

Confidence Intervals for Coefficient of Variation of Lognormal Distribution with Restricted Parameter Space

Sa-aat Niwitpong

Department of Applied Statistics
King Mongkut's University of Technology
North Bangkok, Thailand
snw@kmutnb.ac.th

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Abstract

This paper presents the new confidence interval for the coefficient of variation of lognormal distribution with restricted parameter. We proved the coverage probability and expected length of our proposed confidence interval.

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1 Introduction

Let $X = (X_1, X_2, \dots, X_n)$ be a random variable having a lognormal distribution, and μ and σ^2 , respectively, are denoted by the mean and the variance of Y where $Y = \ln(X) \sim N(\mu, \sigma^2)$. The probability density function of the lognormal distribution, $LN(\mu, \sigma^2)$, is

$$f(x, \mu, \sigma^2) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right) & ; \text{ for } x > 0 \\ 0 & ; \text{ for } x \leq 0. \end{cases} \quad (1)$$

The mean and variance of lognormal population is $E(X) = \exp(\mu + \sigma^2/2)$ and $Var(X) = \exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)$. We are interesting to construct the confidence interval for the CV of the lognormal population (η) which is denoted by $CV = \sqrt{Var(X)}/E(X)$,

$$\begin{aligned} CV = \eta &= \frac{\sqrt{\exp(2\mu + \sigma^2)(\exp(\sigma^2) - 1)}}{\exp(\mu + \sigma^2/2)} \\ &= \sqrt{\exp(\sigma^2) - 1} \end{aligned} \quad (2)$$

when parameter η is bounded i.e., $r < \eta < s$ where r and s are constants and $r < s$.

2 Confidence interval for the coefficient of variation of Lognormal distribution

Let $S^2 = \frac{1}{(n-1)} \sum_{i=1}^n (Y_i - \bar{Y})^2$ where $Y_i = \ln(X_i) \sim N(\mu, \sigma^2), i = 1, 2, \dots, n$.

The statistic $\chi^2 = \frac{(n-1)S^2}{\sigma^2}$ is distributed as the chi-square distribution with $(n-1)$ degrees of freedom. It is straightforward to see that the $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\frac{(n-1)S^2}{\chi_{(n-1), (1-\alpha/2)}^2} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi_{(n-1), (\alpha/2)}^2}, \quad (3)$$

when $\alpha \in (0, 1)$ and $\chi_{(n-1), (\alpha/2)}^2$ is the $\alpha/2$ quantile of the chi-square distribution with $(n-1)$ degrees of freedom.. Verril [4] proposed the $100(1-\alpha)\%$ two-sided confidence interval for CV of the lognormal distribution based on the exact approach (3) which is

$$CI_V = [L, U] = \left[\sqrt{\exp\left(\frac{(n-1)S^2}{\chi_{(n-1), (1-\alpha/2)}^2}\right)} - 1, \sqrt{\exp\left(\frac{(n-1)S^2}{\chi_{(n-1), (\alpha/2)}^2}\right)} - 1 \right]. \quad (4)$$

3 Confidence interval for the coefficient of variation of Lognormal distribution with restricted parameter space

We begin this section by considering the relation between a bounded mean and a bounded variance leading to a bounded coefficient of variation. Consider,

$$\begin{aligned} a < \mu < b &\rightarrow a^2 < \mu^2 < b^2 \\ &\rightarrow -b^2 < -\mu^2 < -a^2 \\ &\rightarrow \frac{\sum Y_i^2}{N} - b^2 < \frac{\sum Y_i^2}{N} - \mu^2 < \frac{\sum Y_i^2}{N} - a^2 \\ &\rightarrow \sigma_b^2 < \sigma^2 < \sigma_a^2 \end{aligned}$$

where $\sigma_a^2 = \frac{\sum Y_i^2}{N} - a^2$ and $\sigma_b^2 = \frac{\sum Y_i^2}{N} - b^2$. Hence the variance of X is also bounded, i.e. $\sigma_b^2 < \sigma^2 < \sigma_a^2$. Additionally, we have $\sqrt{\exp(\sigma_b^2) - 1} < \sqrt{\exp(\sigma^2) - 1} < \sqrt{\exp(\sigma_a^2) - 1}$ or $r < \eta < s$, where $r = \sqrt{\exp(\sigma_b^2) - 1}$ and $s = \sqrt{\exp(\sigma_a^2) - 1}$.

Following Wang [5] and Niwitpong [2], the confidence interval for η with restricted parameter, $0 < r < \eta < s$ is

$$CI_r = [\max(r, L), \min(s, U)].$$

In the next section, we have proved two Theorems for the approximated coverage probability and the expected length of confidence interval CI_r .

4 Main results

Theorem 1. The approximated coverage probability of CI_r is

$$E[\Phi(W_2) - \Phi(W_1)] \text{ where } W_2 = \frac{\min(s, U)}{\sqrt{A}}, W_1 = \frac{\max(r, L)}{\sqrt{A}} \text{ where}$$

$$A = S.E.of CV \approx \sqrt{\frac{\sigma^4 \exp(2\sigma^2)}{2n(n-1)(\exp(\sigma^2)-1)}}.$$

$E[\cdot]$ is an expectation operator and $\Phi(\cdot)$ is the cumulative distribution function of $N(0, 1)$ and the expected length of CI_r are respectively

$$1.1 \ s - r, \text{ if } \max(r, L) = r \text{ and } \min(s, U) = s,$$

$$1.2 \ \sqrt{(1 + c_1)\sigma^2 + O(c_1^2\sigma^4(n-1)^{-1})} - r, \text{ if } \max(r, L) = r \text{ and } \min(s, U) = U,$$

1.3 $s - \sqrt{c_2\sigma^2 + \sigma^2 + O((c_2^2\sigma^4)(n-1)^{-1})}$, if $\max(r, L) = L$ and $\min(s, U) = s$,

1.4 $\sqrt{(1+c_1)\sigma^2 + O(c_1^2\sigma^4(n-1)^{-1})} - \sqrt{c_2\sigma^2 + \sigma^2 + O((c_2^2\sigma^4)(n-1)^{-1})}$, if $\max(r, L) = L$ and $\min(s, U) = U$,

when $c_1 = \frac{n-1}{\chi_{(n-1),(\alpha/2)}^2}$ and $c_2 = \frac{n-1}{\chi_{(n-1),(1-\alpha/2)}^2}$.

Proof. It is easy to see that the standard error of η is A by using the delta method. Also apply Theorems 1 and 2 of Niwitpong and Niwitpong [3] or Niwitpong [1], the coverage probability of CI_r is

$$\begin{aligned}
 1 - \alpha &= P\left[\max(r, L) \leq \eta \leq \min(s, U)\right] \\
 &= P\left[\frac{\max(r, L)}{A} \leq \frac{\eta}{A} \leq \frac{\min(s, U)}{A}\right] \\
 &= E[I_{\{W_1 < Z < W_2\}}(\xi)], I_{\{W_1 < Z < W_2\}}(\xi) = \begin{cases} 1, & \text{if } \xi \in \{W_1 < Z < W_2\} \\ 0, & \text{otherwise} \end{cases} \\
 &= E[E[I_{\{W_1 < Z < W_2\}}(\xi)]|S^2] \\
 &= E[\Phi(W_2) - \Phi(W_1)].
 \end{aligned}$$

The expected length of CI_r is easy to see that

1.1 if $\max(r, L) = r$ and $\min(s, U) = s$, the expected length of CI_r is $E(s-r) = s - r$,

1.2 if $\max(r, L) = r$ and $\min(s, U) = U$, the expected length of CI_r is

$$\begin{aligned}
 E(U - r) &= E\left(\sqrt{\exp\left(\frac{(n-1)S^2}{\chi_{(n-1),(\alpha/2)}^2}\right)} - 1 - r\right) \\
 &= E(\sqrt{\exp(c_1S^2)} - 1 - r) \\
 &\geq \sqrt{E(\exp(c_1S^2))} - 1 - r \\
 &= \sqrt{E\left(1 + c_1S^2 + \frac{(c_1S^2)^2}{2!} + \frac{(c_1S^2)^3}{3!} + \dots\right)} - 1 - r \\
 &= \sqrt{c_1\sigma^2 + \sigma^2 + \frac{c_1^2\sigma^4}{n-1} + \frac{E(c_1S^2)^3}{3!}} - r \\
 &= \sqrt{(1+c_1)\sigma^2 + O(c_1^2\sigma^4(n-1)^{-1})} - r.
 \end{aligned}$$

1.3 if $\max(r, L) = L$ and $\min(s, U) = s$, the expected length of CI_r is

$$\begin{aligned}
 E(s - L) &= s - E\left(\sqrt{\exp\left(\frac{(n-1)S^2}{\chi_{(n-1), (1-\alpha/2)}^2}\right)} - 1\right) \\
 &= s - E(\sqrt{\exp(c_2 S^2)} - 1) \\
 &\geq s - \sqrt{E(\exp(c_2 S^2))} - 1 \\
 &= s - \sqrt{E\left(1 + c_2 S^2 + \frac{(c_2 S^2)^2}{2!} + \frac{(c_2 S^2)^3}{3!} + \dots\right)} - 1 \\
 &= s - \sqrt{c_2 \sigma^2 + \sigma^2 + \frac{c_2^2 \sigma^4}{n-1} + \frac{E(c_2 S^2)^3}{3!}} \\
 &= s - \sqrt{c_2 \sigma^2 + \sigma^2 + O((c_2^2 \sigma^4)(n-1)^{-1})}.
 \end{aligned}$$

1.4 if $\max(r, L) = L$ and $\min(s, U) = U$, the expected length of CI_r is

$$E(U - L) = \sqrt{(1 + c_1)\sigma^2 + O(c_1^2 \sigma^4 (n-1)^{-1})} - \sqrt{c_2 \sigma^2 + \sigma^2 + O((c_2^2 \sigma^4)(n-1)^{-1})}$$

Hence, Theorem 1 is proved.

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