Confinement of vorticity in two dimensional ideal incompressible exterior flow

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Abstract

In [Math. Meth. Appl. Sci. 19 (1996) 53-62], C. Marchioro examined the problem of vorticity confinement in the exterior of a smooth bounded domain. The main result in Marchioro's paper is that solutions of the incompressible 2D Euler equations with compactly supported nonnegative initial vorticity in the exterior of a connected bounded region have vorticity support with diameter growing at most like $\mathcal{O}(t^{(1/2)+\varepsilon})$, for any $\varepsilon > 0$. In addition, if the domain is the exterior of a disk, then the vorticity support is contained in a disk of radius $\mathcal{O}(t^{1/3})$. The purpose of the present article is to refine Marchioro's results. We will prove that, if the initial vorticity is even with respect to the origin, then the exponent for the exterior of the disk may be improved to 1/4. For flows in the exterior of a smooth, connected, bounded domain we prove a confinement estimate with exponent 1/2 (i.e. we remove the ε) and in certain cases, depending on the harmonic part of the flow, we establish a logarithmic improvement over the exponent 1/2. The main new ingredients in our approach are: (1) a detailed asymptotic description of solutions to the exterior Poisson problem near infinity, obtained by the use of Riemann mappings; (2) renormalized energy estimates and bounds on logarithmic moments of vorticity and (3) a new *a priori* estimate on time derivatives of logarithmic perturbations of the moment of inertia.

1   Introduction

Two-dimensional incompressible ideal flow can be described as the active transport of vorticity, see [3]. Vorticity with changing sign may scatter through the divergent motion of soliton-like vortex pairs, see the discussion in [5] and references there contained, but single-signed vorticity tends to rotate around, and spreads very slowly. Studying the rate at which single-signed vorticity spreads is the problem of vorticity confinement.

In 1996, C. Marchioro presented some results concerning vorticity confinement in the case of exterior domain flow, see [11]. Marchioro observed cubic-root confinement in the case of the exterior of a disk, i.e. single-signed, compactly supported, vorticity has its support contained in a space-time region whose diameter grows like the cubic-root of time. This result follows from the proof of similar cubic-root confinement obtained previously

by Marchioro for full plane flow, see [10]. For flows in the exterior of a general connected domain, Marchioro proved $(1/2+\varepsilon)$-root confinement. The purpose of this article is to refine Marchioro's estimates. Our main result is unqualified square-root confinement for exterior flow. We improve this estimate to a logarithmic refinement of square-root confinement under certain conditions on the harmonic part of the flow. In addition, we prove almost fourth-root vorticity confinement in the exterior of a disk if the initial vorticity is even with respect to its center. Technically, we begin with the construction of a conformal map between the exterior of a general domain and the exterior of the unit disk, which behaves nicely up to the boundary, taken from [4]. This conformal map is used to obtain explicit formulas for the Green's function of the exterior domain, the Biot-Savart law and the harmonic part of the velocity. We then prove *a priori* estimates: first, a renormalized energy estimate, next, in some cases, estimates on logarithmic moments of vorticity, and finally an estimate of linear growth in time for logarithmic perturbations of the moment of inertia. Finally, we use these *a priori* estimates to obtain our confinement results.

From a broad viewpoint, the problem of confinement is related with scaling. Roughly speaking, scaling in an evolution equation is determined by the behavior in time of the radius of effective influence of a small localized perturbation. For a parabolic system, the scaling is $x \sim \sqrt{t}$ and for a hyperbolic system it is $x \sim t$. Incompressible ideal flow has interesting behavior at hyperbolic scaling, i.e. waves, but this requires vortex pairs, and therefore vorticity changing sign, see [5, 6, 7] for details. One important issue is whether there is a natural scaling associated with incompressible, ideal 2D flow with distinguished signed vorticity. Confinement estimates explore this issue of scaling, and therefore, are useful in studying the qualitative behavior of solutions. For example, confinement estimates have been used in the rigorous justification of point vortex dynamics as an asymptotic description of the dynamics of highly concentrated vorticity, see [14, 16] and in the results on vortex scattering in [5, 6]. In addition, the issue of confinement has attracted attention in other contexts, such as confinement for slightly viscous flow, see [13], for axisymmetric flow, see [1, 9, 15], for the Vlasov-Poisson system, see [2], and for the quasigeostrophic system, see [12].

The point of departure on vorticity confinement research is the 1994 article [10], by C. Marchioro, in which he proved cubic-root confinement for flows in the full plane. Marchioro used in an essential way the conservation of the moment of inertia, which is associated to the rotational symmetry of full plane flow. This result was improved, independently by Ph. Serfati, [17] and by D. Iftimie, T. Sideris and P. Gamblin, see [7], to nearly fourth root in time. The improvement relied on using, in addition to the moment of inertia, conservation of the center of vorticity, which is associated to the translational symmetry of the plane. However, scaling should be a robust qualitative property, and not dependent on the presence of symmetry. One of the main points of this article is to further explore the role that symmetry has in the problem of confinement. To this end we break symmetry by considering exterior domain flows, both in the exterior of a disk, where only translational symmetry is broken and in more general exterior domains, where translational and rotational symmetry are broken. We begin with the following two questions: (1) Is it possible to use the center of vorticity to improve Marchioro's estimate for the exterior domain in the same way that Serfati, Iftimie, Sideris and Gamblin did in the full-plane case? (2) Given that, without symmetry, there are no conserved quantities, is it still possible to find quantities which remain bounded in time and that could play the role of the moment of inertia for confinement estimates? Perhaps our most important result is negative – we

have not found coercive quantities like the moment of inertia for which we could prove boundedness or slow growth. We proved boundedness for logarithmic moments of vorticity related to conservation of energy and we proved that certain logarithmic perturbations of the moment of inertia grow at most linearly in time. Using these estimates we obtained small improvements over Marchioro's original estimate. Our results support the roughly parabolic natural scaling for incompressible ideal two-dimensional flow with distinguished signed vorticity, obtained originally by Marchioro.

The remainder of this paper is divided into five sections. In Section 2 we set up the problem and we collect preliminary information on Riemann mappings and the Laplacian in an exterior domain. In Section 3 we derive a renormalized energy estimate and we prove that, under appropriate hypothesis', boundedness of a logarithmic moment of vorticity can be deduced from the energy estimate. In Section 4 we obtain a linear-in-time estimate for logarithmic perturbations of quadratic moments of vorticity. In Section 5 we use the estimates deduced in Sections 3 and 4 to obtain new confinement estimates for flows in the exterior of a general connected domain. Finally, in Section 6 we consider flow induced by even vorticity in the exterior of a disk, obtaining nearly fourth-root confinement.

2 Preliminaries on exterior domain flow

The purpose of this section is to collect information on exterior domain flow, particularly the discussion on Riemann mappings and solutions of the Dirichlet and Poisson problems in an exterior domain developed in [4], supplementing the available results as required.

We begin by discussing the asymptotic behavior, near infinity, of solutions to a 2D exterior domain Poisson equation. Let $\Gamma \subseteq \mathbb{R}^2$ be a smooth Jordan curve dividing the plane into a bounded connected component Ω and an unbounded connected component denoted Ω^c. For $x \in \Gamma$, denote by $\hat{n}(x)$ the unit exterior normal to Ω^c at x. Note that both Ω and Ω^c are open. The Green's function for the Laplacian in Ω^c is denoted by G_{Ω^c}. We consider $\omega \in C_c^\infty(\Omega^c)$ and we introduce:

$$\psi(x) \equiv \int_{\Omega^c} G_{\Omega^c}(x,y)\omega(y)dy. \tag{1}$$

Let $S = \{|x| = 1\}$ and $D^c \equiv \{|x| > 1\}$. Denote the inversion with respect to S by $x \mapsto x^* \equiv x/|x|^2$. The Green's function in the case of the exterior of the unit disk can be written explicitly as:

$$G_{D^c}(x,y) = \frac{1}{2\pi} \log \frac{|x-y|}{|x-y^*||y|}.$$

It is easy to obtain detailed information on the asymptotic behavior of ψ and its derivatives, in the case of the exterior of the disk, by means of the representation formula above. The objective of this section is to obtain similar information for general exterior domains. We begin with a version of Lemma 2.1 of [4].

Lemma 1 *There exists a conformal mapping $T : \Omega^c \to D^c$ extending continuously up to the boundary, mapping Γ to S. Furthermore, there exist β, a positive real number, and h, a bounded holomorphic function in Ω^c, such that*

$$T(z) = \beta z + h(z).$$

In addition, there exists a constant $M_1 > 0$ such that $|h'(z)| \leq M_1/|z|^2$, $|h''(z)| \leq M_1/|z|^3$. Furthermore, if T is regarded as a real mapping between Ω^c and $\{|z| > 1\}$, then we have $\|DT\|_{L^\infty} \leq M_1$ and $\|DT^{-1}\|_{L^\infty} \leq M_1$.

Proof: Most of the facts claimed above are either proved in Lemma 2.1 of [4] or in the remark following it. One modification is that β can be assumed positive, which we can obtain by composing T with a rotation, if necessary. The only other difference is the estimate on h'' which follows by differentiating the relation $h'(z) = -(1/z^2)g'(1/z)$, together with the fact that, by construction, g' and g'' are bounded. ∎

It is easy to verify that, if $x_0 \in D^c$ and ϕ satisfies $\Delta\phi = \delta(x - x_0)$ in a neighborhood of x_0, then $\tilde{\phi} \equiv \phi \circ T$ satisfies $\Delta\tilde{\phi} = \delta(y - T^{-1}(x_0))$ in a neighborhood of $T^{-1}(x_0)$. We use this fact on the Green's function G_{D^c} in order to write:

$$G_{\Omega^c}(x, y) = \frac{1}{2\pi} \log \frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|}. \tag{2}$$

We now formulate precisely the initial value problem for incompressible fluid flow in an exterior domain. In this, we follow Section 3.1 of [4]. Let us denote by $u = u(x, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ the velocity of an incompressible, ideal fluid flow in the exterior domain Ω^c. We assume that u is tangent to Γ and $u \to 0$ when $|x| \to \infty$. The evolution of such a flow is governed by the Euler equations. We write the initial-boundary value problem as:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p & \text{in } \Omega^c \times (0, \infty) \\ \text{div } u = 0 & \text{in } \Omega^c \times [0, \infty) \\ u \cdot \hat{n} = 0 & \text{in } \Gamma \times [0, \infty) \\ \lim_{|x| \to \infty} u = 0 & \text{for } t \in [0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega^c, \end{cases} \tag{3}$$

where $p = p(x, t)$ is the scalar pressure. If u_0 is sufficiently smooth, global well-posedness of this problem was proved by K. Kikuchi in [8].

We introduce $\omega = \text{curl } u$, the vorticity of the flow. Vorticity satisfies the transport equation:

$$\omega_t + u \cdot \nabla\omega = 0, \text{ in } \Omega^c \times (0, \infty).$$

Our purpose is to reformulate problem (3) in terms of vorticity. In order to do this we require a version of the Biot-Savart law, which recovers velocity from vorticity. Since Ω^c is not simply connected, we must use Hodge-deRham theory (see [18]). Recall that a *harmonic vector field* in Ω^c is a divergence-free, curl-free vector field tangent to Γ and vanishing at infinity. By Hodge's Theorem, the vector space of harmonic vector fields, in our setting, is one-dimensional, see Section 2.3 in [4]. Therefore, every harmonic vector field is a multiple of a unique harmonic vector field, denoted H_{Ω^c}, defined by requiring that

$$\int_\Gamma H_{\Omega^c} \cdot ds = 1.$$

(Throughout this paper circulation will be computed in the counterclockwise orientation.) We recall identity (2.11) in [4], which gives an explicit expression for H_{Ω^c} in terms of T. We have

$$H_{\Omega^c}(x) = \frac{1}{2\pi} \nabla^\perp \log |T(x)| = \frac{1}{2\pi} \frac{(T(x)DT(x))^\perp}{|T(x)|^2} = \frac{1}{2\pi} \frac{DT^t(x)(T(x))^\perp}{|T(x)|^2} \tag{4}$$

With this notation we can show that there exists $\alpha \in \mathbb{R}$ such that:

$$u = \nabla^{\perp}\psi + \alpha H_{\Omega^c}, \tag{5}$$

where ψ is the stream function introduced in (1). Indeed, by (8), $\nabla^{\perp}\psi$ vanishes at infinity and, since ψ vanishes on Γ, $\nabla^{\perp}\psi$ is tangent to Γ. Clearly $\nabla^{\perp}\psi$ is divergence-free and its curl is ω. Thus $u - \nabla^{\perp}\psi$ is a harmonic vector field, which must then be a real multiple of H_{Ω^c}, (see [4, Proposition 2.1]).

In the language of Hodge theory, the vector field αH_{Ω^c} is called the *harmonic part* of the flow u. In principle this harmonic part is time-dependent, but as a consequence of Kelvin's Circulation Theorem, it is actually a constant of motion.

Lemma 2 *If u is a solution of (3) then α is constant in time.*

This is Lemma 3.1 in [4]. More precisely, let u_0 be such that curl u_0 is compactly supported and set $\omega_0 \equiv$ curl u_0. Then

$$\alpha \equiv \int_{\Gamma} u_0 \cdot ds + \int_{\Omega^c} \omega_0 dx. \tag{6}$$

With this notation the vorticity formulation of the initial-boundary value problem (3) is:

$$\begin{cases} \omega_t + u \cdot \nabla\omega = 0 & \text{in } \Omega^c \times (0, \infty) \\ u = \nabla^{\perp}\psi + \alpha H_{\Omega^c} & \text{in } \Omega^c \times [0, \infty) \\ \psi(x, t) = \int_{\Omega^c} G_{\Omega^c}(x, y)\omega(y, t)dy & \text{in } \Omega^c \times [0, \infty) \\ \omega(x, 0) = \omega_0(x) & \text{in } \Omega^c. \end{cases} \tag{7}$$

The fact that vorticity is transported by a divergence free vector field implies that its L^p-norm is conserved for any $1 \leq p \leq \infty$.

We will also require information on the time-dependent stream function $\psi = \psi(\cdot, t)$, defined by (1) with $\omega = \omega(\cdot, t)$.

Lemma 3 *There exists a constant $C = C(t) > 0$, depending on the diameter of the support of $\omega(\cdot, t)$ such that*

$$\sup_{x \in \Omega^c} |\psi(x, t)| \leq C(t).$$

Proof: We repeat the argument leading to relation (4.10) in [4], substituting ω_t by ω, to obtain

$$\left| \psi(x, t) + \frac{1}{2\pi} \int_{\Omega^c} \log |T(y)|\omega(y, t)\, dy \right| = \mathcal{O}\left(\frac{1}{|x|} \right).$$

Since $\omega(\cdot, t)$ has compact support, the conclusion follows. ∎

In addition, we recall estimates (2.8) and (4.11) of [4]:

$$|\nabla\psi(x, t)| \leq \frac{C(t)}{|x|^2} \qquad \text{and} \qquad |\nabla\psi_t(x, t)| \leq \frac{C(t)}{|x|^2}, \tag{8}$$

where, again, $C(t) > 0$ depends on the diameter of the support of $\omega(\cdot, t)$.

3 Generalized energy and logarithmic moment

The purpose of this section is to derive a new *a priori* bound on the logarithmic moment of vorticity, in terms of global conserved quantities of the flow. Our point of departure is the exact conservation of an energy-like quantity which we will call *generalized energy*. To define this quantity, let us consider $u = u(x,t)$ a smooth solution of problem (3) with compactly supported vorticity. Using (8) and (5) we conclude that:

$$v(x,t) \equiv u(x,t) - \alpha H_{\Omega^c}(x) = (\nabla^\perp \psi)(x,t) = \mathcal{O}(1/|x|^2), \text{ when } |x| \to \infty. \qquad (9)$$

Definition 4 *We define the generalized energy E by:*

$$E \equiv \int_{\Omega^c} (|v|^2 + 2\alpha H_{\Omega^c} \cdot v)dx.$$

The observation on the asymptotic behavior of v as $|x| \to \infty$ allows us to conclude that E is finite. Indeed v is $O(|x|^{-2})$ and from (4) and Lemma 1 the vector field H_{Ω^c} is $O(|x|^{-1})$ so that the integrand above is $O(|x|^{-3})$ which is an integrable function at infinity.

Proposition 5 *The generalized energy E is a constant of motion for smooth flows on Ω^c.*

Proof: We begin this proof by noting that, thanks to (8), for each fixed time $t \geq 0$ we have

$$|u_t(x,t)| = |v_t(x,t)| = |\nabla^\perp \psi_t(x,t)| = \mathcal{O}(1/|x|^2), \text{ as } |x| \to \infty.$$

The construction of H_{Ω^c} given in (4) implies that $|DH_{\Omega^c}| = \mathcal{O}(1/|x|^2)$ as $|x| \to \infty$. Hence, the same argument used above for u_t yields the conclusion that $|Du|(x,t) = \mathcal{O}(1/|x|^2)$ as well. Equation (3) implies that the same conclusion holds for the behavior of $|\nabla p|$ at infinity. By integrating along rays, recalling that p is only defined up to a constant, we can conclude that $p = \mathcal{O}(1/|x|)$ as $|x| \to \infty$.

We use the velocity formulation (3), and Lemma 2 to compute the time derivative of E as follows:

$$\begin{aligned}
\frac{dE}{dt} &= \int_{\Omega^c} (2v \cdot v_t + 2\alpha H_{\Omega^c} \cdot v_t)dx \\
&= 2 \lim_{R \to \infty} \int_{B(0;R)\backslash\Omega} \left[-v(u \cdot \nabla u + \nabla p) - \alpha H_{\Omega^c}(u \cdot \nabla u + \nabla p) \right]dx, \\
&= -2 \lim_{R \to \infty} \left(\int_{B(0;R)\backslash\Omega} u \cdot (u \cdot \nabla u)dx + \int_{B(0;R)\backslash\Omega} u \cdot \nabla p dx \right) \\
&= -2 \lim_{R \to \infty} (\mathcal{I}_1 + \mathcal{I}_2).
\end{aligned}$$

We estimate these integrals:

$$\mathcal{I}_1 = \int_{B(0;R)\backslash\Omega} \text{div} \frac{u|u|^2}{2}dx = \int_{|x|=R} \frac{|u|^2}{2} u \cdot \frac{x}{R}dS + \int_\Gamma \frac{|u|^2}{2} u \cdot \hat{n}dS = \mathcal{O}\left(\frac{1}{R^2}\right),$$

and

$$\mathcal{I}_2 = \int_{B(0;R)\backslash\Omega} \text{div} \, (up)dx = \int_{|x|=R} p(u \cdot \frac{x}{R})dS + \int_\Gamma p(u \cdot \hat{n})dS = \mathcal{O}\left(\frac{1}{R}\right).$$

Thus, $dE/dt = 0$, as we desired. ∎

Instead of estimating the logarithmic moment of vorticity directly we will estimate a quantity that resembles the logarithmic moment, but which is adapted to the geometry of the domain under consideration. Recall that T is the conformal map that takes Ω^c into the exterior of the unit disk. We introduce the *modified logarithmic moment* of vorticity by:

$$L(t) \equiv \frac{1}{2\pi} \int_{\Omega^c} \left(\log |T(y)| \right) \omega(y, t) dy. \tag{10}$$

Throughout the remainder of this paper we will assume that ω_0 is smooth, nonnegative and compactly supported. Since ω is a solution of a transport equation with smooth velocity, it remains compactly supported and nonnegative for positive time. The total mass of vorticity is a conserved quantity and we denote it by

$$m \equiv \int_{\Omega^c} \omega_0(x) dx = \int_{\Omega^c} \omega(x, t) dx.$$

Theorem 6 *If either $\alpha < 0$ or $\alpha > m$ then there exists a constant $M_2 > 0$ such that $L(t) \leq M_2$.*

Proof: We first rewrite the generalized energy in terms of vorticity. By (9) and (4) we have

$$E = \int_{\Omega^c} |v|^2 + 2\alpha H_{\Omega^c} \cdot v dx = \int_{\Omega^c} \nabla^\perp \psi \cdot v dx + \frac{\alpha}{\pi} \int_{\Omega^c} (\nabla^\perp \log |T(x)|) \cdot v dx$$

$$= -\int_{\Omega^c} \psi \omega dx - \frac{\alpha}{\pi} \int_{\Omega^c} \log |T(x)| \omega dx,$$

where the boundary terms vanish since, on Γ, ψ and $\log |T|$ vanish, and at infinity we have, by Lemma 3, that ψ is bounded and, by (9), that v decays like $O(|x|^{-2})$.

Using (1) and (2), we rewrite the energy in the following way:

$$-2\pi E = \iint \log \frac{|T(x) - T(y)|}{|T(x) - T(y)^*||T(y)|} \omega(x, t) \omega(y, t) \, dx dy + 2\alpha \int \log |T(x)| \omega(x, t) dx,$$

where, in the three integrals, the domain of integration is Ω^c. Therefore we have

$$-2\pi E = \iint \log |T(x) - T(y)| \omega(x, t) \omega(y, t) dx dy + 2(\alpha - m) \int \log |T(x)| \omega(x, t) dx +$$

$$+ \iint \log \frac{|T(x)|}{|T(x) - T(y)^*|} \omega(x, t) \omega(y, t) dx dy. \tag{11}$$

We begin by observing that the last integral on the right-hand-side of (11) is bounded independently of time. Indeed, if $z_1, z_2 \in \mathbb{R}^2$ with $|z_2| > 1$ and $|z_1| > 2$ then we have that

$$2|z_1| \geq |z_1| + |z_2^*| \geq |z_1 - z_2^*| \geq |z_1| - |z_2^*| \geq |z_1|/2.$$

Applying this inequality with $z_1 = T(x)$ and $z_2 = T(y)$ yields

$$\left| \iint \log \frac{|T(x)|}{|T(x) - T(y)^*|} \omega(x,t)\omega(y,t)dxdy \right|$$

$$\leq m^2 \log 2 + \left| \int_{1 \leq |T(x)| \leq 2} \int \log \frac{|T(x)|}{|T(x) - T(y)^*|} \omega(x,t)\omega(y,t)dxdy \right|$$

$$\leq 2m^2 \log 2 + \int_{1 \leq |T(x)| \leq 2} \int |\log|T(x) - T(y)^*|| \, \omega(x,t)\omega(y,t)dxdy$$

$$\leq 2m^2 \log 2 + \|\omega\|_{L^\infty} \int \left(\int_{1 \leq |T(x)| \leq 2} |\log|T(x) - T(y)^*|| \, dx \right) \omega(y,t)dy$$

$$\leq 2m^2 \log 2 + m\|\omega\|_{L^\infty} \left(\sup_{1 \leq |\eta| < 2} |\det DT(T^{-1}(\eta))|^{-1} \right) \int_{|\eta| \leq 3} |\log|\eta|| d\eta,$$

where we used the boundedness of derivatives of T and T^{-1} stated in Lemma 1. This establishes the desired time-independent bound.

Next we decompose $\Omega^c \times \Omega^c$ as

$$\Omega^c \times \Omega^c = (\Omega^c \times \Omega^c \cap \{|T(x) - T(y)| \leq 1\}) \cup (\Omega^c \times \Omega^c \cap \{|T(x) - T(y)| > 1\}) \equiv A_1 \cup A_2.$$

An argument similar to the one carried out above implies that

$$\left| \iint_{A_1} \log|T(x) - T(y)|\omega(x,t)\omega(y,t)dxdy \right|$$

$$\leq m\|\omega\|_{L^\infty} \sup_{|\eta| > 1} |\det DT(T^{-1}(\eta))|^{-1} \int_{|\eta| \leq 1} |\log(|\eta|)|d\eta.$$

These estimates, together with (11) and conservation of energy imply that

$$\mathcal{I}(t) \equiv \iint_{A_2} \log|T(x) - T(y)|\omega(x,t)\omega(y,t)dxdy + 2(\alpha - m) \int \log|T(x)|\omega(x,t)dx \quad (12)$$

is bounded independently of time. Now, if $\alpha > m$ both terms are nonnegative and the logarithmic moment bound follows immediately.

Lastly, we treat the case $\alpha < 0$. Since $|T(x)|, |T(y)| \geq 1$ it follows that

$$\log|T(x) - T(y)| \leq \log|T(x)| + \log|T(y)| + \log 2.$$

Therefore,

$$\left| \iint_{A_2} \log|T(x) - T(y)|\omega(x,t)\omega(y,t)dxdy \right| \leq m^2 \log 2 + 2m \int \log|T(x)|\omega(x,t)dx,$$

which, together with the boundedness of $\mathcal{I}$ concludes the proof. ∎

We conclude this section with an estimate which applies to the extreme case $\alpha = 0$. We do not have a logarithmic moment bound in this case, but we can prove another estimate which we will be able to use in place of a logarithmic moment bound in the analysis that follows.

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Lemma 7 *Assume that $\alpha = 0$. Then there exists a constant $M_3 > 0$ such that*

$$\iint_{\Omega^c \times \Omega^c} \log\left(\frac{|T(x)||T(y)|}{|T(x) - T(y)|}\right) \omega(x,t)\omega(y,t)dxdy \leq M_3.$$

Proof: First recall the definition of $\mathcal{I}(t)$ given in (12). Next observe that, if $\alpha = 0$, then the integral we wish to estimate is equal to

$$-\mathcal{I}(t) - \iint_{A_1} \log|T(x) - T(y)|\omega(x,t)\omega(y,t)dxdy,$$

in this case. Finally, the proof of Theorem 6 shows that both of these terms are bounded independently of the choice of α. $\blacksquare$

Remark 8 *If either $\alpha \leq 0$ or $\alpha > m$, then there exists a constant $M_4 > 0$ such that*

$$\iint_{\Omega^c \times \Omega^c} \left[\log\min(|T(x)|, |T(y)|)\right] \omega(x,t)\omega(y,t)dxdy \leq M_4. \tag{13}$$

Indeed, if $\alpha < 0$ or $\alpha > m$ then the relation above is an immediate consequence of Theorem 6. If $\alpha = 0$, then (13) follows from Lemma 7 after we observe that

$$\log\left(\frac{|T(x)||T(y)|}{|T(x) - T(y)|}\right) \geq \log\left(\frac{\min(|T(x)|, |T(y)|)}{2}\right).$$

4 Other moment bounds

The purpose of this section is to derive bounds on logarithmic perturbations of the moment of inertia, which will later be used to improve Marchioro's confinement estimate. We begin with a technical lemma regarding the conformal map T.

Lemma 9 *Let $x, y \in \Omega^c$. We have*

$$|T(x) \cdot (T(y))^\perp| \leq \min\{|T(x)|, |T(y)|\}|T(x) - T(y)|, \tag{14}$$

and there exists $M_5 > 0$ such that

$$||T'(x)|^2 - |T'(y)|^2| \leq M_5 \frac{|T(x) - T(y)|}{\min\{|T(x)|, |T(y)|\}^2} \tag{15}$$

Proof: Let us first prove (14). We observe that

$$|T(x) \cdot (T(y))^\perp| = |(T(x) - T(y)) \cdot (T(y))^\perp| \leq |T(x) - T(y)||T(y)|.$$

Similarly,

$$|T(x) \cdot (T(y))^\perp| = |T(x) \cdot (T(x) - T(y))^\perp| \leq |T(x) - T(y)||T(x)|.$$

Hence, (14) follows from both these inequalities.

Next we prove (15). We use Lemma 1 to compute

$$
\begin{aligned}
||T'(x)|^2 - |T'(y)|^2| &= |2\beta \mathrm{Re}(h'(x) - h'(y)) + |h'(x)|^2 - |h'(y)|^2| \\
&\leq C|h'(x) - h'(y)| + ||h'(x)|^2 - |h'(y)|^2| \\
&\leq C|h'(x) - h'(y)|(1 + |h'(x)| + |h'(y)|) \\
&\leq C|h'(x) - h'(y)|,
\end{aligned}
$$

since, by Lemma 1, $|h'|$ is bounded.

Therefore, to conclude our proof it is enough to show that $|h'(x) - h'(y)|$ is bounded by the right-hand-side of (15).

We first remark that a uniform bound on the first derivatives of a map defined on Ω^c implies a global Lipschitz bound. Next note that, since, by Lemma 1, h'' is bounded, the argument above implies that h' is globally Lipschitz and therefore $|h'(x) - h'(y)| \leq C|x - y|$. Furthermore, $|x - y| \leq C|T(x) - T(y)|$, with C being the Lipschitz constant associated to T^{-1}, and using again the global bound on the derivatives of T^{-1}. Therefore, if either x or y is contained in a ball of radius R, then

$$
\frac{|h'(x) - h'(y)|}{|T(x) - T(y)|} \leq C \leq C(R)\frac{1}{\min\{|T(x)|, |T(y)|\}^2}.
$$

It remains only to prove this inequality assuming both $|x| > R$ and $|y| > R$. In this case, there exists a smooth path $\gamma : [0, 1] \to \{|z| \geq R\}$, whose length is less than $2|x - y|$, and such that $\gamma(0) = x$, $\gamma(1) = y$. Suppose first that $|x| \leq |y|$. We can also assume that $\gamma(s) \in \{z \mid |x| \leq |z| \leq |y|\}$ for all $s \in [0, 1]$. We then have:

$$
\begin{aligned}
|h'(y) - h'(x)| = \left|\int_0^1 \frac{d}{dt} h'(\gamma(t))dt\right| &\leq \int_0^1 |h''(\gamma(t))||\gamma'(t)|dt \\
&\leq \frac{C}{|x|^2} \mathrm{Length}(\gamma) \leq \frac{C|x - y|}{(\min|x|, |y|)^2} \leq \frac{C|T(x) - T(y)|}{(\min|x|, |y|)^2},
\end{aligned}
$$

where we have used, in the second inequality, the decay estimate for $|h''|$ from Lemma 1.

A similar argument holds for $|y| \leq |x|$. This completes the proof. ■

The final result in this section is an estimate on the growth of logarithmic perturbations of the moment of inertia.

Theorem 10 *There exists a constant $M_6 > 0$ such that*

$$
\int_{\Omega^c} |T(x)|^2 \left(\log|T(x)|\right)\omega(x, t)dx \leq M_6(1 + t),
$$

for all $t \geq 0$. Moreover, if either $\alpha \leq 0$ or $\alpha > m$ then

$$
\int_{\Omega^c} |T(x)|^2 \left(\log^2|T(x)|\right)\omega(x, t)dx \leq M_6(1 + t),
$$

for all $t \geq 0$.

Proof: Let $\sigma : (1, \infty) \to \mathbb{R}$ be a smooth function and define

$$
\mathcal{J}_\sigma(t) \equiv \int_{\Omega^c} \sigma(|T(x)|^2)\omega(x, t)dx.
$$

Clearly it is enough to prove that, for the appropriate choice of σ (either $\sigma(s) = s \log s$ or $\sigma(s) = s \log^2 s$), the time-derivative of $\mathcal{J}_\sigma$ is bounded independently of time. We estimate directly

$$
\begin{aligned}
\mathcal{J}'_\sigma(t) &= -\int_{\Omega^c} \sigma(|T(x)|^2) \, \mathrm{div}(u\omega)(x,t)dx \\
&= \int_{\Omega^c} \nabla(\sigma(|T(x)|^2)) \cdot u(x,t)\omega(x,t)dx \\
&= \int_{\Omega^c} \nabla(\sigma(|T(x)|^2)) \cdot v(x,t)\omega(x,t)dx + \alpha \int_{\Omega^c} \nabla(\sigma(|T(x)|^2)) \cdot H_{\Omega^c}(x)\omega(x,t)dx \\
&\equiv I_1 + I_2.
\end{aligned}
$$

We begin by observing that $I_2 = 0$. Indeed,

$$
H_{\Omega^c}(x) = \frac{1}{2\pi} \nabla^\perp \log(|T(x)|) = \frac{1}{4\pi} \nabla^\perp \log |T(x)|^2.
$$

Clearly, $\sigma(|(T(x)|^2)$ and $\log(|T(x)|^2)$ are functionally dependent, which implies that their gradients are proportional everywhere. Therefore, the integrand in I_2 vanishes identically.

Next, we estimate $|I_1|$. We use the explicit expression of the Biot-Savart kernel in [4], equation (2.5) to write

$$
v(x,t) = \int_{\Omega^c} \left(\frac{[(T(x) - T(y))DT(x)]^\perp}{2\pi|T(x) - T(y)|^2} - \frac{[(T(x) - T(y)^*)DT(x)]^\perp}{2\pi|T(x) - T(y)^*|^2} \right) \omega(y,t)dy.
$$

We write $I_1 \equiv I_{11} - I_{12}$ with

$$
I_{11} \equiv \iint_{\Omega^c \times \Omega^c} \nabla(\sigma(|T(x)|^2)) \cdot \frac{[(T(x) - T(y))DT(x)]^\perp}{2\pi|T(x) - T(y)|^2} \omega(x,t)\omega(y,t)dxdy,
$$

and

$$
I_{12} \equiv \iint_{\Omega^c \times \Omega^c} \nabla(\sigma(|T(x)|^2)) \cdot \frac{[(T(x) - T(y)^*)DT(x)]^\perp}{2\pi|T(x) - T(y)^*|^2} \omega(x,t)\omega(y,t)dxdy,
$$

We estimate I_{12} first. We further decompose

$$
I_{12} = \int_{1 \le |T(x)| \le 2} \int_{\Omega^c} \cdots dydx + \int_{2 \le |T(x)|} \int_{\Omega^c} \cdots dydx \equiv I_{121} + I_{122}.
$$

We have

$$
|I_{121}| \le Cm \sup_{|\eta| \le 1} \left(\int_{1 \le |T(x)| \le 2} \frac{\omega(x,t)}{|T(x) - \eta|} dx \right) \le Cm\|\omega\|_{L^\infty} \sup_{|\eta| \le 1} \left(\int_{1 \le |\zeta| \le 2} \frac{d\zeta}{|\zeta - \eta|} \right)
$$

$$
\le Cm\|\omega\|_{L^\infty} \int_{|\zeta| \le 3} \frac{d\zeta}{|\zeta|} \le Cm\|\omega\|_{L^\infty}.
$$

Next we treat I_{122}. Suppose first that $\sigma(s) = s \log^2 s$. Then

$$
\begin{aligned}
\nabla(\sigma(|T(x)|^2)) &= 2\sigma'(|T(x)|^2)T(x)DT(x) \\
&= 4[\log^2(|T(x)|) + \log(|T(x)|)]T(x)DT(x) \equiv [o(|T(x)|)]T(x)DT(x).
\end{aligned}
$$

Hence, using that $T(x) = \beta x + \mathcal{O}(1)$ and $DT(x) = \beta Id + \mathcal{O}(1/|x|^2)$ (see Lemma 1), we obtain, when $|T(x)| > 2$, that:

$$|I_{122}| = \left| C \int_{|T(x)| \geq 2} \int_{\Omega^c} [o(|T(x)|)] T(x) DT(x) \cdot \frac{[(T(y)^*)DT(x)]^\perp}{2\pi |T(x) - T(y)^*|^2} \omega(x,t)\omega(y,t)dxdy \right|$$

$$\leq m^2 \sup_{\{|T(x)| \geq 2\}} \frac{o(|T(x)|)}{|T(x)|},$$

which shows that $|I_{122}|$ is bounded independently of time.

A similar argument may be used in case $\sigma(s) = s \log s$.

Finally, we must estimate I_{11}. First we observe that if we consider the holomorphic map T as a real map from $\mathbb{R}^2$ to itself, we have that $(vDT)^\perp = v^\perp DT^t$. We also note that, by the Cauchy-Riemann relations, the matrix DT has the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

and therefore, $DT(x)DT^t(x) = \det(DT(x))Id = |T'(x)|^2 Id$. Consequently,

$$\nabla(\sigma(|T(x)|^2)) \cdot [(T(x) - T(y))DT(x)]^\perp$$
$$= 2\sigma'(|T(x)|^2)[T(x)DT(x)] \cdot [(T(x) - T(y))^\perp DT(x)]$$
$$= 2\sigma'(|T(x)|^2)[T(x)DT(x)DT^t(x)] \cdot (T(x) - T(y))^\perp$$
$$= -2\sigma'(|T(x)|^2)|T'(x)|^2 T(x) \cdot T(y)^\perp$$

Plugging this expression in the definition of I_{11} we get

$$I_{11} = -\iint \sigma'(|T(x)|^2)|T'(x)|^2 \frac{T(x) \cdot T(y)^\perp}{\pi |T(x) - T(y)|^2} \omega(x,t)\omega(y,t)dxdy$$
$$= \iint [\sigma'(|T(y)|^2)|T'(y)|^2 - \sigma'(|T(x)|^2)|T'(x)|^2] \frac{T(x) \cdot T(y)^\perp}{2\pi |T(x) - T(y)|^2} \omega(x,t)\omega(y,t)dxdy$$
$$\equiv \iint A(x,y)\omega(x,t)\omega(y,t)dxdy.$$

To obtain the second line in the previous relation, we observed that

$$T(x) \cdot T(y)^\perp = -T(y) \cdot T(x)^\perp$$

and we symmetrized, by adding half of the integrand with half of the same expression but with x and y interchanged.

We see that, to conclude the proof, it is sufficient to prove that if $\sigma'(s) = 1 + \log s$, then $|A|$ is bounded by a constant and, in view of (13), if $\sigma'(s) = \log^2 s + 2\log s$, then $|A|$ is bounded by $C + C \log \min(|T(x)|, |T(y)|)$. Observe next that, since $A(x,y)$ is symmetric in

x and y, we can assume for example that $|T(y)| \le |T(x)|$. Using Lemma 9 we first bound

$$2\pi|A(x,y)| = \left| \left\{ \left[\sigma'(|T(x)|^2) - \sigma'(|T(y)|^2) \right] |T'(x)|^2 \right. \right.$$

$$\left. \left. + \sigma'(|T(y)|^2) \left[|T'(x)|^2 - |T'(y)|^2 \right] \right\} \frac{T(x) \cdot T(y)^\perp}{|T(x) - T(y)|^2} \right|$$

$$\le \left| \sigma'(|T(x)|^2) - \sigma'(|T(y)|^2) \right| \|DT\|_{L^\infty}^2 \frac{|T(y)|}{|T(x) - T(y)|} + \sigma'(|T(y)|^2) \frac{M_5}{|T(y)|}$$

$$\le C \left| \sigma'(|T(x)|^2) - \sigma'(|T(y)|^2) \right| \frac{|T(y)|}{|T(x) - T(y)|} + C,$$

where the last inequality holds since, for both choices $\sigma'(s) = 1 + \log s$ and $\sigma'(s) = \log^2 s + 2\log s$, we have $|\sigma'(T(y))| = o(|T(y)|)$.

Let us now consider the case $\sigma'(s) = 1 + \log s$. We will use the elementary inequality

$$|\log a - \log b| \le \frac{|a - b|}{\min(a,b)}, \quad a, b > 0, \tag{16}$$

to observe that

$$\left| \sigma'(|T(x)|^2) - \sigma'(|T(y)|^2) \right| \frac{|T(y)|}{|T(x) - T(y)|} \le 2$$

which implies that A is bounded.

Finally, if $\sigma'(s) = \log^2 s + 2\log s$, we use the inequality

$$|\log^2 a + 2\log a - \log^2 b - 2\log b| \le 2|a - b| \frac{[1 + \log \min(a,b)]}{\min(a,b)} \tag{17}$$

together with (16) to deduce that

$$\left| \sigma'(|T(x)|^2) - \sigma'(|T(y)|^2) \right| \frac{|T(y)|}{|T(x) - T(y)|} \le 12 + 8\log|T(y)| \le C[1 + \log\min(|T(x)|, |T(y)|)].$$

The proof is completed. ∎

5   Dispersion of Vorticity

In this section, we use the bounds on the logarithmic moments of inertia obtained in Section 4 to deduce some confinements results for the support of vorticity. We start with a useful estimate.

Lemma 11 *Let $S \subset \Omega^c$ and $\zeta : S \to \mathbb{R}^+$ be a function belonging to $L^1(S) \cap L^\infty(S)$. There exists a constant $C > 0$ such that*

$$\int_S \frac{\zeta(y)}{|x - T(y)|} \, dy \le C\|\zeta\|_{L^1(S)}^{1/2} \|\zeta\|_{L^\infty(S)}^{1/2} \qquad \forall x \in \mathbb{R}^2$$

and

$$\int_S \frac{\zeta(y)}{|x - T(y)^*|} \, dy \le C\|\zeta\|_{L^1(S)}^{1/2} \|\zeta\|_{L^\infty(S)}^{1/2} + C\|\zeta\|_{L^1(S)} \qquad \forall |x| > 1.$$

Proof: The case $T = Id$ of the first relation is [7, Lemma 2.1]. The general case follows from this particular case in the following manner:

$$\int_S \frac{\zeta(y)}{|x - T(y)|} \, dy \leq C \int_{T(S)} \frac{1}{|x - z|} \zeta(T^{-1}(z)) dz$$

$$\leq C \|\zeta\|_{L^\infty(S)}^{\frac{1}{2}} \left(\int_{T(S)} \zeta(T^{-1}(z)) dz \right)^{\frac{1}{2}}$$

$$\leq C \|\zeta\|_{L^\infty(S)}^{\frac{1}{2}} \left(\int_S \zeta(y) dy \right)^{\frac{1}{2}}.$$

We used above that the Jacobians of T and T^{-1} lie between two positive constants to deduce the first and third lines; we also used the particular case $T = Id$ to obtain the second line.

The second inequality follows from the first one after we observe that if $|x| \geq 2$ or $|T(y)| \geq 2$ then $|x - T(y)^*| \geq \frac{1}{2}$, while if $|x| \leq 2$ and $|T(y)| \leq 2$ then $|x - T(y)^*| = |x||x^* - T(y)|/|T(y)| \geq |x^* - T(y)|/2$. ∎

We prove now the following theorem.

Theorem 12 *Assume that the initial vorticity ω_0 is nonnegative, bounded, compactly supported and let $\omega(x, t)$ be the solution of (7). There exists a constant $M_7 > 0$ such that the support of $\omega(x, t)$ is confined in the set $|x| \leq M_7(1 + t)^{\frac{1}{2}}$. Furthermore, if either $\alpha \leq 0$ or $\alpha > m$ then the support of $\omega(x, t)$ is confined in the set $|x| \leq M_7(1 + t)^{\frac{1}{2}} [\log(2 + t)]^{-\frac{1}{4}}$.*

Proof: We introduce the following parameter θ: we set $\theta = 2$ if either $\alpha \leq 0$ or $\alpha > m$ and $\theta = 1$ otherwise. According to Theorem 10, there exists a constant $M_6 > 0$ such that

$$\int_{\Omega^c} |T(x)|^2 \log^\theta |T(x)| \omega(x, t) \, dx \leq M_6(1 + t) \tag{18}$$

for all t. The conclusion of Theorem 12 is tantamount to proving that there exists a constant $M_7 > 0$ such that the support of $\omega(x, t)$ is confined in the set $|x| \leq M_7(1+t)^{\frac{1}{2}} [\log(2+t)]^{\frac{1-\theta}{4}}$.

For notational convenience we assume without loss of generality that $t \geq 2$. We will repeatedly use in the sequel that, for $a > 0$ and $b \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\int_2^t s^a (\log s)^b \, ds \leq C t^{a+1} (\log t)^b. \tag{19}$$

It suffices to prove that there exists a sufficiently large constant C_1 such that if $|x| \geq C_1 t^{\frac{1}{2}} (\log t)^{\frac{1-\theta}{4}}$, then

$$\frac{x}{|x|} \cdot u(x, t) \leq C_1 t^{\frac{1}{2}} (\log t)^{-\frac{\theta}{2}} |x|^{-2}.$$

This will imply that a fluid particle cannot escape the region $|x| \leq M_7 t^{\frac{1}{2}} (\log t)^{\frac{1-\theta}{4}}$ for a sufficiently large constant M_7. To see this we reason by contradiction, following the trajectory of a fluid particle in a region like $|x| \geq C_1 t^{\frac{1}{2}} (\log t)^{\frac{1-\theta}{4}}$; we note that $\frac{x}{|x|} \cdot u$ is the radial velocity of said particle and we integrate the resulting ODE using (19) to deduce that the trajectory will remain in the region $|x| \leq C_2 t^{\frac{1}{2}} (\log t)^{-\frac{\theta}{6}}$ for some $C_2 > 0$. The conclusion follows since $C_2 t^{\frac{1}{2}} (\log t)^{-\frac{\theta}{6}} \leq M_7 t^{\frac{1}{2}} (\log t)^{\frac{1-\theta}{4}}$ for some $M_7 > 0$.

Let $\widetilde{\omega}(x,t) = \omega(x,t)\left(\int_{\Omega^c} \omega(x,t)\,dx\right)^{-1}$. Since the function $[1,\infty) \ni s \mapsto \sigma(s) = s^2 \log^\theta s$ is convex and the integral of vorticity is a conserved quantity, we deduce from the Jensen inequality that

$$\sigma\left(\int_{\Omega^c} |T(x)|\widetilde{\omega}(x,t)\,dx\right) \leq \int_{\Omega^c} |T(x)|^2 \log^\theta |T(x)|\widetilde{\omega}(x,t)\,dx \leq M_6(1+t)\|\omega_0\|_{L^1}^{-1}.$$

Next, one easily checks that $\sigma^{-1}(\rho) \sim 2^{\frac{\theta}{2}} \rho^{\frac{1}{2}} (\log \rho)^{-\frac{\theta}{2}}$ as $\rho \to \infty$. Therefore, there exists a constant C_3 such that

$$\int_{\Omega^c} |T(x)|\omega(x,t)\,dx \leq C_3 t^{\frac{1}{2}} (\log t)^{-\frac{\theta}{2}}. \tag{20}$$

We assume now that $|x| \geq C_1 t^{\frac{1}{2}} (\log t)^{\frac{1-\theta}{4}}$ for a sufficiently large constant C_1 to be determined later. We decompose $u = v + \alpha H_{\Omega^c}$ where

$$\begin{aligned}
v(x,t) &= \int_{\Omega^c} \left(\frac{[(T(x)-T(y))DT(x)]^\perp}{2\pi|T(x)-T(y)|^2} - \frac{[(T(x)-T(y)^*)DT(x)]^\perp}{2\pi|T(x)-T(y)^*|^2}\right)\omega(y,t)dy \\
&\equiv \int_{\Omega^c} K(x,y)\omega(y,t)dy,
\end{aligned}$$

where $K(x,y)$ is the kernel from the first integral in the formula above. We first estimate v. From the identity $\left|\frac{a}{|a|^2} - \frac{b}{|b|^2}\right| = \frac{|a-b|}{|a||b|}$, we deduce that

$$|K(x,y)| \leq \|DT\|_{L^\infty} \frac{|T(y)-T(y)^*|}{2\pi|T(x)-T(y)||T(x)-T(y)^*|} \leq \|DT\|_{L^\infty} \frac{2|T(y)|}{\pi|T(x)-T(y)||T(x)|}.$$

We used above that the constant C_1 may be chosen sufficiently large so as to have $|T(x)| > 2 > 2|T(y)^*|$. We infer that

$$\begin{aligned}
|v(x,t)| &\leq \frac{C}{|T(x)|} \int_{\Omega^c} \frac{|T(y)|}{|T(x)-T(y)|}\omega(y,t)dy \\
&= \frac{C}{|T(x)|} \int_{|T(x)-T(y)|>|T(x)|/2} \frac{|T(y)|}{|T(x)-T(y)|}\omega(y,t)dy \\
&\qquad\qquad + \frac{C}{|T(x)|} \int_{|T(x)-T(y)|<|T(x)|/2} \frac{|T(y)|}{|T(x)-T(y)|}\omega(y,t)dy \\
&\equiv I_1 + I_2.
\end{aligned}$$

Next, by (20) we have that

$$I_1 \leq \frac{C}{|T(x)|^2} \int_{\Omega^c} |T(y)|\omega(y,t)dy \leq Ct^{\frac{1}{2}} (\log t)^{-\frac{\theta}{2}}|T(x)|^{-2}$$

and from Lemma 11 we deduce that

$$I_2 \leq C\left(\int_{|T(y)|>|T(x)|/2} \omega(y,t)dy\right)^{\frac{1}{2}}.$$

Next we estimate the radial component of αH_{Ω^c}. From the explicit formula for the harmonic vector field H_{Ω^c} given in relation (4) and from Lemma 1, we infer that

$$\frac{x}{|x|} \cdot H_{\Omega^c}(x) = \frac{x}{|x|} \cdot \left(\frac{x^\perp}{2\pi|x|^2} + O(|x|^{-2})\right) = O(|x|^{-2}).$$

According to Proposition 13 below, if the constant C_1 is large enough then the term I_2 is also $O(|x|^{-2})$, so we finally deduce that

$$\frac{x}{|x|} \cdot u(x,t) \le |v(x,t)| + \alpha \frac{x}{|x|} \cdot H_{\Omega^c}(x) \le Ct^{\frac{1}{2}}(\log t)^{-\frac{\theta}{2}}|x|^{-2},$$

with $C = C_1$.

This completes the proof of Theorem 12 once Proposition 13 below is proved. ∎

Proposition 13 *Under the hypothesis of Theorem 12 we have that, for all $k > 0$ there exists a constant $M_8 > 0$ such that*

$$\int_{|T(x)|>r} \omega(x,t)\,dx \le \frac{M_8}{r^k}$$

for all $r > M_8 t^{\frac{1}{2}}(\log t)^{\frac{1-\theta}{4}}$.

Proof: As before we assume $t \ge 2$.

Let us introduce the following approximation for the mass of vorticity in the region of interest:

$$f_r(t) = \int_{\Omega^c} \eta\Big(\frac{|T(x)|^2 - r^2}{\lambda r^2}\Big)\omega(x,t)\,dx,$$

where $\lambda = \lambda(r) \in (0,1)$ is to be chosen later and $\eta(s) = \frac{e^s}{1+e^s}$ verifies that

$$|\eta'(s)| \le \min\{\eta(s), e^{-|s|}\}, \qquad |\eta''(s)| \le \eta(s). \tag{21}$$

To simplify notation, we set $\eta_r(x) = \eta\Big(\frac{|T(x)|^2 - r^2}{\lambda r^2}\Big)$ and $\eta_r'(x) = \eta'\Big(\frac{|T(x)|^2 - r^2}{\lambda r^2}\Big)$. Clearly, it suffices to prove that $f_r(t) \le r^{-k}$ for $r > M_8 t^{\frac{1}{2}}(\log t)^{\frac{1-\theta}{4}}$ and M_8 sufficiently large.

We differentiate f_r to obtain

$$f_r'(t) = \int_{\Omega^c} \eta_r(x)\partial_t\omega = -\int_{\Omega^c} \eta_r(x)u \cdot \nabla\omega = \int_{\Omega^c} u \cdot \nabla\eta_r \omega$$

$$= \iint_{\Omega^c \times \Omega^c} \nabla\eta_r(x) \cdot K(x,y)\omega(x,t)\omega(y,t)\,dx\,dy.$$

We used above that $\nabla\eta_r \cdot H_{\Omega^c} = 0$ pointwise; this follows after noticing that the first term is proportional to $\nabla|T|$ while the second one is proportional to $\nabla^{\perp}|T|$. Using the explicit formula for $K(x,y)$ we finally obtain that

$$\pi f_r'(t) = \frac{1}{\lambda r^2} \iint_{\Omega^c \times \Omega^c} \eta_r'(x)[T(x)DT(x)] \cdot \big[\widetilde{K}(x,y)DT(x)\big]^{\perp}\omega(x,t)\omega(y,t)\,dx\,dy,$$

where

$$\widetilde{K}(x,y) = \frac{T(x) - T(y)}{|T(x) - T(y)|^2} - \frac{T(x) - T(y)^*}{|T(x) - T(y)^*|^2}.$$

From Lemma 1, one can deduce that $DT = \beta Id + O\Big(\frac{1}{|T(x)|^2}\Big)$. On the other hand, Lemma 11 implies that $\int_{\Omega^c} |\widetilde{K}(x,y)|\omega(y,t)\,dy \le C$. We infer that

$$\pi f_r'(t) - \frac{\beta^2}{\lambda r^2} \iint_{\Omega^c \times \Omega^c} \eta_r'(x)T(x) \cdot \widetilde{K}(x,y)^{\perp}\omega(x,t)\omega(y,t)\,dx\,dy \le \frac{C}{\lambda r^2} \int_{\Omega^c} \frac{\eta_r'(x)}{|T(x)|}\omega(x,t)\,dx$$

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Using (21), for $|T(x)| < r/\sqrt{2}$ we bound $\frac{\eta_r'(x)}{|T(x)|} \le e^{-\frac{1}{2\lambda}}$ while for $|T(x)| \ge r/\sqrt{2}$ we bound $\frac{\eta_r'(x)}{|T(x)|} \le \eta_r(x)\sqrt{2}/r$. This implies that

$$\pi f_r'(t) \le J + \frac{C}{\lambda r^3} f_r(t) + \frac{C}{\lambda r^2} e^{-\frac{1}{2\lambda}}, \tag{22}$$

where

$$J = \frac{\beta^2}{\lambda r^2} \iint_{\Omega^c \times \Omega^c} \eta_r'(x) \frac{T(x) \cdot [T(x) - T(y)]^\perp}{|T(x) - T(y)|^2} \omega(x,t)\omega(y,t)\, dx\, dy$$
$$- \frac{\beta^2}{\lambda r^2} \iint_{\Omega^c \times \Omega^c} \eta_r'(x) \frac{T(x) \cdot [T(x) - T(y)^*]^\perp}{|T(x) - T(y)^*|^2} \omega(x,t)\omega(y,t)\, dx\, dy \tag{23}$$
$$\equiv J_1 + J_2.$$

We bound first J_2. Clearly we may assume, without loss of generality, that $r^2 > 8$. We decompose the integral in J_2 into an integral over $\{|T(x)| \le 2\}$ and an integral over $\{|T(x)| \ge 2\}$ and we write $J_2 = J_{21} + J_{22}$ for each portion.

Now, if $|T(x)| \le 2 < r/\sqrt{2}$ we estimate

$$\left| \eta_r'(x) \frac{T(x) \cdot [T(x) - T(y)^*]^\perp}{|T(x) - T(y)^*|^2} \right| = \left| \eta_r'(x) \frac{T(y)^* \cdot [T(x) - T(y)^*]^\perp}{|T(x) - T(y)^*|^2} \right| \le \frac{e^{-\frac{1}{2\lambda}}}{|T(x) - T(y)^*|}$$

Hence, by Lemma 11, we have

$$|J_{21}| \le C \frac{\beta^2}{\lambda r^2} e^{-\frac{1}{2\lambda}}.$$

Next, if $|T(x)| > 2$ then, since $|T(y)^*| < 1$, we find

$$\frac{\left| T(x) \cdot [T(x) - T(y)^*]^\perp \right|}{|T(x) - T(y)^*|^2} = \frac{\left| T(x) \cdot [T(y)^*]^\perp \right|}{|T(x) - T(y)^*|^2} \le \frac{4}{|T(x)|}.$$

Hence, after further decomposing the integral in J_{22}, we repeat the argument used to obtain (22) to find

$$|J_{22}| \le \frac{C}{\lambda r^2} e^{-\frac{1}{2\lambda}} + \frac{C}{\lambda r^3} f_r(t).$$

Putting these two estimates together we infer that

$$|J_2| \le \frac{C}{\lambda r^3} f_r(t) + \frac{C}{\lambda r^2} e^{-\frac{1}{2\lambda}}. \tag{24}$$

We now turn to the estimate of J_1. Using the change of variables $(x,y) \leftrightarrow (y,x)$ we next write

$$J_1 = \frac{\beta^2}{2\lambda r^2} \iint_{\Omega^c \times \Omega^c} L(x,y)\omega(x,t)\omega(y,t)\, dx\, dy \tag{25}$$

where

$$L(x,y) = [\eta_r'(x) - \eta_r'(y)] \frac{T(x) \cdot [T(x) - T(y)]^\perp}{|T(x) - T(y)|^2}$$

We divide the domain of integration $\Omega^c \times \Omega^c$ in three (overlapping) pieces: $A = \{||T(x)| - |T(y)|| \ge r/4\}$, $B = \{|T(x)| \text{ and } |T(y)| \notin (r/2, 3r/2)\}$ and $C = \{|T(x)| \text{ and } |T(y)| \in [r/4, 5r/4]\}$. Clearly $\Omega^c \times \Omega^c = A \cup B \cup C$.

For $(x, y) \in A$ we use (14) and (21) to bound

$$|L(x, y)| \leq [\eta'_r(x) + \eta'_r(y)] \min[|T(x)|, |T(y)|]\frac{4}{r} \leq [\eta_r(x)|T(y)| + \eta_r(y)|T(x)|]\frac{4}{r},$$

so that, in view of (20)

$$\frac{\beta^2}{2\lambda r^2} \iint_A |L(x, y)|\omega(x, t)\omega(y, t) \, dx \, dy \leq C\frac{f_r}{\lambda r^3}t^{\frac{1}{2}}(\log t)^{-\frac{\theta}{2}}. \tag{26}$$

In the region B, as a consequence of (21) both $\eta'_r(x)$ and $\eta'_r(y)$ are less than $e^{-\frac{3}{4\lambda}}$. Therefore, by Lemma 11

$$\frac{\beta^2}{2\lambda r^2} \iint_B |L(x, y)|\omega(x, t)\omega(y, t) \, dx \, dy$$

$$\leq C\frac{e^{-\frac{3}{4\lambda}}}{\lambda r^2} \int_{\Omega^c} |T(x)|\left(\int_{\Omega^c} \frac{1}{|T(x) - T(y)|}\omega(y, t) \, dy\right)\omega(x, t) \, dx$$

$$\leq C\frac{e^{-\frac{3}{4\lambda}}}{\lambda r^2} \int_{\Omega^c} |T(x)|\omega(x, t) \, dx \tag{27}$$

$$\leq C\frac{e^{-\frac{3}{4\lambda}}}{\lambda r^2}t^{\frac{1}{2}}(\log t)^{-\frac{\theta}{2}}.$$

Next, by the mean value theorem, relation (21) and since η is increasing, one has that

$$|\eta'_r(x) - \eta'_r(y)| = \frac{\left||T(x)|^2 - |T(y)|^2\right|}{\lambda r^2}|\eta''(\xi)| \leq \frac{\left||T(x)|^2 - |T(y)|^2\right|}{\lambda r^2}[\eta_r(x) + \eta_r(y)],$$

for some ξ between $\frac{|T(x)|^2 - r^2}{\lambda r^2}$ and $\frac{|T(y)|^2 - r^2}{\lambda r^2}$. Therefore, in view of (14), for $(x, y) \in C$ we can bound

$$|L(x, y)| \leq \frac{\left||T(x)|^2 - |T(y)|^2\right|}{\lambda r^2}[\eta_r(x) + \eta_r(y)]\frac{\min[|T(x)|, |T(y)|]}{|T(x) - T(y)|} \leq \frac{25}{8\lambda}[\eta_r(x) + \eta_r(y)]$$

so that

$$\frac{\beta^2}{2\lambda r^2} \iint_C |L(x, y)|\omega(x, t)\omega(y, t) \, dx \, dy \leq \frac{25\beta^2}{8\lambda^2 r^2}f_r(t) \int_{|T(x)| > \frac{r}{4}} \omega(x, t) \, dx \leq \frac{Ctf_r}{\lambda^2 r^4 \log^\theta r} \tag{28}$$

where we used (18) to bound the mass of vorticity in the region $|T(x)| > r/4$ by $M_6(1 + t)16r^{-2}\left[\log(r/4)\right]^{-\theta}$.

Collecting relations (22) to (28) we finally deduce the following differential inequality verified by f_r:

$$f'_r(t) \leq Cf_r\left(\frac{t^{\frac{1}{2}}}{\lambda r^3(\log t)^{\frac{\theta}{2}}} + \frac{t}{\lambda^2 r^4 \log^\theta r}\right) + C\frac{e^{-\frac{1}{2\lambda}}t^{\frac{1}{2}}}{\lambda r(\log t)^{\frac{\theta}{2}}}, \qquad t \geq 2.$$

Since $\omega(\cdot, 2)$ is compactly supported, choosing r sufficiently large one has that $f_r(2) \leq e^{-\frac{1}{2\lambda}}\|\omega_0\|_{L^1}$. Applying the Gronwall lemma in the inequality above for $t \geq 2$ and using (19), we deduce that there exists a constant C_1 such that

$$f_r(t) \leq C_1\left(1 + \frac{t^{\frac{3}{2}}}{\lambda r(\log t)^{\frac{\theta}{2}}}\right) \exp\left(\frac{C_1 t^{\frac{3}{2}}}{\lambda r^3(\log t)^{\frac{\theta}{2}}} + \frac{C_1 t^2}{\lambda^2 r^4 \log^\theta r} - \frac{1}{2\lambda}\right)$$

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We finally choose $\lambda = \frac{1}{4n \log r}$ and r sufficiently large such that

$$\frac{C_1 t^{\frac{3}{2}}}{\lambda r^3 (\log t)^{\frac{\theta}{2}}} + \frac{C_1 t^2}{\lambda^2 r^4 \log^\theta r} \leq \frac{1}{4\lambda}.$$

It is a straightforward calculation to check that the above restriction is implied by the condition $r > C_2 t^{\frac{1}{2}} (\log t)^{\frac{1-\theta}{4}}$ for some large constant C_2. For this choice one has that

$$\exp\Big(\frac{Ct^{\frac{3}{2}}}{\lambda r^3 (\log t)^{\frac{\theta}{2}}} + \frac{Ct^2}{\lambda^2 r^4 \log^\theta r} - \frac{1}{2\lambda}\Big) \leq r^{-n}$$

For a large enough constant n this term dominates $C_1 + \frac{C_1 t^{\frac{3}{2}}}{\lambda r (\log t)^{\frac{\theta}{2}}}$ and we obtain $f_r(t) \leq r^{-k}$ as desired. ∎

6 Even vorticity on the exterior of the disk

We consider now the case when $\Omega = D(0,1)$. The conformal map T is simply the identity, so that the Biot-Savart law becomes:

$$2\pi u(x,t) = \int \Big[\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y^*)^\perp}{|x-y^*|^2}\Big] \omega(y,t)\, dy + \alpha \frac{x^\perp}{|x|^2}, \tag{29}$$

where α is a real constant and $y^* = \frac{y}{|y|^2}$. It is an easy calculation to show that the moment of inertia is conserved. We prove the following theorem:

Theorem 14 *Suppose that the initial vorticity is nonnegative and even ($\omega_0(-x) = \omega_0(x)\ \forall x$). There exists a constant $M_9 > 0$ such that $\operatorname{supp}\omega(\cdot,t) \subset \big\{|x| \leq M_9\big[(1+t)\log(2+t)\big]^{1/4}\big\}$ for some constant M_9.*

Proof: The proof follows the lines of the argument given in [7] with the modifications induced by the new terms in the Biot-Savart law. Indeed, the two keys facts used in [7] are also true in this situation: both the moment of inertia and the center of mass are conserved quantities (the center of mass is at 0 since the vorticity is even). It suffices to assume that $t \geq 2$ and to prove that there exists a constant C_1 such that

$$\frac{x}{|x|} \cdot u(x,t) \leq \frac{C_1}{|x|^3}, \tag{30}$$

for any $|x| \geq C_1(t \log t)^{1/4}$ such that x is a point in the support of $\omega(\cdot,t)$ whose distance to the origin is the largest possible.

Note that $\omega(-x,t) = \omega(x,t)$ for all x,t. Next, let x be a point as described above. Then,

$$2\pi \frac{x}{|x|} \cdot u(x,t) = \frac{x}{|x|} \cdot \int \Big(\frac{y^{*\perp}}{|x-y^*|^2} - \frac{y^\perp}{|x-y|^2}\Big)\omega(y,t)\, dy = \frac{x}{|x|^3} \cdot \int \big(y^{*\perp} - y^\perp\big)\omega(y,t)\, dy$$

$$+ \frac{x}{|x|} \cdot \int \Big[y^{*\perp}\Big(\frac{1}{|x-y^*|^2} - \frac{1}{|x|^2}\Big) - y^\perp\Big(\frac{1}{|x-y|^2} - \frac{1}{|x|^2}\Big)\Big]\omega(y,t)\, dy.$$

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For symmetry reasons,

$$\int \left(y^{*\perp} - y^{\perp}\right)\omega(y,t)\,dy = 0. \tag{31}$$

We also observe that, for any x and z we have

$$\frac{1}{|x-z|^2} - \frac{1}{|x|^2} = \frac{z \cdot (2x-z)}{|x|^2|x-z|^2}.$$

We deduce that

$$2\pi \frac{x}{|x|} \cdot u(x,t) = \underbrace{\int x \cdot y^{*\perp} \frac{y^* \cdot (2x-y^*)}{|x|^3|x-y^*|^2}\omega(y,t)\,dy}_{I_1} - \underbrace{\int x \cdot y^{\perp} \frac{y \cdot (2x-y)}{|x|^3|x-y|^2}\omega(y,t)\,dy}_{I_2}. \tag{32}$$

Recall that, by assumption, we always have that $|y| \leq |x|$ in the integrands above. To bound the first integral, note that $|y^*| < 1$ so that $|x-y^*| \geq |x|/2$ and $|2x-y^*| \leq 3|x|$. Consequently,

$$I_1 \leq \frac{12}{|x|^3} \int |y^*|^2\omega(y,t)\,dy \leq \frac{12}{|x|^3} \int \omega(y,t)\,dy. \tag{33}$$

We now go to the estimate of I_2 and split the domain of integration in two pieces: $\{|y| < |x|/2\}$ and $\{|x|/2 \leq |y| \leq |x|\}$. For $|y| < |x|/2$, we still have that $|x-y| \geq |x|/2$ and $|2x-y| \leq 3|x|$, so as in the estimate for I_1, we can bound

$$\left|x \cdot y^{\perp} \frac{y \cdot (2x-y)}{|x|^3|x-y|^2}\right| \leq \frac{12|y|^2}{|x|^3} \qquad \forall |y| < \frac{|x|}{2}.$$

Next, for $|x|/2 \leq |y| \leq |x|$ we write

$$\left|x \cdot y^{\perp} \frac{y \cdot (2x-y)}{|x|^3|x-y|^2}\right| = \left|(x-y) \cdot y^{\perp} \frac{y \cdot (2x-y)}{|x|^3|x-y|^2}\right| \leq \frac{|y|^2|2x-y|}{|x|^3|x-y|} \leq \frac{3}{|x-y|}.$$

We infer that

$$\begin{aligned}
|I_2| &\leq \frac{12}{|x|^3} \int |y|^2\omega(y,t)\,dy + 3\int_{|y|\geq|x|/2} \frac{1}{|x-y|}\omega(y,t)\,dy \\
&\leq \frac{12}{|x|^3} \int |y|^2\omega_0(y)\,dy + C\|\omega\|_{L^\infty}^{\frac{1}{2}}\left(\int_{|y|\geq|x|/2} \omega(y,t)\,dy\right)^{\frac{1}{2}},
\end{aligned} \tag{34}$$

where we used that the moment of inertia is conserved and Lemma 11.

Clearly relation (30) now follows from (32), (33), (34) and from Lemma 15 below. This completes the proof of Theorem 14 once Lemma 15 is proved. ∎

Lemma 15 *For every $k \in \mathbb{N}$ there exists a constant $M_{10}(k)$ such that*

$$\int_{|x|\geq r} \omega(x,t)\,dx \leq \frac{M_{10}}{r^k} \quad \textit{for} \quad r \geq M_{10}(t\log t)^{1/4}.$$

Proof: Let
$$f_r(t) = \int \varphi_r(x)\omega(x,t)\,dx,$$

where
$$\varphi_r(x) = \eta\Big(\frac{|x|^2 - r^2}{\lambda r^2}\Big), \quad \eta(s) = \frac{e^s}{1 + e^s},$$

and $\lambda = \lambda(r) \le 1$ is a positive function to be chosen later. It suffices to prove that $f_r(t) \le \frac{C(k)}{r^k}$.

As in the proof of Proposition 13, we deduce that $f_r'(t)$ can be written under the form

$$
\begin{aligned}
2\pi f_r'(t) &= \iint \Big(\frac{y^{*\perp}}{|x - y^*|^2} - \frac{y^\perp}{|x-y|^2}\Big) \cdot \nabla\varphi_r(x)\omega(x,t)\omega(y,t)\,dx\,dy \\
&= \underbrace{\iint \Big(\frac{1}{|x-y^*|^2} - \frac{1}{|x|^2}\Big)y^{*\perp} \cdot \nabla\varphi_r(x)\omega(x,t)\omega(y,t)\,dx\,dy}_{J_1} \\
&\quad - \underbrace{\iint \Big(\frac{1}{|x-y|^2} - \frac{1}{|x|^2}\Big)y^{\perp} \cdot \nabla\varphi_r(x)\omega(x,t)\omega(y,t)\,dx\,dy}_{J_2}
\end{aligned}
\tag{35}
$$

where we have used relation (31).

If we follow the analysis from relation (8) to the bottom of page 1720 of [7], then we find that the second term can be estimated by

$$J_2 \le C\frac{f_r}{\lambda^2 r^4} + \frac{C}{\lambda r^2}e^{-\frac{1}{2\lambda}}. \tag{36}$$

It remains to estimate

$$
\begin{aligned}
J_1 &= \iint_{|x| \le r/2} \underbrace{\Big(\frac{1}{|x-y^*|^2} - \frac{1}{|x|^2}\Big)(y^* - x)^\perp \cdot \nabla\varphi_r(x)\omega(x,t)\omega(y,t)\,dx\,dy}_{L_1(x,y)} \\
&\quad + \iint_{|x| \ge r/2} \underbrace{\frac{y^* \cdot (2x - y^*)}{|x|^2|x-y^*|^2}y^{*\perp} \cdot \nabla\varphi_r(x)\omega(x,t)\omega(y,t)\,dx\,dy}_{L_2(x,y)}.
\end{aligned}
$$

From the definition of φ_r, we have that

$$\nabla\varphi_r(x) = \frac{2x}{\lambda r^2}\eta'\Big(\frac{|x|^2 - r^2}{\lambda r^2}\Big). \tag{37}$$

For $|x| \le r/2$, one has that $\frac{|x|^2 - r^2}{\lambda r^2} \le -\frac{1}{2\lambda}$. Since $|\eta'(s)| \le e^s$, it follows that

$$\Big|\eta'\Big(\frac{|x|^2 - r^2}{\lambda r^2}\Big)\Big| \le e^{-\frac{1}{2\lambda}}$$

so that, using also that $|x|, r \ge 1$,

$$|L_1(x,y)| \le \Big(\frac{1}{|x-y^*|^2} + \frac{1}{|x|^2}\Big)|x - y^*|\frac{2|x|}{\lambda r^2}e^{-\frac{1}{2\lambda}} \le \frac{Ce^{-\frac{1}{2\lambda}}}{\lambda r|x - y^*|} + \frac{C}{\lambda}e^{-\frac{1}{2\lambda}}$$

which, in view of Lemma 11, implies that

$$\Big|\iint_{|x|\le r/2} L_1(x,y)\omega(x,t)\omega(y,t)\,dx\,dy\Big| \le \frac{C}{\lambda}e^{-\frac{1}{2\lambda}}. \tag{38}$$

To estimate $L_2(x,y)$, we use again equation (37) and the facts that $|y^*| < 1$ and $|\eta'| \le \eta$ to deduce that, for $|x| \ge r/2$,

$$\begin{aligned}
|L_2(x,y)| &= \Big|\frac{y^*\cdot(2x-y^*)}{|x|^2|x-y^*|^2}y^{*\perp}\cdot\frac{2x}{\lambda r^2}\eta'\Big(\frac{|x|^2-r^2}{\lambda r^2}\Big)\Big| \\
&\le \frac{C}{|x|^2\lambda r^2}\eta\Big(\frac{|x|^2-r^2}{\lambda r^2}\Big) \\
&\le \frac{C}{\lambda r^4}\varphi_r(x).
\end{aligned}$$

We infer that

$$\Big|\iint_{|x|\ge r/2} L_2(x,y)\omega(x,t)\omega(y,t)\,dx\,dy\Big| \le \frac{C}{\lambda r^4}f_r(t). \tag{39}$$

Relations (38) and (39) now imply that

$$|J_1| \le \frac{C}{\lambda}e^{-\frac{1}{2\lambda}} + \frac{C}{\lambda r^4}f_r(t).$$

Combining this with (35) and (36) we get

$$f'(r) \le \frac{Cf_r}{\lambda^2 r^4} + \frac{C}{\lambda}e^{-\frac{1}{2\lambda}}.$$

After integration, we obtain

$$f_r(t) \le (f_r(2) + \lambda r^4 e^{-\frac{1}{2\lambda}})e^{\frac{Ct}{\lambda^2 r^4}} \le e^{\frac{Ct}{\lambda^2 r^4}-\frac{C_1}{\lambda}}(C + \lambda r^4).$$

Lemma 15 now follows by choosing $\lambda\log r$ sufficiently small as we did at the end of the proof of Proposition 13. ∎

Let us conclude with the remark that, although we assumed that ω_0 was smooth, our results hold, with minor modifications, if $\omega_0 \in L^p_c$ for some $p > 2$.

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