Inventiones math. 21, 213–221 (1973)

Configuration-Spaces and Iterated Loop-Spaces

Graeme Segal (Oxford)

§ 1. Introduction

The object of this paper is to prove a theorem relating "configuration-spaces" to iterated loop-spaces. The idea of the connection between them seems to be due to Boardman and Vogt [2]. Part of the theorem has been proved by May [6]; the general case has been announced by Giffen [4], whose method is to deduce it from the work of Milgram [7].

Let C_n be the space of finite subsets of $\mathbb{R}^n$. It is topologized as the disjoint union $\coprod_{k \geq 0} C_{n,k}$, where $C_{n,k}$ is the space of subsets of cardinal k, regarded as the orbit-space of the action of the symmetric group Σ_k on the space $\tilde{C}_{n,k}$ of ordered subsets of cardinal k, which is an open subset of $\mathbb{R}^{nk}$.

There is a map from C_n to $\Omega^n S^n$, the space of base-point preserving maps $S^n \to S^n$, where S^n is the n-sphere. One description of it (at least when $n > 1$) is as follows. Think of a finite subset c of $\mathbb{R}^n$ as a set of electrically charged particles, each of charge $+1$, and associate to it the electric field E_c it generates. This is a map E_c: $\mathbb{R}^n - c \to \mathbb{R}^n$ which can be extended to a continuous map E_c: $\mathbb{R}^n \cup \infty \to \mathbb{R}^n \cup \infty$ by defining $E_c(\xi) = \infty$ if $\xi \in c$, and $E_c(\infty) = 0$. Then E_c can be regarded as a base-point-preserving map $S^n \to S^n$, where the base-point is ∞ on the left and 0 on the right. Notice that the map $c \mapsto E_c$ takes $C_{n,k}$ into $\Omega^n S^n_{(k)}$, the space of maps of degree k.

Our object is to prove that C_n is an approximation to $\Omega^n S^n$, in the sense that the two spaces have composition-laws which are respected by the map $C_n \to \Omega^n S^n$, and the induced map of classifying-spaces is a homotopy-equivalence. In view of the "group-completion" theorem of Barratt-Priddy-Quillen [1, 8] one can say equivalently that $C_{n,k} \to \Omega^n S^n_{(k)}$ induces an isomorphism of integral homology up to a dimension tending to ∞ with k. But to make precise statements it is convenient to introduce a modification of the space C_n.

If $u \leq v$ in $\mathbb{R}$, let $\mathbb{R}^n_{u,v}$ denote the open set $]u, v[\times \mathbb{R}^{n-1}$ in $\mathbb{R}^n$. Then C_n is homotopy-equivalent to the space

$$C'_n = \{(c, t) \in C_n \times \mathbb{R}: \; t \geq 0, c \subset \mathbb{R}^n_{0,t}\},$$

which has an associative composition-law given by juxtaposition, i.e. $(c,t).(c',t')=(c\cup T_t c', t+t')$, where T_t: $\mathbb{R}^n_{0,t'}\to\mathbb{R}^n_{t,t+t'}$ is translation. As a topological monoid C'_n has a classifying-space BC'_n.

Theorem 1. $BC'_n \simeq \Omega^{n-1}S^n$, *the* $(n-1)$*-fold loop-space of* S^n.

More generally, let X be a space with a good base-point denoted by 0. (That means that there exists a homotopy h_t: $X\to X$ $(0\leqq t\leqq 1)$ such that $h_0=$ identity, $h_t(0)=0$, and $h_1^{-1}(0)$ is a neighbourhood of 0.) Let $C_n(X)$ be the space of finite subsets of $\mathbb{R}^n$ "labelled by X" in the following sense. A point of $C_n(X)$ is a pair (c,x), where c is a finite subset of $\mathbb{R}^n$, and x: $c\to X$ is a map. But (c,x) is identified with (c',x') if $c\subset c'$, $x'|c=x$, and $x'(\xi)=0$ when $\xi\notin c$.

$C_n(X)$ is topologized as a quotient of the disjoint union

$$\coprod_{k\geqq 0}(\tilde{C}_{n,k}\times X^k)/\Sigma_k.$$

As before, $C_n(X)$ is homotopy-equivalent to a topological monoid $C'_n(X)\subset C_n(X)\times\mathbb{R}_+$, consisting of triples (c,x,t) such that $t\geqq 0$ and $c\subset\mathbb{R}^n_{0,t}$.

Theorem 2. $BC'_n(X)\simeq\Omega^{n-1}S^nX$.

Here S^nX is the n-fold reduced suspension of X. Theorem 1 is Theorem 2 for the case $X=S^0$. If X is path-connected so is $C'_n(X)$, and then $\Omega BC'_n(X)\simeq C'_n(X)\simeq C_n(X)$, and one has

Theorem 3. *If* X *is path-connected* $C_n(X)\simeq\Omega^nS^nX$.

This has been proved by May [6].

Some other special cases are

(a) If $n=1$, $C'_n(X)$ is equivalent to the free monoid MX on X, in the sense that there is a homomorphism $C'_1(X)\to MX$ which is a homotopy-equivalence. Thus one obtains the theorem of James [5] that $BMX\simeq SX$.

(b) If $n=2$, $C_{2,k}$ is the classifying-space for the braid group Br_k on k strings. Thus one has $B(\coprod_{k\geqq 0}B(Br_k))\simeq\Omega S^2$.

(c) Because $\tilde{C}_{n,k}$ is $(n-2)$-connected, being the complement of some linear subspaces of codimension n in $\mathbb{R}^{nk}$, one has $C_{n,k}\to B\Sigma_k$ as $n\to\infty$. This gives the theorem of Barratt-Priddy-Quillen that $B(\coprod_{k\geqq 0}B\Sigma_k)\simeq\Omega^{\infty-1}S^\infty$.

The theorems above do not mention a specific map between the configuration-spaces and the loop-spaces. I shall return to this question in § 3.

§ 2. Proofs

Theorem 2 is obtained by induction from

Proposition (2.1). $BC'_n(X) \simeq C_{n-1}(SX)$.

For $C_{n-1}(SX)$ is connected, so $C_{n-1}(SX) \simeq C'_{n-1}(SX) \simeq \Omega BC'_{n-1}(SX) \simeq \Omega C_{n-2}(S^2 X) \simeq \cdots \simeq \Omega^{n-1} C_0(S^n X) = \Omega^{n-1} S^n X$.

The proof of (2.1) is based on the idea of a "partial monoid".

Definition (2.2). A *partial monoid* is a space M with a subspace $M_2 \subset M \times M$ and a map $M_2 \to M$, written $(m, m') \mapsto m \cdot m'$, such that

(a) there is an element 1 in M such that $m.1$ and $1.m$ are defined for all m in M, and $1.m = m.1 = m$.

(b) $m.(m' m'') = (m.m').m''$ for all m, m', m'' in M, in the sense that if one side is defined then the other is too, and they are equal.

A partial monoid M has a classifying-space BM, defined as follows. Let $M_k \subset \underbrace{M \times \cdots \times M}_{\leftarrow k \rightarrow}$ be the space of composable k-tuples. The M_k form a (semi)simplicial space, in which d_i: $M_k \to M_{k-1}$ and s_i: $M_k \to M_{k+1}$ are defined by

$$\begin{aligned} d_i(m_1, \ldots, m_k) &= (m_2, \ldots, m_k) && \text{if } i=0 \\ &= (m_1, \ldots, m_i . m_{i+1}, \ldots m_k) && \text{if } 0<i<k \\ &= (m_1, \ldots, m_{k-1}) && \text{if } i=k \\ s_i(m_1, \ldots, m_k) &= (m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_k) && \text{if } 0 \leqq i \leqq k. \end{aligned}$$

BM is defined as the realization of this simplicial space [9]. If M is actually a monoid (i.e. if $M_2 = M \times M$) then BM is the usual classifying-space. On the other hand if M has trivial composition (i.e. $M_2 = M \vee M$) then $BM = SM$, the reduced suspension of M.

The space $C_{n-1}(X)$ can be regarded as a partial monoid, in which (c, x) and (c', x') are composable if and only if c and c' are disjoint, and then $(c, x).(c', x') = (c \cup c', x \cup x')$.

Proposition (2.3). $BC_{n-1}(X) \cong C_{n-1}(SX)$.

Proof. Write $M = C_{n-1}(X)$. By definition BM is a quotient of the disjoint union of the spaces $M_k \times \Delta^k$ for $k \geqq 0$. Regard the standard simplex Δ^k as $\{(t_1, \ldots, t_k) \in \mathbb{R}^k$: $0 \leqq t_1 \leqq \cdots \leqq t_k \leqq 1\}$. Define $M_k \times \Delta^k \to C_{n-1}(SX)$ by $((c_1, x_1), \ldots, (c_k, x_k);\ t_1, \ldots, t_k) \mapsto (c, \tilde{x})$, where $c = U c_i$ and $\tilde{x}: c \to SX$ takes $P \in c_i$ to $(t_i, x_i(P)) \in SX$.

These maps induce a map $BM \to C_{n-1}(SX)$ which is obviously surjective. It is injective because each point of BM is representable in the above form with $0 < t_1 < \cdots < t_k < 1$, all c_i non-empty, and $x_i(c_i) \subset X - \{0\}$. It is a homeomorphism because one can define a continuous inverse-map, observing that $C_{n-1}(SX)$ is a quotient of the dispoint union for $k \geqq 0$ of the spaces $\tilde{C}_{n-1,k} \times X^k \times [0,1]^k$, which map to $M_k \times [0,1]^k$.

The partial monoid $C_{n-1}(X)$ is related to the monoid $C_n'(X)$ by means of a sub-partial-monoid of the latter. Call an element (c, x, t) of $C_n'(X)$ *projectable* if c is mapped injectively by the projection

$$pr_2 \colon \mathbb{R}^n_{0,t} \to \mathbb{R}^{n-1}.$$

The projectable elements form a subspace which the composition-law of $C_n'(X)$ makes into a partial monoid. Projection defines a homomorphism $C_n''(X) \to C_{n-1}(X)$, and elements of $C_n''(X)$ are composable if and only if their images in $C_{n-1}(X)$ are. Furthermore $C_n''(X) \to C_{n-1}(X)$ is a homotopy-equivalence (with inverse $(c, x) \mapsto (s(c), x, 1)$, where s is any cross-section of $pr_2 \colon \mathbb{R}^n_{0,t} \to \mathbb{R}^{n-1}$); and so is the map $C_n''(X)_k \to C_{n-1}(X)_k$ of spaces of composable k-tuples. Because the simplicial spaces in question are good (see Appendix 2), this implies that $BC_n''(X) \xrightarrow{\simeq} BC_{n-1}(X)$. So to prove (2.1) it is enough to prove

Proposition (2.4). *The inclusion* $C_n''(X) \to C_n'(X)$ *induces a homotopy-equivalence of classifying-spaces.*

To prove (2.4) I shall use an alternative description of the classifying-space of a partial monoid M. Let $\mathscr{C}(M)$ be the topological category [9] whose objects are the elements of M, and whose morphisms from m to m' are pairs of elements $m_1, m_2 \in M$ such that $m_1 . m . m_2 = m'$. Thus $\mathrm{ob}\,\mathscr{C}(M) = M$, and $\mathrm{mor}\,\mathscr{C}(M) = M_3$. Let $|\mathscr{C}(M)|$ be the realization or "classifying space" of $\mathscr{C}(M)$ in the sense of [9].

Proposition (2.5). $|\mathscr{C}(M)| \simeq BM$.

This is a particular case of a subdivision theorem for arbitrary simplicial spaces, proved in Appendix 1. For the proof of (2.4) I shall use a modification of the category $\mathscr{C}(C_n'(X))$. Let Q be the ordered space whose points are 4-tuples $(u, v; c, x)$, with $u, v \in \mathbb{R}$, $u \leqq 0 \leqq v$, c a finite subset of $\mathbb{R}^n_{u,v}$, and $x \colon c \to X$, subject to the usual equivalence-relation. It is ordered by defining $(u, v; c.x) \leqq (u', v'; c'.x')$ if $[u, v] \subset [u', v']$, $c = c' \cap ([u, v] \times \mathbb{R}^{n-1})$, and $x'|c = x$. Thinking of the topological ordered set Q as a topological category, define a functor $\pi \colon Q \to \mathscr{C}(C_n'(X))$ by $(u, v; c, x) \mapsto (T_{-u} c, x, v-u)$.

Lemma (2.6). $|\pi| \colon |Q| \to |\mathscr{C}(C_n(X))|$ *is shrinkable* [3] (i.e. *it has a section* s *such that* $s|\pi| \simeq$ *identity by a homotopy* h_t *for which* $|\pi| h_t = |\pi|$).

Proof. Using the homomorphism of monoids $C_n'(X) \to \mathbb{R}_+$ which takes (c, x, t) to t one can regard Q as the fibre-product (of categories) $\mathscr{C}(C_n'(X)) \times_{\mathscr{C}(\mathbb{R}_+)} \mathscr{I}$, where $\mathscr{I}$ is the space of intervals $[u, v]$ with $u \leqq 0 \leqq v$ ordered by inclusion. Forming the nerve commutes with fibre-products. But $|\mathscr{I}| \to |\mathscr{C}(\mathbb{R}_+)|$ is easily seen to be shrinkable; so $|\pi|$ is shrinkable.

Continuing the proof of (2.4), if P is the sub-ordered-space of Q consisting of all $(u, v; c, x)$ with c projectable then $\pi(P)=\mathscr{C}(C_n''(X))$; so it will be enough to show that $|P|\to|Q|$ is a homotopy-equivalence. Heuristically this is so because P is co-initial in Q, i.e. for each $q\in Q$ there is a $p\in P$ such that $p\leqq q$, and if $p_1\leqq q$ and $p_2\leqq q$ there is a $p_3\in P$ such that $p_3\leqq p_1$ and $p_3\leqq p_2$. But some further conditions are needed, and I shall use the following ad hoc lemma.

Proposition (2.7). *Let Q be a good ordered space such that*

(a) $q_1\cap q_2=\inf(q_1, q_2)$ *is defined whenever there exists $q\in Q$ such that $q_1\leqq q$ and $q_2\leqq q$, and*

(b) $q_1\cap q_2$ *depends continuously on (q_1, q_2) where defined.*

Let Q_0 be an open subspace of Q such that if $q\in Q$ and $p\in Q_0$ and $q\leqq p$ then $q\in Q_0$. Suppose there is a numerable covering [3] $U=\{U_\alpha\}$ *of Q, and maps f_α: $U_\alpha\to Q_0$ such that $f_\alpha(q)\leqq q$ for all $q\in U_\alpha$.*

Then $|Q_0|\to|Q|$ is a homotopy-equivalence.

In the application of (2.7) Q is as above. Using the fact that X has a good base-point, choose a homotopy h_t: $X\to X$ such that h_0 is the identity and $h_1^{-1}(0)$ is a neighbourhood of 0. This induces h_t: $Q\to Q$. Let $Q_0=h_1^{-1}(P)$, a neighbourhood of P. One might call Q_0 the "almost projectable" elements of Q. Obviously $|P|\to|Q_0|$ is a homotopy-equivalence, with inverse induced by h_1. But Q_0 and Q satisfy the hypotheses of (2.7) – it is proved in Appendix 2 that Q is good. Thus (2.4) is proved modulo (2.7).

The proof of (2.7) would be almost trivial if one could choose the maps f_α compatibly so as to get a continuous map $Q\to Q_0$. In general it depends on the fact that one does not change the homotopy-type of a topological category by breaking apart the space of objects into the sets of a numerable covering and introducing isomorphism between the reduplicated objects. To be precise, if C is a topological category, and $U=\{U_\alpha\}_{\alpha\in A}$ is a numerable covering of $\mathrm{ob}(C)$, then the *disintegration of C by U* is the topological category $\tilde{C}$ whose objects are pairs (x, α) with $\alpha\in A$ and $x\in U_\alpha$, and whose morphisms $(x, \alpha)\to(x', \alpha')$ are the morphisms from x to x' in C. Thus $\mathrm{ob}(\tilde{C})=\coprod_{\alpha\in A} U_\alpha$ and $\mathrm{mor}(\tilde{C})=\coprod_{\alpha,\beta\in A} V_{\alpha\beta}$, where $V_{\alpha\beta}$ consists of the elements of $\mathrm{mor}(C)$ with source in U_α and target in U_β.

Proposition (2.8). *If C is a topological category, U is a numerable covering of* $\mathrm{ob}(C)$, *and $\tilde{C}$ is the disintegration of C by U, then the projection $|\tilde{C}|\to|C|$ is shrinkable.*

Proof. Let $\{C_k\}_{k\geqq 0}$ be the simplicial space associated to C (i.e. $C_0=\mathrm{ob}(C)$, $C_1=\mathrm{mor}(C)$, etc.), and let $\{\tilde{C}_k\}_{k\geqq 0}$ be that associated to $\tilde{C}$. Then $\tilde{C}_k=(\tilde{C}_0)^{k+1}\times_{(C_0)^{k+1}} C_k$. Now for any space Y there is a simplicial space $\{Y^{k+1}\}_{k\geqq 0}$ whose realization is a contractible space called EY [9].

Because realization commutes with fibre-products $|\tilde{C}| = E\tilde{C}_0 \times_{EC_0} |C|$, and it is enough to show that $E\tilde{C}_0 \to EC_0$ is shrinkable. But $\{EU_\alpha\}_{\alpha\in A}$ is a numerable covering of EC_0 over which $E\tilde{C}_0 \to EC_0$ is shrinkable, so the result follows from [3].

Proof of (2.7). Let p: $\tilde{Q} \to Q$ be the disintegration associated to U. $\tilde{Q}$ is a preordered space. The f_α define a map f: $\tilde{Q} \to Q_0$ such that $f(\xi) \leqq p(\xi)$ for all $\xi \in \tilde{Q}$. Let $\operatorname{chn}(\tilde{Q})$ be the space of finite chains of $\tilde{Q}$ ordered by inclusion (cf. Appendix 2). Define F: $\operatorname{chn}(\tilde{Q}) \to Q_0$ by $(\xi_0 \leqq \cdots \leqq \xi_k) \mapsto \inf(f(\xi_0), \ldots, f(\xi_k))$. This is order-preserving, and $F(\sigma) \leqq r \operatorname{chn}(p)(\sigma)$ for $\sigma \in \operatorname{chn}(Q)$, where r: $\operatorname{chn}(\tilde{Q}) \to \tilde{Q}$ is $(\xi_0 \leqq \cdots \leqq \xi_k) \mapsto \xi_0$. As $|r|$ is a homotopy-equivalence by Appendix 2, and $|p|$, and so $|\operatorname{chn}(p)|$, is one by (2.8), and $|F| \simeq |r|\,|\operatorname{chn}(p)|$ by [9] ( ), it follows that the composite $|\operatorname{chn}(\tilde{Q})| \to |Q_0| \to |Q|$ is a homotopy-equivalence. Similarly the composite $|\operatorname{chn}(\tilde{Q}_0)| \to |\operatorname{chn}(\tilde{Q})| \to |Q_0|$ is a homotopy-equivalence; and so $|Q_0| \to |Q|$ is one, as desired.

§ 3. The Map $C_n(X) \to \Omega^n S^n X$

Despite its picturesqueness the electrostatic map described in §1 is not very convenient in practice. It is homotopic, however, to the following map. Let $D_n(X)$ be the space of finite sets of disjoint open unit disks in $\mathbb{R}^n$, labelled by X. (This is a closed subset of $C_n(X)$, and obviously a deformation retract of it.) Choose a fixed map f of degree 1 from a standard disk D to S^n, taking the boundary to the base-point. Then associate to a point $(\{i_\alpha: D \to \mathbb{R}^n\}_{\alpha\in c}, x: c \to X)$ of $D_n(X)$ the map ϕ: $\mathbb{R}^n \cup \infty \to S^n X$ defined by

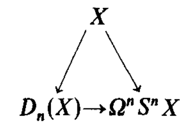
$$\begin{aligned} \phi(\xi) &= (f\, i_\alpha^{-1}(\xi), x(\alpha)) \quad \text{if } \xi \in i_\alpha(D), \text{ and} \\ &= \text{base-point otherwise}. \end{aligned}$$

Regard ϕ as a point of $\Omega^n S^n X$.

$D_n(X)$ can be regarded as a partial monoid in n different ways, the composition in each case being superimposition, but two sets of disks being composable for the i-th law if and only if they are separated by a hyperplane perpendicular to the i-th coordinate direction. These composition-laws are compatible in the sense that each is a homomorphism for the others, so one can use each of them in turn and thus define an n-fold classifying-space $B^n D_n(X)$. There is a map $X \to D_n(X)$ taking x to the unit disk at the origin labelled by x. Giving X n trivial composition-laws (so that $B^n X = S^n X$), $X \to D_n(X)$ is a homomorphism for all n laws, and induces $S^n X \to B^n D_n(X)$. Evidently what was proved in §2 was that $S^n X \to B^n D_n(X)$.

On the other hand the space of maps ϕ: $\mathbb{R}^n \cup \infty \to S^n X$ also has n partial composition-laws. Define support $(\phi) = \phi^{-1}(S^n X - \{0\})$. Then ϕ_1

and ϕ_2 are composable for the i-th law if their supports are separated by a hyperplane perpendicular to the i-th axis, and the composite is in any case given by "union". The map $X \to \Omega^n S^n X$ is an n-fold homomorphism, so induces $S^n X \to B^n \Omega^n S^n X$, which one knows classically to be a homotopy-equivalence. But the map $D_n(X) \to \Omega^n S^n X$ defined above is an n-fold homomorphism, and the diagram

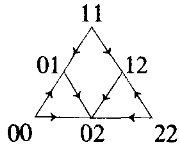


commutes. So $D_n(X) \to \Omega^n S^n X$ induces an isomorphism of classifying-spaces, as desired.

Appendix 1. The Edgewise Subdivision of a Simplicial Space

This is more or less due to Quillen.

The standard n-simplex $\Delta^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : 0 \leqq t_1 \leqq \cdots \leqq t_n \leqq 1\}$ can be subdivided into 2^n n-simplexes corresponding to the 2^n possible orders in which the 2^n numbers $(t_1, \ldots, t_n; 1-t_n, \ldots, 1-t_1)$ can occur in $[0, 1]$. When $n=2$ the diagram is



In general one puts a new vertex P_{ij} at the mid-point of each edge $P_{ii}P_{jj}$ of Δ^n, the original vertices being denoted $P_{ii} (0 \leqq i \leqq n)$. And $P_{i_0 j_0}, \ldots, P_{i_k j_k}$ span a simplex of the subdivision if $i_0 \geqq \cdots \geqq i_k$ and $j_0 \leqq \cdots \leqq j_k$.

This subdivision of a simplex, being functorial for simplicial maps, induces a subdivision of $|A|$ for any simplicial space A. Thereby $|A|$ is expressed as the realization of a simplicial space B such that $B_n = A_{2n+1}$. To be precise, let Δ be the category of finite ordered sets, so that A is a contravariant functor from Δ to spaces. There is a functor $T: \Delta \to \Delta$ which takes the set $\{\alpha_0, \ldots, \alpha_n\}$ with $n+1$ elements to the set $\{\alpha_0, \ldots, \alpha_n, \alpha'_n, \ldots, \alpha'_0\}$ with $2n+2$ elements, ordered as written. Then B is $A \cdot T$.

Proposition (A.1). *For any simplicial space A, $|A| \cong |A \cdot T|$.*

Proof. To write down maps $|A| \leftrightarrows |A \cdot T|$, observe that the 2^n simplexes into which Δ^n is subdivided can be indexed $\Delta^n{}_\theta$, where θ runs through a set of 2^n maps $[2n+1] \to [n]$; and $\Delta^n{}_\theta$ is the image of the composition $\theta_* i$, where $i: \Delta^n \to \Delta^{2n+1}$ is $(t_1, \ldots, t_n) \mapsto (\frac{1}{2}t_1, \ldots, \frac{1}{2}t_n, \frac{1}{2}, 1-\frac{1}{2}t_n, \ldots, 1-\frac{1}{2}t_1)$.

Then the maps $A_{2n+1}\times\Delta^n\to\Delta_{2n+1}\times\Delta^{2n+1}$ given by $(a,\xi)\mapsto(a,i\xi)$ induce $|A\cdot T|\to|A|$; and the maps $A_n\times\Delta^n{}_\theta\to A_{2n+1}\times\Delta^n$ given by $(a,\theta_* i\xi)\mapsto(\theta^* a,\xi)$ induce its inverse.

Appendix 2. Good Simplicial Spaces

Goodness is a condition on a simplicial space which ensures that its realization has convenient properties. (May [6] uses "strictly proper" for a similar idea.) I have discussed the condition at greater length in [10].

First observe that for any simplicial space $A=\{A_n\}$ there are n distinguished subsets $A_{n,i}$ $(1\leqq i\leqq n)$ in A_n, homeomorphic images of A_{n-1} by the degeneracy maps $s_i\colon A_{n-1}\to A_n$.

Definition. A simplicial space A is *good* if for each n there exists a homotopy $f_t\colon A_n\to A_n (0\leqq t\leqq 1)$ such that

(i) $f_0=$ identity

(ii) $f_t(A_{n,i})\subset A_{n,i}$ for $1\leqq i\leqq n$

(iii) $f_1^{-1}(A_{n,i})$ is a neighbourhood of $A_{n,i}$ in A_n for $1\leqq i\leqq n$.

Two results about good simplicial spaces are used in this paper. They are proved in [10].

Proposition (A.1). *If $\phi\colon A\to B$ is a map of good simplicial spaces, and $\phi_n\colon A_n\to B_n$ is a homotopy-equivalence for each n then $|f|\colon |A|\to|B|$ is a homotopy-equivalence.*

To state the second result one first associates to a simplicial space A a topological category simp(A). Its objects are pairs (n,a), where $n\geqq 0$ and $a\in A_n$, and its morphisms $(n,a)\to(m,b)$ are morphisms $\theta\colon[n]\to[m]$ in Δ such that $\theta^* b=a$. Thus

$$\mathrm{ob}(\mathrm{simp}(A))=\coprod_{n\geqq 0}A_n,\quad\text{and}\quad \mathrm{mor}(\mathrm{simp}(A))=\coprod_{\theta:[n]\to[m]}A_m.$$

There is a natural map $|\mathrm{simp}(A)|\to|A|$.

Proposition (A.2). *If A is a good simplicial space, $|\mathrm{simp}(A)|\to|A|$ is a homotopy-equivalence.*

In the case occurring in this paper A is the simplicial space arising from an ordered space Q. Then simp(A) is precisely chn(Q), the space of finite chains $q_0\leqq\cdots\leqq q_n$ in Q, ordered by inclusion; and $|\mathrm{simp}(A)|\to|A|$ is induced by the order-reversing map $(q_0\leqq\cdots\leqq q_n)\mapsto q_0$.

I call a monoid, partial monoid, category, ordered space, etc., *good* if it gives rise to a good simplicial space. One needs to know that the condition holds for all the examples arising in the paper. Notice first:

1. A monoid is good if and only if it is locally contractible at 1.

2. A neighbourhood of the identity in a good monoid is a good partial monoid.

3. The edgewise subdivision of a good simplicial space is good.

Now we must consider in turn $C_{n-1}(X)$, $C'_n(X)$, $C''_n(X)$, Q. Because X has a good base-point there is a homotopy h_t: $X \to X$ such that $h_0 = id$, $h_1^{-1}(0) = U$, a neighbourhood of 0. The map h_t: $C_{n-1}(X) \to C_{n-1}(X)$ contracts a neighbourhood of the identity through partial-monoid-homomorphisms, proving $C_{n-1}(X)$ is good. On the other hand $C'_n(U)$ is a contractible neighbourhood of the identity in $C'_n(X)$, so the latter is good. For the same reason $h_1^{-1}\, C''_n(X)$ is good. But this is homotopy-equivalent to $C''_n(X)$ *as partial monoid*, so $C''_n(X)$ is good. Finally Q is good because $Q \to \mathscr{C}(C'_n(X))$ is shrinkable.

References

1. Barratt, M.G., Priddy, S.: On the homology of non-connected monoids and their associated groups. Comment. Math. Helvet. **47**, 1–14 (1972).
2. Boardman, J. M., Vogt, R. M.: Homotopy-everything H-spaces. Bull. Amer. Math. Soc. **74**, 1117–1122 (1968).
3. Dold, A.: Partitions of unity in the theory of fibrations. Ann. of Math. **78**, 223–255 (1963).
4. Giffen, C.: To appear.
5. James, I. M.: Reduced product spaces. Ann. of Math. **62**, 170–197 (1955).
6. May, J.P.: The geometry of iterated loop spaces. Springer Lecture Notes in Mathematics **271**. Berlin-Heidelberg-New York 1972.
7. Milgram, R.J.: Iterated loop spaces. Ann. of Math. **84**, 386–403 (1966).
8. Quillen, D.G.: The group completion of a simplicial monoid. To appear.
9. Segal, G.B.: Classifying spaces and spectral sequences. Publ. Math. I.H.E.S. Paris **34**, 105–112 (1968).
10. Segal, G.B.: Categories and cohomology theories. Topology.

Graeme Segal
Mathematical Institute
University of Oxford
24–29. St. Giles
Oxford OX1 3LB, U.K.

(Received March 15, 1973)