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Conflict-Free Coloring for Rectangle Ranges Using $O(n^{.382})$ Colors

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Abstract

Given a set of points $P \subseteq \mathbb{R}^2$, a *conflict-free coloring* of P w.r.t. rectangle ranges is an assignment of colors to points of P, such that each non-empty axis-parallel rectangle T in the plane contains a point whose color is distinct from all other points in $P \cap T$. This notion has been the subject of recent interest, and is motivated by frequency assignment in wireless cellular networks: one naturally would like to minimize the number of frequencies (colors) assigned to base stations (points), such that within any range (for instance, rectangle), there is no interference. We show that any set of n points in $\mathbb{R}^2$ can be conflict-free colored with $O(n^{\beta^*+o(1)})$ colors in expected polynomial time, where $\beta^* = \frac{3-\sqrt{5}}{2} < 0.382$.

Keywords. Frequency assignment in wireless networks, conflict-free coloring, axis-parallel rectangles, boundary sets, monotone sequences

1 Introduction

The study of conflict-free coloring is motivated by the frequency assignment problem in wireless networks. A wireless network is a heterogeneous network consisting of *base stations* and *agents*. The base stations have a fixed location, and are interlinked via a fixed backbone network, while the agents are typically mobile and can connect to the base stations via radio links. The base stations are assigned fixed frequencies to enable links to agents. The agents can connect to any base station, provided that the radio link to that particular station has good reception. Good reception is only possible if i) the base station is located within range, and ii) no other base station within range of the agent has the same frequency assignment (to avoid interference). Thus the fundamental problem of frequency-assignment in cellular networks is to assign frequencies to base stations, such that an agent can always find a base station with unique frequency among the base stations in its range. Naturally, due to cost, flexibility and other restrictions, one would like to minimize the total number of assigned frequencies.

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The study of the above problem was initiated in [9], and continued in a series of recent papers [3, 4, 5, 6, 8, 11, 12, 15, 16]. For a recent survey on the problem and its applications, refer to [17]. The conflict-free coloring problem can be formally described as follows. Let $P \subseteq \mathbb{R}^2$ be a set of points and $\mathcal{R}$ be a set of ranges (e.g. the set of all discs or rectangles in the plane). A *conflict-free* coloring (CF-coloring in short) of P w.r.t. the range $\mathcal{R}$ is an assignment of a color to each point $p \in P$ such that for any range $T \in \mathcal{R}$ with $T \cap P \neq 0$, the set $T \cap P$ contains a point of unique color. Naturally, the goal is to assign a conflict-free coloring to the points of P with the *smallest* number of colors possible.

The work in [9] presented a general framework for computing a conflict-free coloring for several types of ranges. In particular, for the case where the ranges are discs in the plane, they present a polynomial time coloring algorithm that uses $O(\log n)$ colors for conflict-free coloring and this bound is shown to be tight. This result was then extended in [12] by considering the case where the ranges are axis-parallel rectangles in the plane. This seems much harder than the disc case, and the work in [12] presented a simple algorithm that uses $O(\sqrt{n})$ colors. As mentioned in [12], this can be further improved to $O(\sqrt{n \log\log n / \log n})$ using the sparse neighborhood property of the conflict-free graph, as independently observed by Noga Alon, Timothy Chan, and János Pach and Geza Tóth [2, 15]. Prior to this paper, this was the best known upper bound for CF-coloring axis-parallel rectangles. A lower bound of $\Omega(\log n)$ trivially follows from the lower bound for intervals. A related notion is that of the *delaunay graph* of a point set P with respect to axis-parallel rectangles, defined as the graph on the vertex set P, whose two points $p, q \in P$ are connected by an edge if and only if there is an axis-parallel rectangle that contains p and q, but no other points of P. Chen et al. [7] show that there exists a set of n points for which the maximum size of an independent set in the conflict-free graph is $O(n \log^2 \log n / \log n)$.

Recent works have shown that one can obtain better upper bounds for special cases of this problem. In [12], it was shown that for the case of random points in a unit square, $O(\log^4 n)$ colors suffice, and for points lying in an *exact* uniform $\sqrt{n} \times \sqrt{n}$ grid, $O(\log n)$ colors are sufficient. Chen [5] showed that polylogarithmic number of colors suffice for the case of *nearly equal* rectangle ranges. Elbassioni and Mustafa [8] asked the following question: Given a set of points P in the plane, can we insert new points Q such that the conflict free coloring of $P \cup Q$ requires fewer colors? They showed that by inserting $O(n^{1-\epsilon})$ points, $P \cup Q$ can be conflict free colored using $\tilde{O}(n^{3(1+\epsilon)/8})$ colors.

While the CF-coloring problem is closed for disc ranges, the upper bounds are very far from the currently known lower bounds for axis-parallel rectangular ranges. It remains very interesting to reduce this gap between upper and lower bounds, and this is, in fact, the main open problem posed in [12]. In this paper, we improve the upper bound significantly.

Theorem 1.1 *Any set of n points in $\mathbb{R}^2$ can be conflict-free colored with respect to rectangle ranges using $O(n^{\beta^* + O(\frac{1}{\sqrt{\log n}})})$ colors, in expected polynomial time, where $\beta^* = \frac{3-\sqrt{5}}{2} < 0.382$.*

An immediate corollary of Theorem 1.1 is that the delaunay graph of any set of points in the plane with respect to axis-paralled rectangles has an independence number $\Omega(n^{0.618})$.

Our main tool for proving this theorem is a probabilistic coloring technique, introduced in [8], that can be used to get a coloring with weaker properties, which we call *quasi-conflict-free* coloring.

This will be combined with boundary sets, monotone sequences, and careful griding of the point set, in a recursive way, to obtain the claimed result. We start with some definitions and preliminaries in Section 2. To illustrate our ideas, we sketch a simple $\tilde{O}(n^{6/13})$ conflict free coloring algorithm in Section 3. The main algorithm will be given in Section 4. We describe the quasi-conflict-free coloring technique in a slightly more general form in Section 5. Section 4 contains the analysis of the main algorithm.

2 Preliminaries

By $\mathcal{R} \subseteq 2^{\mathbb{R}^2}$, we denote the set of all *axis-parallel* rectangles. Let P be a set of points in $\mathbb{R}^2$.

Definition 2.1 (Conflict-free coloring) *A coloring of P is a function $\chi : P \mapsto \mathcal{N}$ from P to some finite set $\mathcal{N}$. A rectangle $T \in \mathcal{R}$ is said to be* conflict-free *with respect to a coloring χ if either $T \cap P = \emptyset$, or there exists a point $p \in P \cap T$ such that $\chi(p) \neq \chi(p')$ for all points $p' \in P \cap T$, distinct from p. A coloring χ is said to be* conflict-free *(w.r.t. $\mathcal{R}$) if every rectangle $T \in \mathcal{R}$ is conflict-free w.r.t. χ.*

In this paper, we shall say that a given procedure is an $f(n)$-CF-coloring algorithm if it conflict-free colors any set of points of size n with at most $f(n)$ colors.

Definition 2.2 (Boundary sets) *For a point $p = (p^x, p^y) \in \mathbb{R}^2$, define $W_1(p) = \{q \in \mathbb{R}^2 | \ q^x \geq p^x, q^y \geq p^y\}$ to be the upper-right quadrant defined by p. Similarly, let $W_2(p), W_3(p)$ and $W_4(p)$ be the upper-left, lower-right and lower-left quadrants respectively. Define the* boundary set *of type i for P, denoted by $D_i(P), 1 \leq i \leq 4$, as follows:*

$$D_i(P) = \{p \in P \mid W_i(p) \cap P = \{p\}\}.$$

Definition 2.3 (Monotonic sets) *Let $P = \{p_1, p_2, \ldots, p_k\}$ be a set of points that is sorted by x coordinate. P is (resp. monotonic non-increasing) if $p^y_j \geq p^y_i$ (resp. $p^y_j \leq p^y_i$) for all $1 \leq i < j \leq k$.*

It is easy to see that the boundary set of type 2 and 3 (resp. type 1 and 4) are monotonic non-decreasing (resp. non-increasing); see Figure 1.

Definition 2.4 (r-Grid) *Let $r \in \mathbb{Z}_{>0}$ be a positive integer. An r-grid on P (see Figure 2), denoted by $G_r = G_r(P)$, is an $r \times r$ axis-parallel grid containing all points of P. For $i = 1, \ldots, r$, denote by R_i and C_i the subsets of P lying in the ith row and column of G_r, respectively. Denote by $B(G_r)$, the maximum number of points of P in a row or a column of G_r. For $1 \leq h \leq 2r-1$, let M^1_h (resp. M^2_h) be the set of grid cells lying along a diagonal h of positive slope (resp. negative slope) in G_r. For $l = 2, 3$ (resp. $l = 1, 4$), let $\mathcal{D}^h_l = \cup_{(i,j) \in M^1_h} D_l(R_i \cap C_j)$ (resp. $\mathcal{D}^h_l = \cup_{(i,j) \in M^2_h} D_l(R_i \cap C_j)$) be the union of boundary sets of type l over grid cells in M^1_h (resp. M^2_h). Let $\mathcal{D}_l = \cup_{(i,j) \in G_r} D_l(R_i \cap C_j)$ be the union of boundary sets of type l over all the grid cells in G_r.*

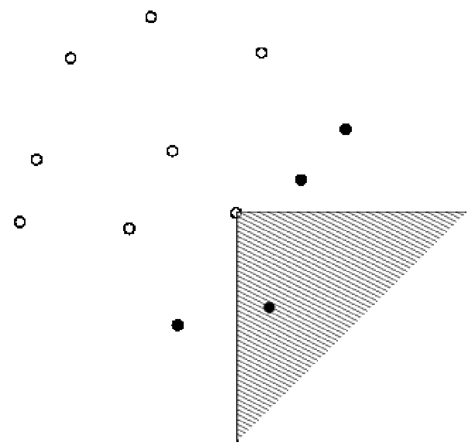

Figure 1: Boundary sets: the shaded region represents the lower right quadrant, and the solid black points represent the boundary set $D_3(P)$ of type 3.

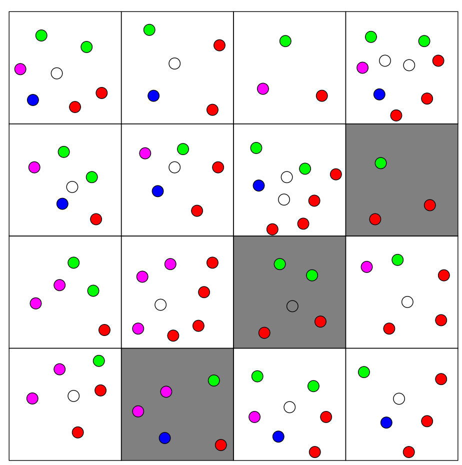

Figure 2: r-grid G_r: $r = 4$, $B(G_r) = 24$, the four types of boundary sets are shown as solid circles in four different colors, and the remaining points are shown as hollow circles. The shaded cells represent the set M_h^1, for $h = 3$. Note that some points may be in many different boundary sets. In this figure, a point belonging to multiple boundary sets is colored by the color of either one of them.

Note that, for $l = 2, 3$ and $1 \le h \le 2r - 1$, $\mathcal{D}_l^h$ is monotonic non-decreasing, since the grid cells in M_h^1, which lie along the diagonal of positive slope, are horizontally and vertically separated and hence the union of $D_l(R_i \cap C_j)$ (which are monotonic non-decreasing) is also monotonic non-decreasing. By a similar argument, for $l = 1, 4$ with M_h^2 and $1 \le h \le 2r - 1$, $\mathcal{D}_l^h$ is monotonic non-increasing.

Definition 2.5 (Quasi-conflict-free coloring) *Given a grid G_r on P, we call a coloring $\chi : P \mapsto \mathcal{N}$* quasi-conflict-free *with respect to G_r, if every axis-parallel rectangle which contains points only from the same row or the same column of G_r is conflict-free.*

Let G_r be an r-grid on a point set P such that $B(G_r) = B$. It is shown in [8] that there exists a quasi-conflict-free coloring of G_r requiring $\tilde{O}(B^{3/4})$ colors [1].

3 A simple conflict-free coloring algorithm using $\tilde{O}(n^{6/13})$ colors

In this section, we sketch a simple algorithm for CF-coloring P in order to illustrate the main ideas. This algorithm CF-colors P using $\tilde{O}(n^{6/13})$ colors. We deliberately skip some technical details in order to make the main idea as clear as possible. The later sections contain a more detailed analysis.

It will be useful first to illustrate the idea behind the $O(n^{1/2})$-CF-coloring algorithm in [12]. By the Erdős-Szkeres Theorem [10], the set of points P, regarded as sequence when ordered by the x-coordinate, has a monotone subsequence of size $\sqrt{n}$. Clearly, the set I consisting of every other point in this monotone sequence defines an independent set in the conflict-free graph of P. We color all the points in I with one color, and then recurse on the rest of the points with a different set of colors. One can easily argue that the resulting coloring will be conflict free since I is an independent set, and that the total number of colors needed is $O(n^{1/2})$.

Let $\mathcal{A}$ be an $O(n^{1/2})$ conflict-free coloring algorithm (as the one described above). To reduce the number of colors needed below $O(n^{1/2})$, we proceed as follows. Set $t = n^{7/13}$. As long as the current point set contains a monotonic sequence of size t, we color alternate points in that sequence with the same color, remove them, and continue with the remaining points using new colors. Since we remove $\Omega(t)$ points every time, the number of colors used in this process is $O(n^{6/13})$. Let Q be the set of points left after this step and let $m = |q|$. Now, let $r = m^{\frac{5}{13}}$. Grid Q using G_r such that each row and column has $B = m^{\frac{8}{13}}$ points of P. Compute the boundary sets $\mathcal{D}_l(Q), 1 \le l \le 4$ and let $D = \cup_{l=1}^4 D_l(Q)$ and $Q' = Q \setminus D$. We quasi-CF color Q' with $\tilde{O}(B^{3/4})$ colors using the algorithm of [8] (which uses $\mathcal{A}$ as subroutine). Then, we CF-color D using $\mathcal{A}$ with a different set of colors.

Lemma 3.1 *The above coloring of P is conflict-free.*

Proof. Let $T \in \mathcal{R}$ be a rectangle such that $T \cap P \neq \emptyset$. We show that T contains a point of unique color among the points in $T \cap P$.

We consider 3 cases:

[1] Just as O-notation hides constant factors, $\tilde{O}$ hides the poly-logarithmic factors

Case 1. A monotone sequence of size t is found and we colored every other point in the sequence (set I) with one color: if $(T \cap P) \setminus I \neq \emptyset$, then by induction and the fact that I and $P \setminus I$ are colored with distinct sets of colors, we know that $T \cap P$ contains a point of a unique color. If $T \cap P \subseteq I$, then $|T \cap P| = 1$ (since I consists of every other point in a monotone sequence) and the statement trivially holds.

Case 2. $T \cap D \neq \emptyset$: The CF-coloring of D guarantees that there is a point p of unique color among points in $T \cap D$. Since D and $Q' = Q \setminus D$ are colored with distinct sets of colors, p is a point of unique color among points in $T \cap P$ also.

Case 3. $T \cap D = \emptyset$: Let (i, j) be a grid cell of G_r defined by the intersection of row R_i and collumn C_j. If T contains at least one corner of some grid cell (i, j), $T \cap D_l(R_i \cap C_j) \neq \emptyset$ for some $l \in \{1, \dots, 4\}$ contradicting the fact that $T \cap D = \emptyset$. Therefore in this case, T lies completely within one row or one column of G_r. Since $T \cap P \neq \emptyset$ and $T \cap D = \emptyset$, we have $T \cap Q' \neq \emptyset$. The quasi-CF coloring of Q' guarantees that there is a point p of unique color among the points in $T \cap Q'$. p is also a point of unique color among points in $T \cap P$. □

We now bound the total number of colors used by our algorithm. As argued before, the number of colors used in the first step (removing monotonic sequences of size t) is $\Omega(n^{6/13})$. Quasi-CF-coloring of Q requires $\tilde{O}(n^{\frac{8}{13} \times \frac{3}{4}}) = \tilde{O}(n^{6/13})$ colors. To bound the number of colors used in CF-coloring D, we first bound the size of D: $|\mathcal{D}_l^h| \leq t$ for all h and l, because each $\mathcal{D}_l^h$ is a monotonic sequence. Since $D = \cup_{l,h} \mathcal{D}_l^h$ over $1 \leq h \leq 2m^{5/13} - 1$, and $1 \leq l \leq 4$, we have $|D| = O(n^{12/13})$. Thus, the CF-coloring of D (using the $O(n^{1/2})$-coloring algorithm $\mathcal{A}$) requires $O(n^{6/13})$ colors. The total number of colors used by our algorithm is thus $\tilde{O}(n^{6/13})$.

4 Improved Conflict Free Coloring

In the algorithm described in Section 3, we used an $O(n^{1/2})$-"black-box" $\mathcal{A}$ for CF-coloring the boundary set D and the quasi-CF-coloring of P'. As a result we obtained an $\tilde{O}(n^{6/13})$ CF-coloring algorithm. We can improve this coloring further by using this $\tilde{O}(n^{6/13})$ as a new black-box for CF-coloring the boundary set D and quasi-CF-coloring of P'. An easy calculation shows that the number of colors used is asymptotically smaller than $\tilde{O}(n^{6/13})$.

This *bootstrapping* approach can be taken to the limit. This results in a sequence of strictly improved algorithms, $\mathcal{A} = \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots$ For $k = 1, 2, \dots,$ the structure of $\mathcal{A}_k$ is similar to the algorithm described in Section 3: Grid the point set P using G_r, where $r = n^{1-\alpha_k}$, for some α_k; Partition P into boundary set D and $P' = P \setminus D$ and use algorithm $\mathcal{A}_{k-1}$ for CF-coloring D and quasi-CF-coloring P'. We choose the parameter α_k such that both the CF-coloring of D and quasi-CF-coloring of P' balance-out into using an $\tilde{O}(n^{\beta_k})$ colors, for some β_k as small as possible.

Ideally, one would like to always recursively apply algorithm $\mathcal{A}_\infty$ to get a bound of $\tilde{O}(n^{\beta_\infty})$ on the number of colors (assuming these limits exist). However, there is a technical problem with such a recursion: the sublinearity of the bound on the number of colors implies that the power of the logarithmic factor increases exponentially[2] with k. To solve this problem, we can stop the

[2]This is essentially a byproduct of the fact that $n_1^\beta + n_2^\beta > (n_1 + n_2)^\beta$, for $0 < \beta < 1$.

recursion at a level of $O(\log \frac{1}{\epsilon})$, settling at a bound of $\tilde{O}(n^{\beta_\infty+\epsilon})$, for any arbitrarily small constant $\epsilon > 0$. Analysing this approach[3] is however technically complicated, and we present an alternate method here, which is asymptotically better in terms of the number of colors, but with possibly worse constants.

In the rest of the paper, logarithms are with base 2. Let $\beta^* = (3 - \sqrt{5})/2$, $\alpha^* = 1 - \beta^*$, $c = 2^{19}$ and $n_0 = 2^{(14c)^2}$. Define functions $\alpha(n) = \alpha^* - 5c/\sqrt{\log n}$, $\beta(n) = \beta^* + 9c/\sqrt{\log n}$ and $\gamma(n) = \alpha^* - 7c/\sqrt{\log n}$.

Let P be a set of n points. If $n \leq n_0$, we use any CF-coloring algorithm to color P. Otherwise, we use the same approach as in Section 3. Namely, if P contains a monotonic chain of points of size $m = 2n^{\gamma(n)}$ then we color alternate points of the chain with one color and recursively color the rest of the points in P using a new set of colors. Otherwise (the size of any monotonic chain in P is at most m), we construct a grid G so that each row and column of G contains at most $n^{\alpha(n)}$ points. Let D be the set of all points belonging to the boundary sets of the cells of G. We conflict free-color D recursively using our CF-coloring procedure, and quasi-CF-color the rest of the points using a different set of colors. In the quasi conflict-free coloring algorithm, we use a recursive call to the conflict-free coloring procedure. (However, since we are calling the quasi conflict-free coloring algorithm only for smaller-size point sets, there is no circularity here.) The coloring procedure is given as Algorithm 1.

Algorithm 1 Procedure $\mathcal{A}(P, S)$:

Input: A point set $P \subseteq \mathbb{R}^2, |P| = n$, a set of colors S
Output: A CF-coloring $\chi : P \mapsto S$
if $n \leq n_0$ **then**
  return a coloring of P using the $O(\sqrt{n})$-coloring algorithm
else
  Set $\alpha = \alpha^* - 5c/\sqrt{\log n}$, $\gamma = \alpha^* - 7c/\sqrt{\log n}$ and $r \leftarrow \lceil n^{1-\alpha} \rceil$
  if $\exists$ a monotonic sequence L of size $2n^\gamma$ in P **then**
    Let I be the set consisting of every other point of L
    Color every point of I with the same color $i \in S$, i.e. set $\chi'(p) \leftarrow i$ for all $p \in I$
    $\chi'' \leftarrow \mathcal{A}(P \setminus I, S \setminus \{i\})$
    return $\chi' + \chi''$
  else
    Grid P using G_r
    Compute the boundary set D w.r.t. G_r
    $\chi' \leftarrow$QCFC$(P \setminus D, \mathcal{A}, G_r, S)$
    $\chi'' \leftarrow \mathcal{A}(D, S \setminus \mathrm{range}(\chi'))$
    return $\chi := \chi' + \chi''$

The structure of the the above algorithm is the same as the algorithm described in Section 3. Hence, by Lemma 3.1 the coloring returned by the algorithm is conflict-free.

[3]We refer the interested reader to the conference version of this paper [1] for the details of such a bootstrapping approach

In the next section, we bound the number of colors needed by the quasi-CF-coloring algorithm. We use this result in Section 6 to anaylse the number of colors needed by Algorithm 1

5 Generalized quasi-conflict free coloring

Given an r-grid G_r on point set P, we start by coloring the points of each column, using a CF-coloring algorithm $\mathcal{A}$ as a black-box. We use the same set of colors for all columns. Then randomly and independently for each column, we redistribute the colors on the different color classes of the column. Finally, a recoloring step is applied on each monochromatic set of points in each row, again using algorithm $\mathcal{A}$ as the CF-coloring procedure. The color assigned to a point is the concatenation of its first and second colorings. A formal description of this procedure is given as Algorithm 2.

Algorithm 2 Procedure QCFC$(P, \mathcal{A}, G_r, S)$:

Input: A point set $P \subseteq \mathbb{R}^2$, an $f(\cdot)$-CF-coloring algorithm $\mathcal{A}$, an r-grid G_r on P, and a set of possible colors S

Output: A quasi-CF-coloring $\chi : P \mapsto S$ of P w.r.t. G_r

Let $h = f(B(G_r))$; $N = \{1, \ldots, h\}$
for $j = 1, \ldots, r$ **do**
    $\chi_j \leftarrow \mathcal{A}(C_j, N)$
    Let $\pi \in \mathbb{S}_h$ be a random permutation
    for all $p \in C_j$ **do**
        $\chi'_j(p) \leftarrow \pi(\chi_j(p))$
$\chi' \leftarrow \sum_{j=1}^{r} \chi'_j$
for $i = 1, \ldots, r$ **do**
    for $\ell = 1, \ldots, h$ **do**
        $P_i^\ell \leftarrow \{p \in R_i : \chi'(p) = \ell\}$
        $\chi''_{i,\ell} \leftarrow \mathcal{A}(P_i^\ell, S)$
$\chi'' \leftarrow \sum_{i=1}^{r} \sum_{\ell=1}^{h} \chi''_{i,\ell}$
return $\chi := \chi' \times \chi''$ (mapped to S)

The following is a straightforward generalization of Theorem 3 in [8], in which $\mathcal{A}$ is used as the CF-procedure (instead of the $\sqrt{n}$-procedure used in [8]).

Theorem 5.1 *Given any point set $P \subseteq \mathbb{R}^2$, a grid G_r with $B = B(G_r)$ on P, and an $f(\cdot)$-conflict-free coloring algorithm $\mathcal{A}$ such that $B \geq 4$ and*

$$r \cdot f(B)(\log B)^{(-\log B)/8} \leq \frac{1}{2}, \tag{1}$$

procedure QCFC returns a quasi-conflict-free coloring of G_r using

$$q(B) = f(B) f\left(\frac{B \log B}{f(B)}\right) \tag{2}$$

colors, in expected polynomial time.

6 Analysis

We now show an improved bound on the number of colors required for conflict free coloring a set of n points. Namely, we show that any set of points n can always be conflict-free colored with $f(n) := n^{\beta(n)}$ colors. The function $f(n)$ is clearly monotonically increasing and is chosen so that it satisfies the following.

Claim 6.1 *For $n > n_0$, $f(n), \alpha(n)$ and $\gamma(n)$ satisfy the following inequalities:*

$$f(n) \geq 1 + f(n - n^{\gamma(n)}) \tag{3}$$

$$f(n) \geq f(16 \cdot n^{1-\alpha(n)+\gamma(n)}) + f(n^{\alpha(n)}) \cdot f(\frac{n^{\alpha(n)} \log n^{\alpha(n)}}{f(n^{\alpha(n)})}) \tag{4}$$

$$n^{1-\alpha(n)} \cdot f(n^{\alpha(n)}) \cdot (\log n^{\alpha(n)})^{(-\log n^{\alpha(n)})/8} \leq \frac{1}{2} \tag{5}$$

We defer the proof of the above inequalities and first show the following.

Theorem 6.1 *Any set of n points can be conflict-free colored using $f(n)$ colors.*

Proof. We show that Algorithm 1 requires $f(n)$ colors to CF-color any point set P of size n. The proof is by induction on n. The theorem is trivially true for $n \leq n_0$ since for such n, $\beta(n) > 1$ and therefore $f(n) > n$. Let P be a set of $n > n_0$ points and assume that for point sets of smaller size the statement is true. If P contains a monotonic chain of points of size $u = 2n^{\gamma(n)}$ then the algorithm colors alternate points of the chain with one color and recursively colors the rest of the points in P using a new set of colors. Thus we have colored the point set using $1 + f(n - n^{\gamma(n)})$ colors which by the first inequality in Claim 6.1 is at most $f(n)$. On the other hand, if the size of any monotonic chain in P is at most u, then we construct a grid G so that each row and column of G contains at most $n^{\alpha(n)}$ points. There are $n^{1-\alpha(n)}$ rows and columns in G. Let D be the set of all points belonging to the boundary sets of the cells of G. Since D can be partitioned into at most $8 \cdot n^{1-\alpha(n)}$ monotonic sets, we have $|D| \leq u \cdot 8 \cdot n^{1-\alpha(n)} \leq 16 \cdot n^{1-\alpha(n)+\gamma(n)}$. We conflict free color D using $f(16 \cdot n^{1-\alpha(n)+\gamma(n)})$ colors and quasi conflict-free color the rest of the points using a different set of colors. For this, we invoke the algorithm described in Section 5 with the grid G. Since by (5), condition (1) is satisfied, we are guaranteed by Theorem 5.1 to use at most $f(n^{\alpha(n)}) \cdot f(\frac{n^{\alpha(n)} \log n^{\alpha(n)}}{f(n^{\alpha(n)})})$ colors for the quasi-conflict free coloring step. By the second inequality in Claim 6.1 the total number of colors used is at most $f(n)$. □

Proof of Claim 6.1. For brevity of notation, we denote, respectively, $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ by

α, β and γ, whenever convenient. Let us start with the first inequality.

$$
\begin{aligned}
& f(n) - f(n-n^\gamma) = n^\beta - m^{\beta(m)} \quad \text{(where } m = n - n^\gamma \text{)} \\
& = 2^{\beta(n)\log n} - 2^{\beta(m)\log m} = 2^{\beta(m)\log m}(2^{\beta(n)\log n - \beta(m)\log m} - 1) \\
& \geq f(m)\cdot(2^{\beta^*\log(n/m)} - 1) \quad \text{(using the expressions for } \beta(m) \text{ and } \beta(n) \text{ and that } m < n \text{)} \\
& \geq f(m)\cdot \beta^* \log(n/m)/2 \quad \text{(since } 2^x - 1 \geq x/2 \text{ for all } x \text{)} \\
& = 0.5\beta^* \cdot f(m) \cdot \log(1 + n^\gamma/m) \\
& \geq 0.5\beta^* \cdot f(m) \cdot \frac{n^\gamma}{m} \quad \text{(since } \log_2(1+x) \geq x \text{ for } 0 \leq x \leq 1 \text{ and } n^\gamma \leq m \text{ for } n > n_0\text{)} \\
& \geq 0.5\beta^* \cdot f(m) \cdot \frac{m^\gamma}{m} \quad \text{(since } m < n \text{)} \\
& = 0.5\beta^* \cdot m^{(\beta^* + 9c/\sqrt{\log m}) + (\alpha^* - 7c/\sqrt{\log n}) - 1} = 0.5\beta^* \cdot m^{9c/\sqrt{\log m} - 7c/\sqrt{\log n}} \\
& \geq 0.5\beta^* \cdot m^{2c/\sqrt{\log m}} \quad \text{(since } m < n\text{)} \\
& \geq 1 \quad \text{(for } n > n_0\text{)}
\end{aligned}
$$

The first inequality follows by rearranging the terms. We prove the second inequality in two parts. We show that the quantities $f(16 \cdot n^{1-\alpha+\gamma})$ and $f(n^\alpha) \cdot f(\frac{n^\alpha \log n^\alpha}{f(n^\alpha)})$ are both at most $f(n)/2$. It follows that their sum is at most $f(n)$. We first observe some simpler inequalities that we need. For any $\lambda > 0$,

$$f(n^\lambda) = (n^\lambda)^{\beta^* + 9c/\sqrt{\log n^\lambda}} = n^{\lambda\beta^* + 9c\sqrt{\lambda}/\sqrt{\log n}} \tag{6}$$

Using the above with $\lambda = \alpha$,

$$f(n^\alpha) = n^{\alpha\beta^* + 9c\sqrt{\alpha}/\sqrt{\log n}} = n^{\alpha^*\beta^* + (9c\sqrt{\alpha} - 5c\beta^*)/\sqrt{\log n}} \tag{7}$$

It follows from the above that

$$f(n^\alpha) \leq n^{\alpha^*\beta^* + (9c\sqrt{\alpha^*} - 5c\beta^*)/\sqrt{\log n}} \quad \text{(since } \alpha^* \geq \alpha \text{)} \tag{8}$$

$$f(n^\alpha) \geq n^{\alpha^*\beta^*} \quad \text{(since } (9\sqrt{\alpha} - 5\beta^*) \geq 0 \text{ for } n > n_0 \text{)} \tag{9}$$

From the above, we get

$$
\begin{aligned}
& \frac{n^\alpha \log n^\alpha}{f(n^\alpha)} \leq \frac{n^\alpha \log n^\alpha}{n^{\alpha^*\beta^*}} \leq n^{\alpha^*(1-\beta^*) - 5c/\sqrt{\log n} + \log\log n^\alpha/\log n} \\
& \leq n^{\alpha^*(1-\beta^*) - (5c-1)/\sqrt{\log n}} \quad \text{(since } 1/\sqrt{\log n} \geq \log\log n^\alpha/\log n \text{ for } n > n_0 \text{)}
\end{aligned} \tag{10}
$$

Therefore,

$$
\begin{aligned}
& f(\frac{n^\alpha \log n^\alpha}{f(n^\alpha)}) \leq f(n^{\alpha^*(1-\beta^*) - (5c-1)/\sqrt{\log n}}) \\
& = n^{\tau\beta^* + 9c\sqrt{\tau}/\sqrt{\log n}} \quad \text{(using (6) with } \lambda = \tau, \text{ where } \tau = \alpha^*(1-\beta^*) - (5c-1)/\sqrt{\log n} \text{)} \\
& \leq n^{\alpha^*\beta^*(1-\beta^*) - (5c-1)\beta^*/\sqrt{\log n} + 9c\sqrt{\alpha^*(1-\beta^*)}/\sqrt{\log n}} \\
& = n^{\alpha^*\beta^*(1-\beta^*) + (9c\alpha^* - (5c-1)\beta^*)/\sqrt{\log n})} \quad \text{(since } 1 - \beta^* = \alpha^* \text{)}
\end{aligned} \tag{11}
$$

Using (8) and (11),

$$f(n^\alpha)\cdot f(\frac{n^\alpha \log n^\alpha}{f(n^\alpha)}) \le n^{\alpha^*\beta^*(2-\beta^*)+(9c(\alpha^*+\sqrt{\alpha^*})-(10c-1)\beta^*)/\sqrt{\log n}}$$
$$\le n^{\beta^*+(9c-1)/\sqrt{\log n}} \text{ (since } (9c(\alpha^*+\sqrt{\alpha^*})-(10c-1)\beta^*) \le 9c-1 \text{ and } \alpha^*\beta^*(2-\beta^*)=\beta^* \text{)}$$
$$= n^{\beta-1/\sqrt{\log n}}$$
$$\le n^\beta/2 \qquad \text{(for } n > n_0 \text{)} \tag{12}$$

On the other hand,

$$f(16\cdot n^{1-\alpha+\gamma}) = f(16\cdot n^{1-2c/\sqrt{\log n}}) \le f(n^{1-c/\sqrt{\log n}}) \quad \text{(since } n^{c/\sqrt{\log n}} \ge 16 \text{ for } n > n_0 \text{)}$$
$$= n^{\beta^*(1-c/\sqrt{\log n})+9c\sqrt{1-c/\sqrt{\log n}}/\sqrt{\log n}} \quad \text{(using (6))}$$
$$\le n^{\beta^*-c\beta^*/\sqrt{\log n}+9c/\sqrt{\log n}} = n^{\beta-c\beta^*/\sqrt{\log n}}$$
$$\le n^\beta/2 \qquad \text{(for } n > n_0 \text{)} \tag{13}$$

From (12) and (13), we conclude the second inequality in the Claim.

Finally, it remains to verify the third inequality:

$$n^{1-\alpha}\cdot f(n^\alpha)\cdot(\log n^\alpha)^{(-\log n^\alpha)/8} \le n^{1-\alpha+\alpha^*\beta^*+(9c\sqrt{\alpha^*}-5c\beta^*)/\sqrt{\log n}-\frac{\alpha\log\log n^\alpha}{8}} \quad \text{(Using (8))}$$
$$= n^{1-\alpha^*+5c/\sqrt{\log n}+\alpha^*\beta^*+(9c\sqrt{\alpha^*}-5c\beta^*)/\sqrt{\log n}-\frac{\alpha\log\log n^\alpha}{8}}$$
$$\le n^{0.62+10.2c/\sqrt{\log n}-\frac{1}{32}\log\log n^{1/4}} \quad \text{(since } \alpha \ge \tfrac{1}{4} \text{ for } n > n_0 \text{)}$$
$$\le n^{1.35-\frac{1}{32}\log\log n^{1/4}} \quad \text{(since } \sqrt{\log n} \ge 14c \text{ for } n > n_0 \text{)}$$
$$\le n^{1.35-43.6/32} \quad \text{(since } \log\log n^{1/4} \ge 43.6 \text{ for } n > n_0 \text{)}$$
$$\le n^{-0.01} < \frac{1}{2} \quad \text{(for } n > n_0 \text{)}$$

The claim follows. □

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A Proof of Theorem 5.1

Let $\chi_i, \chi', \chi'', h, P_i^\ell$ be as defined in the procedure, and $\chi = \chi' \times \chi''$ be the coloring returned in Step 13. The theorem follows from the following two claims.

Claim A.1 ([8]) χ *is quasi-conflict-free.*

Proof. Let $T \in \mathcal{R}$ be any rectangle that lies completely inside a row or a column of G_r, such that $T \cap P \neq \emptyset$. If T contains only points belonging to a single column C_j of G_r, then the fact that algorithm $\mathcal{A}$ returns a conflict-free coloring of C_j and the definition of χ'_j imply that T contains a point $p \in T \cap C_j$ such that $\chi'_j(p) \neq \chi'_j(p')$ for all $p' \in T \cap P$, $p' \neq p$. Then $\chi'(p)$ and hence $\chi(p)$ is different in the first coordinate from $\chi(p')$ for every $p' \in T \cap P$, $p' \neq p$. Now assume that T contains only points belonging to a single row i of G_r. Since $T \cap P \neq \emptyset$, there is an $\ell \in [h]$ such that $T \cap P_i^\ell \neq \emptyset$. Since $\mathcal{A}$ returns a conflict-free coloring $\chi''_{i,\ell}$ of P_i^ℓ, there is a point $p \in T \cap P_i^\ell$, such that $\chi''_{i,\ell}(p) \neq \chi''_{i,\ell}(p')$ for all $p' \in T \cap P_i^\ell$, $p' \neq p$. Thus if $p' \in T \cap R_i$, then either $p' \in P_i^{\ell'}$ for $\ell' \neq \ell$ in which case $\chi'(p') \neq \chi'(p)$, or $p' \in P_i^\ell$ but $\chi''(p') \neq \chi''(p)$. In both cases $\chi(p') \neq \chi(p)$. □

Claim A.2 *With probability at least* $1/2$, $|\operatorname{range}(\chi)| \leq q(B)$ *given by (2).*

Proof. Fix a row $i \in [r]$. For a column $j \in [r]$ and a color $\ell \in [h]$, let $A_{i,j}^\ell = \{p \in R_i \cap C_j : \chi_j(p) = \ell\}$ be the set of points in cell (i, j) assigned color ℓ by the initial (column) coloring χ_j. Since the Algorithm 1 produces a coloring such that all color classes have a size bounded by $2B/h$,

$$|A_{i,j}^\ell| \leq 2B/h. \tag{14}$$

Recall that, for any $j \in [r]$ and $\ell \in [h]$, all the points $p \in A_{i,j}^\ell$ get the same random color $\chi'_j(p)$ in Step 6. Thus we can think of the coloring in Step 6 as of permuting randomly the colors to the sets $A_{i,j}^\ell$, and may use $\chi'(A_{i,j}^\ell)$ to denote the color assigned in Step 6 to all points in $A_{i,j}^\ell$.

For $j \in [r]$ and $\ell, \ell' \in [h]$, let $Y_{i,j}^{\ell',\ell}$ be an indicator random variable that takes value 1 if and only if $\chi'_j(A_{i,j}^{\ell'}) = \ell$, i.e., if all the points in column j assigned initially color ℓ' are reassigned color ℓ by χ'_j (if $\mathcal{A}_{i,j}^{\ell'}$ is empty, then the corresponding random variable is 0 with probability 1). Let $Y_i^\ell = |P_i^\ell| = \sum_{j \in [r],\, \ell' \in [h]}^{r} |A_{i,j}^{\ell'}| Y_{i,j}^{\ell',\ell}$ be the random variable giving the number of points of row i colored ℓ by χ'. Then an easy calculation shows that

$$\mathbb{E}[Y_i^\ell] = \sum_{j \in [r],\, \ell' \in [h]} |A_{i,j}^{\ell'}| \cdot \mathbb{E}[Y_{i,j}^{\ell',\ell}] = \sum_{j \in [r],\, \ell' \in [h]} \frac{|A_{i,j}^{\ell'}|}{h} \leq \frac{B}{h}, \tag{15}$$

since the total number of points in row i of G_r is at most B.

Note that the variable Y_i^ℓ is the sum of negatively correlated random variables[4], and thus applying the Chernoff bound[5], we get by (15) and (14)

$$\Pr[Y_i^\ell \geq \frac{B}{h} \cdot \log B] \leq e^{-\frac{\log B \ln(\log B)}{8}}. \quad (16)$$

Thus, the probability that there exist i and ℓ such that $Y_i^\ell > \frac{B \log B}{h}$ is at most

$$rh(\log B)^{-(\log B)/8} \leq \frac{1}{2},$$

by (1). Therefore with probability at least $1/2$, $|P_i^\ell| \leq B \log B / h$ for all i and ℓ. Since algorithm $\mathcal{A}$ has guarantee $f(\cdot)$, with constant probability, the total number of colors needed is

$$|\operatorname{range}(\chi)| \leq \sum_{\ell=1}^{h} f(|P_i^\ell|) \leq h \cdot f(B \log B / h) = q(B),$$

as claimed. □

[4] That is, for any subset $\{X_i : i \in S\}$ of these variables, $\Pr[\bigwedge_{i \in S}(X_i = 1)] \leq \prod_{i \in S} \Pr[X_i = 1]$.

[5] In particular, the following version [14]: Let $X = \sum_{i=1}^{n} a_i X_i$ be the weighted sum of negatively correlated random variables $X_i \in \{0, 1\}$. Then $\Pr[X \geq (1+\theta)\mu] \leq e^{-\frac{\mu}{4a}(1+\theta)\ln(1+\theta)}$, for $\theta \geq 1$, $a \geq \max\{a_1, \ldots, a_n\}$ and $\mu \geq \mathbb{E}[X]$.