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Research Article

Conformable Integral Inequalities of the Hermite-Hadamard Type in terms of GG- and GA-Convexities

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In the article, we present several conformable fractional integrals' versions of the Hermite-Hadamard type inequalities for GG- and GA-convex functions and provide their applications in special bivariate means.

1. Introduction

Let $I \in \mathbb{R}$ be an interval and let $h : I \longrightarrow \mathbb{R}$ be a convex function. Then the well known HH (Hermite-Hadamard) inequality [1] states that

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} h(x)\,dx \le \frac{h(\kappa_1)+h(\kappa_2)}{2} \tag{1}$$

for all $\kappa_1, \kappa_2 \in I$. It is well known that the convexity has been playing a key role in mathematical programming, engineering, and optimization theory. Recently, many generalizations and extensions for the classical convexity can be found in the literature [2–14]. In [15, 16], Niculescu defined the GA- and GG-convex functions as follows.

Definition 1 (see [15]). A function $h : I \longrightarrow [0,\infty)$ is said to be GA-convex if the inequality

$$h\left(\kappa_1^t\kappa_2^{1-t}\right) \le th(\kappa_1) + (1-t)h(\kappa_2) \tag{2}$$

holds for all $\kappa_1, \kappa_2 \in I$ and $t \in [0,1]$.

Definition 2 (see [16]). A function $h : I \longrightarrow [0,\infty)$ is said to be GG-convex if the inequality

$$h\left(\kappa_1^t\kappa_2^{1-t}\right) \le h(\kappa_1)^t\, h(\kappa_2)^{1-t} \tag{3}$$

holds for all $\kappa_1, \kappa_2 \in I$ and $t \in [0,1]$.

Zhang, Ji, and Qi established Lemma 3 and Theorems 4–7.

Lemma 3 (see [17]). *Let $\kappa_1, \kappa_2 \in \mathbb{R}^+$ and $h : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ be a differentiable function on (κ_1, κ_2). Then the identity*

$$\kappa_2 h(\kappa_2) - \kappa_1 h(\kappa_1) - \int_{\kappa_1}^{\kappa_2} h(x)\,dx = (\log\kappa_2 - \log\kappa_1)\int_0^1 \left(\kappa_2^{2t}\kappa_1^{2(1-t)}\right) h'\left(\kappa_2^t\kappa_1^{1-t}\right)dt \tag{4}$$

holds if $h' \in L([\kappa_1, \kappa_2])$.

Theorem 4 (see [17]). *If the function h satisfies the conditions of Lemma 3 and, additionally, if $|h'(x)|$ is GA-convex, then we have the following inequality:*

$$\left|\kappa_2 h(\kappa_2) - \kappa_1 h(\kappa_1) - \int_{\kappa_1}^{\kappa_2} h(x)\,dx\right| \le \frac{|h'(\kappa_2)|}{2}\left[\kappa_2^2 - L\left(\kappa_2^2,\kappa_1^2\right)\right] + \frac{|h'(\kappa_1)|}{2}\left[L\left(\kappa_2^2,\kappa_1^2\right) - \kappa_1^2\right], \tag{5}$$

where and in what follows $L(\kappa_1,\kappa_2) = (\kappa_2-\kappa_1)/(\log\kappa_2-\log\kappa_1)$ is the logarithmic mean of κ_1 and κ_2.

Theorem 5 (see [17]). *If the function h satisfies the conditions of Lemma 3 and, additionally, if* $|h'(x)|^{\gamma}$ $(\gamma > 1)$ *is GA-convex, then one has*

$$\left|\kappa_2 h(\kappa_2) - \kappa_1 h(\kappa_1) - \int_{\kappa_1}^{\kappa_2} h(x)\,dx\right| \le (\log\kappa_2 - \log\kappa_1)^{1-1/\gamma} \left(L\left(\kappa_2^2, \kappa_1^2\right)\right)^{1-1/\gamma} \times \left(\frac{|h'(\kappa_2)|^{\gamma}\left[\kappa_2^2 - L\left(\kappa_2^2, \kappa_1^2\right)\right] + |h'(\kappa_1)|^{\gamma}\left[L\left(\kappa_2^2, \kappa_1^2\right) - \kappa_1^2\right]}{2}\right)^{1/\gamma}. \quad (6)$$

Theorem 6 (see [17]). *If the function h satisfies the conditions of Lemma 3, then the inequality*

$$\left|\kappa_2 h(\kappa_2) - \kappa_1 h(\kappa_1) - \int_{\kappa_1}^{\kappa_2} h(x)\,dx\right| \le (\log\kappa_2 - \log\kappa_1)^{1-1/\gamma} \times \left(\frac{|h'(\kappa_2)|^{\gamma}\left[\kappa_2^{2\gamma} - L\left(\kappa_2^{2\gamma}, \kappa_1^{2\gamma}\right)\right] + |h'(\kappa_1)|^{\gamma}\left[L\left(\kappa_2^{2\gamma}, \kappa_1^{2\gamma}\right) - \kappa_1^{2\gamma}\right]}{2\gamma}\right)^{1/\gamma} \quad (7)$$

holds if $|h'(x)|^{\gamma}$ $(\gamma > 1)$ *is GA-convex.*

Theorem 7 (see [17]). *If* $\vartheta, \gamma > 1$ *with* $\vartheta^{-1} + \gamma^{-1} = 1$ *and the function h satisfies the conditions of Lemma 3 and, additionally, if* $|h'|^{\gamma}$ *is GA-convex, then we have*

$$\left|\kappa_2 h(\kappa_2) - \kappa_1 h(\kappa_1) - \int_{\kappa_1}^{\kappa_2} h(x)\,dx\right| \le (\log\kappa_2 - \log\kappa_1) \cdot \left(L\left(\kappa_2^{2\vartheta}, \kappa_1^{2\vartheta}\right)\right)^{1/\vartheta} \left(A\left(|h'(\kappa_2)|^{\gamma}, |h'(\kappa_1)|^{\gamma}\right)\right)^{1/\gamma}, \quad (8)$$

where $A(x, y) = (x + y)/2$ *is the arithmetic mean of x and y.*

The conformable fractional derivative of order $0 < \alpha \le 1$ at $s > 0$ for a function $h : [0, \infty) \longrightarrow \mathbb{R}$ was defined in [18] as follows:

$$D_{\alpha}(h)(s) = \lim_{\epsilon \longrightarrow 0} \frac{h\left(s + \epsilon s^{1-\alpha}\right) - h(s)}{\epsilon}. \quad (9)$$

h is said to be α-fractional differentiable if the conformable fractional derivative of h of order α exists. The fractional derivative at 0 is defined as $h^{\alpha}(0) = \lim_{s \longrightarrow 0^+} h^{\alpha}(s)$. Theorem 8 for the conformable fractional derivative can be found in the literature [18].

Theorem 8 (see [18]). *Let* $\alpha \in (0, 1]$ *and* h_1, h_2 *be* α*-differentiable at* $t > 0$*. Then*

(i) $(d_{\alpha}/d_{\alpha}t)(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$

(ii) $(d_{\alpha}/d_{\alpha}t)(c) = 0$ *if* $c \in \mathbb{R}$ *is a constant*

(iii) $(d_{\alpha}/d_{\alpha}t)(\kappa_1 h_1(t) + \kappa_2 h_2(t)) = \kappa_1 (d_{\alpha}/d_{\alpha}t)(h_1(t)) + \kappa_2 (d_{\alpha}/d_{\alpha}t)(h_2(t))$ *for all* $\kappa_1, \kappa_2 \in \mathbb{R}$

(iv) $(d_{\alpha}/d_{\alpha}t)(h_1(t)h_2(t)) = h_1(t)(d_{\alpha}/d_{\alpha}t)(h_2(t)) + h_2(t)(d_{\alpha}/d_{\alpha}t)(h_1(t))$

(v) $(d_{\alpha}/d_{\alpha}t)(h_1(t)/h_2(t)) = (h_2(t)(d_{\alpha}/d_{\alpha}t)(h_1(t)) - h_1(t)(d_{\alpha}/d_{\alpha}t)(h_2(t)))/(h_2(t))^2$

(vi) $(d_{\alpha}/d_{\alpha}t)((h_1^{\circ} h_2)(t)) = h_1'(h_2(t))(d_{\alpha}/d_{\alpha}t)(h_2(t))$ *if* h_1 *is differentiable at* $h_2(t)$*.*

In addition,

$$\frac{d_{\alpha}}{d_{\alpha}t}(h_1(t)) = t^{1-\alpha}\frac{d}{dt}(h_1(t)) \quad (10)$$

if h_1 *is differentiable.*

Definition 9 (see [18], conformable fractional integral). Let $\alpha \in (0, 1]$ and $0 \le \kappa_1 < \kappa_2$. A function $h : [\kappa_1, \kappa_2] \longrightarrow \mathbb{R}$ is α-fractional integrable on $[\kappa_1, \kappa_2]$ if the integral

$$\int_{\kappa_1}^{\kappa_2} h(x)\,d_{\alpha}x := \int_{\kappa_1}^{\kappa_2} h(x)\,x^{\alpha-1}dx \quad (11)$$

exists and is finite. All α-fractional integrable functions on $[\kappa_1, \kappa_2]$ are indicated by $L_{\alpha}([\kappa_1, \kappa_2])$.

Remark 10.

$$I_{\alpha}^{\kappa_1}(h_1)(s) = I_1^{\kappa_1}\left(s^{\alpha-1}h_1\right) = \int_{\kappa_1}^{s} \frac{h_1(x)}{x^{1-\alpha}}dx, \quad (12)$$

where the integral is the usual Riemann improper integral and $\alpha \in (0, 1]$.

Recently, the conformable integrals and derivatives have been the subject of intensive research; many remarkable properties and inequalities involving the conformable integrals and derivatives can be found in the literature [19–37].

Anderson [38] provided the conformable integral version of the HH inequality as follows.

Theorem 11 (see [38]). *If $\alpha \in (0,1]$ and $h : [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha h$ is increasing, then the inequality*

$$\frac{\alpha}{\kappa_2^\alpha - \kappa_1^\alpha}\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x \le \frac{h(\kappa_1)+h(\kappa_2)}{2} \quad (13)$$

holds. Moreover if h is decreasing on $[\kappa_1,\kappa_2]$, then we have

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{\alpha}{\kappa_2^\alpha - \kappa_1^\alpha}\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x. \quad (14)$$

If $\alpha = 1$, then this reduces to the classical HH inequality.

In this paper, we shall establish the Hermite-Hadamard type inequalities for GA and GG-convex functions via conformable fractional integrals and give their applications in the special bivariate means.

2. Main Results

In order to establish our main results, we need a lemma which we present in this section.

Lemma 12. *Let $\kappa_1,\kappa_2 \in \mathbb{R}^+$, $\alpha \in (0,1]$, and $h : [\kappa_1,\kappa_2] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function on (κ_1,κ_2). Then the identity*

$$\begin{aligned}&\kappa_2^\alpha h(\kappa_2) - \kappa_1^\alpha h(\kappa_1) - \alpha\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x \\ &= (\log\kappa_2 - \log\kappa_1) \\ &\cdot \int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{2\alpha} D_\alpha(h)\left(\kappa_2^t\kappa_1^{1-t}\right) t^{1-\alpha} d_\alpha t\end{aligned} \quad (15)$$

holds if $D_\alpha(h) \in L_\alpha([\kappa_1,\kappa_2])$.

Proof. Using integration by parts, we have

$$\begin{aligned}I &= \int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{2\alpha} D_\alpha(h)\left(\kappa_2^t\kappa_1^{1-t}\right) dt \\ &= \int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{\alpha+1} h'\left(\kappa_2^t\kappa_1^{1-t}\right) dt.\end{aligned} \quad (16)$$

By the change of the variable $x = \kappa_2^t\kappa_1^{1-t}$ and integration by parts, we have

$$\begin{aligned}I &= \frac{1}{\log\kappa_2 - \log\kappa_1}\int_{\kappa_1}^{\kappa_2} x^\alpha h'(x)\,dx \\ &= \frac{1}{\log\kappa_2 - \log\kappa_1}\Big[\kappa_2^\alpha h(\kappa_2) - \kappa_1^\alpha h(\kappa_1) \\ &\quad - \alpha\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x\Big].\end{aligned} \quad (17)$$

Now multiplying by $(\log\kappa_2 - \log\kappa_1)$, we obtain the required result. □

Remark 13. Let $\alpha = 1$, then Lemma 12 reduces to Lemma 3.

Theorem 14. *If the function h satisfies the conditions of Lemma 12 and, additionally, if $|h'(x)|$ is GG-convex, then we have*

$$\begin{aligned}&\left|\kappa_2^\alpha h(\kappa_2) - \kappa_1^\alpha h(\kappa_1) - \alpha\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x\right| \\ &\le (\log\kappa_2 - \log\kappa_1)\, L\left(\kappa_2^{\alpha+1}\left|h'(\kappa_2)\right|, \kappa_1^{\alpha+1}\left|h'(\kappa_1)\right|\right).\end{aligned} \quad (18)$$

Proof. It follows from the GG-convexity and Lemma 12 that

$$\begin{aligned}&\left|\kappa_2^\alpha h(\kappa_2) - \kappa_1^\alpha h(\kappa_1) - \alpha\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x\right| \\ &\le (\log\kappa_2 - \log\kappa_1)\int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{\alpha+1}\left|h'\left(\kappa_2^t\kappa_1^{1-t}\right)\right| dt \\ &\le (\log\kappa_2 - \log\kappa_1) \\ &\cdot \int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{\alpha+1}\left|h'(\kappa_2)\right|^t\left|h'(\kappa_1)\right|^{1-t} dt.\end{aligned} \quad (19)$$

The desired result can be obtained by evaluating the above integral. □

Theorem 15. *If $\vartheta,\gamma > 1$ with $\vartheta^{-1} + \gamma^{-1} = 1$ and the function h satisfies the conditions of Lemma 12 and, additionally, if $|h'|^\gamma$ is GG-convex, then one has*

$$\begin{aligned}&\left|\kappa_2^\alpha h(\kappa_2) - \kappa_1^\alpha h(\kappa_1) - \alpha\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x\right| \\ &\le (\log\kappa_2 - \log\kappa_1)\left(L\left(\kappa_2^{(\alpha+1)\vartheta}, \kappa_1^{(\alpha+1)\vartheta}\right)\right)^{1/\vartheta} \\ &\cdot \left(L\left(\left|h'(\kappa_2)\right|^\gamma, \left|h'(\kappa_1)\right|^\gamma\right)\right)^{1/\gamma}.\end{aligned} \quad (20)$$

Proof. It follows from Lemma 12, the property of the modulus, the GG-convexity of $|h'|^\gamma$, and the Hölder inequality that

$$\begin{aligned}&\left|\kappa_2^\alpha h(\kappa_2) - \kappa_1^\alpha h(\kappa_1) - \alpha\int_{\kappa_1}^{\kappa_2} h(x)\,d_\alpha x\right| \\ &\le (\log\kappa_2 - \log\kappa_1)\int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{\alpha+1}\left|h'\left(\kappa_2^t\kappa_1^{1-t}\right)\right| dt \\ &\le (\log\kappa_2 - \log\kappa_1)\left(\int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{(\alpha+1)\vartheta} dt\right)^{1/\vartheta} \\ &\cdot \left(\int_0^1 \left|h'\left(\kappa_2^t\kappa_1^{1-t}\right)\right|^\gamma dt\right)^{1/\gamma} \le (\log\kappa_2 - \log\kappa_1) \\ &\cdot \left(\int_0^1 \left(\kappa_2^t\kappa_1^{1-t}\right)^{(\alpha+1)\vartheta} dt\right)^{1/\vartheta} \\ &\cdot \left(\int_0^1 \left|h'(\kappa_2)\right|^{\gamma t}\left|h'(\kappa_1)\right|^{(1-t)\gamma} dt\right)^{1/\gamma}.\end{aligned} \quad (21)$$

The desired result can be obtained by evaluating the above integral. □

Theorem 16. *If the function h satisfies the conditions of Lemma 12 and, additionally, if $|h'(x)|^{\gamma}$ $(\gamma > 1)$ is GG-convex, then we have the inequality*

$$\begin{aligned}&\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha} x\right| \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \\ &\quad \cdot \left(L\left(\kappa_2^{(\alpha+1)\gamma} \left|h'(\kappa_2)\right|^{\gamma}, \kappa_1^{(\alpha+1)\gamma} \left|h'(\kappa_1)\right|^{\gamma}\right)\right)^{1/\gamma}.\end{aligned} \tag{22}$$

Proof. By using Lemma 12 we clearly see that

$$\begin{aligned}&\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha} x\right| \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right| dt.\end{aligned} \tag{23}$$

Since $\gamma > 1$, we can choose $\vartheta > 1$ such that $\vartheta^{-1} + \gamma^{-1} = 1$. Applying the Hölder integral inequality and the GG-convexity of $|h'|^{\gamma}$ we have

$$\begin{aligned}&\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha} x\right| \leq (\log \kappa_2 \\ &\quad - \log \kappa_1) \left(\int_0^1 dt\right)^{1/\vartheta} \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)\gamma}\right. \\ &\quad \left.\cdot \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right|^{\gamma} dt\right)^{1/\gamma} \leq (\log \kappa_2 - \log \kappa_1) \left(\int_0^1 dt\right)^{1/\vartheta} \\ &\quad \cdot \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)\gamma} \left|h'(\kappa_2)\right|^{\gamma t} \left|h'(\kappa_1)\right|^{(1-t)\gamma} dt\right)^{1/\gamma}.\end{aligned} \tag{24}$$

The desired result can be obtained by evaluating the above integral. □

Theorem 17. *If the function h satisfies the conditions of Lemma 12 and, additionally, if $|h'(x)|^{\gamma}$ $(\gamma > 1)$ is GG-convex, then we have*

$$\begin{aligned}&\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha} x\right| \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \left(L\left(\kappa_2^{(\alpha+1)}, \kappa_1^{(\alpha+1)}\right)\right)^{1-1/\gamma} \\ &\quad \cdot \left(L\left(\kappa_2^{(\alpha+1)} \left|h'(\kappa_2)\right|^{\gamma}, \kappa_1^{(\alpha+1)} \left|h'(\kappa_1)\right|^{\gamma}\right)\right)^{1/\gamma}.\end{aligned} \tag{25}$$

Proof. From the GG-convexity of $|h'|^{\gamma}$, the power mean inequality, and the property of the modulus together with Lemma 12 we get

$$\begin{aligned}&\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha} x\right| \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right| dt \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} dt\right)^{1-1/\gamma} \\ &\quad \cdot \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right|^{\gamma} dt\right)^{1/\gamma} \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} dt\right)^{1-1/\gamma} \\ &\quad \cdot \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} \left|h'(\kappa_2)\right|^{\gamma t} \left|h'(\kappa_1)\right|^{(1-t)\gamma} dt\right)^{1/\gamma}.\end{aligned} \tag{26}$$

The desired result can be obtained by evaluating the above integral. □

Theorem 18. *If the function h satisfies the conditions of Lemma 12 and, additionally, if $|h'(x)|$ is GA-convex, then*

$$\begin{aligned}&\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha} x\right| \\ &\quad \leq \frac{\left|h'(\kappa_2)\right|}{\alpha+1} \left[\kappa_2^{\alpha+1} - L\left(\kappa_2^{\alpha+1}, \kappa_1^{\alpha+1}\right)\right] \\ &\quad + \frac{\left|h'(\kappa_1)\right|}{\alpha+1} \left[L\left(\kappa_2^{\alpha+1}, \kappa_1^{\alpha+1}\right) - \kappa_1^{\alpha+1}\right].\end{aligned} \tag{27}$$

Proof. It follows from the GA-convexity of $|h'|$ and the property of the modulus together with Lemma 12 that

$$\begin{aligned}&\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha} x\right| \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right| dt \\ &\quad \leq (\log \kappa_2 - \log \kappa_1) \\ &\quad \cdot \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} \left[t \left|h'(\kappa_2)\right| + (1-t) \left|h'(\kappa_1)\right|\right] dt \\ &\quad = (\log \kappa_2 - \log \kappa_1) \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} t \left|h'(\kappa_2)\right| dt \\ &\quad + (\log \kappa_2 - \log \kappa_1) \\ &\quad \cdot \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} (1-t) \left|h'(\kappa_1)\right| dt.\end{aligned} \tag{28}$$

The desired result can be obtained by evaluating the above integrals. □

Remark 19. By setting $\alpha = 1$ in inequality (27), we regain inequality (5).

Theorem 20. *If the function h satisfies the conditions of Lemma 12 and, additionally, if* $|h'(x)|^{\gamma}$ $(\gamma > 1)$ *is GA-convex, then we have the following inequality:*

$$\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha}x\right| \leq (\log \kappa_2 - \log \kappa_1)^{1-1/\gamma} \left(L\left(\kappa_2^{(\alpha+1)}, \kappa_1^{(\alpha+1)}\right)\right)^{1-1/\gamma} \times \left(\frac{\left|h'(\kappa_2)\right|^{\gamma}\left[\kappa_2^{\alpha+1} - L\left(\kappa_2^{\alpha+1}, \kappa_1^{\alpha+1}\right)\right] + \left|h'(\kappa_1)\right|^{\gamma}\left[L\left(\kappa_2^{\alpha+1}, \kappa_1^{\alpha+1}\right) - \kappa_1^{\alpha+1}\right]}{\alpha+1}\right)^{1/\gamma}. \tag{29}$$

Proof. From the GA-convexity of $|h'|^{\gamma}$, the power mean inequality, the property of the modulus, and Lemma 12 we clearly see that

$$\begin{aligned}\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha}x\right| &\leq (\log \kappa_2 - \log \kappa_1) \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right| dt \\ &\leq (\log \kappa_2 - \log \kappa_1) \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} dt\right)^{1-1/\gamma} \cdot \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right|^{\gamma} dt\right)^{1/\gamma} \\ &\leq (\log \kappa_2 - \log \kappa_1) \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} dt\right)^{1-1/\gamma} \times \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{(\alpha+1)} \cdot \left[t\left|h'(\kappa_2)\right|^{\gamma} + (1-t)\left|h'(\kappa_1)\right|^{\gamma}\right] dt\right)^{1/\gamma}.\end{aligned} \tag{30}$$

The desired result can be obtained by evaluating the above integrals. ☐

Remark 21. By setting $\alpha = 1$ in inequality (29), we regain inequality (6).

Theorem 22. *If the function h satisfies the conditions of Lemma 12 and, additionally, if* $|h'(x)|$ $(\gamma > 1)$ *is GA-convex, then we have the following inequality:*

$$\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha}x\right| \leq (\log \kappa_2 - \log \kappa_1)^{1-1/\gamma} \times \left(\frac{\left|h'(\kappa_2)\right|^{\gamma}\left[\kappa_2^{\gamma(\alpha+1)} - L\left(\kappa_2^{\gamma(\alpha+1)}, \kappa_1^{\gamma(\alpha+1)}\right)\right] + \left|h'(\kappa_1)\right|^{\gamma}\left[L\left(\kappa_2^{\gamma(\alpha+1)}, \kappa_1^{\gamma(\alpha+1)}\right) - \kappa_1^{\gamma(\alpha+1)}\right]}{\gamma(\alpha+1)}\right)^{1/\gamma}. \tag{31}$$

Proof. With the help of the GA-convexity of $|h'|^{\gamma}$, the power mean inequality, the property of the modulus, and Lemma 12, we can write

$$\begin{aligned}\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha}x\right| &\leq (\log \kappa_2 - \log \kappa_1) \int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\alpha+1} \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right| dt \\ &\leq (\log \kappa_2 - \log \kappa_1) \left(\int_0^1 dt\right)^{1-1/\gamma} \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\gamma(\alpha+1)} \cdot \left|h'\left(\kappa_2^t \kappa_1^{1-t}\right)\right|^{\gamma} dt\right)^{1/\gamma} \\ &\leq (\log \kappa_2 - \log \kappa_1) \left(\int_0^1 dt\right)^{1-1/\gamma} \cdot \left(\int_0^1 \left(\kappa_2^t \kappa_1^{1-t}\right)^{\gamma(\alpha+1)} \cdot \left[t\left|h'(\kappa_2)\right|^{\gamma} + (1-t)\left|h'(\kappa_1)\right|^{\gamma}\right] dt\right)^{1/\gamma}.\end{aligned} \tag{32}$$

The desired result can be obtained by evaluating the above integrals. ☐

Remark 23. By setting $\alpha = 1$ in inequality (31), we regain inequality (7).

Theorem 24. *If* $\vartheta, \gamma > 1$ *with* $\vartheta^{-1} + \gamma^{-1} = 1$ *and the function h satisfies the conditions of Lemma 12 and, additionally, if* $|h'|^{\gamma}$ *is GA-convex, then we have the following inequality:*

$$\left|\kappa_2^{\alpha} h(\kappa_2) - \kappa_1^{\alpha} h(\kappa_1) - \alpha \int_{\kappa_1}^{\kappa_2} h(x)\, d_{\alpha}x\right| \leq (\log \kappa_2 - \log \kappa_1) \left(L\left(\kappa_2^{(\alpha+1)\vartheta}, \kappa_1^{(\alpha+1)\vartheta}\right)\right)^{1/\vartheta}$$

$$\cdot\left(A\left(\left|h'(\kappa_2)\right|^{\gamma},\left|h'(\kappa_1)\right|^{\gamma}\right)\right)^{1/\gamma}, \tag{33}$$

where $A(x,y)$ represents the arithmetic mean of x and y.

Proof. With the help of the GA-convexity of $|h'|^{\gamma}$, the Hölder integral inequality, the property of the modulus, and Lemma 12, we can write

$$\begin{aligned}
&\left|\kappa_2^{\alpha}h(\kappa_2)-\kappa_1^{\alpha}h(\kappa_1)-\alpha\int_{\kappa_1}^{\kappa_2}h(x)\,d_{\alpha}x\right| \\
&\leq(\log\kappa_2-\log\kappa_1)\int_0^1\left(\kappa_2^t\kappa_1^{1-t}\right)^{\alpha+1}\left|h'\left(\kappa_2^t\kappa_1^{1-t}\right)\right|dt \\
&\leq(\log\kappa_2-\log\kappa_1)\left(\int_0^1\left(\kappa_2^t\kappa_1^{1-t}\right)^{(\alpha+1)\vartheta}dt\right)^{1/\vartheta} \\
&\cdot\left(\int_0^1 t\left|h'(\kappa_2)\right|^{\gamma}+(1-t)\left|h'(\kappa_1)\right|^{\gamma}dt\right)^{1/\gamma} \\
&=(\log\kappa_2-\log\kappa_1)\left(\kappa_1^{(\alpha+1)\vartheta}\int_0^1\left(\frac{\kappa_2^{(\alpha+1)\vartheta}}{\kappa_1^{(\alpha+1)\vartheta}}\right)^t dt\right)^{1/\vartheta} \\
&\cdot\left(\int_0^1 t\left|h'(\kappa_2)\right|^{\gamma}+(1-t)\left|h'(\kappa_1)\right|^{\gamma}dt\right)^{1/\gamma}.
\end{aligned} \tag{34}$$

The desired result can be obtained by evaluating the above integrals. □

Remark 25. By setting $\alpha = 1$ in inequality (33), we regain inequality (8).

3. Applications to Special Means

A bivariate function $M:(0,\infty)\times(0,\infty)\longmapsto(0,\infty)$ is said to be a bivariate mean if $\min\{\kappa_1,\kappa_2\}\leq M(\kappa_1,\kappa_2)\leq\max\{\kappa_1,\kappa_2\}$ for all $\kappa_1,\kappa_2\in(0,\infty)$. Recently, the bivariate mean has attracted the attention of many researchers; in particular, many remarkable inequalities for the bivariate means and their related special functions can be found in the literature [39–42].

In this section, we use the results obtained in Section 2 to present several applications to the arithmetic mean

$$A(\kappa_1,\kappa_2)=\frac{\kappa_1+\kappa_2}{2},\quad \kappa_1,\kappa_2>0, \tag{35}$$

logarithmic mean

$$L(\kappa_1,\kappa_2)=\frac{\kappa_2-\kappa_1}{\log\kappa_2-\log\kappa_1},\quad \kappa_1\neq\kappa_2,\ \kappa_1,\kappa_2\in\mathbb{R}^+, \tag{36}$$

and (α,μ)-th generalized logarithmic mean

$$\begin{aligned}
&L_{(\alpha,\mu+1)}(\kappa_1,\kappa_2) \\
&=\left[\frac{\alpha\left(\kappa_2^{\mu+\alpha+1}-\kappa_1^{\mu+\alpha+1}\right)}{\left(\kappa_2^{\alpha}-\kappa_1^{\alpha}\right)(\mu+\alpha+1)(\mu+1)}\right]^{1/(\mu+1)}, \\
&\kappa_1\neq\kappa_2,\ \mu\neq-1,-\alpha-1,\ \alpha\in(0,1],\ \mu\in\mathbb{R}.
\end{aligned} \tag{37}$$

Proposition 26. *Let $\alpha\in(0,1]$, $\kappa_1,\kappa_2\in\mathbb{R}^+$, and $\mu>0$. Then*

$$\begin{aligned}
&\left|\frac{\left(\kappa_2^{\alpha}-\kappa_1^{\alpha}\right)}{\alpha(\mu+1)}L_{(\alpha,\mu+1)}^{\mu+1}(\kappa_1,\kappa_2)\right| \\
&\leq\frac{|\kappa_2|^{\mu}}{\alpha+1}\left[\kappa_2^{\alpha+1}-L\left(\kappa_2^{\alpha+1},\kappa_1^{\alpha+1}\right)\right] \\
&+\frac{|\kappa_1|^{\mu}}{\alpha+1}\left[L\left(\kappa_2^{\alpha+1},\kappa_1^{\alpha+1}\right)-\kappa_2^{\alpha+1}\right].
\end{aligned} \tag{38}$$

Proof. Let

$$h(x)=\frac{x^{\mu+1}}{\mu+1} \tag{39}$$

for $x>0$. Then $|h'(x)|^{\gamma}$ is a GA-convex function on $\mathbb{R}^+$ for $\gamma\geq 1$. Let $\gamma=1$. Then making use of function (39) in Theorem 18, we obtain the required result. □

Proposition 27. *Let $\kappa_1,\kappa_2\in\mathbb{R}^+$, $\gamma>1$, $\mu>0$, and $\alpha\in(0,1]$. Then*

$$\begin{aligned}
\left|\frac{\left(\kappa_2^{\alpha}-\kappa_1^{\alpha}\right)}{\alpha(\mu+1)}L_{(\alpha,\mu+1)}^{\mu+1}(\kappa_1,\kappa_2)\right| &\leq(\log\kappa_2-\log\kappa_1)^{1-1/\gamma}\left(L\left(\kappa_2^{(\alpha+1)},\kappa_1^{(\alpha+1)}\right)\right)^{1-1/\gamma} \\
&\times\left(\frac{|\kappa_2|^{\mu\gamma}\left[\kappa_2^{\alpha+1}-L\left(\kappa_2^{\alpha+1},\kappa_1^{\alpha+1}\right)\right]+|\kappa_1|^{\mu\gamma}\left[L\left(\kappa_2^{\alpha+1},\kappa_1^{\alpha+1}\right)-\kappa_1^{\alpha+1}\right]}{\alpha+1}\right)^{1/\gamma}.
\end{aligned} \tag{40}$$

Proof. Using function (39) in Theorem 20, we obtain the required result. □

Proposition 28. *Let $\kappa_1,\kappa_2\in\mathbb{R}^+$, $\gamma>1$, $\mu>0$, and $\alpha\in(0,1]$. Then*

$$\left|\frac{\left(\kappa_2^{\alpha}-\kappa_1^{\alpha}\right)}{\alpha(\mu+1)}L_{(\alpha,\mu+1)}^{\mu+1}(\kappa_1,\kappa_2)\right|\leq(\log\kappa_2-\log\kappa_1)^{1-1/\gamma}$$

$$\times\left(\frac{|\kappa_2|^{\mu\gamma}\left[\kappa_2^{\gamma(\alpha+1)}-L\left(\kappa_2^{\gamma(\alpha+1)},\kappa_1^{\gamma(\alpha+1)}\right)\right]+|\kappa_1|^{\mu\gamma}\left[L\left(\kappa_2^{\gamma(\alpha+1)},\kappa_1^{\gamma(\alpha+1)}\right)-\kappa_1^{\gamma(\alpha+1)}\right]}{\gamma(\alpha+1)}\right)^{1/\gamma}. \tag{41}$$

Proof. Using function (39) in Theorem 22, we obtain the required result. □

Proposition 29. *Let* $\kappa_1, \kappa_2 \in \mathbb{R}^+$, $\mu > 0$, $\alpha \in (0,1]$, *and* $\vartheta, \gamma > 1$ *with* $\vartheta^{-1} + \gamma^{-1} = 1$. *Then*

$$\left|\frac{(\kappa_2^\alpha-\kappa_1^\alpha)}{\alpha(\mu+1)}L_{(\alpha,\mu+1)}^{\mu+1}(\kappa_1,\kappa_2)\right| \le (\log\kappa_2-\log\kappa_1) \cdot\left(L\left(\kappa_2^{\vartheta(\alpha+1)},\kappa_1^{\vartheta(\alpha+1)}\right)\right)^{1/\vartheta}\left(A\left(|\kappa_2|^{\mu\gamma},|\kappa_1|^{\mu\gamma}\right)\right)^{1/\gamma}. \tag{42}$$

Proof. Using function (39) in Theorem 24, we obtain the required result. □

4. Conclusions

In the article, we derive the conformable fractional integrals' versions of the Hermite-Hadamard type inequalities for GG- and GA-convex functions. Our approach is based on an identity involving the conformable fractional integrals, the Hölder inequality, and the power mean inequality. The proven results generalized some previously obtained results. As applications, we provide several inequalities for some special bivariate means. The present idea may stimulate further research in the theory of inequalities for other generalized integrals, for example, as presented in [35–37].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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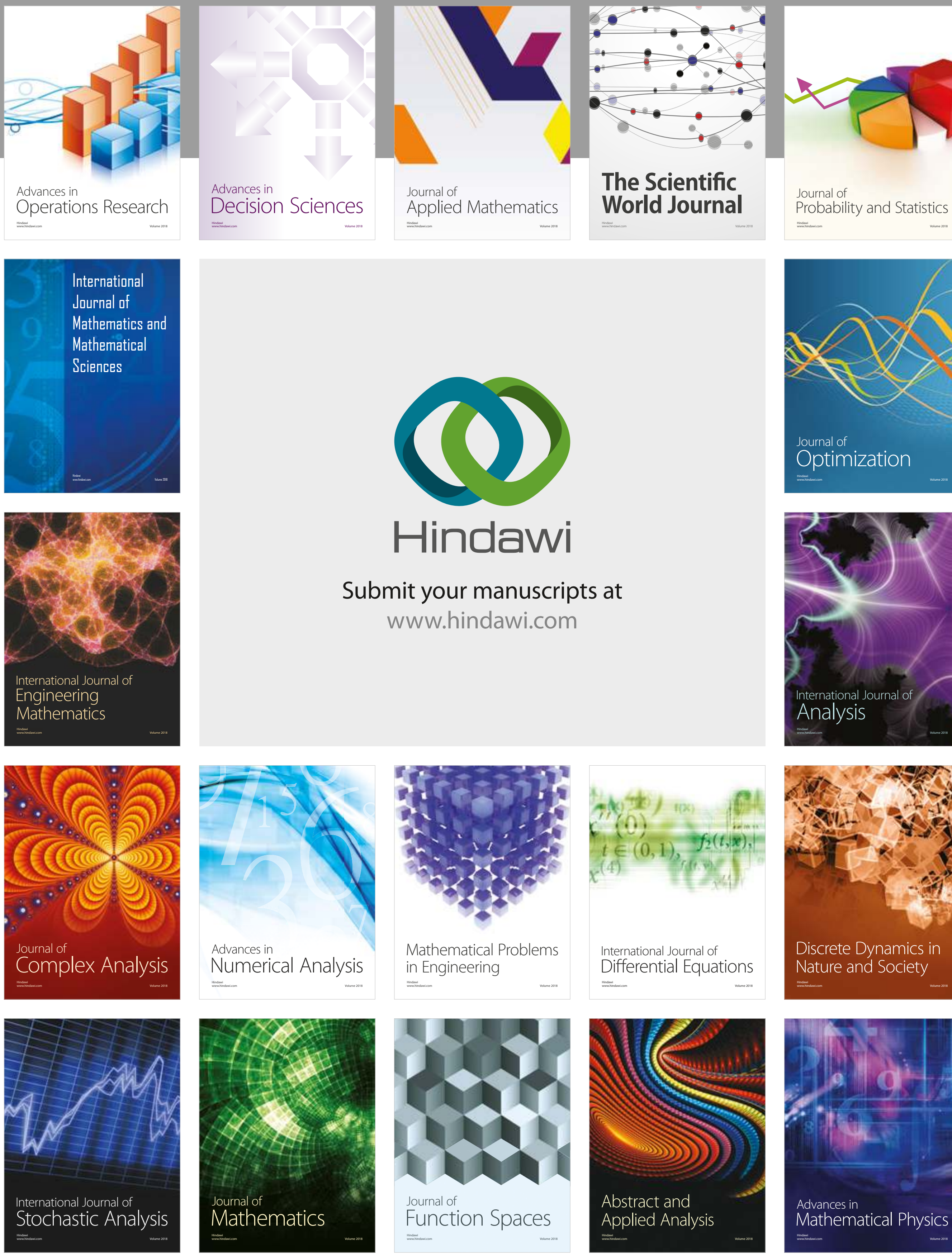
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