

CONFIDENCE INTERVALS FOR LINEAR FUNCTIONS OF THE NORMAL MEAN AND VARIANCE¹

BY CHARLES E. LAND

Oregon State University

1. Introduction and summary. If $Y = g(X)$ is normal (μ, σ^2) , where g is a one-to-one real function and X is a random variable whose expectation exists, we may write $EX = f(\mu, \sigma^2)$. The practical importance of this observation is that we often are concerned with testing hypotheses about, and constructing confidence intervals for, known functions of both the mean and variance of a normal distribution. This may happen when we use a statistical model, such as the lognormal distribution, that is related to the normal distribution by a transformation of variables. A slightly different case occurs when a transformation of data is made before applying a statistical method, such as analysis of variance or regression analysis, that involves the assumption of normality for the transformed data. Some familiar examples in this context are

- (i) $Y = X^{\frac{1}{2}}, EX = \mu^2 + \sigma^2$;
- (ii) $Y = X^{\frac{3}{2}}, EX = \mu^3 + 3\mu\sigma^2$;
- (iii) $Y = \arcsin(X^{\frac{1}{2}}), EX = \frac{1}{2}(1 - \cos(2\mu) \exp(-2\sigma^2))$;
- (iv) $Y = \operatorname{arcsinh}(X^{\frac{1}{2}}), EX = \frac{1}{2}(\cosh(2\mu) \exp(2\sigma^2) - 1)$;
- (v) $Y = \log(X), EX = \exp(\mu + \frac{1}{2}\sigma^2)$.

The theory of statistical inference in terms of $\mu = EY$ alone or $\sigma^2 = \operatorname{Var} Y$ alone is not easily extended to problems of inference in terms of EX or $\operatorname{Var} X$, parametric functions of both μ and σ^2 . Minimum variance unbiased estimators (MVUE's) for EX and $\operatorname{Var} X$ were obtained by Finney (1941) for the case $Y = \log X$. Solutions for a much wider class of transformations were obtained by Neyman and Scott (1960) and Hoyle (1968). However there have been no analogous achievements with respect to hypothesis tests and confidence interval estimates for EX and $\operatorname{Var} X$. The present paper, in which uniformly most accurate unbiased confidence interval procedures of level $1 - \alpha$ are derived for linear functions of μ and σ^2 , is an approach to these problems. The results of this paper define an optimal solution for EX when $Y = \log X$, since in this case the parametric function of interest is a monotone function of $\mu + \frac{1}{2}\sigma^2$. The results also provide a basis for approximate confidence interval solutions for other parametric functions of μ and σ^2 .

Received July 22, 1969; revised November 16, 1970.

¹ This paper is essentially part of a thesis submitted in partial fulfillment of the requirements for the Ph.D. degree in Statistics at the University of Chicago. The research was supported by USPHS Training Grant No. 2T1 GM 201-07.

It is helpful to consider the problem in terms of confidence regions in the half-plane of points (μ, σ^2) . For any transformation $Y = g(X)$ likely to be of practical significance, a confidence interval for $f(\mu, \sigma^2) = EX$ or $\text{Var } X$ is a region in this half-plane, bounded by one or two contours of the form $f(\mu, \sigma^2) = m$.

Kanofsky (1969) has proposed a method of simultaneous confidence estimation for all functions of μ and σ^2 . He constructs a trapezoidal-shaped confidence region of level $1 - \alpha$ for μ and σ^2 , and for an arbitrary function $h(\mu, \sigma^2)$, defines a confidence set for this function as the set of values m such that the curve $h(\mu, \sigma^2) = m$ intersects this confidence region. If one is only interested in a single function, the procedure is conservative. However, for most such functions this is the only method based on exact distribution theory, to my knowledge, that has been proposed.

The usual approach to confidence interval estimation for EX or $\text{Var } X$ has been to rely on approximate methods. For example, a common method of confidence interval estimation for EX is to transform a level $1 - \alpha$ confidence interval for $EY = E(g(X))$, say (μ_1, μ_2) , by the inverse transform. Then $(g^{-1}(\mu_1), g^{-1}(\mu_2))$ would be an approximate level $1 - \alpha$ confidence interval for EX if g is monotone increasing. More sophisticated versions of this method have been proposed by Patterson (1966) and Hoyle (1968).

A more direct approach is to use an estimator T of $f(\mu, \sigma^2)$ and an estimator V of the variance of T . T is then assumed to be approximately normally distributed with mean $(f(\mu, \sigma^2))$ and variance equal to the observed value of V . For example, the sample mean \bar{X} is an estimate of EX , and $S_X^2/(n(n-1)) = \sum (X_i - \bar{X})^2/(n(n-1))$ is an estimate of the variance of \bar{X} (e.g., see Aitchison and Brown (1957) Section 5.62). Hoyle (1968) has suggested letting T be the MVUE of EX , and V the MVUE of $\text{Var } T$, which he has given for a number of transformations.

In this paper an optimal exact confidence interval procedure is presented for linear functions of μ and σ^2 . That is, the procedure gives uniformly most accurate unbiased joint confidence regions of level $1 - \alpha$ for μ and σ^2 , bounded by one or two contours of form $\mu + \lambda\sigma^2 = m$, for arbitrary λ . This provides an optimal confidence interval procedure for EX when $Y = \log(X)$ is normal. Also it provides the basis for a new approximate confidence interval method for EX in the general case $Y = g(X)$. That is, by a proper choice of the linear coefficient λ , it seems reasonable that a confidence region bounded by one or two contours of form $f(\mu, \sigma^2) = m$ might be approximated with some success by a confidence region bounded by contours of the form $\mu + \lambda\sigma^2 = m$. Certainly the degree of approximation possible should be better than that obtainable using only vertical bounding contours, as when a confidence interval for μ is transformed to give an approximate confidence interval for EX . Also, if the contours $f(\mu, \sigma^2) = m$ are fairly straight within a convex joint confidence region of level $1 - \alpha$ for μ and σ^2 , it is not unreasonable to hope that an approximate confidence region should be possible that would have a true level near $1 - \alpha$, and that would be less conservative than a level $1 - \alpha$ region for $f(\mu, \sigma^2)$ determined by Kanofsky's method.

The main result of the paper is the derivation in Section 2 of uniformly most powerful unbiased level α hypothesis tests for linear functions of μ and σ^2 . The

theoretical interest of this section is mainly in the analytic detail of how a well-known theorem applies to this somewhat unusual case. A numerical example follows, illustrating the use of the tables of critical values given in the Appendix. It is not obvious that the confidence procedures defined by these tests in Section 4 define confidence sets that are intervals, an extremely desirable property both for ease of calculation and for practical usefulness of the confidence sets. The proof in Section 5 that the one-sided tests define one-sided confidence intervals provided that ν , the number of degrees of freedom available for the estimate of σ^2 , is at least two, is the second major result of the paper. In Section 6 it is shown that this property does not obtain when $\nu = 1$. The analogous result in the two-sided case is proved only for $\nu = 2$ in Section 7. However it is conjectured that, as in the one-sided case, the desired property also holds for all larger values of ν .

The final section contains a brief discussion of applications of the method to confidence interval estimation for EX when $Y = g(X)$ is normal. It is shown that essentially the only direct application is to the case where $Y = \log(X)$, and that there are no nontrivial direct applications where EX is a function of μ or σ^2 alone. The construction of normal tolerance limits involves confidence interval estimation of functions of the form $\mu + \delta\sigma$ (Owen (1958)). However it is shown here that there are essentially no transformations to normality such that EX is a function of $\mu + \delta\sigma$ for some δ . A more complete discussion of approximate applications of the method is left for a subsequent paper.

2. Hypothesis tests involving $\mu + \lambda\sigma^2$. Let \mathbf{W} be a spherical normal random vector of dimension n , with covariance matrix $\sigma^2 I$. Let $E\mathbf{W}$ be restricted to a subspace with orthonormal basis $\{\alpha_1, \dots, \alpha_k\}$, which we extend to an orthonormal basis $\{\alpha_1, \dots, \alpha_n\}$ for the vector space of \mathbf{W} . Let $\mathbf{U} = (U_1, \dots, U_n)'$ be the canonical form of \mathbf{W} with respect to this basis (Scheffé (1959) Section 1.6). The probability-density function of \mathbf{U} is

$$f(\mathbf{u}; \boldsymbol{\beta}, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}n} \exp \left\{ -\frac{1}{2}(\mathbf{u}'\mathbf{u} - 2\mathbf{u}'\boldsymbol{\beta} + \boldsymbol{\beta}'\boldsymbol{\beta})/\sigma^2 \right\},$$

where $\boldsymbol{\beta} = E\mathbf{U} = (\beta_1, \dots, \beta_k, 0, \dots, 0)'$. This density can also be written in its exponential form, with parameters $\zeta = -1/(2\sigma^2)$ and $\eta_i = \beta_i/\sigma^2, i = 1, \dots, k$. In this form, the density function is

$$(2.1) \quad f^*(\mathbf{u}; \boldsymbol{\eta}, \zeta) = c(\boldsymbol{\eta}, \zeta) \exp \left\{ v\zeta + \sum_{i=1}^k u_i \eta_i \right\},$$

where $v = \mathbf{u}'\mathbf{u}$, for a suitable function $c(\boldsymbol{\eta}, \zeta)$.

Let $\mu = \mathbf{a}'E\mathbf{W}$, where \mathbf{a} is an arbitrary n -vector. Without loss of generality, we can assume that $\alpha_1 = r^{\frac{1}{2}}\mathbf{a}$, for some positive number r . Then $\mu = r^{-\frac{1}{2}}\beta_1$, and $Y = r^{-\frac{1}{2}}U_1$ is the Gauss-Markov estimator of μ . Setting $\theta = \mu/\sigma^2$, we can re-write (2.1) as

$$f^*(\mathbf{u}; \theta, \boldsymbol{\eta}, \zeta) = c(\boldsymbol{\eta}, \zeta) \exp \left\{ r\gamma\theta + v\zeta + \sum_{i=2}^k u_i \eta_i \right\}.$$

Let λ be an arbitrary number, and consider the problem of testing $H(m): \mu + \lambda\sigma^2 = m$ against any one of the alternatives $A_1(m): \mu + \lambda\sigma^2 < m$, $A_2(m): \mu + \lambda\sigma^2 > m$, and $A_3(m): \mu + \lambda\sigma^2 \neq m$. By a translation of \mathbf{W} the problem

reduces to the case $m = 0$. For this case, we may rewrite these hypotheses in terms of $\theta = \mu/\sigma^2$, as $H: \theta = -\lambda$, $A_1: \theta < -\lambda$, $A_2: \theta > -\lambda$, and $A_3: \theta \neq -\lambda$.

A theorem that applies to this problem, concerning the existence of uniformly most powerful unbiased (UMPU) tests, is to be found in Lehmann ((1959) Section 4.4). The tests are defined in terms of the conditional distribution of Y given U_2, \dots, U_k , and $V = \mathbf{U}'\mathbf{U}$. The transformation

$$(2.2) \quad T = r^{\frac{1}{2}} Y / (S^2/v)^{\frac{1}{2}},$$

where

$$S^2 = V - \sum_{i=1}^k U_i^2 = \sum_{i=k+1}^n U_i^2,$$

and $v = n - k$, is monotone in Y for fixed U_2, \dots, U_k and V . Also, unlike Y , which is bounded by the inequality $rY^2 \leq V$, T has an unbounded conditional distribution. S^2/v , of course, is the usual estimator of σ^2 , and Y has variance σ^2/r . According to the theorem, the UMP unbiased level α tests of H against A_1 , A_2 , and A_3 respectively are given by the rules R_1 : "Reject H in favor of A_1 if $T < t(\alpha)$," R_2 : "Reject H in favor of A_2 if $T > t(1 - \alpha)$," and R_3 : "Reject H in favor of A_3 if $T < t_1(\alpha)$ or $T > t_2(\alpha)$," where the functions t , t_1 , and t_2 depend on U_2, \dots, U_k , and V . If $f(\tau; -\lambda)$ denotes the conditional density, under H , of T given U_2, \dots, U_k and V , $t(\alpha)$ is defined by

$$(2.3) \quad \int_{-\infty}^{t(\alpha)} f(\tau; -\lambda) d\tau = \alpha,$$

and $t_1(\alpha)$ and $t_2(\alpha)$ are defined by the pair of equations

$$(2.4) \quad \int_{t_1}^{t_2} f(\tau; -\lambda) d\tau = 1 - \alpha,$$

$$(2.5) \quad \int_{t_1}^{t_2} \tau(v + \tau^2)^{-\frac{1}{2}} f(\tau; -\lambda) d\tau = (1 - \alpha) \int_{-\infty}^{\infty} \tau(v + \tau^2)^{-\frac{1}{2}} f(\tau; -\lambda) d\tau.$$

The joint density of Y, U_2, \dots, U_k , and V may be obtained from the joint distribution of S^2 and U_1, \dots, U_k . Since these statistics are mutually independent, with S^2/σ^2 a chi-square variate with v degrees of freedom and U_i normal (β_i, σ^2) , $i = 1, \dots, k$, we have

$$(2.6) \quad f(y, u_2, \dots, u_k, v) \propto (s^2)^{\frac{1}{2}v-1} \exp \{v\zeta + ry\theta + \sum_{i=2}^k u_i \eta_i\}$$

where $s^2 = v - ry^2 - \sum_{i=2}^k u_i^2$, for $v > 0$ and $ry^2 + \sum_{i=2}^k u_i^2 < v$. This gives the conditional density for Y ,

$$(2.7) \quad f(y | u_2, \dots, u_k, v) \propto (v^* - ry^2)^{\frac{1}{2}v-1} \exp \{r\theta y\}$$

where $ry^2 \leq v^* = v - \sum_{i=2}^k u_i^2$. This conditional density depends only on r, θ, v , and v^* . The conditional density for the transformed statistic T , accordingly, is (from (2.2) and (2.7))

$$(2.8) \quad f(\tau | v^*; \theta, v) \propto (v + \tau^2)^{-\frac{1}{2}(v+1)} \exp \{\theta(rv^*)^{\frac{1}{2}} \tau / (v + \tau^2)^{\frac{1}{2}}\},$$

for $-\infty < \tau < \infty$.

Since $V^* = S^2 + rY^2$ is a random variable whose mean, for small μ , is roughly proportional to $v+1$, and since $r = \sigma^2/\text{Var}(Y)$ can be expected to be roughly proportional to $v+1$, it is convenient to define

$$(2.9) \quad Z = (rV^*)^{\frac{1}{2}}/(v+1) = r^{\frac{1}{2}}(S^2 + rY^2)^{\frac{1}{2}}/(v+1)$$

as a variate whose magnitude should not depend much on the sample size. In what follows, we shall speak of “the conditional distribution of T given $Z = z$.” Assuming $H:\theta = -\lambda$, and defining a “noncentrality parameter” $\xi = -\lambda z$, this conditional distribution may be referred to as “the conditional t distribution with v degrees of freedom and noncentrality ξ ,” with density

$$(2.10) \quad f_v(\tau | \xi) \propto (v + \tau^2)^{-\frac{1}{2}(v+1)} \exp \{ (v+1)\xi\tau/(v + \tau^2)^{\frac{1}{2}} \},$$

for $-\infty < \tau < \infty$, $-\infty < \xi < \infty$, and $v = 1, 2, \dots$.

Substituting the density (2.10) into the integral equations (2.3) to (2.5), the critical values may be determined. The critical value obtained from (2.3) is denoted by $t(v, \xi, \alpha)$, and the critical values obtained from (2.4) and (2.5) are denoted by $t_1(v, \xi, \alpha)$ and $t_2(v, \xi, \alpha)$. It is apparent from (2.10) that $f_v(\tau | \xi) = f_v(-\tau | -\xi)$, giving the symmetry relations for the critical values,

$$(2.11) \quad t(v, \xi, \alpha) = -t(v, -\xi, 1-\alpha),$$

$$(2.12) \quad t_1(v, \xi, \alpha) = -t_2(v, -\xi, \alpha).$$

Tables of critical values for one and two-sided tests have been computed for selected values of v , ξ , and α . The complete tables and the method of their construction are given elsewhere (Land (1969)). A brief subcollection of the one and two-sided tables is given in Tables 1 and 2. Briefly, the one-sided tables were computed by the step-by-step Runge-Kutta-Gill numerical integration of a system of differential equations based on the conditional density (2.10), closely following the method used by Johnson *et al* ((1963) Section 4.1) to compute quantiles for certain Pearson distributions. The two-sided tables were computed only for even values of v , by a cruder method in which the integrals in equations (2.4) and (2.5) were computed directly. The integration limits were adjusted until the ratios of left-hand to right-hand sides in both equations were arbitrarily close to one. Numerical problems prevented the calculation of critical values for some combinations of large v and small ξ .

An examination of (2.10) shows that the conditional t distribution with v degrees of freedom and noncentrality ξ is asymptotically normal as $v \rightarrow \infty$. More specifically $T - (v+1)\xi v^{-\frac{1}{2}}$ is asymptotically normal with zero mean and unit variance. The critical value tables, however, indicate that this convergence is slow for large ξ . Nevertheless, the Cornish-Fisher approximations of order $1/v$ and $1/v^2$ for the quantiles $t(v, \xi, \alpha)$ have been found to work reasonably well for v above 25 and ξ less than 100 (Land (1968)).

3. A numerical example. Let $y = 3.7$ and $s^2/10 = 10.0$ be the sample mean and variance of a random sample of size 11 from a normal (μ, σ^2) distribution. In order to test the null hypothesis that $\mu + \sigma^2/2$ is equal to 6.0 against the alternative that it is larger, we first compute $t_{6.0} = (3.7 - 6.0)/(10.0/11)^{\frac{1}{2}} = -2.412$, and $z_{6.0} = ((10/11)(10.0) + (3.7 - 6.0)^2)^{\frac{1}{2}} = 3.728$. Since $\xi_{6.0} = (-1/2)z_{6.0} = -1.864$, we

TABLE 1

Critical values $t(\nu, \xi, \alpha)$ for one-sided tests, for selected values of ν , ξ , and α

| ξ | α | | | | | |
|------------|----------|--------|--------|--------|--------|---------|
| | .01 | .05 | .10 | .90 | .95 | .99 |
| $\nu = 2$ | | | | | | |
| 0 | -6.965 | -2.920 | -1.886 | 1.886 | 2.920 | 6.965 |
| .1 | -5.929 | -2.446 | -1.540 | 2.245 | 3.416 | 8.053 |
| .2 | -4.967 | -2.004 | -1.217 | 2.610 | 3.922 | 9.169 |
| .5 | -2.668 | -0.939 | -0.426 | 3.680 | 5.415 | 12.474 |
| 1. | -0.737 | .024 | .349 | 5.202 | 7.559 | 17.255 |
| 2. | .338 | .818 | 1.107 | 7.447 | 10.746 | 24.403 |
| 5. | 1.359 | 1.887 | 2.249 | 11.869 | 17.057 | 38.613 |
| 10. | 2.249 | 2.922 | 3.398 | 16.830 | 24.153 | 54.621 |
| 20. | 3.398 | 4.306 | 4.957 | 23.832 | 34.180 | 77.256 |
| 50. | 5.575 | 6.970 | 7.978 | 37.712 | 54.063 | 122.161 |
| 100. | 7.978 | 9.932 | 11.349 | 53.347 | 76.467 | 172.766 |
| $\nu = 5$ | | | | | | |
| 0 | -3.365 | -2.015 | -1.476 | 1.476 | 2.015 | 3.365 |
| .1 | -2.886 | -1.658 | -1.158 | 1.798 | 2.377 | 3.852 |
| .2 | -2.427 | -1.314 | -0.851 | 2.116 | 2.736 | 4.337 |
| .5 | -1.258 | -0.434 | -0.059 | 3.003 | 3.745 | 5.711 |
| 1. | -0.079 | .515 | .830 | 4.214 | 5.138 | 7.639 |
| 2. | 1.029 | 1.557 | 1.872 | 6.008 | 7.227 | 10.575 |
| 5. | 2.577 | 3.208 | 3.609 | 9.583 | 11.428 | 16.548 |
| 10. | 4.059 | 4.874 | 5.404 | 13.599 | 16.170 | 23.331 |
| 20. | 6.026 | 7.129 | 7.852 | 19.267 | 22.877 | 32.946 |
| 50. | 9.795 | 11.493 | 12.612 | 30.497 | 36.180 | 52.048 |
| 100. | 13.977 | 16.358 | 17.929 | 43.146 | 51.171 | 73.587 |
| $\nu = 10$ | | | | | | |
| 0 | -2.764 | -1.812 | -1.372 | 1.372 | 1.812 | 2.764 |
| .1 | -2.305 | -1.414 | -0.995 | 1.750 | 2.213 | 3.225 |
| .2 | -1.862 | -1.029 | -0.631 | 2.121 | 2.606 | 3.680 |
| .5 | -0.715 | -0.025 | .325 | 3.138 | 3.689 | 4.945 |
| 1. | .546 | 1.130 | 1.451 | 4.502 | 5.159 | 6.689 |
| 2. | 1.953 | 2.528 | 2.867 | 6.509 | 7.347 | 9.329 |
| 5. | 4.173 | 4.892 | 5.331 | 10.482 | 11.721 | 14.683 |
| 10. | 6.397 | 7.336 | 7.917 | 14.927 | 16.639 | 20.747 |
| 20. | 9.393 | 10.669 | 11.464 | 21.186 | 23.579 | 29.332 |
| 50. | 15.178 | 17.148 | 18.379 | 33.573 | 37.329 | 46.371 |
| 100. | 21.618 | 24.382 | 26.112 | 47.514 | 52.813 | 65.576 |

TABLE 1—continued

| ξ | α | | | | | |
|-------------|----------|--------|--------|---------|---------|---------|
| | .01 | .05 | .10 | .90 | .95 | .99 |
| $\nu = 20$ | | | | | | |
| 0 | -2.528 | -1.725 | -1.325 | 1.325 | 1.725 | 2.528 |
| .1 | -1.992 | -1.224 | -0.838 | 1.813 | 2.226 | 3.066 |
| .2 | -1.472 | -0.739 | -0.366 | 2.290 | 2.716 | 3.592 |
| .5 | -0.113 | .537 | .881 | 3.584 | 4.054 | 5.039 |
| 1. | 1.462 | 2.062 | 2.394 | 5.300 | 5.843 | 7.003 |
| 2. | 3.383 | 4.010 | 4.372 | 7.800 | 8.475 | 9.939 |
| 5. | 6.629 | 7.436 | 7.913 | 12.706 | 13.685 | 15.831 |
| 10. | 9.976 | 11.039 | 11.673 | 18.166 | 19.509 | 22.463 |
| 20. | 14.531 | 15.982 | 16.851 | 25.835 | 27.705 | 31.826 |
| 50. | 23.377 | 25.622 | 26.969 | 40.988 | 43.917 | 50.380 |
| 100. | 33.248 | 36.401 | 38.294 | 58.032 | 62.161 | 71.277 |
| $\nu = 50$ | | | | | | |
| 0 | -2.403 | -1.676 | -1.299 | 1.299 | 1.676 | 2.403 |
| .1 | -1.645 | -0.938 | -0.569 | 2.029 | 2.414 | 3.162 |
| .2 | -0.910 | -0.222 | .140 | 2.740 | 3.133 | 3.902 |
| .5 | 1.029 | 1.672 | 2.019 | 4.660 | 5.080 | 5.915 |
| 1. | 3.371 | 4.002 | 4.351 | 7.175 | 7.647 | 8.598 |
| 2. | 6.409 | 7.099 | 7.488 | 10.797 | 11.370 | 12.538 |
| 5. | 11.801 | 12.713 | 13.235 | 17.829 | 18.646 | 20.324 |
| 10. | 17.491 | 18.703 | 19.400 | 25.605 | 26.718 | 29.012 |
| 20. | 25.307 | 26.967 | 27.923 | 36.495 | 38.040 | 41.228 |
| 50. | 40.556 | 43.131 | 44.616 | 57.979 | 60.394 | 65.382 |
| 100. | 57.610 | 61.229 | 63.316 | 82.125 | 85.527 | 92.556 |
| $\nu = 100$ | | | | | | |
| 0 | -2.364 | -1.660 | -1.290 | 1.290 | 1.660 | 2.364 |
| .1 | -1.332 | -0.642 | -0.277 | 2.304 | 2.679 | 3.397 |
| .2 | -0.329 | .348 | .708 | 3.290 | 3.671 | 4.404 |
| .5 | 2.325 | 2.975 | 3.326 | 5.947 | 6.348 | 7.128 |
| 1. | 5.592 | 6.246 | 6.605 | 9.404 | 9.848 | 10.720 |
| 2. | 9.939 | 10.669 | 11.075 | 14.346 | 14.879 | 15.937 |
| 5. | 17.823 | 18.802 | 19.351 | 23.881 | 24.633 | 26.135 |
| 10. | 26.232 | 27.537 | 28.272 | 34.384 | 35.407 | 37.452 |
| 20. | 37.830 | 39.622 | 40.632 | 49.072 | 50.489 | 53.325 |
| 50. | 69.513 | 63.296 | 64.866 | 78.018 | 80.230 | 84.663 |
| 100. | 85.908 | 89.820 | 92.028 | 110.537 | 113.653 | 119.897 |

TABLE 1—*continued*

| ξ | α | | | | | |
|--------------|----------|---------|---------|---------|---------|---------|
| | .01 | .05 | .10 | .90 | .95 | .99 |
| $\nu = 200$ | | | | | | |
| 0 | -2.345 | -1.653 | -1.286 | 1.286 | 1.653 | 2.345 |
| .1 | -0.912 | -0.229 | .134 | 2.706 | 3.077 | 3.779 |
| .2 | .481 | 1.155 | 1.514 | 4.088 | 4.462 | 5.175 |
| .5 | 4.178 | 4.836 | 5.190 | 7.802 | 8.192 | 8.940 |
| 1. | 8.783 | 9.458 | 9.826 | 12.612 | 13.039 | 13.865 |
| 2. | 15.015 | 15.778 | 16.197 | 19.449 | 19.958 | 20.948 |
| 5. | 26.474 | 27.505 | 28.076 | 32.574 | 33.287 | 34.683 |
| 10. | 38.781 | 40.161 | 40.926 | 46.993 | 47.960 | 49.854 |
| 20. | 55.805 | 57.703 | 58.756 | 67.130 | 68.468 | 71.092 |
| 50. | 89.152 | 92.102 | 93.741 | 106.788 | 108.875 | 112.973 |
| 100. | 126.514 | 130.662 | 132.966 | 151.327 | 154.266 | 160.036 |
| $\nu = 500$ | | | | | | |
| 0 | -2.334 | -1.648 | -1.283 | 1.283 | 1.648 | 2.334 |
| .1 | -0.092 | .588 | .950 | 3.517 | 3.884 | 4.576 |
| .2 | 2.086 | 2.760 | 3.120 | 5.689 | 6.058 | 6.757 |
| .5 | 7.884 | 8.552 | 8.911 | 11.517 | 11.898 | 12.623 |
| 1. | 15.182 | 15.877 | 16.253 | 19.032 | 19.445 | 20.235 |
| 2. | 25.189 | 25.985 | 26.417 | 29.658 | 30.147 | 31.084 |
| 5. | 43.802 | 44.886 | 45.478 | 49.958 | 50.639 | 51.951 |
| 10. | 63.912 | 65.366 | 66.162 | 72.201 | 73.123 | 74.899 |
| 20. | 91.797 | 93.800 | 94.895 | 103.230 | 104.504 | 106.962 |
| 50. | 146.494 | 149.609 | 151.314 | 164.299 | 166.286 | 170.119 |
| 100. | 207.813 | 212.194 | 214.592 | 232.865 | 235.662 | 241.059 |
| $\nu = 1000$ | | | | | | |
| 0 | -2.330 | -1.646 | -1.282 | 1.282 | 1.646 | 2.330 |
| .1 | .828 | 1.508 | 1.870 | 4.435 | 4.801 | 5.489 |
| .2 | 3.898 | 4.573 | 4.934 | 7.501 | 7.868 | 8.561 |
| .5 | 12.080 | 12.754 | 13.116 | 15.720 | 16.097 | 16.811 |
| 1. | 22.430 | 23.136 | 23.517 | 26.293 | 26.700 | 27.474 |
| 2. | 36.713 | 37.526 | 37.965 | 41.202 | 41.681 | 42.595 |
| 5. | 63.423 | 64.536 | 65.139 | 69.612 | 70.279 | 71.552 |
| 10. | 92.363 | 93.858 | 94.669 | 100.700 | 101.600 | 103.322 |
| 20. | 132.541 | 134.601 | 135.719 | 144.041 | 145.285 | 147.665 |
| 50. | 211.404 | 214.608 | 216.349 | 229.314 | 231.254 | 234.965 |
| 100. | 299.840 | 304.348 | 306.797 | 325.041 | 327.771 | 332.995 |

TABLE 2

Critical values $t_1(v, \xi, \alpha)$ and $t_2(v, \xi, \alpha)$ for two-sided tests, for selected values of $v, \xi,$ and α

| ξ | α | | | | | |
|------------|----------|---------|--------|--------|--------|--------|
| | .010 | | .050 | | .100 | |
| $\nu = 2$ | | | | | | |
| 0 | -9.925 | 9.925 | -4.303 | 4.303 | -2.920 | 2.920 |
| .1 | -8.924 | 10.925 | -3.851 | 4.755 | -2.596 | 3.244 |
| .2 | -7.937 | 11.916 | -3.404 | 5.202 | -2.276 | 3.565 |
| .5 | -5.202 | 14.758 | -2.168 | 6.493 | -1.389 | 4.495 |
| 1. | -2.063 | 18.977 | -0.754 | 8.461 | -0.361 | 5.933 |
| 2. | -0.150 | 26.189 | .298 | 11.839 | .519 | 8.373 |
| 5. | .949 | 41.422 | 1.320 | 18.777 | 1.546 | 13.322 |
| 10. | 1.755 | 58.593 | 2.200 | 26.583 | 2.483 | 18.879 |
| 20. | 2.750 | 82.872 | 3.333 | 37.614 | 3.712 | 26.728 |
| 50. | 4.593 | 131.041 | 5.476 | 59.492 | 6.054 | 42.287 |
| 100. | 6.608 | 185.324 | 7.840 | 84.143 | 8.649 | 59.815 |
| $\nu = 4$ | | | | | | |
| 0 | -4.604 | 4.604 | -2.776 | 2.776 | -2.132 | 2.132 |
| .1 | -4.103 | 5.105 | -2.435 | 3.118 | -1.840 | 2.424 |
| .2 | -3.611 | 5.600 | -2.099 | 3.455 | -1.553 | 2.712 |
| .5 | -2.267 | 7.015 | -1.184 | 4.424 | -0.770 | 3.540 |
| 1. | -0.782 | 9.118 | -0.143 | 5.865 | .144 | 4.768 |
| 2. | .434 | 12.534 | .873 | 8.163 | 1.111 | 6.701 |
| 5. | 1.803 | 19.590 | 2.265 | 12.853 | 2.539 | 10.615 |
| 10. | 3.008 | 27.612 | 3.585 | 18.162 | 3.937 | 15.030 |
| 20. | 4.564 | 38.987 | 5.333 | 25.677 | 5.807 | 21.271 |
| 50. | 7.504 | 61.588 | 8.676 | 40.593 | 9.405 | 33.649 |
| 100. | 10.745 | 87.071 | 12.385 | 57.404 | 13.405 | 47.594 |
| $\nu = 10$ | | | | | | |
| 0 | -3.169 | 3.169 | -2.228 | 2.228 | -1.812 | 1.812 |
| .5 | -1.121 | 5.293 | -0.456 | 4.058 | -0.140 | 3.533 |
| 1. | .197 | 7.057 | .726 | 5.556 | .998 | 4.929 |
| 2. | 1.624 | 9.786 | 2.114 | 7.839 | 2.383 | 7.035 |
| 5. | 3.780 | 15.362 | 4.367 | 12.445 | 4.704 | 11.252 |
| 10. | 5.892 | 21.689 | 6.649 | 17.640 | 7.089 | 15.989 |
| 20. | 8.711 | 30.652 | 9.734 | 24.979 | 10.331 | 22.669 |
| 50. | 14.129 | 48.447 | 15.703 | 39.527 | 16.625 | 35.900 |
| 100. | 20.149 | 68.506 | 22.353 | 55.916 | 23.648 | 50.798 |

TABLE 2—*continued*

| ξ | α | | | | | |
|-------------|----------|---------|--------|---------|--------|---------|
| | .010 | | .050 | | .100 | |
| $\nu = 20$ | | | | | | |
| 0 | -2.845 | 2.845 | -2.086 | 2.086 | -1.725 | 1.725 |
| 1. | 1.156 | 7.318 | 1.689 | 6.208 | 1.968 | 5.705 |
| 2. | 3.069 | 10.327 | 3.609 | 8.921 | 3.903 | 8.293 |
| 5. | 6.233 | 16.398 | 6.913 | 14.332 | 7.293 | 13.416 |
| 10. | 9.459 | 23.245 | 10.348 | 20.397 | 10.849 | 19.138 |
| 20. | 13.829 | 32.918 | 15.039 | 28.943 | 15.723 | 27.188 |
| 50. | 22.294 | 52.094 | 24.161 | 45.858 | 25.220 | 43.108 |
| 100. | 31.729 | 73.694 | 34.349 | 64.899 | 35.836 | 61.021 |
| $\nu = 50$ | | | | | | |
| 0 | -2.678 | 2.678 | -2.009 | 2.009 | -1.676 | 1.676 |
| 2. | 6.101 | 12.888 | 6.704 | 11.791 | 7.027 | 11.271 |
| 5. | 11.398 | 20.825 | 12.187 | 19.245 | 12.616 | 18.501 |
| 10. | 16.958 | 29.698 | 18.002 | 27.536 | 18.573 | 26.520 |
| 20. | 24.578 | 42.184 | 26.006 | 39.177 | 26.788 | 37.765 |
| 50. | 39.426 | 66.878 | 41.640 | 62.171 | 42.853 | 59.963 |
| 100. | 56.024 | 94.665 | 59.133 | 88.031 | 60.838 | 84.921 |
| $\nu = 100$ | | | | | | |
| 0 | -2.626 | 2.626 | -1.984 | 1.984 | -1.660 | 1.660 |
| 2. | 9.637 | 16.278 | 10.277 | 15.293 | 10.618 | 14.814 |
| 5. | 17.413 | 26.613 | 18.271 | 25.215 | 18.730 | 24.538 |
| 10. | 25.687 | 38.103 | 26.828 | 36.197 | 27.441 | 35.277 |
| 20. | 37.083 | 54.228 | 38.648 | 51.584 | 39.489 | 50.309 |
| 50. | 59.353 | 86.075 | 61.782 | 81.942 | 63.089 | 79.949 |
| 100. | 84.278 | 121.886 | 87.692 | 116.064 | 89.529 | 113.256 |

reject the null hypothesis at level α if $t_{6,0}$ is greater than $t(10, -1.864, 1-\alpha)$. Using the symmetry relation (2.11) we obtain from Table 1 the critical values $t(10, -0.5, .95) = 0.025$, $t(10, -1.0, .95) = -1.130$, $t(10, -2.0, .95) = -2.528$, and $t(10, -5.0, .95) = -4.982$. Four-point Lagrangian interpolation on these numbers with respect to $\log \xi$ gives the approximate value $t(10, -1.864, .95) = -2.372$. (This interpolation method, applied to Table 1, gives interpolated values accurate to within three significant digits, according to comparisons using more detailed tables. For $\nu = 10$ and ξ around ± 1.75 , the accuracy is within .003.) Similarly, we have $t(10, -0.5, .90) = -0.325$, $t(10, -1.0, .90) = 1.451$, $t(10, -2.0, .90) = -2.867$, and $t(10, -5.0, .90) = -5.331$, from which we obtain the interpolated value $t(10, -1.864, .90) = -2.707$. Thus we have $t(10, -1.864, .90) < t_{6,0} < t(10, -1.864, .95)$, so that the null hypothesis is rejected at level .10, but not at level .05.

If we vary the above procedure, computing $t_m = (3.7-m)/(10.0/11)^{\frac{1}{2}}$ and $z_m = [(10/11)(10.0) + (3.7-m)^2]^{\frac{1}{2}}$ for different values of m , we find that the hypo-

thesis that $\mu + \sigma^2/2 = m$ is rejected at level .05 in favor of the alternative that $\mu + \sigma^2/2 > m$ for $m < 5.62$, and accepted for $m > 5.62$.

4. Confidence intervals for $\mu + \lambda\sigma^2$. The interval $[5.62, \infty)$ obtained in the above example as the set of values m for which $t_m < t(14, -\frac{1}{2}z_m, .95)$ is a level .95 confidence interval for $\mu + \frac{1}{2}\sigma^2$, corresponding to the given sample mean and variance. More generally, if we define

$$(4.1) \quad T_m = (r)^{\frac{1}{2}}(Y-m)/(S^2/v)^{\frac{1}{2}}$$

and

$$(4.2) \quad Z_m = (rS^2 + r^2(Y-m)^2)^{\frac{1}{2}}/(v+1),$$

and define $P_i(Y, S^2)$ as the set of values m such that the rule $R_i (i = 1, 2, 3)$ applied to T_m and Z_m at level α does not reject the null hypothesis that $\mu + \lambda\sigma^2 = m$, then $P_i(Y, S^2)$ is a level $1 - \alpha$ confidence procedure for $\mu + \lambda\sigma^2$. Because the rules R_i define UMPU level α tests, the confidence procedures $P_i(Y, S^2)$ are uniformly most accurate unbiased level $1 - \alpha$, in the sense used by Lehmann (1959).

It remains to show that the confidence sets $P_i(y, s^2)$ are intervals. Although these confidence sets are defined in terms of UMPU tests of hypotheses of form $H(-\lambda): \theta = -\lambda$ against one and two-sided alternatives, where θ is one of the parameters of a multiparameter exponential family, the result does not follow from this fact. In particular, the argument in Lehmann ((1959) pages 179-80), showing that in such cases the UMPU tests, do define confidence intervals for θ , does not apply in the present case since θ is not the parameter of interest.

Given two functions $g(m)$ and $h(m)$, the sets $\{m: g(m) < h(m)\}$, $\{m: g(m) = h(m)\}$, and $\{m: g(m) > h(m)\}$ are, respectively, a right-infinite interval, a point, and a left-infinite interval if and only if

$$(4.3) \quad g(m') \leq h(m') \Rightarrow g(m'') < h(m'') \text{ for } m' < m''.$$

Setting $g(m) = t_m$, the confidence sets $P_i(y, s^2) (i = 1, 2)$ are intervals if and only if (4.3) holds for $h(m) = t(v, -\lambda z_m, \alpha)$, and the sets $P_3(y, s^2)$ are intervals if and only if (4.3) holds for $h(m) = t_i(v, -\lambda z_m, \alpha)$ for both $i = 1$ and $i = 2$.

Introducing the monotone transformation, given z ,

$$(4.4) \quad w(t; z) = ((v+1)/r)zt/(v+t^2)^{\frac{1}{2}},$$

we have, by (4.1), $w(t_m; z_m) = y - m$, so that it is necessary to show that the derivatives with respect to m of the transformed critical values, considered as functions of m , are greater than -1 . Since, by (4.2),

$$\frac{\partial}{\partial m} z_m = -(r/(v+1))r(y-m)/(rs^2 + r^2(y-m)^2)^{\frac{1}{2}},$$

which has absolute value less than $r/(v+1)$, it is enough to show that the derivatives with respect to z of

$$(4.5) \quad p(z) = w(t(v, -\lambda z, \alpha); z)r/(v+1)$$

and

$$(4.6) \quad p_i(z) = w(t_i(v, -\lambda z, \alpha); z)r/(v+1), \quad i = 1, 2,$$

are between -1 and $+1$.

Section 5 contains a proof that $|p'(z)| \leq 1$ for $v \geq 2$, and in Section 6 it is shown that this is not true in general for $v = 1$. The two-sided case is discussed in Section 7.

5. The one-sided case, $v \geq 2$. Let $p = p(z)$, $\beta = -(v+1)\lambda$, and $k = (v/2) - 1$. From (4.4) and (4.5) we have

$$(5.1) \quad p = zt(v, -\lambda z, \alpha)/(v + t^2(v, -\lambda z, \alpha))^{\frac{1}{2}}.$$

Note that while $-\infty < t(v, -\lambda z, \alpha) < \infty$, we have $-z < p < z$, and that $t(v, -\lambda z, \alpha) = v^{\frac{1}{2}}p/(z^2 - p^2)^{\frac{1}{2}}$. It follows, therefore, from (2.3) that

$$(5.2) \quad \int_{-z}^p g(u) du = \alpha,$$

where $g(u)$ is an appropriate probability density function obtained by a change of variable formula from the conditional density of T given $Z = z$. From (5.1) and (2.10) we obtain this density as $g(u) = g_k(u)/G_k(z)$, where

$$(5.3) \quad g_k(u) = (z^2 - u^2)^k \exp \{\beta u\}$$

and

$$(5.4) \quad G_k(u) = \int_{-z}^u g_k(v) dv,$$

for $-z < u < z$. Therefore we can write (5.2) in the form

$$(5.5) \quad G_k(p) = \alpha G_k(z).$$

For $v = 2$, (5.5) reduces to

$$p = \log (\alpha R + (1 - \alpha)R^{-1})/\beta$$

where $R = \exp [\beta z]$. Therefore

$$p' = (\alpha R - (1 - \alpha)R^{-1})/(\alpha R + (1 - \alpha)R^{-1}),$$

which has absolute value not greater than one, for $0 < \alpha < 1$.

For $v > 2$, we can differentiate both sides of (5.5) with respect to z to obtain

$$p'g_k(p) + 2kzG_{k-1}(p) = 2kzG_{k-1}(z)\alpha.$$

By (5.5) this reduces to

$$(5.6) \quad p'g_k(p) = -2kz(G_{k-1}(p) - G_k(p)M),$$

where $M = G_{k-1}(z)/G_k(z)$.

For $0 \leq b \leq 2z$, let $C(b)$ be the partial derivative with respect to z of $G_k(b-z)/G_k(z)$. Then we have

$$(5.7) \quad C(b)G_k(z) = -g_k(b-z) + 2kz(G_{k-1}(b-z) - MG_k(b-z)).$$

For any fixed value of z , if b is chosen so that $b-z = p$, (5.6) and (5.7) imply that

$$C(b)G_k(z) = -(1+p')g_k(b-z).$$

Since $g_k(u)$ is positive for $-z < u < z$, it follows that for any fixed z , $p' \geq -1$ if and only if $C(b) \leq 0$ for $b = z+p$. Remembering that (5.1) defines p as a function of α and λ as well as z , it follows from the symmetry relation (2.11) that if $p'(z) = p' \geq -1$ for $0 < \alpha < 1$, $z > 0$, and $-\infty < \lambda < \infty$, then $p'(z) \leq 1$ for the same range of α , z , and λ . Therefore it is enough to show that $C(b) \leq 0$ for $0 < b < 2z$, $z > 0$, and $-\infty < \beta < \infty$.

From (5.7), $C(b)$ is a differentiable function of b , and $C(0) = C(2z) = 0$. Thus there must exist a zero of $C'(b)$ in the interval $0 < b < 2z$. Differentiating (5.7) and simplifying, we get

$$(5.8) \quad C'(b)G_k(z) = g_{k-1}(b-z)b(2k - (2z-b)(\beta + 2kzM)).$$

Thus the only possible zero of $C'(b)$ is at $b^* = 2(z - k/(\beta + 2kzM))$. Since $b^* > 0$, it follows that $k - z(\beta + 2kzM) < 0$. But by (5.8), $C'(b^*/2)G_k(z) = g(b^*/2 - z)(b^*/2)(k - z(\beta + 2kzM))$, and thus $C'(b^*/2) < 0$, which implies that $C(b^*) < 0$. Therefore $C(b) < 0$ for $0 < b < 2z$.

6. The one-sided case, $v = 1$. For $v = 1$, equation (5.5) may be rewritten as

$$(6.1) \quad H(p/z) = \alpha H(1)$$

where $H(u) = \int_{-1}^u h(v) dv$ and $h(u) = (1-u^2)^{-\frac{1}{2}} \exp(\beta zu)$ for $-1 < u < 1$, where $\beta = -2\lambda$. Differentiating (6.1) with respect to z gives

$$(p' - p/z)h(p/z)/z + \beta H^*(p/z) = \alpha \beta H^*(1),$$

where $H^*(u) = \int_{-1}^u v h(v) dv$.

By (6.1) this reduces to

$$(6.2) \quad p'h(p/z) = (p/z)h(p/z) + \beta z(H(p/z)A - H^*(p/z)),$$

where $A = H^*(1)/H(1)$.

As in Section 5, $|p'| \leq 1$ for $0 < \alpha < 1$, $0 < z$, and $-\infty < \beta < \infty$ if and only if $p' \geq -1$ for the same values of α , z , and β . Again as in Section 5, let b be any fixed number between zero and $2z$. Defining $D(b)$ as the derivative with respect to z of $H(b/z-1)/H(1)$, we have

$$(6.3) \quad D(b)H(1) = -(b/z^2)h(b/z-1) + \beta \int_{-1}^{b/z-1} (u-A)h(u) du.$$

For fixed z , if b is chosen so that $p(z) = b-z$, we have, by (6.2) and (6.3) $D(b)H(1) = -(1+p')h(p/z)/z$. Since $h(u)$ is positive for $-1 < u < 1$, it follows that $p' \geq -1$ for $0 < \alpha < 1$, $z > 0$, and $-\infty < \beta < \infty$ if and only if $D(b)$ is negative for $0 < b < 2z$.

By (6.3) it is easy to show that $D(b) \rightarrow 0$ as $b \rightarrow 0$, and that $D(b) \rightarrow -\infty$ as $b \rightarrow 2z$. By differentiating (6.3) with respect to b and simplifying we obtain

$$(6.4) \quad D'(b)H(1) = h(b/z - 1)[-1 - \beta(2z - b)(1 + A)]/(2z^2 - bz).$$

The only possible zero of $D'(b)$ is at $b^* = 2z + 1/(\beta(1 + A))$, assuming $\beta \neq 0$ (if $\beta = 0$, $D'(b) < 0$ for $0 < b < 2z$). If $b^* > 2z$, or $b^* < 0$, then $D(b) < 0$ for $0 < b < 2z$. However, if $0 < b^* < 2z$ then either b^* represents an inflexion point of $D(b)$, which is negative for $0 < b < 2z$, or $D(b) > 0$ at least for $0 < b < b^*$.

In order that $0 < b^* < 2z$, it is necessary and sufficient that $-2z < \beta(1 + A)^{-1} < 0$. $A = A(\beta z)$ can be expressed as the ratio of two Bessel functions, $A(\beta z) = I_1(v)/I_0(\beta z)$ (since $I_0'(v) = I_1(v)$), where, for $m > -\frac{1}{2}$ and real v ,

$$(6.5) \quad I_m(v) = (v/2)^m \pi^{-\frac{1}{2}} (1/\Gamma(m + \frac{1}{2})) \int_{-1}^1 (1 - u^2)^{m - \frac{1}{2}} \exp(vu) du.$$

Since z is always positive, it is clear that $A(\beta z)$ has the same sign as β . Therefore $(\beta(1 + A))^{-1}$ is positive for positive β , which means that $b^* > 2z$. If $\beta < 0$, $A(\beta z)$ is negative, and $\beta(1 + A(\beta z)) < 0$ if and only if $A(\beta z) > -1$. Also, $-2z < (\beta(1 + A(\beta z)))^{-1}$ if and only if $A(\beta z) > -1 - (2z\beta)^{-1}$ or $A(\beta z) < -1$. Therefore $0 < b^* < 2z$ if and only if $A(\beta z) > -1 - (2z\beta)^{-1}$. From (6.5) and the preceding discussion we obtain the relation $A(-v) = -A(v)$, from which it follows that $0 < b^* < 2z$ for some negative β if there exists a positive number v such that $A(v) = I_1(v)/I_0(v) < 1 - (2v)^{-1}$. An examination of tables of $I_0(v)$ and $I_1(v)$ (Abramovitz and Stegun (1964) Table 9.8) shows that this inequality obtains for many values of v . For example, $I_1(1)/I_0(1) = .4464$, which is less than $1 - \frac{1}{2}$. Thus for $\beta\xi = -1$, we must have $0 < b^* < 2z$.

An examination of (6.4) readily shows that $D'(b^* + \delta)$ is positive for $\delta < 0$ and negative for $\delta > 0$, which implies that $D(b)$ has a maximum, which must be positive, at $b = b^*$. Thus there exist combinations of α, λ, y, s^2 such that the confidence sets $P_1(y, s^2)$ and $P_2(y, s^2)$ for $\mu + \lambda\sigma^2$ are not intervals.

7. The two-sided case. Let $g_k(u)$ and $G_k(u)$ be defined as in Section 5 for $-z < u < z$, and let $H_k(u) = \int_{-z}^u v g_k(v) dv$. Let p_1 and p_2 be defined in terms of the critical values $t_1(v, \xi, \alpha)$ and $t_2(v, \xi, \alpha)$, respectively, according to (5.1). Under this transformation of variables, equations (2.4) and (2.5) may be rewritten as

$$(7.1) \quad G_k(p_2) - G_k(p_1) = (1 - \alpha)G_k(z),$$

$$(7.2) \quad H_k(p_2) - H_k(p_1) = (1 - \alpha)H_k(z).$$

For $v = 2(k = 0)$, equations (7.1) and (7.2) may be evaluated by direct integration to give, after simplification,

$$(7.3) \quad e^{\beta p_2} - e^{\beta p_1} = (1 - \alpha)(e^{\beta z} - e^{-\beta z})$$

and $e^{\beta p_2}(\beta p_2 - 1) - e^{\beta p_1}(\beta p_1 - 1) = (1 - \alpha)(e^{\beta z}(\beta z - 1) + e^{-\beta z}(\beta z + 1))$, which reduces by (7.3), to

$$(7.4) \quad p_2 e^{\beta p_2} - p_1 e^{\beta p_1} = z(1 - \alpha)(e^{\beta z} + e^{-\beta z}).$$

If we differentiate both sides of (7.3) with respect to z we obtain

$$(7.5) \quad p_2' e^{\beta p_2} - p_1' e^{\beta p_1} = (1 - \alpha)(e^{\beta z} + e^{-\beta z}).$$

By (7.4) this reduces to

$$(7.6) \quad (p_2' - p_2/z) e^{\beta p_2} = (p_1' - p_1/z) e^{\beta p_1}.$$

Similarly, differentiation of (7.4) gives

$$p_2'(1 + \beta p_2) e^{\beta p_2} - p_1'(1 + \beta p_1) e^{\beta p_1} = (1 - \alpha)(1 + \beta z) e^{\beta z} + (1 - \beta z) e^{-\beta z},$$

which reduces, by (7.3) and (7.5), to

$$(7.7) \quad (p_2' p_2 - z) e^{\beta p_2} = (p_1' p_1 - z) e^{\beta p_1}.$$

Equations (7.6) and (7.7) may be used to solve for p_1' and p_2' in terms of $p_1, p_2,$ and z . If we multiply both sides of (7.6) by p_2 and subtract from (7.7), we obtain, after simplification,

$$(7.8) \quad p_1' = (p_1 p_2 - z^2 + (z^2 - p_2^2) e^{\beta(p_2 - p_1)}) / (z(p_2 - p_1)).$$

If (7.6) is multiplied by p_1 and subtracted from (7.7), we get

$$(7.9) \quad p_2' = (p_1 p_2 - z^2 + (z^2 - p_1^2) e^{\beta(p_1 - p_2)}) / (z(p_2 - p_1)).$$

In particular, it follows from (7.8) and (7.9) that $|p_i'| \leq 1, i = 1, 2,$ for $v = 2,$ if and only if

$$(7.10) \quad \frac{z + p_1}{z + p_2} \leq e^{\beta(p_2 - p_1)} \leq \frac{z - p_1}{z - p_2}.$$

If both sides of (7.3) are multiplied by z and added to the corresponding sides of (7.4), we obtain

$$(p_2 + z) e^{\beta p_2} - (p_1 + z) e^{\beta p_1} = 2z(1 - \alpha) e^{\beta z},$$

while subtraction gives

$$(p_2 - z) e^{\beta p_2} - (p_1 - z) e^{\beta p_1} = 2z(1 - \alpha) e^{-\beta z}.$$

These two equations can be restated, by dividing through by $\exp(\beta p_1)$ and $(p_2 + z)$ or $(p_2 - z),$ in the forms

$$(7.11) \quad e^{\beta(p_2 - p_1)} \frac{p_1 + z}{p_2 + z} = \frac{2z(1 - \alpha)}{p_2 + z} e^{\beta(z - p_1)}$$

and

$$(7.12) \quad e^{\beta(p_2 - p_1)} \frac{p_1 - z}{p_2 - z} = \frac{2z(1 - \alpha)}{p_2 - z} e^{-\beta(z + p_1)}.$$

Since z is positive and p_2 is less than $z,$ the right-hand side of (7.11) is positive, while that of (7.12) is negative. Thus for $v = 2,$ (7.10) holds for any α and any $\beta,$ which means that the two-sided tests define confidence intervals for $\mu + \lambda \sigma^2,$ for any $\lambda.$

For $v > 2$, a development analogous to that of equations (7.3) through (7.10) may be obtained, using integration by parts to express $H_k(u)$ in terms of $g_k(u)$, $G_k(u)$, and $G_{k-1}(u)$ (a device not possible in the case $v = 2$ since $G_{-1}(u)$ is not defined). As in the case $v = 2$, it is possible to solve for p_1' and p_2' in terms of p_1 , p_2 , and z . From this we obtain the result that $|p_i'| \leq 1$, for $i = 1, 2$, if and only if

$$(7.13) \quad \frac{z+p_1}{z+p_2} \leq \frac{g_k(p_2)}{g_k(p_1)} \leq \frac{z-p_1}{z-p_2},$$

obviously the general form of (7.10). However, at this writing no proof of (7.13) has been obtained for $k > 0$. It should be noted that if the conditional density of $f_v(t | \xi)$ were symmetric about its mean, we would have $t_1(v, \xi, \alpha) = t(v, \xi, \alpha/2)$ and $t_2(v, \xi, \alpha) = t(v, \xi, 1-\alpha/2)$, and by the results of Section 5, the two-sided tests would define confidence intervals. As v increases, $f_v(t | \xi)$ becomes more nearly symmetric, in fact $f_v(t | \xi)$ approaches a normal density as $v \rightarrow \infty$. It has been shown above that confidence intervals are defined by the two-sided tests when $v = 2$, for all α and all ξ . A plausible conjecture is that the same is true for $v > 2$, when $f_v(t | \xi)$ is more nearly symmetric than in the case $v = 2$. This conjecture is supported by the observation that in all cases where two-sided critical values have been calculated for $v > 2$, no example has been found where the inequalities (7.13) do not hold.

8. Applications. Exact applications of the method developed in this paper to problems involving transformations to normality are essentially limited to transformations of the form $Y = \log(X)$. To see this, suppose that $X = h(Y)$, where Y is normal (μ, σ^2) . Suppose also that h is twice-differentiable, and that $E(h(Y))$ and $E(h'(Y))$ both exist. The method can be applied to confidence interval estimation of EX only if EX depends on μ and σ^2 as a function of $a\mu + b\sigma^2$ for some a and b , that is, only if

$$(8.1) \quad f(a\mu + b\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} h(\mu + \sigma w) \exp(-\frac{1}{2}w^2) dw.$$

By letting σ^2 approach zero it is clear that we must have $f(a\mu) = h(\mu)$, and that a cannot be zero. Therefore there is no loss of generality in writing $f(a\mu + b\sigma^2) = h(\mu + \lambda\sigma^2)$, where $\lambda = b/a$. With this substitution, differentiation of (8.1) with respect to σ yields

$$(8.2) \quad 2\lambda\sigma h'(\mu + \lambda\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} wh'(\mu + \sigma w) \exp(-\frac{1}{2}w^2) dw.$$

Differentiation of (8.1) with respect to μ gives

$$h'(\mu + \lambda\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} h'(\mu + \sigma w) \exp(-\frac{1}{2}w^2) dw,$$

which may be rewritten as

$$(8.3) \quad h'(\mu + \lambda\sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} h'(y) \exp[-\frac{1}{2}(y-\mu)^2/\sigma^2] dy.$$

By differentiating (8.3) with respect to μ we obtain

$$h''(\mu + \lambda\sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (y-\mu)\sigma^{-2}h'(y) \exp[-\frac{1}{2}(y-\mu)^2/\sigma^2] dy,$$

which may be rewritten as

$$(8.4) \quad \sigma h''(\mu + \lambda\sigma^2) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} wh'(\mu + \sigma w) \exp(-\frac{1}{2}w^2) dw.$$

Together, (8.2) and (8.4) imply that $h''(\mu + \lambda\sigma^2) = 2\lambda h'(\mu + \lambda\sigma^2)$ for all μ and σ^2 , which implies that $h''(y) = 2\lambda h'(y)$ for all y . If $\lambda = 0$, this means that $h(y)$ is linear, in other words a transformation of a normal variate to another normal variate. If $\lambda \neq 0$, it means that $h(y) = \alpha + \beta \exp(2\lambda y)$ for some numbers α and β , so that the normalizing transform is of the form $Y = (2\lambda)^{-1} \log((X - \alpha)/\beta)$.

The results may be summarized as the following theorem.

THEOREM. *If Y is a normal (μ, σ^2) random variable, and h a twice-differentiable real function such that the expected values of $h(Y)$ and $h'(Y)$ exist for $-\infty < \mu < \infty$ and $\sigma^2 \geq 0$, then*

- (i) *$E(h(Y))$ depends nontrivially on both μ and σ^2 unless h is a constant or a linear function of y , and*
- (ii) *$E(h(Y))$ depends on μ and σ^2 as a function of $\mu + \lambda\sigma^2$, for some $\lambda \neq 0$, if and only if $h(y) = \alpha + \beta \exp(2\lambda y)$ for some α and β .*

COROLLARY. *$E(h(Y))$ cannot be a function of $\mu + \delta\sigma$. (Take $\lambda = \delta/\sigma$. The function h cannot depend on σ^2 .)*

For non-logarithmic transformations to normality the contours $EX = f(\mu, \sigma^2) = \theta$ are not parallel lines, to which the method of this paper may be directly applied. However, it is easy to imagine ways in which an approximate confidence interval for EX could be obtained from a confidence interval for $\mu + \lambda\sigma^2$, for some λ . We are familiar with the idea of using a confidence region for a vector-valued parameter to define approximate confidence intervals for real functions of the parameter (e.g., Scheffé (1961), Halperin and Mantel (1963), Halperin (1964 and 1965), and Kanofsky (1969)). The kind of confidence region for the vector-valued parameter (μ, σ^2) that is discussed in this paper is one that defines exactly a confidence interval for $\mu + \lambda\sigma^2$.

The details of a method for using the confidence regions defined in this paper to obtain an approximate confidence interval for EX are the choice of the value of λ in the function $\mu + \lambda\sigma^2$, and the rule by which values of EX are associated with the confidence region defined by a confidence interval for $\mu + \lambda\sigma^2$. For a particular transformation, and thus a particular function $EX = f(\mu, \sigma^2)$, these questions might be decided by examining a plot of the contours $f(\mu, \sigma^2) = p$ in a neighborhood of the point at which μ and σ^2 are equal to their estimated values $\hat{\mu}$ and $\hat{\sigma}^2$, respectively. For example, λ might be chosen as the value characterizing the line tangent to the contour $f(\mu, \sigma^2) = f(\hat{\mu}, \hat{\sigma}^2)$ at the point $\mu = \hat{\mu}, \sigma^2 = \hat{\sigma}^2$. That is, we would have $\lambda = f_{\mu}/f_{\sigma^2}$, where f_{μ} and f_{σ^2} represent the partial derivatives of $f(\mu, \sigma^2)$ with respect to μ and σ^2 , respectively, evaluated at $\mu = \hat{\mu}, \sigma^2 = \hat{\sigma}^2$.

The rule for defining an approximate confidence interval for $f(\mu, \sigma^2)$ in terms of an exact one for $\mu + \lambda\sigma^2$ might be expressed as a rule associating a line $\mu + \lambda\sigma^2 = m(p)$ with each contour $f(\mu, \sigma^2) = p$. The set of values p for which $m(p)$ is contained in the confidence interval for $\mu + \lambda\sigma^2$ would be an approximate confidence interval for $f(\mu, \sigma^2)$, provided that the function $m(p)$ is monotone. A possible rule would be to choose $m(p)$ so that the line $\mu + \lambda\sigma^2 = m(p)$ simultaneously intersects the curve $f(\mu, \sigma^2) = p$ and the perpendicular line $\mu - \sigma^2/\lambda = \hat{\mu} - \hat{\sigma}^2/\lambda$.

If $\log(X)$ is normal, approximate confidence interval methods for EX can be evaluated by comparing approximate confidence limits with the corresponding optimal exact limits, computed from the same data. This is not possible for other normalizing transformations, for which no optimal exact methods are available. However, comparisons can be made with conservative exact confidence limits computed according to Kanofsky's method, and estimates of the coverage probabilities for given values of μ and σ^2 can be obtained by Monte Carlo simulation. A subsequent paper is planned in which approximate methods of the kind discussed in this section are compared with other approximate confidence interval estimation procedures.

Acknowledgment. I would like to thank Professor William Kruskal for his help and encouragement in directing the Ph. D. thesis on which this paper is based. My thanks also go to the referee for some useful suggestions.

REFERENCES

- [1] ABRAMOVITZ, M. and STEGUN, I. A. (1964). *Handbook of Mathematical Functions*. National Bureau of Standards Applied Mathematics Series 55. U.S. Govt. Printing Office, Washington, D.C.
- [2] AITCHISON, J. and BROWN, J. A. C. (1957). *The Lognormal Distribution*. Cambridge Univ. Press.
- [3] HALPERIN, M. (1964). Interval estimation of non-linear parametric functions II. *J. Amer. Statist. Assoc.* **59** 168–81.
- [4] HALPERIN, M. (1965). Interval estimation of non-linear parametric functions III. *J. Amer. Statist. Assoc.* **60** 1191–9.
- [5] HALPERIN, M. and MANTEL, N. (1963). Interval estimation of nonlinear parametric functions. *J. Amer. Statist. Assoc.* **58** 611–27.
- [6] HOYLE, M. H. (1968). The estimation of variances after using a Gaussianizing transformation. *Ann. Math. Statist.* **39** 1125–43.
- [7] JOHNSON, N. L., NIXON, E., AMOS, D. E., and PEARSON, E. S. (1963). Tables of percentage points of Pearson curves, for given $(\beta_1)^{\frac{1}{2}}$ and β_2 , expressed in standard measure. *Biometrika* **50** 459–98.
- [8] KANOFSKY, PAUL (1969). Derivation of simultaneous confidence intervals for parametric functions from a parametric confidence region. *Sankhyā Ser. A* **36** 379–86.
- [9] LAND, C. E. (1968). Tests and confidence intervals from transformed data. Ph. D. dissertation, University of Chicago.
- [10] LAND, C. E. (1969). Critical values for hypothesis tests about linear functions of the normal mean and variance. Copy deposited in the Unpublished Mathematical Tables File of the journal *Mathematics of Computation* (October, 1971).

- [11] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.
- [12] NEYMAN, J. and SCOTT, E. L. (1960). Correction for bias introduced by a transformation of variables. *Ann. Math. Statist.* **31** 643–55.
- [13] OWEN, D. B. (1958). Tables of factors for one-sided tolerance limits for a normal distribution. Sandia Corp. Monograph, available from the Office of Technical Services, Dept. of Commerce, Washington 25, D.C.
- [14] PATTERSON, R. L. (1966). Difficulties involved in the estimation of a population mean using transformed sample data. *Technometrics* **8** 535–7.
- [15] SCHEFFÉ, H. (1961). Simultaneous interval estimation of linear functions of parameters. *Bull. Inst. Internat. Statist.* **38** 245–53.
- [16] WHEELER, D. J. (1959). A note on the Runge-Kutta method of integrating ordinary differential equations. *Comput. J.* **2** 23.