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Research Note

Confidence Limits for the Precision Parameter κ

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Summary

Confidence limits are calculated for the precision parameter κ used in the analysis of palaeomagnetic data and for the angular standard deviation σ. A set of tables for 95 per cent and 99 per cent confidence limits is presented.

Most analyses of palaeomagnetic data are made using the statistical method of Fisher (1953) in which magnetic vectors are represented by unit vectors with angular probability density

$$\frac{\kappa}{4\pi \sinh \kappa} \exp \{\kappa \cos \delta\} \tag{1}$$

where δ is the angle from the mean direction and κ is a precision parameter which describes the dispersion of the vectors. In most palaeomagnetic research the main objective of the statistical analysis is to determine the mean direction of the vectors and a confidence limit about this mean. In recent work there has been increased interest in the precision parameter κ itself. If the individual vectors determined palaeomagnetically correspond to individual directions of the Earth's magnetic field in the past, the parameter κ then provides an inverse measure of the angular dispersion produced by geomagnetic secular variation. A need has developed for calculating confidence limits in order to determine whether observed values of κ are consistent with different models to describe variations of κ with latitude (Creer 1962; Cox 1962; Irving & Ward 1964).

An analysis of variance may be used similar to that employed by Watson (1956) and Watson & Irving (1957) to develop numerous other useful statistical tests. The best estimate k of κ if the true polar angle is unknown is, for $\kappa > 3$,

$$k = \frac{N-1}{N-R} \tag{2}$$

where N is the number of unit vectors and R the length of their vector sum. For large κ, the statistic

$$2\kappa(N-R) = \frac{\kappa}{k} 2(N-1) \tag{3}$$

is distributed approximately like χ^2 with $2(N-1)$ degrees of freedom (Watson 1956). Hence for a given level of significance α,

$$\frac{k_u}{k} = \frac{P(\chi^2_{\alpha/2,\,2(N-1)})}{2(N-1)} \tag{4}$$

where k_u is the upper confidence limit and $P(\chi^2_{\alpha/2,\,2(N-1)})$ is the cumulative χ^2 distribution from $\chi^2 = \alpha/2$ to $\chi^2 = \infty$ for $2(N-1)$ degrees of freedom as found in standard tables. Similarly

$$\frac{k_1}{k} = \frac{P(\chi^2_{1-\alpha/2,\,2(N-1)})}{2(N-1)} \tag{5}$$

is the lower confidence limit. These ratios, which give confidence limits smaller than those found using F ratios (Gough, Opdyke & McElhinny, 1964) are tabulated for easy reference in Tables 1 and 2.

Table 1

95 per cent confidence limits for the precision parameter κ ($k_1 \leqslant \kappa \leqslant k_u$) and angular standard deviation σ ($s_1 \leqslant \sigma \leqslant s_u$)

N	N_0	$\frac{k_u}{k}$	$\frac{k_1}{k}$	$\frac{s_u}{s}$	$\frac{s_1}{s}$
4	5	2·047	0·325	1·755	0·699
5	6	1·945	0·367	1·651	0·717
6	7	1·866	0·402	1·577	0·732
7	8	1·802	0·432	1·522	0·745
8	9	1·750	0·457	1·479	0·756
9	10	1·709	0·480	1·444	0·765
10	11	1·674	0·499	1·415	0·773
11	12	1·639	0·517	1·391	0·781
12	13	1·610	0·533	1·370	0·788
13	14	1·586	0·547	1·352	0·794
14	15	1·566	0·559	1·337	0·799
19	20	1·484	0·610	1·280	0·821
24	25	1·427	0·647	1·243	0·837
29	30	1·387	0·675	1·217	0·849
34	35	1·358	0·697	1·198	0·858
39	40	1·333	0·715	1·183	0·866
44	45	1·312	0·729	1·171	0·873
49	50	1·294	0·742	1·161	0·879
59	60	1·268	0·763	1·145	0·888
69	70	1·248	0·779	1·133	0·895
79	80	1·232	0·793	1·123	0·901
89	90	1·218	0·804	1·115	0·906
99	100	1·205	0·813	1·109	0·911
119	120	1·187	0·829	1·098	0·918
139	140	1·171	0·842	1·090	0·924
159	160	1·161	0·851	1·084	0·928
179	180	1·151	0·859	1·079	0·932
199	200	1·144	0·867	1·074	0·935
249	250	1·127	0·880	1·066	0·942
499	500	1·090	0·914	1·046	0·958

Table 2

99 per cent confidence limits for the precision parameter κ and the angular standard deviation σ

N	N_0	$\frac{k_u}{k}$	$\frac{k_1}{k}$	$\frac{s_u}{s}$	$\frac{s_1}{s}$
4	5	2·520	0·216	2·154	0·630
5	6	2·360	0·256	1·976	0·651
6	7	2·234	0·291	1·854	0·669
7	8	2·143	0·321	1·764	0·683
8	9	2·064	0·348	1·695	0·696
9	10	2·001	0·372	1·640	0·707
10	11	1·945	0·393	1·595	0·717
11	12	1·897	0·412	1·558	0·726
12	13	1·856	0·429	1·526	0·734
13	14	1·821	0·445	1·499	0·741
14	15	1·787	0·460	1·475	0·748
19	20	1·669	0·518	1·390	0·774
24	25	1·590	0·559	1·337	0·793
29	30	1·532	0·593	1·299	0·808
34	35	1·487	0·618	1·272	0·820
39	40	1·455	0·640	1·250	0·829
44	45	1·424	0·658	1·233	0·838
49	50	1·401	0·673	1·219	0·845
59	60	1·361	0·695	1·199	0·857
69	70	1·332	0·716	1·182	0·867
79	80	1·309	0·733	1·168	0·874
89	90	1·290	0·747	1·157	0·880
99	100	1·274	0·759	1·148	0·886
119	120	1·249	0·779	1·133	0·895
139	140	1·230	0·794	1·122	0·902
159	160	1·214	0·807	1·113	0·908
179	180	1·201	0·817	1·106	0·912
199	200	1·190	0·826	1·100	0·917
249	250	1·170	0·844	1·089	0·925
499	500	1·118	0·888	1·061	0·946

If the true mean direction is known, as for example the axial dipole field direction in the case of analysing results from young rocks, then the best estimate of κ is given not by equation (2) but rather by

$$k = \frac{N_0}{N_0 - R_0 \cos\delta_0} \tag{6}$$

where N_0 is the number of unit vectors, R_0 is the length of their vector sum, and δ_0 is the angle between the vector sum and the known polar angle. For this case the statistic

$$2\kappa(N_0 - R_0 \cos\delta_0) = \frac{\kappa}{k} 2N_0 \tag{7}$$

is distributed like χ^2 with $2N_0$ degrees of freedom (Watson 1956), so that k_u/k and k_1/k may be found as in equations (4) and (5) substituting $2(N_0)$ for $2(N-1)$. The second columns in Tables 1 and 2 give the number N_0 of magnetic directions for this case.

The angular standard deviation σ may be used rather than κ in analysing directions. If δ_1 is the angular deviation between an individual magnetic direction and the mean, then the estimate s^2 of the population variance σ^2 obtained from standard univariate analysis is

$$s^2 = \frac{1}{N-1} \sum_{i=i}^{N} \delta_i^2 \tag{8}$$

for small δ_i,

$$\delta_1^2 \cong 2(1-\cos\delta_i) \tag{9}$$

hence,

$$s^2 \cong \frac{1}{N-1} 2\sum_{i=i}^{N} (1-\cos\delta_i) = \frac{2(N-R)}{(N-1)} = 2/k. \tag{10}$$

The upper and lower confidence limits on s are

$$\frac{s_u}{s} = \left[\frac{2(N-1)}{P(\chi_{1-\alpha/2,\, 2(N-1)})}\right]^{\frac{1}{2}} \tag{11}$$

and

$$\frac{s_1}{s} = \left[\frac{2(N-1)}{P(\chi_{\alpha/2,\, 2(N-1)})}\right]^{\frac{1}{2}}. \tag{12}$$

These ratios are tabulated in Tables 1 and 2, N_0 again indicating the number of magnetic vectors when the true mean direction is known.

As an example, on the island of Hawaii (Doell & Cox 1965) a between-site precision of $k = 29$ was found for $N = 112$ magnetic directions. Interpolating between $N = 99$ and $N = 119$ in Table 1, $k_u = (29)\ (1{\cdot}193) = 35$ and $k_1 = (29)\ (0{\cdot}823) = 24$. Hence at the 95 per cent confidence level, $24 \leqslant k \leqslant 35$. Similarly for

$$s = \sqrt{(2/k)} = 15{\cdot}1^\circ,$$

$$s_u = (1{\cdot}102) \quad (15{\cdot}1) = 16{\cdot}7^\circ$$

$$s_1 = (0{\cdot}916) \quad (15{\cdot}1) = 13{\cdot}8^\circ$$

so that the 95 per cent confidence interval is $13{\cdot}8^\circ \leqslant s \leqslant 16{\cdot}7^\circ$.

These results are of some interest in designing experiments to measure secular variation palaeomagnetically. Even if a large number of samples are collected at each site to reduce experimental error, the intrinsic uncertainty at 95 per cent confidence in the final determination of κ from 100 sites is about ± 20 per cent, and that in σ is about ± 10 per cent. It is rarely feasible to collect samples from more than 100 sites, so these figures indicate approximate practical limitations of using palaeomagnetic results to discriminate between models for secular variation.

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