuous, piecewise analytic function on $[0, T]$.*

Proof. See the cited proposition for a proof of the assertion about switching times. To prove the piecewise analyticity assertion, it suffices to note that if $x(t, x^0) \in \mathcal{X}_j$ for some j and all t in a subinterval $[\, t_{i-1}, t_i \,]$, where t_{i-1} and t_i are any two consecutive switching times, then $x(t, x^0) = e^{A_j (t - t_{i-1})} x(t_{i-1}, x^0)$ for all t in this subinterval. Hence $x(t, x^0)$ is an analytic function for $t \in (t_{i-1}, t_i)$. Since there are finitely many such subintervals, the piecewise analyticity of $x(\bullet, x^0)$ follows. ∎

For the bimodal system (2.9), we can say more; see Proposition 5.3. For now, we specialize Theorem 3.7 to the CLS (2.18), obtaining the following corollary.

COROLLARY 3.8. *Let $\mathcal{C}$ be given by* (2.19)*. Assume that $B\mathrm{SOL}(\mathcal{C}, Cx, D)$ is a singleton for all $x \in \mathbb{R}^n$. For every $x^0 \in \mathbb{R}^n$ and $T > 0$, there exist a pair of functions $(z, \lambda) : [0, T] \mapsto \mathcal{C} \times \mathbb{R}^r_+$ with $Bz(t)$ being continuous, a partition*

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \tag{3.2}$$

of the interval $[0, T]$, and index subsets $\alpha_j \subseteq \{1, \ldots, r\}$ each with complement $\bar{\alpha}_j$ for $j = 1, \ldots, N$, such that on each subinterval $[t_j, t_{j+1}]$, the triple $(x(t, x^0), z(t), \lambda(t))$ satisfies the linear differential algebraic equation

$$\begin{aligned} \dot{x}(t, x^0) &= Ax(t, x^0) + Bz(t), \\ 0 &= Cx(t, x^0) + Dz(t) - G^T \lambda(t), \\ 0 &= G_{\alpha_j \bullet} z(t) \quad \textit{and} \quad \lambda_{\bar{\alpha}_j}(t) = 0. \end{aligned}$$

It is interesting to compare the above corollary with the non-Zenoness results in [33], which address only the LCS:

$$\dot{x} = Ax + Bz \quad \text{and} \quad 0 \leqslant z \perp Cx + Dz \geqslant 0.$$

There are some obvious similarities and subtle differences that are worth noting. The most obvious similarity is that all the results assert the finite number of switch times of some kind. The major difference lies in the treatment of the algebraic variable z. In Corollary 3.8, which is based on a hybrid system approach, z is treated implicitly; whereas in the treatment of [33], which originates from the P-matrix case and focuses on the fundamental triple of index sets

$$\begin{aligned} \alpha(t) &\equiv \{\, j \mid z_j(t) \,>\, 0 \,=\, (\, Cx(t,x^0) + Dz(t)\,)_j \,\}, \\ \beta(t) &\equiv \{\, j \mid z_j(t) \,=\, 0 \,=\, (\, Cx(t,x^0) + Dz(t)\,)_j \,\}, \\ \gamma(t) &\equiv \{\, j \mid z_j(t) \,=\, 0 \,<\, (\, Cx(t,x^0) + Dz(t)\,)_j \,\}, \end{aligned} \tag{3.3}$$

the switch times are defined with regard to a given trajectory $z(t)$. As a result of this difference in the points of view, the former corollary asserts the *existence* of a trajectory $z(t)$ satisfying the specified switching property; in contrast, the results in [33] start with a fixed but arbitrary trajectory $z(t)$ and establish the finite number of switch times for the pair $(x(t,x^0), z(t))$. The latter treatment has a price associated with it, namely, a restriction placed on the triple (B, C, D); such a restriction is not needed here. In the special case where D is a P-matrix (thus the trajectory $z(t)$ is unique), Theorem 9 in [33] is stronger than Corollary 3.8 here (for the LCS) in that the former asserts the constancy of the triple of index sets $(\alpha(t), \beta(t), \gamma(t))$ on the subintervals, whereas the latter pays no attention to the degenerate index set $\beta(t)$. As it is well known from finite-dimensional complementarity theory [14], the elements of $\beta(t)$ are most critical when one is interested in the sensitivity analysis of the system subject to parameter perturbations. The detailed exploration of this issue in a dynamic setting is beyond the scope of the present paper.

Not surprisingly, we can also establish the constancy of index sets for the CLS (2.5) similar to that for the P-matrix case of the LCS. We first establish the following proposition that pertains to an individual state.

PROPOSITION 3.9. *Let $x(t,x^0)$ be a solution trajectory of the CLS* (2.5) *over a time interval $[0,T]$. The following two statements hold.*

(a) *For every $t_* \in [0,T)$, there exists $\varepsilon_+ > 0$ such that $\mathcal{J}(x(t,x^0)) = \mathcal{J}(x(t_*,x^0))$ for all $t \in [t_*, t_* + \varepsilon_+]$.*

(b) *For every $t_* \in (0,T]$, there exists $\varepsilon_- > 0$ such that $\mathcal{J}(x(t,x^0)) = \mathcal{J}^{\,\mathrm{r}}(x(t_*,x^0))$ for all $t \in [t_* - \varepsilon_-, t_*)$.*

Proof. We prove only statement (a); the proof of (b) is similar. Write $\xi \equiv x(t_*,x^0)$. For each $i \in \mathcal{J}(\xi)$, Lemma 2.4 implies that there exist $\varepsilon_i > 0$ and $\mathcal{X}_i \in \Xi$ such that $x(t,x^0) \in \mathcal{X}_i$ for all $t \in [t_*, t_* + \varepsilon_i]$. Hence $x(t,x^0) \in \mathcal{Y}_i$ or, equivalently, $i \in \mathcal{J}(x(t,x^0))$ for all $t \in [t_*, t_* + \varepsilon_i/2]$. By letting $\varepsilon_+ \equiv \min_{1 \leqslant i \leqslant |\mathcal{J}(\xi)|} \varepsilon_i/2$, where $|\mathcal{J}(\xi)|$ denotes the cardinality of the set $\mathcal{J}(\xi)$, it follows that $\mathcal{J}(\xi) \subseteq \mathcal{J}(x(t,x^0))$ for all $t \in [t_*, t_* + \varepsilon_+]$. Conversely, consider an index $j \notin \mathcal{J}(\xi)$. If $(C_j\xi, C_jA_i\xi, \dots, C_jA_i^{n-1}\xi) \succcurlyeq 0$ for some $i \in \mathcal{J}(\xi)$, then $C_jx(t,x^0) = C_je^{A_i(t-t_*)}\xi \geqslant 0$ for all $t \geqslant t_*$ sufficiently near t_*. Hence $j \in \mathcal{J}(\xi)$, which is a contradiction. Thus, $(C_j\xi, C_jA_i\xi, \dots, C_jA_i^{n-1}\xi) \not\succcurlyeq 0$ for all $i \in \mathcal{J}(\xi)$. Hence, an index ℓ_i exists such that $(C_j\xi, C_jA_i\xi, \dots, C_jA_i^{n-1}\xi)_{\ell_i} \prec 0$. By the non-Zenoness property, $\xi \in \mathcal{Y}_{i_0}$ for some $i_0 \in \mathcal{J}(\xi)$, which implies that $x(t,x^0) = e^{A_{i_0}(t-t_*)}\xi$ for all $t > t_*$ sufficiently near t_*. Hence $C_jx(t,x^0) = C_je^{A_{i_0}(t-t_*)}\xi$ for all such t. Since the tuple $(C_j\xi, C_jA_{i_0}\xi, \dots, C_jA_{i_0}^{n-1}\xi)_{\ell_{i_0}}$ is nonzero and its first nonzero component is negative, it follows that $(C_jx(t,x^0))_{\ell_{i_0}} < 0$ for all $t > t_*$ sufficiently near t_*. Hence $j \notin \mathcal{J}(x(t,x^0))$ for all such t. Consequently, we must have $\mathcal{J}(\xi) \supseteq \mathcal{J}(x(t,x^0))$, and thus $\mathcal{J}(\xi) = \mathcal{J}(x(t,x^0))$, for all $t \in [t_*, t_* + \varepsilon_+]$, provided that $\varepsilon_+ > 0$ is further restricted if necessary. $\square$

Extending the above proposition to a compact interval and using the reverse-time trajectory (2.8), we have the following result.

COROLLARY 3.10. *Let $x(t, x^0)$ be a solution trajectory of the CLS* (2.5) *on an open time interval containing* $[0, T]$. *There exists a partition* (3.2) *of the interval* $[0, T]$ *such that for every $i = 0, 1, \ldots, N-1$, $\mathcal{J}(x(t, x^0))$ is a constant for all $t \in [t_i, t_{i+1})$.*

Proof. By Proposition 3.9, we deduce that for every $t \in [0, T]$, there exists $\varepsilon_t > 0$ such that $\mathcal{J}(x(t', x^0)) = \mathcal{J}(x(t, x^0))$ for every $t' \in [t, t + \varepsilon_t]$ and $\mathcal{J}(x(\widetilde{t}, x^0)) = \mathcal{J}^{\,\mathrm{r}}(x(t, x^0))$ for every $\widetilde{t} \in [t - \varepsilon_t, t)$. We can now employ the same covering argument as in [33, Proposition 8] to complete the proof of the corollary. ∎

Switching times can also be expressed in terms of forward-time and backward-time index sets shown as follows.

PROPOSITION 3.11. *Let $x(t, x^0)$ be a solution trajectory of the CLS* (2.5). *Then a time $t_* > 0$ is a switching time if and only if $\mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0)) = \varnothing$.*

Proof. "Sufficiency." Suppose $\mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0)) = \varnothing$ but t_* is not a switching time. Then by Definition 3.6, there exist $i \in \{1, \ldots, m\}$ and $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_i$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$. This implies that $x(t_*, x^0) \in \mathcal{Y}_i \cap \mathcal{Y}_i^{\mathrm{r}}$ by taking derivatives of the forward-time trajectory at t_* and of the reverse-time trajectory at t_*, respectively. Thus $i \in \mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$. This results in a contradiction.

"Necessity." Suppose t_* is a switching time but $\mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0)) \neq \varnothing$. Let $i \in \mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$. Then $x(t_*, x^0) \in \mathcal{Y}_i \cap \mathcal{Y}_i^{\mathrm{r}}$. By Lemma 2.4 and the reverse-time argument, we deduce the existence of $\varepsilon > 0$ such that $x(t, x^0) \in \mathcal{X}_i$ for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$. This contradicts the assumption that t_* is a switching time. ∎

One interesting observation about the CLS (2.5) is that a state trajectory may have boundary crossing, i.e., crossing a boundary of one cone and entering another cone, at a *nonswitching time* in the sense of Definition 3.6. We illustrate this observation by the following example.

Example 3.12. Consider a 3-dimensional CLS with the polyhedral subdivision:

$$\mathcal{X}_1 = \mathbb{R}_- \times \mathbb{R}^2, \quad \mathcal{X}_2 = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+, \quad \mathcal{X}_3 = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_-,$$

where $\mathbb{R}_\pm$ denote the nonnegative and nonpositive rays on the real line, respectively, and

$$A_1 = A_2 = A_3 = A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to show that for any $x^0 \in \mathbb{R}^3$, $x(t, x^0) = e^{At} x^0 = \left(x_1^0\, e^t, x_2^0, x_3^0 + x_2^0 t \right)$ for all t. Now consider $x^0 = \left(0, 1, -1 \right)$. Thus $x(t, x^0) = \left(0, 1, t-1 \right)$. Hence, $x(t, x^0) \in \mathcal{X}_1$ for all t and $x(t, x^0) \in \mathcal{X}_2$ for all $t \geqslant 1$, but $x(t, x^0) \notin \mathcal{X}_2$ for all $t < 1$. Consequently, $t_* = 1$ is *not* a switching time, but $x(t, x^0)$ crosses the boundary of $\mathcal{X}_2$ and $\mathcal{X}_3$ at $t_* = 1$.

We further illustrate this property via the index sets. Recall from Proposition 3.9 that for any given t_*, the index set $\mathcal{J}(x(t, x^0))$ remains constant for all t sufficiently close to t_*, both in the forward-time direction and in the backward-time direction. Note that the two constant index sets $\mathcal{J}(x(t_*, x^0))$ and $\mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$ may not be equal in general. In fact, expressing in terms of these index sets for this example, we have $\mathcal{J}^{\,\mathrm{r}}(x(1, x^0)) = \{1, 3\}$ and $\mathcal{J}(x(1, x^0)) = \{1, 2\}$. Notice that $\mathcal{J}^{\,\mathrm{r}}(x(1, x^0)) \cap \mathcal{J}(x(1, x^0)) = \{\,1\,\}$, but $\mathcal{J}^{\,\mathrm{r}}(x(1, x^0)) \neq \mathcal{J}(x(1, x^0))$.

The following proposition, however, shows that the LCS with the P-property, which is a special class of CLSs discussed previously, does not have the problem shown above and therefore exhibits relatively "simpler" switching behavior than general CLSs.

PROPOSITION 3.13. *Consider the LCS* (2.17) *with the P-property. If, for any* $t_* > 0$, $\mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$ *is nonempty, then* $\mathcal{J}(x(t_*, x^0)) = \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$. *In other words, if* t_* *is not a switching time, then* $\mathcal{J}(x(t_*, x^0)) = \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$.

Proof. It is shown in [33] that the LCS satisfying the P-property possesses the strong non-Zenoness at each state; i.e., for any t_*, there exist $\varepsilon_t > 0$ and two triples of index sets, $(\alpha_+, \beta_+, \gamma_+)$ and $(\alpha_-, \beta_-, \gamma_-)$, such that

$$\big(\,\alpha(t),\, \beta(t), \gamma(t)\,\big) = (\alpha_-, \beta_-, \gamma_-) \quad \forall\, t \in [t_* - \varepsilon_t, t_*),$$

$$\big(\,\alpha(t),\, \beta(t), \gamma(t)\,\big) = (\alpha_+, \beta_+, \gamma_+) \quad \forall\, t \in (t_*, t_* + \varepsilon_t],$$

where the index triple (α, β, γ) is defined in (3.3) for the associated LCP. For notational convenience, we denote each complementary cone in (2.17) by $\mathcal{X}_\delta = \{x \in \mathbb{R}^n \mid C_\delta\, x \geqslant 0\}$, where

$$C_\delta \;\equiv\; \begin{bmatrix} -D_{\delta\delta}^{-1} & 0 \\ -D_{\bar{\delta}\delta}(D_{\delta\delta})^{-1} & I_{\bar{\delta}\bar{\delta}} \end{bmatrix} \begin{bmatrix} C_{\delta\bullet} \\ C_{\bar{\delta}\bullet} \end{bmatrix}$$

and δ is a subset of $\{1, \ldots, m\}$. By the uniqueness of the solution of the LCS, it is clear that for all $t \in [t_* - \varepsilon_t, t_*)$, $x(t, x^0)$ is only in the cones $\mathcal{X}_\delta$'s with $\delta = \alpha_- \cup \beta_-^1$ and $\bar{\delta} = \gamma_- \cup \big(\beta_- \setminus \beta_-^1\big)$, where β_-^1 is a subset of β_-. Hence,

$$\mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0)) \;=\; \{\, \delta \;=\; \alpha_- \cup \beta_-^1 \;|\;\; \beta_-^1 \;\subseteq\; \beta_- \,\}. \tag{3.4}$$

Similarly, we have

$$\mathcal{J}(x(t_*, x^0)) \;=\; \{\, \delta \;=\; \alpha_+ \cup \beta_+^1 \;|\;\; \beta_+^1 \;\subseteq\; \beta_+ \,\}. \tag{3.5}$$

Let $\delta_i \in \mathcal{J}(x(t_*, x^0)) \cap \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$ with $\mathcal{X}_{\delta_i} = \{x \in \mathbb{R}^n \mid C_{\delta_i}\, x \geqslant 0\}$ and $\dot{x} = A_{\delta_i} x$ being the corresponding cone and dynamics, respectively. By the time-continuity of the state trajectory, it is easy to verify that there exists an $\varepsilon' > 0$ such that $x(t, x^0) \in \mathcal{X}_{\delta_i}$ for all $t \in [\, t_* - \varepsilon', t_* + \varepsilon'\,]$. Letting $\xi^* = x(t_*, x^0)$, we have

$$\big(\,(C_{\delta_i}\xi^*)_{\alpha_-},\, (C_{\delta_i}(-A_{\delta_i})\xi^*)_{\alpha_-}, \ldots, (C_{\delta_i}(-A_{\delta_i})^{n-1}\xi^*)_{\alpha_-}\,\big) \succ 0,$$

$$\big(\,(C_{\delta_i}\xi^*)_{\beta_-},\, (C_{\delta_i}(-A_{\delta_i})\xi^*)_{\beta_-}, \ldots, (C_{\delta_i}(-A_{\delta_i})^{n-1}\xi^*)_{\beta_-}\,\big) = 0,$$

$$\big(\,(C_{\delta_i}\xi^*)_{\gamma_-},\, (C_{\delta_i}(-A_{\delta_i})\xi^*)_{\gamma_-}, \ldots, (C_{\delta_i}(-A_{\delta_i})^{n-1}\xi^*)_{\gamma_-}\,\big) \succ 0,$$

which further implies that

$$\begin{aligned}
&\big(\,(C_{\delta_i}\xi^*)_{\alpha_-},\, (C_{\delta_i}(A_{\delta_i})\xi^*)_{\alpha_-}, \ldots, (C_{\delta_i}(A_{\delta_i})^{n-1}\xi^*)_{\alpha_-}\,\big) \neq 0,\\
&\big(\,(C_{\delta_i}\xi^*)_{\beta_-},\, (C_{\delta_i}(A_{\delta_i})\xi^*)_{\beta_-}, \ldots, (C_{\delta_i}(A_{\delta_i})^{n-1}\xi^*)_{\beta_-}\,\big) = 0,\\
&\big(\,(C_{\delta_i}\xi^*)_{\gamma_-},\, (C_{\delta_i}(A_{\delta_i})\xi^*)_{\gamma_-}, \ldots, (C_{\delta_i}(A_{\delta_i})^{n-1}\xi^*)_{\gamma_-}\,\big) \neq 0.
\end{aligned}$$

The second equality shows that $\beta_+ \supseteq \beta_-$, and the first and third inequalities show that $\beta_+ \subseteq \beta_-$. Hence $\beta_+ = \beta_-$. Moreover, we deduce from $\delta_i \in \mathcal{J}(x(t_*, x^0))$ that

$$\big(\,(C_{\delta_i}\xi^*)_{\alpha_-},\, (C_{\delta_i}(A_{\delta_i})\xi^*)_{\alpha_-}, \ldots, (C_{\delta_i}(A_{\delta_i})^{n-1}\xi^*)_{\alpha_-}\,\big) \succ 0,$$

$$\big(\,(C_{\delta_i}\xi^*)_{\gamma_-},\, (C_{\delta_i}(A_{\delta_i})\xi^*)_{\gamma_-}, \ldots, (C_{\delta_i}(A_{\delta_i})^{n-1}\xi^*)_{\gamma_-}\,\big) \succ 0.$$

Thus, by the uniqueness of the solution pair $\big(x(t), z(t)\big)$ at each t, we deduce that there is an $\varepsilon_+ > 0$ such that $z_{\alpha_-}(t) > 0$ and $(Cx(t) + Dz(t))_{\gamma_-} > 0$ for all $t \in (t_*, t_* + \varepsilon_+]$. This suggests $\alpha_- \subseteq \alpha_+$ and $\gamma_- \subseteq \gamma_+$. Since $z_{\beta_-}(t) = z_{\beta_+}(t) = \big(Cx(t)+Dz(t)\big)_{\beta_-} = \big(Cx(t)+Dz(t)\big)_{\beta_+} = 0$ for all $t \in [t_*, t_*+\varepsilon_+]$, we must have $\alpha_+ \equiv \alpha_-$ and $\gamma_+ \equiv \gamma_-$. By (3.4) and (3.5), we conclude that $\mathcal{J}(x(t_*, x^0)) = \mathcal{J}^{\,\mathrm{r}}(x(t_*, x^0))$. The second statement thus easily follows from Proposition 3.11. $\square$

4. Observability of CLSs. In this section, we treat another fundamental property of the CLS, namely, observability with respect to a linear output. In the recent paper [24], we have treated this property rather extensively for the LCS (2.15); the treatment herein extends the previous analysis in several major ways. One, we deal with a general conewise linear system; two, Theorem 4.5 when specialized to the LCS closes a gap that was unresolved in [24]; three, we also treat other observability notions in detail. To be fair to [24], the approach used there is based on a general result for a nonlinear ODE with a nondifferentiable right-hand side and is applicable to nonlinear systems such as the nonlinear complementarity system; in contrast, the approach used below takes full advantage of the (piecewise) linear structure of the CLS. Most importantly, the notion of lexicographic ordering that has played a fundamental role in [24] remains the key to the present extended treatment.

Throughout the rest of this paper, let $H \in \mathbb{R}^{r\times n}$ be a given matrix that induces the linear output $Hx(t,\xi)$ associated with the solution trajectory $x(t,\xi)$ of (2.5). With respect to this matrix H, we formally introduce the observability concepts (see Definitions 4.2 and 4.3) to be analyzed subsequently, all of which are based on the following indistinguishability definition, which is classical in systems theory.

DEFINITION 4.1. *We say that a pair of states $(\xi, \eta) \in \mathbb{R}^{n+n}$ is*

- short-time indistinguishable *if $\varepsilon > 0$ exists such that $Hx(t,\xi) = Hx(t,\eta)$ for all $t \in [0,\varepsilon]$;*
- T-time indistinguishable *for a given $T > 0$ if $Hx(t,\xi) = Hx(t,\eta)$ for all $t \in [0,T]$;*
- long-time indistinguishable *if $Hx(t,\xi) = Hx(t,\eta)$ for all $t \geqslant 0$.*

Clearly, long-time indistinguishability $\Rightarrow$ T-time indistinguishability $\Rightarrow$ short-time indistinguishability for any pair of states.

DEFINITION 4.2. *We say that a state $\xi \in \mathbb{R}^n$ is*

- short-time locally observable *if there exists a neighborhood $\mathcal{N}$ of ξ such that no pair (ξ,η) with $\eta \in \mathcal{N} \setminus \{\xi\}$ is short-time indistinguishable;*
- short-time globally observable *if there exists no state $\eta \neq \xi$ such that the pair (ξ,η) is short-time indistinguishable;*
- T-time locally observable *for a given $T > 0$ if there exists a neighborhood $\mathcal{N}$ of ξ such that no pair (ξ,η) with $\eta \in \mathcal{N} \setminus \{\xi\}$ is T-time indistinguishable;*
- T-time globally observable *for a given $T > 0$ if there exists no state $\eta \neq \xi$ such that the pair (ξ,η) is T-time indistinguishable;*
- long-time locally observable *if there exists a neighborhood $\mathcal{N}$ of ξ such that no pair (ξ,η) with $\eta \in \mathcal{N} \setminus \{\xi\}$ is long-time indistinguishable;*
- long-time globally observable *if there exists no state $\eta \neq \xi$ such that the pair (ξ,η) is long-time indistinguishable.*

Clearly, the following implications hold for any state $\xi \in \mathbb{R}^n$:

$$\begin{array}{ccc}
\text{short-time global observability} & \Rightarrow & \text{short-time local observability} \\
\Downarrow & & \Downarrow \\
T\text{-time global observability} & \Rightarrow & T\text{-time local observability} \\
\Downarrow & & \Downarrow \\
\text{long-time global observability} & \Rightarrow & \text{long-time local observability.}
\end{array}$$

The above definitions pertain to individual states. At the system level, we have the following concepts. For simplicity, we define only the short-time version of the concepts.

DEFINITION 4.3. *The CLS* (2.5) *is said to be*

- short-time locally observable *if all states are short-time locally observable.*
- short-time globally observable *if all states are short-time globally observable.*

Recall that for a given pair of matrices $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{m\times n}$, the unobservable space of (N, M), denoted $\overline{O}(N, M)$, is the set of vectors $\xi \in \mathbb{R}^n$ such that $NM^j\xi = 0$ for all $j = 0, 1, 2, \ldots$. By the well-known Cayley–Hamilton theorem in linear algebra, it follows that $\xi \in \overline{O}(N, M)$ if and only if $NM^j\xi = 0$ for all $j = 0, 1, 2, \ldots, n-1$. Elements of the space $\overline{O}(N, M)$ are said to be *unobservable* with respect to the pair (N, M). If $\overline{O}(N, M)$ consists only of the zero vector, then (N, M) is called an *observable pair.* We remark that a vector $\xi \in \overline{O}(H, A)$ if and only if for every $t_* \geqslant 0$, a scalar $\varepsilon_* > 0$ exists such that $He^{A(t-t_*)}x(t_*, \xi) = 0$ for all $t \in [t_*, t_* + \varepsilon_*]$. More generally, for any two vectors u and v, any scalar $\varepsilon > 0$, and two square matrices A and B,

$$\left\{He^{A(t-t_*)}u = He^{B(t-t_*)}v \ \forall t \in [t_*, t_* + \varepsilon]\right\} \Leftrightarrow \left[HA^k u = HB^k v \ \forall k = 0, \ldots, n-1\right].$$

This follows easily by differentiating the expression involving the exponential functions and then substituting $t = t_*$. This equivalence allows one to check the left-hand condition, which involves a continuum of times t, by a finite set of linear equations.

4.1. Short-time observability. We begin our investigation of various observability properties with the discussion of state short-time observability.

THEOREM 4.4. *A state ξ is short-time globally observable for the CLS* (2.5) *if and only if there exists no triple (η, i, j) satisfying $\eta \neq \xi$, $i \in \mathcal{J}(\eta)$, $j \in \mathcal{J}(\xi)$, and*

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} \in \overline{O}\left(\begin{bmatrix} H & -H \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \right). \tag{4.1}$$

Proof. Suppose one such triple (η, i, j) exists. From Lemma 2.4, we know that there exists a positive number ε such that $x(t, \eta)$ and $x(t, \xi)$ lie in the cones $\mathcal{X}_i$ and $\mathcal{X}_j$ for all $t \in [0, \varepsilon]$, respectively. Therefore, $x(t, \eta) = \exp(A_i t)\eta$ and $x(t, \xi) = \exp(A_j t)\xi$ on the same interval. This, together with (4.1), implies that $Hx(t, \xi) = Hx(t, \eta)$ for all $t \in [0, \varepsilon]$. Hence, the pair (ξ, η) is short-time indistinguishable. Consequently, ξ is not short-time globally observable. Conversely, suppose that ξ is short-time globally unobservable. There must exist a state $\eta \neq \xi$ such that the pair (ξ, η) is short-time indistinguishable, i.e., $Hx(t, \xi) = Hx(t, \eta)$ on an interval $[0, \varepsilon']$ for some $\varepsilon' > 0$. Let $\eta \in \mathcal{Y}_i$ and $\xi \in \mathcal{Y}_j$ for some $i, j \in \{1, \ldots, m\}$. By Lemma 2.4, there exists a positive number ε such that $H\exp(A_i t)\eta = H\exp(A_j t)\xi$ on the interval $[0, \varepsilon]$. Taking derivatives and evaluating at $t = 0$ show that the membership (4.1) holds. This leads to a contradiction. $\square$

Toward the characterization of short-time local observability, we define, for each $i = 1, \ldots, m$ and each subset $\mathcal{L}$ of $\{1, \ldots, m_i\}$,

$$\mathcal{Y}_{i,\mathcal{L}} \equiv \left\{ x \,\middle|\, \left((C_i x)_\ell, (C_i A_i x)_\ell, \ldots, (C_i A_i^{n-1} x)_\ell\right) \succcurlyeq 0, \ \forall \, \ell \in \mathcal{L} \right\} \supseteq \mathcal{Y}_i.$$

The equality $\mathcal{Y}_{i,\mathcal{L}} = \mathcal{Y}_i$ holds when $\mathcal{L} = \{1, \ldots, m_i\}$; by convention, we let $\mathcal{Y}_{i,\varnothing} = \mathbb{R}^n$. To obtain a local version of Theorem 4.4, we need to define several index sets associated with a given state $\xi \in \mathbb{R}^n$. The first one is

$$\begin{aligned}\mathcal{K}(\xi) &\equiv \left\{ i \in \mathcal{I}(\xi) \,\middle|\, \begin{pmatrix} \xi \\ \xi \end{pmatrix} \in \overline{O}\left(\begin{bmatrix} H & -H \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \right) \text{ for some } j \in \mathcal{J}(\xi) \right\} \\ &= \left\{ i \in \mathcal{I}(\xi) \,\middle|\, \exists\, j \in \mathcal{J}(\xi) \text{ such that } HA_i^k \xi = HA_j^k \xi \;\forall\, k \geqslant 0 \right\}.\end{aligned}$$

By part (d) of Lemma 2.5 and the above definition, the following inclusions are clear:

$$\mathcal{J}(\xi) \subseteq \mathcal{K}(\xi) \subseteq \mathcal{I}(\xi) \quad \forall\, \xi \in \mathbb{R}^n.$$

As it turns out (see Theorem 4.5), the pieces $\mathcal{X}_i$ for $i \notin \mathcal{K}(\xi)$ play no role in the short-time local observability of ξ. Indeed, the set $\mathcal{K}(\xi)$ is the key to a complete characterization of the short-time local observability of ξ; this set was not discovered in [24] for the LCS.

For each $i \in \mathcal{I}(\xi)$, define the index set $\mathcal{I}_{i0}(\xi) \equiv \{\ell \mid (C_i\xi)_\ell = 0\} \subseteq \{1, \ldots, m_i\}$. Note that if $\mathcal{I}_{i0}(\xi) = \varnothing$ (a case which we call *nondegenerate*), we must have $\xi \in \operatorname{int} \mathcal{X}_i$ by (2.7), which implies $\mathcal{I}(\xi) = \mathcal{J}(\xi) = \mathcal{K}(\xi) = \{i\}$. If $\mathcal{J}(\xi)$ is a proper subset of $\mathcal{K}(\xi)$, we define, for each $i \in \mathcal{K}(\xi) \setminus \mathcal{J}(\xi)$,

$$\vartheta_i(\xi) \equiv \{\, \ell \;\left|\, \left((C_i\xi)_\ell,\ (C_iA_i\xi)_\ell, \ldots, (C_iA_i^{n-1}\xi\,)_\ell\right) \prec 0 \,\right.\},$$

which must be nonempty. Since $\mathcal{K}(\xi) \subseteq \mathcal{I}(\xi)$, we have $C_i\xi \geqslant 0$ for all $i \in \mathcal{K}(\xi) \setminus \mathcal{J}(\xi)$, which implies $(C_i\xi)_{\vartheta_i(\xi)} = 0$; thus $\vartheta_i(\xi) \subseteq \mathcal{I}_{i0}(\xi)$. For each $\ell \in \vartheta_i(\xi)$, we let μ_ℓ^i be the first nonnegative integer k such that $(C_iA_i^k\xi)_\ell < 0$. We must have $1 \leq \mu_\ell^i \leq n-1$ for all $\ell \in \vartheta_i(\xi)$. Let $\overline{\vartheta}_i(\xi)$ be the complement of $\vartheta_i(\xi)$ in $\{1, \ldots, m_i\}$. Clearly,

$$\xi \in \mathcal{Y}_{i,\overline{\vartheta}_i(\xi)} = \left\{ x \,\middle|\, \left((C_ix)_\ell,\ (C_iA_ix)_\ell, \ldots, (C_iA_i^{n-1}x)_\ell\right) \succcurlyeq 0 \quad \forall\, \ell \in \overline{\vartheta}_i(\xi) \right\}.$$

Finally, we define

$$\widetilde{\mathcal{Y}}_{i,\vartheta_i(\xi)} \equiv \left\{ x \,\middle|\, \left((C_ix)_\ell,\ (C_iA_ix)_\ell, \ldots, (C_iA_i^{\mu_\ell^i-1}x)_\ell\right) \succ 0 \quad \forall\, \ell \in \vartheta_i(\xi) \right\}$$

for each $i \in \mathcal{K}(\xi) \setminus \mathcal{J}(\xi)$. Note that $0 \notin \widetilde{\mathcal{Y}}_{i,\vartheta_i(\xi)}$. Moreover, it is easy to see that the following implication holds:

$$\eta \in \widetilde{\mathcal{Y}}_{i,\vartheta_i(\xi)} \;\Leftrightarrow\; \xi + \tau\eta \in \mathcal{Y}_{i,\vartheta_i(\xi)} \quad \forall\, \tau > 0 \text{ sufficiently small.} \tag{4.2}$$

With the above preparation, we are ready to establish a necessary and sufficient condition for a given state of the CLS (2.5) to be short-time locally observable.

THEOREM 4.5. *A state ξ is short-time locally observable for the CLS* (2.5) *if and only if*

$$\begin{aligned}\overline{O}(H, A_i) \cap (\,\mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)} - \xi\,) &= \{\,0\,\} \quad \forall\, i \in \mathcal{J}(\xi), \\ \overline{O}(H, A_i) \cap \widetilde{\mathcal{Y}}_{i,\vartheta_i(\xi)} \cap (\,\mathcal{Y}_{i,\overline{\vartheta}_i(\xi)} - \xi\,) &= \varnothing \quad \forall\, i \in \mathcal{K}(\xi) \setminus \mathcal{J}(\xi).\end{aligned} \tag{4.3}$$

Proof. "Sufficiency." Suppose that the state ξ is not short-time locally observable. Since $\cup_{i=1}^m \mathcal{Y}_i = \mathbb{R}^n$, by Lemma 2.5(a), there exist an index $i \in \mathcal{I}(\xi)$ and a sequence $\{\xi^\nu\}$ converging to ξ such that $\xi \neq \xi^\nu \in \mathcal{Y}_i$ and the pair (ξ, ξ^ν) is short-time indistinguishable for all ν. We claim that, for all ν sufficiently large, the nonzero vector $\eta^\nu \equiv \xi^\nu - \xi$ violates one of the two conditions in (4.3). Let $j \in \mathcal{J}(\xi)$ such that $\xi \in \mathcal{Y}_j$. By the proof of Theorem 4.4, we deduce that, for all ν,

$$\begin{pmatrix} \xi^\nu \\ \xi \end{pmatrix} \in \overline{O}\left(\begin{bmatrix} H & -H \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \right).$$

By taking the limit $\nu \to \infty$, we get

$$\begin{pmatrix} \xi \\ \xi \end{pmatrix} \in \overline{O}\left(\begin{bmatrix} H & -H \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \right).$$

Thus, i must belong to $\mathcal{K}(\xi)$. This implies that, for all nonnegative integers k, $HA_i^k\xi^\nu = HA_j^k\xi = HA_i^k\xi$; thus η^ν belongs to $\overline{O}(H, A_i)$. Since $\xi \neq \xi^\nu \in \mathcal{Y}_i \subseteq \mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)}$, we see that the first condition in (4.3) is violated if $i \in \mathcal{J}(\xi)$. Now suppose i belongs to $\mathcal{K}(\xi) \setminus \mathcal{J}(\xi)$. To see that this contradicts the second condition in (4.3), it remains to verify that $\eta^\nu \in \widetilde{\mathcal{Y}}_{i,\vartheta_i(\xi)}$ and $\xi^\nu \in \mathcal{Y}_{i,\overline{\vartheta}_i(\xi)}$. The latter membership is obvious because $\xi^\nu \in \mathcal{Y}_i$ for all ν. To prove the former membership, suppose that an index $\bar{\ell} \in \vartheta_i(\xi)$ exists satisfying $((C_i\eta^\nu)_{\bar{\ell}}, (C_iA_i\eta^\nu)_{\bar{\ell}}, \ldots, (C_iA_i^{\mu^i_{\bar{\ell}}-1}\eta^\nu)_{\bar{\ell}}) \preceq 0$, where $\mu^i_{\bar{\ell}}$ is the first nonnegative integer k such that $(C_iA_i^k\xi)_{\bar{\ell}} < 0$ defined before. Since $((C_i\xi)_{\bar{\ell}}, (C_iA_i\xi)_{\bar{\ell}}, \ldots, (C_iA_i^{\mu^i_{\bar{\ell}}-1}\xi)_{\bar{\ell}}) = 0$, we deduce that $((C_i\xi^\nu)_{\bar{\ell}}, (C_iA_i\xi^\nu)_{\bar{\ell}}, \ldots, (C_iA_i^{\mu^i_{\bar{\ell}}-1}\xi^\nu)_{\bar{\ell}}) \preceq 0$. Hence $(C_iA_i^k\xi^\nu)_{\bar{\ell}} = 0$ for all $k = 0, 1, \ldots, \mu^i_{\bar{\ell}} - 1$. But since $(C_iA_i^{\mu^i_{\bar{\ell}}}\xi)_{\bar{\ell}} < 0$, which implies $(C_iA_i^{\mu^i_{\bar{\ell}}}\xi^\nu)_{\bar{\ell}} < 0$ for all ν sufficiently large, it follows that $((C_i\xi^\nu)_{\bar{\ell}}, (C_iA_i\xi^\nu)_{\bar{\ell}}, \ldots, (C_iA_i^{\mu^i_{\bar{\ell}}}\xi^\nu)_{\bar{\ell}}) \prec 0$, which contradicts $\xi^\nu \in \mathcal{Y}_i$.

"Necessity." We show in what follows that the violation of either one of the two conditions in (4.3) leads to a contradiction to short-time local observability of ξ. Suppose that there exist an index $i \in \mathcal{J}(\xi)$ and a nonzero vector $\eta \in \overline{O}(H, A_i) \cap (\mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)} - \xi)$. We have $\eta + \xi \in \mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)}$. Since $\xi \in \mathcal{Y}_i \subseteq \mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)}$ and $\mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)}$ is convex, it follows that $\xi + \tau\eta \in \mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)}$ for all $\tau \in [0, 1]$. Thus, for each such τ, there exists $\varepsilon_\tau > 0$ such that $(C_ie^{A_it}[\xi + \tau\eta])_{\mathcal{I}_{i0}(\xi)} \geq 0$ for all $t \in [0, \varepsilon_\tau]$. Since $(C_i\xi)_j > 0$ for all $j \in \{1, \ldots, m_i\} \setminus \mathcal{I}_{i0}(\xi)$, it follows that $(C_ie^{A_it}[\xi + \tau\eta])_j \geq 0$ for all such j and all $(t, \tau) > 0$ sufficiently small. Consequently, for every $\tau > 0$ sufficiently small, there exists $\varepsilon_\tau > 0$ such that $x(t, \xi + \tau\eta) = e^{A_it}[\xi + \tau\eta]$ for all $t \in [0, \varepsilon_\tau]$. Since $\eta \in \overline{O}(H, A_i)$, we deduce $Hx(t, \xi + \tau\eta) = Hx(t, \xi)$ for all such pairs (τ, t). Hence the pair $(\xi + \tau\eta, \xi)$ is short-time indistinguishable, contradicting the short-time local observability of ξ.

Next, suppose that a nonzero vector $\eta \in \overline{O}(H, A_i) \cap \widetilde{\mathcal{Y}}_{i,\vartheta_i(\xi)} \cap (\mathcal{Y}_{i,\overline{\vartheta}_i(\xi)} - \xi)$ exists for some $i \in \mathcal{K}(\xi) \setminus \mathcal{J}(\xi)$. By (4.2), we deduce that $\xi + \tau\eta \in \mathcal{Y}_{i,\vartheta_i(\xi)}$ for all $\tau > 0$ sufficiently small. Moreover, since $\xi + \eta \in \mathcal{Y}_{i,\overline{\vartheta}_i(\xi)}$ and $\xi \in \mathcal{Y}_{i,\overline{\vartheta}_i(\xi)}$, it follows that $\xi + \tau\eta \in \mathcal{Y}_{i,\overline{\vartheta}_i(\xi)}$ for all $\tau \in [0, 1]$. Consequently, $\xi + \tau\eta \in \mathcal{Y}_i$ for all $\tau > 0$ sufficiently small. We can now apply the same argument as before to deduce a contradiction to the short-time local observability of ξ. $\square$

The next result is an immediate corollary of Theorem 4.5 that pertains to a nondegenerate state $\xi \in \operatorname{int} \mathcal{X}_i$ for some i. No proof is needed.

Corollary 4.6. *A nondegenerate state $\xi \in \operatorname{int} \mathcal{X}_i$ is short-time locally observable for the CLS* (2.5) *if and only if $\overline{O}(H, A_i) = \{0\}$.*

We apply Theorem 4.5 to the bimodal CLS (2.9) with the two pieces $\mathcal{X}_1 = \{x | c^Tx \geqslant 0\}$ and $\mathcal{X}_2 = \{x | c^Tx \leqslant 0\}$, and the two matrices $A_1 = A + bc^T$ and $A_2 = A$. Let $\xi \in \mathbb{R}^n$ be arbitrary. The cases where $c^T\xi > 0$ and $c^T\xi < 0$ are covered by Corollary 4.6. We focus on the case where $c^T\xi = 0$. In this case, we have $\mathcal{I}(\xi) = \{1, 2\}$, $\mathcal{I}_{10}(\xi) = \{1\}$, $\mathcal{I}_{20}(\xi) = \{2\}$,

$$\mathcal{Y}_{1,\mathcal{I}_{10}(\xi)} = \{ x \,|\, (c^Tx, c^T(A + bc^T)x, \ldots, c^T(A + bc^T)^{n-1}x) \succcurlyeq 0 \} = \mathcal{Y}_1$$

and

$$\mathcal{Y}_{2,\mathcal{I}_{20}(\xi)} = \{x | (c^Tx, c^TAx, \ldots, c^TA^{n-1}x) \preceq 0\} = \mathcal{Y}_2.$$

The tuple $Y(\xi) \equiv (c^T\xi, c^TA\xi, \ldots, c^TA^{n-1}\xi)$ plays a central role in the following corollary of Theorem 4.5.

COROLLARY 4.7. *The following statements hold for the bimodal CLS* (2.9).

(a) *If* $Y(\xi) \succ 0$, *then* ξ *is short-time locally observable if and only if* $\overline{O}(H, A + bc^T) = \{0\}$.

(b) *If* $Y(\xi) \prec 0$, *then* ξ *is short-time locally observable if and only if* $\overline{O}(H, A) = \{0\}$.

(c) *If* $Y(\xi) = 0$, *then* ξ *is short-time locally observable if and only if* $\overline{O}(H, A + bc^T) = \{0\}$ *and* $\overline{O}(H, A) = \{0\}$.

Proof. It suffices to consider the case where $c^T\xi = 0$. We prove (a) only as the proofs of (b) and (c) are similar. Suppose $Y(\xi) \succ 0$, which implies $\mathcal{J}(\xi) = \{1\}$ and $\xi \notin \overline{O}(c^T, A)$. It can be verified by the definition of $\mathcal{Y}_{1,\mathcal{I}_{10}(\xi)}$ that, for any $\eta \in \mathbb{R}^n$, either $\xi + \tau\eta \in \mathcal{Y}_{1,\mathcal{I}_{10}(\xi)}$ for all $\tau > 0$ sufficiently small or $\xi - \tau\eta \in \mathcal{Y}_{1,\mathcal{I}_{10}(\xi)}$ for all $\tau > 0$ sufficiently small. Hence, for any linear subspace S of $\mathbb{R}^n$, $S \cap (\mathcal{Y}_{1,\mathcal{I}_{10}(\xi)} - \xi) = \{0\}$ if and only if $S = \{0\}$. From this observation, it follows that the first condition in (4.3) is equivalent to

$$\overline{O}(H, A + bc^T) \cap (\mathcal{Y}_{1,\mathcal{I}_{10}(\xi)} - \xi) = \{0\} \Leftrightarrow \overline{O}(H, A + bc^T) = \{0\}. \tag{4.4}$$

There are two subcases to consider: (i) $b \notin \overline{O}(H, A)$ and (ii) $b \in \overline{O}(H, A)$. In subcase (i), it is easy to show, using $\xi \notin \overline{O}(c^T, A)$, that there must exist a positive integer k such that $H(A + bc^T)^k\xi \neq HA^k\xi$. This means that $\mathcal{K}(\xi) = \mathcal{J}(\xi) = \{1\}$. In subcase (ii), we have $H(A + bc^T)^kv = HA^kv$ for all nonnegative integers k and all $v \in \mathbb{R}^n$; thus $\overline{O}(H, A + bc^T) = \overline{O}(H, A)$, which further implies $\mathcal{K}(\xi) = \{1, 2\}$ and $\overline{\vartheta}_2(\xi) = \varnothing$. In both subcases, assertion (a) follows readily from Theorem 4.5 using (4.4), and $\overline{O}(H, A) \cap \widetilde{\mathcal{Y}}_{2,\vartheta_2(\xi)} = \overline{O}(H, A) \setminus \{0\}$. $\square$

Corollary 4.7 recovers Proposition 19 in [24], which was obtained by specializing a theory for nonsmooth systems that in turn was based on a differential approach. The purpose of including the above proof of Corollary 4.7 is to illustrate the application of Theorem 4.5 in the case of a bimodal CLS. The corollary also identifies the key vector $Y(\xi)$ that was not explicitly employed in [24]. It follows from this corollary that if both $\overline{O}(H, A) \neq \{0\}$ and $\overline{O}(H, A + bc^T) \neq \{0\}$, then the bimodal CLS (2.9) has no short-time locally observable state; see Theorem 4.9 below for a general result.

We can employ the state short-time local/global observability characterizations to deduce some corresponding system short-time local/global observability results. The first such result pertains to short-time global observability and requires no proof.

THEOREM 4.8. *The CLS* (2.5) *is short-time globally observable if and only if*

$$\overline{O}\left(\begin{bmatrix} H & -H \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \right) \bigcap (\mathcal{Y}_i \times \mathcal{Y}_j) \subseteq \{ (\xi, \xi) \mid \xi \in \mathbb{R}^n \} \tag{4.5}$$

for all i *and* j.

It turns out that the characterization of system short-time local observability is quite simple, involving only linear subspace conditions that are easily verifiable.

THEOREM 4.9. *The CLS* (2.5) *is short-time locally observable if and only if*

$$\overline{O}(H, A_i) = \{0\}, \quad i \in \{1, \ldots, m\}. \tag{4.6}$$

Proof. The sufficiency is clear. For the necessity, let ξ be an arbitrary interior point of the cone $\mathcal{X}_i$ for each $i = 1, \ldots, m$. For such a vector, we have $\mathcal{J}(\xi) = \mathcal{I}(\xi) = \mathcal{K}(\xi) = \{i\}$ and $\mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)} = \mathbb{R}^n$. The necessity of (4.6) now follows readily. $\square$

4.2. Finite verification. The characterizations of short-time observation beg the question of whether the necessary and sufficient conditions in Theorems 4.4, 4.5, and 4.8 can be verified by a finite procedure (it is obvious for Theorem 4.9). Note that we are not concerned about the computational complexity of the procedure, knowing that any such procedure is very likely to be exponential in the case of the LCS. We begin with the first condition in (4.3).

For each $i \in \mathcal{J}(\xi)$ and each $\ell \in \mathcal{I}_{i0}(\xi)$ for a given $\xi \in \mathbb{R}^n$, let μ_ℓ^i be the observability degree of the pair $((C_i)_{\ell\bullet}, A_i)$ at ξ (i.e., μ_ℓ^i is the first positive integer k such that $(C_i)_{\ell\bullet} A_i^k \xi > 0$); we set $\mu_\ell^i = n$ if $(C_i)_{\ell\bullet} A_i^k \xi = 0$ for all k. We claim that for each $i \in \mathcal{J}(\xi)$,

$$\overline{O}(H, A_i) \cap (\mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)} - \xi) = \{0\} \Leftrightarrow \overline{O}(H, A_i) \bigcap \left(\bigcap_{\ell \in \mathcal{I}_{i0}(\xi)} \mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)}^\ell \right) = \{0\}, \tag{4.7}$$

where $\mathcal{Y}_{i,\mathcal{L}}^\ell \equiv \left\{ v | ((C_i)_{\ell\bullet} v, (C_i)_{\ell\bullet} A_i v, \ldots, (C_i)_{\ell\bullet} A_i^{\mu_\ell^i - 1} v) \succcurlyeq 0 \right\}$ for any $\mathcal{L} \subseteq \{1, \ldots, m_i\}$ and each $\ell \in \mathcal{L}$. The claim (4.7) is a direct consequence of the first statement of the following lemma; the second statement of the lemma is used in the subsequent development (see Proposition 5.10).

LEMMA 4.10. *Let $\mathcal{L} \subseteq \{1, \ldots, m_i\}$, $\xi \in \mathcal{Y}_{i,\mathcal{L}}$, and $\ell \in \mathcal{L}$. It holds that*

(a) *$v \in \cap_{\ell \in \mathcal{L}} \mathcal{Y}_{i,\mathcal{L}}^\ell \Leftrightarrow \xi + \tau v \in \mathcal{Y}_{i,\mathcal{L}}$ for all $\tau > 0$ sufficiently small;*

(b) *for any $v \in \cap_{\ell \in \mathcal{L}} \mathcal{Y}_{i,\mathcal{L}}^\ell$, there exist $\tau_0 > 0$ and $\varepsilon_0 > 0$ (possibly dependent on τ_0) such that $\left(C_i e^{A_i t}[\xi + \tau v]\right)_{\mathcal{L}} \geqslant 0$ for all $(t, \tau) \in [0, \varepsilon_0] \times (0, \tau_0]$.*

Proof. The first statement is obvious, following from an argument similar to (4.2). We next show the second statement. Consider the case where the vector v is nonzero and $\mu_\ell^i < n$ for all ℓ; the other cases can easily be shown in a similar fashion. For each $\ell \in \mathcal{L}$, $v \in \bigcap_{\ell \in \mathcal{L}} \mathcal{Y}_{i,\mathcal{L}}^\ell$ implies that there exists an integer $0 \leqslant k \leqslant \mu_i^\ell - 1$ such that $(C_i A_i^k v)_\ell > 0$ and $(C_i A_i^j v)_\ell = 0$ for all $j = 0, \ldots, k-1$. Hence,

$$\left(C_i e^{A_i t}[\xi + \tau v]\right)_\ell = \tau \sum_{s=k}^{\mu_\ell^i - 1} \frac{(C_i A_i^s v)_\ell}{s!} t^s + \sum_{j=\mu_\ell^i}^{\infty} \frac{(C_i A_i^j [\xi + \tau v])_\ell}{j!} t^j.$$

The first summation is positive for all $\tau > 0$ and $t > 0$ sufficiently small, and since $(C_i A_i^{\mu_\ell^i} \xi)_\ell > 0$, it follows that for some positive ε_ℓ and τ_ℓ, the second summation

$$\sum_{j=\mu_\ell^i}^{\infty} \frac{(C_i A_i^j [\xi + \tau v])_\ell}{j!} t^j = t^{\mu_\ell^i} \left\{ \frac{(C_i A_i^{\mu_\ell^i} \xi)_\ell}{j!} + O(t) + \tau \left[\frac{(C_i A_i^{\mu_\ell^i} v)_\ell}{j!} + O(t) \right] \right\} \geqslant 0$$

for all $(t, \tau) \in [0, \varepsilon_\ell] \times (0, \tau_\ell]$. Hence, $\left(C_i e^{A_i t}[\xi + \tau v]\right)_\ell \geqslant 0$ for all $(t, \tau) \in [0, \varepsilon_\ell] \times (0, \tau_\ell]$. Finally, letting $\varepsilon_0 = \min_{\ell \in \mathcal{L}} \varepsilon_\ell$ and $\tau_0 = \min_{\ell \in \mathcal{L}} \tau_\ell$, we obtain the desired result. ∎

As explained in [24] in the context of the "semi-unobservable cones," checking the right-hand condition in (4.7) can be accomplished by solving finitely many linear programs; hence so can the first condition in (4.3). Indeed, a vector v belongs to $\mathcal{Y}_{i,\mathcal{I}_{i0}(\xi)}^\ell$ if and only if either $(C_i)_{\ell\bullet} A_i^k v = 0$ for all $k \in \{0, 1, \ldots, \mu_\ell^i - 1\}$ or there exists $k' \in \{0, 1, \ldots, \mu_\ell^i - 1\}$ such that $(C_i)_{\ell\bullet} A_i^k v = 0$ for all $k \in \{0, 1, \ldots, k' - 1\}$ and $(C_i)_{\ell\bullet} A_i^{k'} v > 0$. Hence, one can easily formulate finitely many linear inequality systems to determine if the right-hand condition in (4.7) holds. A similar procedure can be applied to check the conditions (4.1), (4.5), and the second condition in (4.3).

4.3. *T*-time and long-time observability. It has been pointed out at the beginning of section 4 that if a state $\xi = x(0, \xi)$ is short-time locally observable, then ξ is T-time locally observable for all $T > 0$, including $T = \infty$. The following result extends this observation to the case where a certain state $x(t, \xi)$ along the nominal solution trajectory is short-time locally observable for some $t \in (0, T)$.

PROPOSITION 4.11. *Let $0 < T \leqslant \infty$ and $\xi \in \mathbb{R}^n$ be given. If for some t_0 in $[0, T)$, the state $x(t_0, \xi)$ is short-time locally observable, then ξ is T-time locally observable.*

Proof. The short-time local observability of the state $x^0 \equiv x(t_0, \xi)$ means that there exists a neighborhood $\widetilde{\mathcal{N}}$ of x^0 such that for all $\widetilde{x} \in \widetilde{\mathcal{N}}$,

$$[\, Hx(t, x^0) \,=\, Hx(t, \widetilde{x}) \;\forall\, t \,\geqslant\, 0 \text{ sufficiently small} \,] \;\Rightarrow\; x^0 \,=\, \widetilde{x}.$$

Since the CLS is an ODE with a globally Lipschitz continuous right-hand side, there is a constant $L > 0$ such that $\| x(t, \xi) - x(t, \xi') \| \leqslant e^{Lt} \| \xi - \xi' \|$ for all $t > 0$. Hence, a neighborhood $\mathcal{N}_0$ of ξ exists such that $x(t_0, \xi') \in \widetilde{\mathcal{N}}$ for all $\xi' \in \mathcal{N}_0$. Let $\xi' \in \mathcal{N}_0$ be such that $Hx(t, \xi) = Hx(t, \xi')$ for all $t \in [0, T]$. Hence, we have, for all $\tau \in [0, T - t_0]$,

$$Hx(\tau, x^0) \,=\, Hx(\tau, x(t_0, \xi)) \,=\, Hx(t_0 + \tau, \xi) \,=\, Hx(t_0 + \tau, \xi') \,=\, Hx(\tau, x(t_0, \xi')).$$

Therefore, it follows that $x^0 = x(t_0, \xi) = x(t_0, \xi')$. Hence by considering the reverse-time system (2.8) starting at x^0 and noting that both ξ and ξ' are states on this reverse-time trajectory at time t_0, we easily obtain $\xi = \xi'$. $\square$

Proposition 4.11 can be used to show, via the example below, that T-time local observability of a state does not imply short-time local observability of the given state.

Example 4.12. Consider the LCS(A, B, C, D) with the P-property where

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},\; B = \begin{bmatrix} 0 & b_{12} \\ 0 & 0 \end{bmatrix},\; C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\; D = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\; H = \begin{bmatrix} h_1 & 0 \end{bmatrix}$$

with $\lambda_2 > \lambda_1 > 0$, and both b_{12} and h_1 nonzero. As shown in (2.17), the LCS is in the form of the CLS with four pieces:

$$\mathcal{X}_1 = \{ x \in \mathbb{R}^2 \mid x_1 \geq 0,\, x_1 + x_2 \geq 0 \}, \qquad \mathcal{X}_2 = \{ x \in \mathbb{R}^2 \mid x_1 \leq 0,\, x_2 \geq 0 \},$$

$$\mathcal{X}_3 = \{ x \in \mathbb{R}^2 \mid 2x_1 + x_2 \leq 0,\, x_2 \leq 0 \}, \qquad \mathcal{X}_4 = \{ x \in \mathbb{R}^2 \mid 2x_1 + x_2 \geq 0,\, x_1 + x_2 \leq 0 \},$$

which correspond to $\alpha = \varnothing$, $\alpha = \{1\}$, $\alpha = \{1, 2\}$, $\alpha = \{2\}$ in (2.17), respectively, and their respective state matrices are

$$A_1 \,=\, A_2 \,=\, \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad A_3 \,=\, \begin{bmatrix} \lambda_1 & -\frac{1}{2}\, b_{12} \\ 0 & \lambda_2 \end{bmatrix}, \quad A_4 \,=\, \begin{bmatrix} \lambda_1 & -b_{12} \\ 0 & \lambda_2 \end{bmatrix}.$$

Moreover, $\overline{O}(H, A_i) = \operatorname{span}\{(0, 1)\}$ for $i = 1, 2$ and $\overline{O}(H, A_i) = \{0\}$ for $i = 3, 4$. Consider $\xi = (\xi_1, \xi_2)$ with $\xi_1 > 0 > \xi_2$ and $\xi_1 + \xi_2 > 0$. Since $\xi \in \operatorname{int} \mathcal{X}_1$ and $\overline{O}(H, A_1) \neq \{0\}$, Corollary 4.6 implies that ξ is not short-time locally observable. However, it can be seen that $t_1 > 0$ exists such that $x(t, \xi) \in \operatorname{int} \mathcal{X}_4$ for all $t > t_1$ sufficiently close to t_1. Hence, the condition $\overline{O}(H, A_4) = \{0\}$ implies the short-time local observability at $\widehat{x} = x(t_*, \xi)$ for some $t_* > t_1$. Consequently, ξ is T-time locally observable for any $T \in (t_1, \infty]$ by Proposition 4.11.

In light of Proposition 4.11, the challenge in establishing the T-time local observability of a given state $\xi \in \mathbb{R}^n$ occurs when none of the states $x(t, \xi)$ for $t \in [0, T)$ is short-time locally observable. In general, this is a rather difficult case to analyze fully, due to the mode switchings along the nominal state trajectory $x(t,\xi)$ and perturbed state trajectories in the interval $[0,T]$. Our approach to dealing with this challenge is to invoke a result in [24] that pertains to an ODE with a B-differentiable right-hand side, which includes the CLS (2.5) as a special case. In what follows, after presenting a slight improvement of this result, we identify a class of initial states for which necessary and sufficient conditions for T-time local observability can be derived; see Proposition 4.13.

As mentioned in section 2, the solution map $\xi \mapsto x(t,\xi)$ is B-differentiable for all fixed $t \geqslant 0$; in particular, the directional derivative

$$x'_\xi(t,\xi;\eta) \equiv \lim_{\tau\downarrow 0} \frac{x(t,\xi+\tau\eta) - x(t,\xi)}{\tau}$$

of $x(t,\cdot)$ at $\xi \in \mathbb{R}^n$ along any direction $\eta \in \mathbb{R}^n$ exists and satisfies a certain first-order time-dependent variational ODE. In terms of such a derivative, define the set

$$\mathcal{Z}_T^\xi \equiv \left\{ v \in \mathbb{R}^n \,|\, Hx'_\xi(t,\xi;v) = 0 \;\forall\, t \in [\,0,T\,] \right\},$$

which is a closed, albeit not necessarily convex, cone. It was proved in [24, Theorem 10] that $\mathcal{Z}_T^\xi = \{0\}$ is a sufficient condition for the T-time local observability of ξ.

We next derive an improvement of the above result via the introduction of the set Ω_T^ξ consisting of all T-time indistinguishable states from ξ. Thus $\eta \in \Omega_T^\xi$ if and only if $Hx(t,\xi) = Hx(t,\eta)$ for all $t \in [0,T]$. Clearly, $\xi \in \Omega_T^\xi$. As such we can speak of the *feasible cone*, denoted $\mathcal{F}(\Omega_T^\xi,\xi)$, and the *tangent cone*, denoted $\mathcal{T}(\Omega_T^\xi,\xi)$, of Ω_T^ξ at ξ. Specifically, v is an element of the former cone if a $\bar\tau > 0$ exists such that $Hx(t,\xi+\tau v) = Hx(t,\xi)$ for all $(t,\tau) \in [0,T] \times [0,\bar\tau]$; w is an element of the latter cone if a sequence of vectors $\{\eta^k\} \subset \Omega_T^\xi$ converging to ξ and a sequence of positive scalars $\{\tau_k\}$ converging to zero exist such that $w = \lim_{k\to\infty} \frac{\eta^k - \xi}{\tau_k}$. We have

$$\mathcal{F}(\Omega_T^\xi,\xi) \subseteq \mathcal{T}(\Omega_T^\xi,\xi) \subseteq \mathcal{Z}_T^\xi. \tag{4.8}$$

Indeed, the first inclusion holds with Ω_T^ξ replaced by any set containing ξ; the second inclusion holds by the approximation

$$Hx(t,\eta) = Hx(t,\xi) + Hx'_\xi(t,\xi;\eta-\xi) + o_t(\|\,\eta-\xi\,\|), \tag{4.9}$$

where the error function $o_t(\tau)$ satisfies $\lim_{\tau\downarrow 0} \frac{o_t(\tau)}{\tau} = 0$. In turn, (4.9) is the consequence of the B-differentiability of the solution map $x(t,\cdot)$. We have the following result.

PROPOSITION 4.13. *Let $0 < T \leqslant \infty$ and $\xi \in \mathbb{R}^n$ be given. The following implications hold for the solution trajectory $x(t,\xi)$ of the CLS* (2.5)*:*

$$\mathcal{T}(\Omega_T^\xi,\xi) = \{0\} \;\Rightarrow\; \xi \textit{ is } T\textit{-time locally observable} \;\Rightarrow\; \mathcal{F}(\Omega_T^\xi,\xi) = \{0\}. \tag{4.10}$$

Hence, the following two statements are equivalent.

(a) $\mathcal{F}(\Omega_T^\xi,\xi) = \mathcal{T}(\Omega_T^\xi,\xi)$ *and ξ is T-time locally observable.*

(b) $\mathcal{F}(\Omega_T^\xi,\xi) = \mathcal{T}(\Omega_T^\xi,\xi) = \{0\}$.

Proof. Based on the same proof as in [24, Theorem 10], the first implication in (4.10) follows from (4.9). The equivalence of the two statements (a) and (b) is obvious. ∎

Admittedly, the condition $\mathcal{F}(\Omega_T^\xi, \xi) = \mathcal{T}(\Omega_T^\xi, \xi)$, while simple, is practically not easy to verify, due to the difficulty of complete characterization of the set Ω_T^ξ, except in special cases; see Theorem 5.6 and Corollary 5.7. Thus, rather than investigating this condition in its full generality, we devote section 5 to a detailed study of the bimodal CLS (2.9).

Before ending the discussion on the general CLS, we state the following sufficient condition for a state to be long-time locally observable.

COROLLARY 4.14. *Suppose that there exists* $T \in (0, \infty)$ *such that* $\mathcal{T}(\Omega_T^\xi, \xi) = \{0\}$*; then* ξ *is long-time locally observable for the CLS* (2.5).

Proof. This is obvious because $\Omega_\infty^\xi \subseteq \Omega_T^\xi$ for any finite $T > 0$. Thus the assumption implies $\mathcal{T}(\Omega_\infty^\xi, \xi) = \{0\}$ which holds because $\mathcal{T}(\Omega_\infty^\xi, \xi) = \{0\} \subseteq \mathcal{T}(\Omega_T^\xi, \xi) = \{0\}$. The long-time local observability of ξ now follows from Proposition 4.13. ∎

5. Bimodal CLSs.

Currently, the results for T-time and long-time observability of a general CLS are limited to those in subsection 4.3. Further results, in particular, complete characterizations and finite verifications, appear difficult. Nevertheless, much more can be obtained for the bimodal CLS (2.9), whose detailed analysis is the subject of this section that is divided into several subsections. As we will see, even this simplified case is not easy to analyze, and some unsolved issues remain.

5.1. *T*-time local observability.

For the analysis to be of interest, we make the blanket assumption throughout this section that $b \neq 0$ and do not repeat the assumption. We begin by giving two necessary conditions for the bimodal system (2.9) to have a T-time locally observable state for any $T \in [0, \infty]$. The first condition (a) is a minor variant of Proposition 13 in [24] specialized to the bimodal CLS. A proof of this part can be found in the reference.

PROPOSITION 5.1. *For the bimodal system* (2.9) *to have a* T*-time locally observable state, for any* $T \in [0, \infty]$*, it is necessary that* (a) $\overline{O}(H, A) \cap \overline{O}(c^T, A) = \{0\}$ *and* (b) $b \notin \overline{O}(H, A)$.

Proof. We prove only (b). Assume that $0 \neq b \in \overline{O}(H, A)$. By induction, it can be shown that $HA^k = H(A+bc^T)^k$ for all nonnegative k. (Indeed, assume that this holds for some i; we have $HA^{i+1} = H(A + bc^T)^i A = H(A + bc^T)^{i+1} - HA^i bc^T = H(A + bc^T)^{i+1}$, completing the inductive step.) Hence, $\overline{O}(H, A) = \overline{O}(H, A + bc^T)$. Thus $He^{At}\xi = He^{(A+bc^T)t}\xi$ for all $t \geqslant 0$ and all $\xi \in \mathbb{R}^n$, which implies $Hx(t, \xi) = He^{At}\xi$ for all $t \geqslant 0$ and all $\xi \in \mathbb{R}^n$. Since $0 \neq b \in \overline{O}(H, A) = \overline{O}(H, A + bc^T)$, we can similarly deduce $He^{At}\xi = He^{At}(\xi + \tau b) = He^{(A+bc^T)t}(\xi + \tau b)$ for all $t \geqslant 0$, all $\xi \in \mathbb{R}^n$, and all $\tau \in \mathbb{R}$. Consequently, $Hx(t, \xi) = Hx(t, \xi + \tau b)$ for all such triples (t, ξ, τ). Since $b \neq 0$, no state ξ can be T-time globally/locally observable. ∎

It is useful to record a corollary of the above proposition, which becomes yet another necessary condition for (2.9) to have a T-time locally observable state.

PROPOSITION 5.2. *Under conditions* (a) *and* (b) *of Proposition* 5.1, *it holds that* $\overline{O}(H, A) \cap \overline{O}(H, A + bc^T) = \{0\}$*. Conversely, if the latter holds, then* $b \notin \overline{O}(H, A)$.

Proof. Assume by way of contradiction that $0 \neq v \in \overline{O}(H, A) \cap \overline{O}(H, A + bc^T)$. Then $v \notin \overline{O}(c^T, A)$ by (a). Hence, a first nonnegative integer ℓ exists such that $c^T A^\ell v \neq 0$. Similarly, since $b \notin \overline{O}(H, A)$, we have the first nonnegative integer k satisfying $HA^k b \neq 0$. By expanding $(A + bc^T)^{k+\ell+1}$, it can easily be verified that $H(A + bc^T)^{k+\ell+1}v = (HA^k b)(c^T A^\ell v) \neq 0$ by the choice of k and ℓ. This contradicts $v \in \overline{O}(H, A + bc^T)$. The second assertion of the proposition can be easily argued as follows. If $b \in \overline{O}(H, A)$, then by the proof of Proposition 5.1, we have $0 \neq b \in \overline{O}(H, A) = \overline{O}(H, A) \cap \overline{O}(H, A + bc^T)$. ∎

We next establish two basic properties of a solution trajectory to the bimodal system (2.9). Stated in Proposition 5.3, the first property pertains to an individual trajectory $x(t,\xi)$; the second property pertains to perturbed trajectories $x(t,\eta)$ where the initial condition η is sufficiently close to ξ.

PROPOSITION 5.3. *If $x(t,\xi)$ is a solution of the bimodal CLS* (2.9) *such that $c^T x(t,\xi)$ is not identically zero on the interval $[0,T]$ with $T \in (0,\infty)$, then a partition* (3.2) *of the interval $[0,T]$ exists such that $c^T x(t,\xi)$ does not have a zero, and thus is persistently positive or negative, in each of the open subintervals (t_{i-1},t_i) for all $i = 1,\dots,N$. Moreover, for every positive $\varepsilon < \frac{1}{2}\min_{1\leqslant i\leqslant N}(t_i - t_{i-1})$, there exists a neighborhood $\mathcal{N}$ of ξ such that for any $\eta \in \mathcal{N}$, $\min_{1\leqslant i\leqslant N}\min_{t\in[t_{i-1}+\varepsilon,t_i-\varepsilon]}\big(c^T x(t,\xi)\big)\big(c^T x(t,\eta)\big) > 0$.*

Proof. By the partition in Theorem 3.7, we deduce the existence of finitely many time instants t_i for $i = 1,\dots,N$ with $t_0 = 0$ and $t_N = T$ such that $c^T x(t,\xi) = c^T e^{A_i(t-t_{i-1})}x(t_{i-1},\xi)$ for all $t \in [t_{i-1},t_i]$, where A_i is either A or $A + bc^T$. Since the right-hand function is analytic on the real line, it has finitely many zeros in the compact subinterval $[t_{i-1},t_i]$, unless the function is identically zero. In the latter case, $c^T x(t,\xi)$ is identically zero on $[t_{i-1},t_i]$, which implies that $x(t_{i-1},\xi) \in \overline{O}(c^T,A_i) = \overline{O}(c^T,A)$. Proceeding forward and backward in time, we can establish that $\xi \in \overline{O}(c^T,A)$ so that $c^T x(t,\xi)$ is identically zero on the entire interval $[0,T]$, contradicting the assumption. Thus, by refining the partition (3.2) of the interval $[0,T]$ if necessary, we readily obtained the desired conclusions of the proposition. ∎

Noticing that the right-hand side of the bimodal ODE (2.9) is the sum of a linear function and the max function whose directional derivative is trivial to write down, we can invoke the results in [26] to obtain the directional derivative $x'_\xi(t,\xi;\eta)$ of the solution function $x(t,\cdot)$ at a vector $\xi \in \mathbb{R}^n$ along a direction $\eta \in \mathbb{R}^n$. To describe this derivative succinctly, consider the function $g^\xi : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$g^\xi(t,y) \equiv \begin{cases} Ay & \text{if } c^T x(t,\xi) < 0,\\ Ay + b\max(0,c^T y) & \text{if } c^T x(t,\xi) = 0,\\ (A + bc^T)y & \text{if } c^T x(t,\xi) > 0 \end{cases}$$

and the time-dependent ODE

$$\dot{y}(t) = g^\xi(t,y), \quad y(0) = \eta. \tag{5.1}$$

Note that the function $g^\xi(t,y)$ is only piecewise continuous in t. In fact, by Proposition 5.3, if $c^T x(t,\xi)$ is not identically zero on the interval $[0,T]$, where $T \in [0,\infty)$, there exists a partition (3.2) of the interval $[0,T]$ such that $c^T x(t,\xi)$ has no zero in (t_{i-1},t_i) and $c^T x(t,\xi) = 0$ for all $t \in \{t_1,\dots,t_{N-1}\}$. The case where $c^T x(t,\xi)$ is identically equal to zero can be made to be part of this treatment by taking $N = 0$ (the vacuous partition). Note that the case where $c^T x(t,\xi)$ does not change sign, but can have (isolated) zeros, in $[0,T]$ is clearly permitted. We call the subinterval (t_{i-1},t_i) *positive* (*negative*) if $c^T x(t,\xi)$ is positive (negative, respectively) throughout (t_{i-1},t_i). Note that this terminology refers to the given trajectory $x(t,\xi)$. In terms of the partition (3.2), we can write

$$g^\xi(t,y) \equiv \begin{cases} Ay & \text{if } t \text{ is in a negative subinterval } (t_{i-1},t_i),\\ Ay + b\max(0,c^T y) & \text{if } t \in \{\,t_1,\dots,t_{N-1}\,\},\\ (A + bc^T)y & \text{if } t \text{ is in a positive subinterval } (t_{i-1},t_i), \end{cases}$$

which shows that $g^\xi(t,y)$ is in general discontinuous at the times t_i, $i = 1,\dots,N-1$. By Theorem 3.2 in [21], the time-varying ODE (5.1) has a unique solution, which we denote $y^\xi(t;\eta)$, for every initial η that is continuous on $[0,T]$; moreover, by Theorem 7 in [26], this unique solution is equal to $x'_\xi(t,\xi;\eta)$. Therefore, we obtain

$$x'_\xi(t,\xi;\eta) = \begin{cases} e^{A(t-t_i)}x'_\xi(t_{i-1},\xi;\eta) & \text{if } t \text{ is in a negative subinterval } (t_{i-1},t_i), \\ e^{(A+bc^T)(t-t_i)}x'_\xi(t_{i-1},\xi;\eta) & \text{if } t \text{ is in a positive subinterval } (t_{i-1},t_i). \end{cases}$$

From this expression, we deduce that $x(t,\xi)+x'_\xi(t,\xi;\eta-\xi) = y^\xi(t;\eta)$ for all $t \in [0,T]$. This equality remains valid if $c^T x(t,\xi)$ is identically equal to zero on the entire interval $[0,T]$. Note that $y^\xi(t;\eta) \neq x(t,\eta)$ in general. As proved in Proposition 5.5 below, these two functions will coincide if the states ξ and η are T-time mode consistent with respect to the bimodal CLS (2.9) as defined below.

DEFINITION 5.4. *Two states ξ and η are said to be* mode consistent *on an interval $\mathcal{I}$ with respect to the bimodal CLS* (2.9) *if* $(c^T x(t,\xi))(c^T x(t,\eta)) \geqslant 0$ *for all times* $t \in \mathcal{I}$. *If $\mathcal{I} = [0,T]$, we say that ξ and η are T*-time mode consistent.

According to Proposition 5.3, η is mode consistent with ξ on the subintervals $[t_{i-1}+\varepsilon, t_i-\varepsilon]$ for all $i = 1,\dots,N$, provided that η is sufficiently near ξ. The difficulty with analyzing the perturbed trajectory $x(t,\eta)$ lies in the subintervals $(t_i-\varepsilon, t_i+\varepsilon)$ which are ε-neighborhoods of the critical times t_i. Due to its importance, we introduce the notation $\mathcal{M}_T^\xi$ to denote the set of all states that are T-time mode consistent with ξ. The next result asserts that $\mathcal{M}_T^\xi \cap \Omega_T^\xi = \mathcal{M}_T^\xi \cap (\,\xi + \mathcal{Z}_T^\xi\,)$.

PROPOSITION 5.5. *Two T-time mode consistent states ξ and η of the bimodal CLS* (2.9) *are T-time indistinguishable if and only if $Hx'_\xi(t,\xi;\eta-\xi) = 0$ for all $t \in [0,T]$.*

Proof. Under the mode consistency assumption, we have $x(t,\eta) = x(t,\xi) + x'_\xi(t,\xi;\eta-\xi)$ for all $t \in [0,T]$. The desired equivalence follows readily. ∎

Based on Propositions 5.1 and 5.5, the following result pertains to a special class of initial states ξ such that the trajectory $x(t,\xi)$ remains on the boundary of the two pieces at all times.

THEOREM 5.6. *Suppose $c^T x(t,\xi) = 0$ for all $t \in [0,T]$ with $0 < T \leqslant \infty$. The following six statements are equivalent.*

(a) *ξ is T-time globally observable for the bimodal CLS* (2.9).

(b) $\mathcal{Z}_T^\xi = \{0\}$.

(c) *$Hx(t,\eta) = 0$ for all $t \in [0,T]$ implies $\eta = 0$; i.e., the zero state is T-time globally (or locally) observable.*

(d) $\mathcal{F}(\Omega_T^\xi,\xi) = \mathcal{T}(\Omega_T^\xi,\xi) = \{0\}$.

(e) *ξ is T-time locally observable.*

(f) $\overline{O}(H,A) \cap \overline{O}(H,A+bc^T) = \{0\}$,

$$\begin{aligned} &\overline{O}(H,A) \cap \left\{ v \mid c^T e^{At}v \leqslant 0 \ \forall\, t \in [0,T] \right\} = \{0\}, \\ &\overline{O}(H,A+bc^T) \cap \left\{ v \mid c^T e^{(A+bc^T)t}v \geqslant 0 \ \forall\, t \in [0,T] \right\} = \{0\}. \end{aligned} \tag{5.2}$$

Proof. Under the assumption, all states are T-time mode consistent with ξ. Therefore, the equivalence of (a) and (b) follows from Proposition 5.5. Moreover, $x'_\xi(t,\xi;\eta) = x(t,\eta)$ for all $t \in [0,T]$. Therefore, (b) and (c) are equivalent. Statement (b) clearly implies (d) by (4.8); (d) implies (e) by Proposition 4.13. Finally, we show that (e) implies (b). Suppose (e) holds, but $\mathcal{Z}_T^\xi$ has a nonzero vector v. Then $Hx'_\xi(t,\xi;v) = 0$ for all $t \in [0,T]$. Let $\eta \equiv \xi + \tau v$, where $\tau > 0$ is chosen so that η falls in the neighborhood of ξ where its T-time local observability holds. Since

$Hx'_\xi(t,\xi;\eta-\xi)=0$ for all $t\in[0,T]$, Proposition 5.5 implies that η is T-time indistinguishable from ξ. By the T-time local observability of ξ, we deduce that $\eta-\xi$, and thus v is equal to zero, which is a contradiction. Hence (b) holds. The first five statements of the corollary are thus equivalent.

Suppose that any one of the five equivalent statements (a)–(e) holds. By Propositions 5.1 and 5.2, $\overline{O}(H,A)\cap\overline{O}(H,A+bc^T)=\{0\}$. Moreover, if either condition in (5.2) does not hold, then we have a nonzero vector v belonging to the left-hand set of one of these two expressions. It follows that $Hx(t,v)=0$ for all $t\in[0,T]$, which contradicts (c). Conversely, suppose that (f) holds. It suffices to show that any state η such that $Hx(t,\eta)=0$ for all $t\in[0,T]$ must satisfy $c^Tx(t,\eta)\geqslant 0$ for all $t\in[0,T]$ or $c^Tx(t,\eta)\leqslant 0$ for all $t\in[0,T]$; i.e., $c^Tx(t,\eta)$ remains in one piece over $[0,T]$. Suppose the claim does not hold. Then there is a switching time $t_*\in(0,T)$. This means that $\delta>0$ exists such that $c^Tx(t,\eta)$ is of one nonzero sign in $(t_*-\delta,t_*)$ and of a different nonzero sign in $(t_*,t_*+\delta)$. This implies in particular that $x(t,\eta)\neq 0$ for all $t\geqslant 0$. Without loss of generality, we may assume that $c^Tx(t,\eta)>0$ for all $t\in(t_*-\delta,t_*)$ and $c^Tx(t,\eta)<0$ for all $t\in(t_*,t_*+\delta)$. Hence, the indistinguishability condition $Hx(t,\eta)=0$ for all $t\in[0,T]$ yields $x(t_*,\eta)\in\overline{O}(H,A+bc^T)\cap\overline{O}(H,A)$, which is a contradiction because $x(t_*,\eta)\neq 0$. ∎

It is interesting to note that conditions (c) and (f) are independent of the state ξ. Clearly, elements of the set $\{v\mid c^Te^{At}v\leqslant 0\ \forall\, t\in[0,T]\}$ are vectors v in the half-plane $\mathcal{X}_2=\{v\mid c^Tv\leqslant 0\}$ such that a trajectory, when initiated at v, remains in the same half-plane. Thus the first condition in (5.2) stipulates that the zero vector is the only such vector that also lies in the unobservable space of the pair (H,A), which is the mode to which the trajectory in question belongs. A similar interpretation applies to the second condition. A local version of Theorem 5.6 is as follows.

Corollary 5.7. *Suppose that all states sufficiently near ξ are T-time mode consistent with ξ. The following statements are equivalent.*

(a) *ξ is T-time locally observable for the bimodal CLS* (2.9).

(b) $\mathcal{Z}_T^\xi=\{0\}$.

(c) $\mathcal{F}(\Omega_T^\xi,\xi)=\mathcal{T}(\Omega_T^\xi,\xi)=\{0\}$.

Proof. (a) $\Rightarrow$ (b). This basically follows the same proof as in the previous proof. Suppose $Hx'_\xi(t,\xi;v)=0$ for all $t\in[0,T]$. Let $\eta\equiv\xi+\tau v$, where $\tau>0$ is chosen so that η is T-time mode consistent with ξ and that η falls in the neighborhood of ξ where its T-time local observability holds. Proposition 5.5 implies that η is T-time indistinguishable from ξ. By the T-time local observability of ξ, we then have $v=0$. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) require no proof. ∎

The condition that $c^Tx(t,\xi)=0$ for all $t\in[0,T]$ with $0<T\leqslant\infty$ in Theorem 5.6 is equivalent to $\xi\in\overline{O}(c^T,A)$, which is further equivalent to $x(t,\xi)\in\overline{O}(c^T,A)$ for all $t\in[0,T]$. The T-time (local or global) observability of such a state is completely resolved by the corollary. Note that for a state $\xi\in\overline{O}(c^T,A)$, the set $\mathcal{M}_T^\xi=\mathbb{R}^n$. The next result treats the case where $\xi\notin\overline{O}(c^T,A)$; the key condition in this result is the generalization of condition (c) in Theorem 5.6 to condition (b) in the theorem below.

Theorem 5.8. *Suppose that $\xi\notin\overline{O}(c^T,A)$. The two conditions,*

(a) $\overline{O}(H,A)\cap\overline{O}(H,A+bc^T)=\{0\}$,

(b) *a neighborhood $\mathcal{N}$ of ξ exists such that $\mathcal{M}_T^\xi\cap(\,\xi+\mathcal{Z}_T^\xi\,)\cap\mathcal{N}=\{\xi\}$,*

are necessary for ξ to be T-time locally observable for the bimodal CLS (2.9) *for any $T\in(0,\infty]$, and sufficient for any $T\in(0,\infty)$.*

We first establish a lemma that is the key to the proof of the sufficiency part of Theorem 5.8.

LEMMA 5.9. *If* $\xi \notin \overline{O}(c^T, A)$*, then, under condition* (a) *in Theorem* 5.8*, for every finite* $T > 0$*, there exists a neighborhood* $\mathcal{N}_0$ *of* ξ *such that* $\Omega_T^\xi \cap \mathcal{N}_0 \subseteq \mathcal{M}_T^\xi$.

Proof. Since $\xi \notin \overline{O}(c^T, A)$, it follows that $x(t,\xi) \notin \overline{O}(c^T, A)$ for all $t \geqslant 0$. Furthermore, by Proposition 5.2, $b \notin \overline{O}(H, A)$. Consequently, for each $t \geqslant 0$, there exists an integer $\ell_t \geqslant 0$ such that $HA^{\ell_t}x(t,\xi) \neq H(A + bc^T)^{\ell_t}x(t,\xi)$. We claim that a neighborhood $\widetilde{\mathcal{N}}$ of ξ exists such that for every $t \in [0, T]$ and every $\eta \in \widetilde{\mathcal{N}}$, an integer $\ell \in \{0, 1, \dots, n-1\}$ exists such that

$$HA^\ell x(t,\xi) \neq H(A + bc^T)^\ell x(t,\eta) \quad \text{and} \quad HA^\ell x(t,\eta) \neq H(A + bc^T)^\ell x(t,\xi).$$

Indeed, if no such neighborhood exists, then there exist a sequence of vectors $\{\eta^k\}$ converging to ξ and a sequence of times $\{t_k\} \subset [0, T]$ such that for each $\ell \in \{0, 1, \dots, n-1\}$, either $HA^\ell x(t_k,\xi) = H(A+bc^T)^\ell x(t_k,\eta^k)$ or $HA^\ell x(t_k,\eta^k) = H(A+bc^T)^\ell x(t_k,\xi)$. The sequence $\{t_k\}$ accumulates to some time t_* in $[0, T]$; for any such time, we must have $HA^\ell x(t_*,\xi) = H(A + bc^T)^\ell x(t_*,\xi)$ for all nonnegative integers ℓ. This contradiction establishes the claim.

Let $t_i, i = 1, \dots, N$ be finitely many time instants in $[0, T]$ described in Proposition 5.3 such that $c^T x(t_i,\xi) = 0$. Let ε be a positive scalar and $\widehat{\mathcal{N}}$ be a neighborhood of ξ such that every $\eta \in \widehat{\mathcal{N}}$ is mode consistent with ξ on all the subintervals $[t_{i-1} + \varepsilon, t_i - \varepsilon]$ for all $i = 1, \dots, N$; the existence of ε and $\widehat{\mathcal{N}}$ is due to Proposition 5.3. Let $\mathcal{N}_0 \equiv \widetilde{\mathcal{N}} \cap \widehat{\mathcal{N}}$, where $\widetilde{\mathcal{N}}$ is established above. By way of contradiction, suppose that there exists a sequence of vectors $\{\eta^k\} \in \mathcal{N}_0$ converging to ξ such that each η^k is T-time indistinguishable from but not T-time mode consistent with ξ. Hence, there exists $\widetilde{t}_k \in (t_{i_k} - \varepsilon, t_{i_k} + \varepsilon)$ for some $i_k \in \{1, \dots, N-1\}$ or $\widetilde{t}_k \in [0, \varepsilon)$ or $\widetilde{t}_k \in (T - \varepsilon, T]$ such that $(c^T x(\widetilde{t}_k,\xi))(c^T x(\widetilde{t}_k,\eta^k)) < 0$ for every k. We may assume without loss of generality that $\widetilde{t}_k \in [0, T)$ is such that for some $\delta_k > 0$, $c^T x(t,\xi) > 0$ and $c^T x(t,\eta^k) < 0$ for all $t \in (\widetilde{t}_k, \widetilde{t}_k + \delta_k)$ (if $\widetilde{t}_k = T$, we consider the interval $(\widetilde{t}_k - \delta_k, \widetilde{t}_k)$ and use the reverse-time argument). Hence, we have $x(t,\xi) = e^{(A+bc^T)(t-\widetilde{t}_k)}x(\widetilde{t}_k,\xi)$ and $x(t,\eta) = e^{A(t-\widetilde{t}_k)}x(\widetilde{t}_k,\eta^k)$ for all such t. Thus $He^{(A+bc^T)(t-\widetilde{t}_k)}x(\widetilde{t}_k,\xi) = He^{A(t-\widetilde{t}_k)}x(\widetilde{t}_k,\eta^k)$ for all $t \in (\widetilde{t}_k, \widetilde{t}_k + \delta_k)$, which yields $H(A + bc^T)^\ell x(\widetilde{t}_k,\xi) = HA^\ell x(\widetilde{t}_k,\eta^k)$ for all nonnegative integers ℓ. But this contradicts the claim established above because $\eta^k \in \widetilde{\mathcal{N}}$. $\square$

Proof of Theorem 5.8. Suppose that ξ is T-time locally observable for $T \in [0,\infty]$. By Propositions 5.1 and 5.2, condition (a) is necessary. To prove (b), choose a neighborhood $\mathcal{N}$ of ξ so that $\Omega_T^\xi \cap \mathcal{N} = \{\xi\}$. Since $\mathcal{M}_T^\xi \cap (\xi + \mathcal{Z}_T^\xi) = \Omega_T^\xi$ by Proposition 5.5, we have

$$\mathcal{M}_T^\xi \cap (\xi + \mathcal{Z}_T^\xi) \cap \mathcal{N} = \mathcal{M}_T^\xi \cap \Omega_T^\xi \cap \mathcal{N} = \{\xi\}.$$

Conversely, let $T < \infty$. Assume that (a) and (b) hold and let $\mathcal{N}$ be the neighborhood of ξ described in (b). By Lemma 5.9, a neighborhood $\mathcal{N}_0$ of ξ exists such that $\Omega_T^\xi \cap \mathcal{N}_0 \subseteq \mathcal{M}_T^\xi$. So, by (b),

$$\Omega_T^\xi \cap \mathcal{N} \cap \mathcal{N}_0 \subseteq \Omega_T^\xi \cap \mathcal{M}_T^\xi \cap \mathcal{N} \cap \mathcal{N}_0 = (\xi + \mathcal{Z}_T^\xi) \cap \mathcal{M}_T^\xi \cap \mathcal{N} \cap \mathcal{N}_0 = \{\xi\}.$$

Hence ξ is T-time observable within the neighborhood $\mathcal{N} \cap \mathcal{N}_0$. $\square$

It is important to note that the sufficient part of Theorem 5.8 requires T to be finite. We will discuss more about the case where $T = \infty$ in subsection 5.3.

5.2. More on mode consistency. From condition (b) in Theorem 5.8, the important role of mode consistency in T-time local observability is amply evident. In what follows, we discuss this condition in greater detail. Specifically, our goal is to generalize the two conditions in (5.2) to the case where $\xi \notin \overline{O}(c^T, A)$. As it turns out, such a generalization is not trivial because we need to deal with various mode transitions, which necessitate the introduction of the "mode-transition matrices" $\Phi_\xi^{\mathrm{n}}(t)$ associated with the nominal trajectory $x(t,\xi)$.

Recall that Proposition 5.3 implies the existence of a partition of the finite interval $[0,T]$, where $T \in (0,\infty)$, such that $c^T x(t,\xi)$ is persistently positive or negative in (t_{i-1}, t_i) for all $i = 1,\dots,N$. In other words, $c^T x(t,\xi) = 0$ only at t_i's. For each i, define

$$(\mathbf{c}^i, S_i) \equiv \begin{cases} (-c^T, A) & \text{if } (t_i, t_{i+1}) \text{ is a negative subinterval,} \\ (c^T, A + bc^T) & \text{if } (t_i, t_{i+1}) \text{ is a positive subinterval,} \end{cases} \qquad i = 0,\dots,N-1,$$

and $\mathcal{Y}(S_i) \equiv \left\{ x \,\middle|\, \left(\mathbf{c}^i x,\ \mathbf{c}^i S_i x, \dots, \mathbf{c}^i S_i^{n-1} x \right) \succcurlyeq 0 \right\}$. The nominal state trajectory on $[0,T]$ can be written as $x(t,\xi) = \Phi_\xi^{\mathrm{n}}(t)\xi$, where

$$\Phi_\xi^{\mathrm{n}}(t) \equiv \begin{cases} \text{the identity matrix} & \text{for } t = 0, \\ e^{S_k(t-t_k)} \displaystyle\prod_{i=1}^{k} e^{S_{i-1}(t_i - t_{i-1})} & \text{for } t \in [t_k, t_{k+1}],\ k = 0,\dots,N-1, \end{cases}$$

is a *mode-transition matrix*. It is clear that $\Phi_\xi^{\mathrm{n}}(t)$ is invertible for all t.

PROPOSITION 5.10. *Let $T \in (0,\infty)$. If $\xi \notin \overline{O}(H,A)$, then Theorem* 5.8(b) *is equivalent to the following: with $\widehat{x}_i = \Phi_\xi^{\mathrm{n}}(t_i)\xi$ for $i = 0,\dots,N$, and $S_{-1} = S_N \equiv 0$,*

$$\bigcap_{i=0}^{N} \left\{ \left[\Phi_\xi^{\mathrm{n}}(t_i)\right]^{-1} \left[\overline{O}(H,S_i) \cap \left(\mathcal{Y}(S_i) - \widehat{x}_i\right) \cap \overline{O}(H,-S_{i-1}) \cap \left(\mathcal{Y}(-S_{i-1}) - \widehat{x}_i\right) \right] \right\} = \{0\}.$$

Proof. Suppose that Theorem 5.8(b) does not hold. Then for any neighborhood $\mathcal{N}$ of ξ, there is $\eta \in \mathcal{N}$ other than ξ such that $\eta \in \mathcal{M}_T^\xi \cap (\xi + \mathcal{Z}_T^\xi)$. Hence, by Proposition 5.5, it follows that ξ and η are T-time mode consistent states and $Hx_\xi'(t,\xi;\eta-\xi) = 0$ for all $t \in [0,T]$. This implies that $x(t,\eta) = \Phi_\xi^{\mathrm{n}}(t)\eta$ and $Hx_\xi'(t,\xi;\eta-\xi) = H\Phi_\xi^{\mathrm{n}}(t)[\eta-\xi] = 0$ on $[0,T]$. Let $v = \eta - \xi$. The mode consistency condition further implies that for all $i = 1,\dots,N$,

$$\Phi_\xi^{\mathrm{n}}(t_i)\eta = \Phi_\xi^{\mathrm{n}}(t_i)[v+\xi] = \Phi_\xi^{\mathrm{n}}(t_i)v + \widehat{x}_i \in \mathcal{Y}(S_i).$$

Similarly, $\Phi_\xi^{\mathrm{n}}(t_i)\eta = \Phi_\xi^{\mathrm{n}}(t_i)v + \widehat{x}_i \in \mathcal{Y}(-S_{i-1})$. Moreover, for any $t \in (t_i, t_{i+1})$ and $i = 0,\dots,N-1$, $H\Phi_\xi^{\mathrm{n}}(t)v = He^{S_i(t-t_i)}\Phi_\xi^{\mathrm{n}}(t_i)v$. Hence, $H\Phi_\xi^{\mathrm{n}}(t)v = 0$ for all $t \in [0,T]$ implies that $\Phi_\xi^{\mathrm{n}}(t_i)v \in \overline{O}(H,S_i)$. Using the reverse-time argument, one can also deduce $\Phi_\xi^{\mathrm{n}}(t_i)v \in \overline{O}(H,-S_{i-1})$. Consequently, we have

$$0 \neq v \in \bigcap_{i=1}^{N} \left\{ \left[\Phi_\xi^{\mathrm{n}}(t_i)\right]^{-1} \left[\overline{O}(H,S_i) \cap \left(\mathcal{Y}(S_i) - \widehat{x}_i\right) \cap \overline{O}(H,-S_{i-1}) \cap \left(\mathcal{Y}(-S_{i-1}) - \widehat{x}_i\right) \right] \right\},$$

which is a contradiction.

Conversely, suppose Theorem 5.8(b) holds, but there is a nonzero vector v belonging to the above intersection. We therefore have $\Phi_\xi^{\mathrm{n}}(t_i)[\xi+v] \in \mathcal{Y}(S_i) \cap \mathcal{Y}(-S_{i-1})$ for all $i = 0, \ldots, N-1$. Since $\Phi_\xi^{\rm n}(t_i)\xi \in \mathcal{Y}(S_i) \cap \mathcal{Y}(-S_{i-1})$, it is clear that $\Phi_\xi^{\rm n}(t_i)[\xi + \kappa v] \in \mathcal{Y}(S_i) \cap \mathcal{Y}(-S_{i-1})$ for all $0 \leqslant \kappa \leqslant 1$, following the same argument in the proof of Theorem 4.5. For each i, we can apply the second statement of Lemma 4.10 to obtain two positive pairs $(\varepsilon_{i+}, \tau_{i+})$ and $(\varepsilon_{i-}, \tau_{i-})$ such that $\mathbf{c}^i e^{S_i(t-t_i)} \Phi_\xi^{\rm n}(t_i)[\xi + \kappa v] \geqslant 0$ for all $(t, \kappa) \in [t_i, t_i + \varepsilon_{i+}] \times (0, \tau_{i+}]$ and $\mathbf{c}^i e^{S_{i-1}(t_i - t)} \Phi_\xi^{\rm n}(t_i)[\xi + \kappa v] \geqslant 0$ for all $(t, \kappa) \in [t_i - \varepsilon_{i-}, t_i] \times (0, \tau_{i-}]$. Define $\varepsilon' = \min_{1 \leqslant i \leqslant N} \varepsilon_{i\pm}$ and $\tau' = \min_{1 \leqslant i \leqslant N} \tau_{i\pm}$. As a result, for all $t \in \cup_{1 \leqslant i \leqslant N}[t_i - \varepsilon', t_i + \varepsilon']$ and all $0 \leqslant \kappa \leqslant \tau'$,

$$\left(c^T x(t, \xi)\right)\left(c^T x(t, \xi + \kappa v)\right) \geqslant 0. \tag{5.3}$$

By Proposition 5.3, a neighborhood $\mathcal{N}$ of ξ and a positive number $\tau_0 \leqslant \tau'$ exist such that $\xi + \kappa v \in \mathcal{N}$ for all $\kappa \in [0, \tau_0]$ and that $\left(c^T x(t, \xi)\right)\left(c^T x(t, \xi + \kappa v)\right) > 0$ for all $t \in \cup_{1 \leqslant i \leqslant N}[t_{i-1} + \varepsilon', t_i - \varepsilon']$. Since (5.3) also holds for $0 \leqslant \kappa \leqslant \tau_0$, we have $\left(c^T x(t, \xi)\right)\left(c^T x(t, \xi + \kappa v)\right) \geqslant 0$ for all $t \in [0, T]$ and $0 \leqslant \kappa \leqslant \tau_0$. Hence $(\xi, \xi + \kappa v)$ are two T-time mode consistence states for all $0 \leqslant \kappa \leqslant \tau_0$. Thus, for all $0 \leqslant \kappa \leqslant \tau_0$, $Hx(t, \xi + \kappa v) = H\Phi_\xi^{\rm n}(t)[\xi + \kappa v] = H\Phi_\xi^{\rm n}(t)\xi + \kappa H \Phi_\xi^{\rm n}(t) v$. Notice that for any $t \in [t_i, t_{i+1}]$, we have $H\Phi_\xi^{\rm n}(t)v = He^{S_i(t-t_i)}\Phi_\xi^{\rm n}(t_i)v$. Since $\Phi_\xi^{\rm n}(t_i)v \in \overline{O}(H, S_i) \cap \overline{O}(H, -S_i)$, $He^{S_i(t-t_i)}\Phi_\xi^{\rm n}(t_i)v \equiv 0$ for all $t \in [t_i, t_{i+1}]$ and all $i = 0, \ldots, N-1$. Consequently, $H\Phi_\xi^{\rm n}(t)v \equiv 0$ on $[0, T]$. This suggests that $(\xi, \xi + \kappa v)$ are two T-time indistinguishable states for all $0 \leqslant \kappa \leqslant \tau_0$. By Proposition 5.5, $v \in \mathcal{Z}_T^\xi$ and thus $(\xi + \kappa v) \in \mathcal{M}_T^\xi \cap (\xi + \mathcal{Z}_T^\xi) \cap \mathcal{N}$ for all $0 \leqslant \kappa \leqslant \tau_0$. As a result, for any small neighborhood of $\mathcal{V} \subseteq \mathcal{N}$, $(\xi + \kappa v) \in \mathcal{M}_T^\xi \cap (\xi + \mathcal{Z}_T^\xi) \cap \mathcal{V}$ for some $0 < \kappa \leqslant \tau_0$. This contradicts Theorem 5.8(b). $\square$

5.3. Long-time observability. The finiteness of T is needed to ensure the applicability of Lemma 5.9 in the proof of the sufficiency part of Theorem 5.8. In what follows, we establish the long-time (i.e., $T = \infty$) local observability of ξ for the bimodal CLS (2.9) under various conditions, one of which postulates that the trajectory $x(t, \xi)$ has a switching time $t_* > 0$; this is a time for which $\delta > 0$ exists such that $c^T x(t, \xi)$ is of one nonzero sign in $(t_* - \delta, t_*)$ and of a different nonzero sign in $(t_*, t_* + \delta)$. Note that any trajectory $x(t, \xi)$ such that $c^T x(t, \xi)$ changes sign at least once must have a switching time, provided that $\xi \notin \overline{O}(c^T, A)$.

Proposition 5.11. *Any of the conditions* (a), (b), *and* (c) *below is sufficient for ξ to be long-time locally observable for the bimodal system* (2.9)*:*

(a) $\overline{O}(H, A) = \{0\}$ *and there exists $t_* \geqslant 0$ such that $c^T x(t_*, \xi) < 0$;*

(b) $\overline{O}(H, A + bc^T) = \{0\}$ *and there exists $t_* \geqslant 0$ such that $c^T x(t_*, \xi) > 0$;*

(c) $\overline{O}(H, A) \cap \overline{O}(H, A + bc^T) = \{0\}$ *and the trajectory $x(t, \xi)$ has a switching time.*

Proof. Under condition (a) or (b), the state $x(t_*, \xi)$ is short-time, and thus long-time, locally observable, by Corollary 4.7 and Proposition 4.11. To prove the same under condition (c), we assume without loss of generality that for some $\delta > 0$, $c^T x(t, \xi)$ is positive on $[t_* - \delta, t_*)$ and negative on $(t_*, t_* + \delta]$. Localizing the proof of Lemma 5.9 to the compact interval $\mathcal{T}_* \equiv [t_* - \delta, t_* + \delta]$ and using the time invariance of the bimodal system, we deduce the existence of a neighborhood $\mathcal{N}$ of ξ such that any state $\eta \in \mathcal{N}$ that is indistinguishable from ξ in the interval $\mathcal{T}_*$ must be mode consistent with ξ on the same interval. Suppose that ξ is not long-time locally observable. There exists a sequence of vectors $\{\eta^k\} \subset \Omega_\infty^\xi$ converging to ξ such that $\eta^k \neq \xi$ for all k. It follows that for k sufficiently large, η^k must be mode consistent with ξ on the interval $\mathcal{T}_*$.

Hence we have

$$\begin{aligned} He^{(A+bc^T)(t-t_*+\delta)}x(t_*-\delta,\xi) &= He^{(A+bc^T)(t-t_*+\delta)}x(t_*-\delta,\eta^k) \quad \forall\,[\,t_*-\delta,t_*\,), \\ He^{A(t-t_*)}x(t_*,\xi) &= He^{A(t-t_*)}x(t_*,\eta^k) \qquad\qquad\qquad \forall\,(\,t_*,t_*+\delta\,]. \end{aligned}$$

It follows that $x(t_*,\xi)-x(t_*,\eta^k) \in \overline{O}(H,A+bc^T)\cap\overline{O}(H,A)$. Hence $x(t_*,\xi) = x(t_*,\eta^k)$, which yields $\xi = \eta^k$, a contradiction! □

We close this section by presenting numerical examples to illustrate several counterintuitive properties of the long-time observability of an initial state ξ as well as the states along the trajectory $x(t,\xi)$. These examples also demonstrate the difficulties of designing constructive tests for long-time observability. The first example shows that in order for ξ to be long-time locally observable, it is possible that no state on the trajectory $x(t,\xi)$ is short-time observable. This example strengthens Example 4.12 which concerns mainly properties of the initial state.

Example 5.12. Consider a bimodal system in $\mathbb{R}^3$ with

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},\ b = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix},\ c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},\ \text{and } H = \begin{bmatrix} h_1 & h_2 & 0 \end{bmatrix},$$

where $\lambda_3 > \lambda_2 > \lambda_1 > 0$, $b_1 \neq 0$, $c_1c_2c_3 \neq 0$, $h_1h_2 \neq 0$,

$$\lambda_2\,\lambda_3\,(\,\lambda_3-\lambda_2\,) + \lambda_1\,\lambda_2\,(\,\lambda_2-\lambda_1\,) + \lambda_3\,\lambda_1\,(\,\lambda_1-\lambda_3\,) \neq 0, \tag{5.4}$$

and $h_2(\lambda_1-\lambda_2+b_1c_1) = h_1b_1c_2$. Straightforward computations show that

$$\overline{O}(H,A) = \text{span}\,\{(0,0,1)\} \quad \text{and} \quad \overline{O}(H,A+bc^T) = \text{span}\{(h_2,-h_1,0)\}.$$

Notice that $\overline{O}(H,A)\cap\overline{O}(H,A+bc^T) = \{0\}$. Since both of the above two subspaces contain nonzero states, it follows from Corollary 4.7 that the bimodal CLS has no short-time local observable states. However, the condition (5.4) ensures that $\overline{O}(c^T,A) = \{0\}$. Hence for any $\xi \neq 0$, the trajectory $c^Tx(t,\xi)$ can have only isolated zeros. Pick an initial state ξ such that $c^T\xi < 0$ but $c_3\xi_3 > 0$. Then $c^Tx(t,\xi) < 0$ for all $t \geqslant 0$ sufficiently small, but $c^Tx(t,\xi)$ must eventually become positive. Thus the trajectory $x(t,\xi)$ must have a switching time. By Proposition 5.11, such an initial state must be long-time locally observable.

The next example shows that the long-time local observability of a state does not imply its long-time global observability.

Example 5.13. Consider a bimodal system in $\mathbb{R}^2$ with

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda-b_2c_2 \end{bmatrix},\ b = \begin{pmatrix} 0 \\ b_2 \end{pmatrix},\ c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},\ \text{and } H = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

where $\lambda > 0$, $b_2 \neq 0$, $c_1c_2 < 0$, and $b_2c_2 > 0$. It is clear that $\overline{O}(H,A) = \{0\}$. Choose $\xi = (\xi_1,0)$ with $c_1\xi_1 < 0$. Hence, $c^Te^{At}\xi = e^{\lambda t}c_1\xi_1 \leqslant c_1\xi_1 < 0$ for all $t \geq 0$, implying that $x(t,\xi) = e^{At}\xi = (e^{\lambda t}\xi_1, 0)$ for all $t \geqslant 0$. Hence such a ξ is short-time, and thus long-time locally observable. Yet, ξ is not long-time globally observable because by considering $\eta = (0,\xi_1)$, we have $c^T\eta = c_2\xi_1 > 0$. Since $c^Te^{(A+bc^T)t}\eta = e^{\lambda t}c_2\xi_1 > 0$, implying that $x(t,\eta) = (0, e^{\lambda t}\xi_1)$ for all $t \geqslant 0$. Thus, ξ and η are long-time indistinguishable. Hence ξ is not long-time globally observable.

Our last example shows that for a state to be long-time locally observable, it is not necessary for this state and any future state along the nominal trajectory to be T-time locally observable for any finite $T \geqslant 0$. In other words, even if an initial state (and any future state along the nominal trajectory) is not T-time locally observable for any finite $T \geqslant 0$, it is still possible that the CLS is long-time locally observable at the initial state.

Example 5.14. Consider a bimodal system in $\mathbb{R}^3$ with

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \omega \\ 0 & -\omega & \alpha \end{bmatrix}, \; b = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}, \; c = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}, \text{ and } H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

where $\lambda < 0$, $\alpha > 0$, $\omega > 0$, $b_1 \neq 0$, and c_1 and c_2 are both nonzero. Let $\xi = (\xi_1, 0, 0)$ with $c_1\xi_1 < 0$. It follows that $x(t, \xi) = (e^{\lambda t}\xi_1, 0, 0)$ and $c^T x(t, \xi) = e^{\lambda t} c_1 \xi_1 < 0$ for all $t \geqslant 0$. Let $v_I = (0, 1, 0)$ and $v_R = (0, 0, 1)$. It is easy to see that $\overline{O}(H, A) = \operatorname{span}\{v_I, v_R\}$ and $\overline{O}(H, A + bc^T) = \{0\}$. Note that any $\eta \in \overline{O}(H, A)$ with $\|\eta\|_2 = 1$ can be written as $\eta(\phi) = v_I \sin\phi \, + \, v_R \cos\phi$ for some $\phi \in [0, 2\pi)$ and that

$$e^{At}\eta(\phi) \; = \; e^{\alpha t}\big[(-v_I \sin\phi \; + \; v_R \cos\phi)\sin(\omega t) + (v_R \sin\phi \; + \; v_I \cos\phi)\cos(\omega t)\big].$$

Moreover, any initial state that is indistinguishable over the time interval when the corresponding state trajectory is in the mode characterized by A must be of the form $\xi + \tau\eta(\phi)$ for some $\tau \in \mathbb{R}$ and $\phi \in [0, 2\pi)$. Hence, for any given $T \geqslant 0$, there is a $\tau_T > 0$ such that for all $(t, \tau, \phi) \in [0, T] \times [-\tau_T, \tau_T] \times [0, 2\pi)$,

$$\begin{aligned} c^T x(t, \xi + \tau\eta(\phi)) &= c_1\xi_1 e^{\lambda t} \; + \; \tau c_2 e^{\alpha t}\big[-\sin\phi \sin(\omega t) + \cos\phi\cos(\omega t)\big] \\ &= c_1\xi_1 e^{\lambda t} + \tau c_2 e^{\alpha t}\cos(\omega t + \phi) \; < \; 0. \end{aligned}$$

Note that $Hx(t, \xi) = Hx(t, \xi + \tau\eta(\phi))$ for all such triples (t, τ, ϕ). Hence, ξ is not T-time locally observable. The same argument applies to any future state along the nominal trajectory. However, notice that no matter how small $|\tau| > 0$ is and what $\phi \in [0, 2\pi)$ is, $c^T x(t, \xi + \tau\eta(\phi))$ will become positive in some finite time. Hence, $x(t, \xi + \tau\eta(\phi))$ will switch to the mode characterized by $(A + bc^T)$ at some $t_* > 0$; i.e., $c^T x(t_*, \xi + \tau\eta(\phi)) = 0$ but $c^T x(t, \xi + \tau\eta(\phi)) > 0$ for all $t > t_*$ sufficiently close to t_*. Let $\widetilde{x} = x(t_*, \xi + \tau\eta(\phi))$. Note that $0 \neq \widetilde{x} \notin \overline{O}(c^T, A)$, which implies that $H(A+bc^T)^\ell \widetilde{x} \neq HA^\ell x(t_*, \xi)$ for some nonnegative integer ℓ. We have $x(t, \xi+\tau\eta(\phi)) = e^{(A+bc^T)(t-t_*)}\widetilde{x}$ for all $t > t_*$ sufficiently close to t_*. By the proof of Lemma 5.9, $Hx(t, \xi) \equiv Hx(t, \xi + \tau\eta(\phi))$ for all $t \geqslant 0$ only when $\tau = 0$. This establishes that ξ is long-time locally observable.

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