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CONFORMAL AND MINIMAL IMMERSIONS OF COMPACT SURFACES INTO THE 4-SPHERE

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ABSTRACT. We study the twistor map of Penrose, T: $\mathbf{CP}^3 \to S^4$ and show that the complex 2-plane field in $\mathbf{CP}^3$ orthogonal to the fibers of T is a holomorphic nonintegrable 2-plane field. We then show that every horizontal holomorphic curve in $\mathbf{CP}^3$ projects under T to be a minimal surface in S^4. Finally, we use the Riemann-Roch theorem to construct, for any compact Riemann surface M^2, a holomorphic horizontal curve Φ: $M^2 \to \mathbf{CP}^3$ without ramification. It follows that $T \circ \Phi$: $M^2 \to S^4$ is a conformal and minimal immersion.

0. Introduction

The study of minimal surfaces in spheres has received much attention. In [7] Lawson proved that every compact surface except $\mathbf{RP}^2$ could be immersed into S^3 as a minimal surface. However, it is unknown whether every compact *Riemann* surface (= compact surface with a fixed complex structure) can be conformally and minimally immersed into S^3.

In [2], [3] Calabi studied minimal surfaces in Euclidean spheres, and in [4], [5] Chern studied minimal immersions of the two-sphere into S^4 and more general spaces of constant curvature. Given a minimal immersion X: $M^2 \to S^n$, where M^2 is assumed oriented and is given the unique complex structure compatible with the orientation and the metric on M^2 induced by the immersion, they found that they could construct a holomorphic quartic form Q_X on M^2 from the second fundamental form of the immersion. If $M^2 = S^2$, then the Riemann-Roch theorem shows that $Q_X \equiv 0$. Exploiting this fact, Calabi and Chern were able to prove extensive results concerning minimal immersions of S^2 into S^n. In the present paper, immersions X: $M^2 \to S^n$ satisfying $Q_X \equiv 0$ are referred to as *superminimal* immersions. In an unpublished work, the author has shown that the over-determined system of partial differential equations whose solutions are the superminimal immersions is *involutive* in Cartan's sense, so one expects a good local theory.

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In [1] the author considers "complex curves" ϕ: $M^2 \to S^6$. These are smooth maps from a given Riemann surface M^2 into S^6 whose differential at any point is complex linear with respect to the standard almost complex structure on S^6. Such a map ϕ is automatically superminimal on the open set where ϕ is an immersion. One constructs a map π: $Q_5 \to S^6$ where $Q_5 \subseteq \mathbf{CP}^6$ is the complex hyperquadric and a holomorphic differential system $\mathcal{I} \subseteq \Omega^{1,0}Q_5$ so that the holomorphic integral curves of $\mathcal{I}$, Φ: $M^2 \to Q_5$, project to S^6 as complex curves. By using a local normal form for $\mathcal{I}$ due to Élie Cartan, the Riemann-Roch theorem may be applied to show that every Riemann surface M^2 has a holomorphic map Φ: $M^2 \to Q_5$ which is generically 1-1 (and which ramifies over a finite divisor in M^2) and moreover is an integral of $\mathcal{I}$. In this way, we show that every Riemann surface appears as a "complex curve" in S^6. In terms of minimal surfaces, this shows that every Riemann surface occurs as a minimal surface in S^6 with a finite number of classical branch points.

In the present paper, we show that a similar result holds for superminimal surfaces in S^4.

In §1 we study the geometry of the celebrated "twistor map" of Penrose T: $\mathbf{CP}^3 \to S^4$. This section is essentially expository; we are merely collecting the facts we need from an extensive literature and formulating them in terms of the moving frame. In particular, we show that the complex 2-plane field orthogonal to the fibers of T (under the standard Fubini-Study metric on $\mathbf{CP}^3$) is a holomorphic nonintegrable distribution on $\mathbf{CP}^3$; moreover, the metrics induced on each such 2-plane by restriction of the Fubini-Study metric and by pull-back of the standard S^4-metric are the same.

In §2 we show that if Φ: $M^2 \to \mathbf{CP}^3$ is a holomorphic horizontal curve (where M^2 is a Riemann surface), then $T \circ \Phi$: $M^2 \to S^4$ is a superminimal surface (with classical branch points where Φ ramifies). Conversely, every superminimal immersion Ψ: $M^2 \to S^4$ is shown to be of the form $\Psi = T \circ \Phi$ where Φ is a (essentially) unique holomorphic horizontal curve Φ: $M^2 \to S^4$.

Finally, in §3 we derive a "Weierstrass formula" which shows how to produce a holomorphic horizontal curve $\Phi(f, g)$: $M^2 \to \mathbf{CP}^3$ for any pair of meromorphic functions f and g (with $dg \not\equiv 0$) on a Riemann surface M^2. We then use a global theorem, Riemann-Roch, to show that for any compact Riemann surface M^2, we can find meromorphic functions f and g on M^2 so that $\Phi(f, g)$ is an immersion (in fact, we may easily arrange to have $\Phi(f, g)$ an embedding). We conclude that $T \circ \Phi(f, g)$: $M^2 \to S^4$ is a conformal minimal immersion, thus proving our theorem.

1. The structure equations and some relevant geometry

We let $\mathbf{H}$ denote the real division algebra of quaternions. An element of $\mathbf{H}$ can be written uniquely in the form $q = z + jw$ where z, $w \in \mathbf{C}$, and $j \in \mathbf{H}$ satisfies

$$(1.1) \qquad j^2 = -1, \quad zj = j\bar{z}$$

for all $z \in \mathbf{C}$. In this way, we regard $\mathbf{C} \subseteq \mathbf{H}$ as a subalgebra and we give $\mathbf{H}$ the structure of a complex vector space by letting $\mathbf{C}$ act *on the right* (because of (1.1), this specification is necessary).

We let $\mathbf{H}^2$ denote the space of pairs (q_1, q_2) where $q_a \in \mathbf{H}$ for $a = 1, 2$. We will make $\mathbf{H}^2$ into a quaternion vector space by letting $\mathbf{H}$ act *on the right*:

$$(q_1, q_2) \cdot p = (q_1 p, q_2 p).$$

Since $\mathbf{C} \subseteq \mathbf{H}$, this automatically makes $\mathbf{H}^2$ into a complex vector space of dimension 4. In fact, regarding $\mathbf{C}^4$ as the space of 4-tuples (z_0, z_1, z_2, z_3), we make the explicit identification

$$(1.2) \qquad (z_0, z_1, z_2, z_3) \simeq (z_0 + jz_1, z_2 + jz_3).$$

This specific isomorphism is the one we will always mean when we write "$\mathbf{C}^4 = \mathbf{H}^2$".

If $v \in \mathbf{H}^2 - \{(0,0)\}$ is given, let $v\mathbf{C}$ and $v\mathbf{H}$ denote, respectively, the complex line and the quaternion line spanned by v. We have $v\mathbf{C} \subseteq v\mathbf{H}$, moreover, because $\mathbf{H}$ is associative, $(v\mathbf{C})\mathbf{H} = v\mathbf{H}$. It follows that the assignment $v\mathbf{C} \to v\mathbf{H}$ induces a well-defined mapping $T\colon \mathbf{CP}^3 \to \mathbf{HP}^1$. Since $T^{-1}(v\mathbf{H})$ consists of all the complex lines in $v\mathbf{H} \simeq \mathbf{C}^2$, we see that the fibres of T are $\mathbf{CP}^1$'s. As we will see below, T is submersive (and T is clearly surjective), so we have a fibration

$$\begin{array}{ccc} \mathbf{CP}^1 & \longrightarrow & \mathbf{CP}^3 \\ & & \uparrow T \\ & & \mathbf{HP}^1 \end{array} .$$

This is, of course, the famous "twistor" fibration. In order to study its geometry more thoroughly, we will now introduce the structure equations of $\mathbf{H}^2$.

First, we endow $\mathbf{H}^2$ with a quaternion-valued inner product $\langle\ ,\ \rangle\colon \mathbf{H}^2 \times \mathbf{H}^2 \to \mathbf{H}$ defined by

$$(1.3) \qquad \langle (q_1, q_2), (p_1, p_2) \rangle = \bar{q}_1 p_1 + \bar{q}_2 p_2.$$

We have the identities

$$(1.4) \quad \langle v, wq \rangle = \langle v, w \rangle q, \quad \overline{\langle v, w \rangle} = \langle w, v \rangle, \quad \langle vq, w \rangle = \bar{q} \langle v, w \rangle.$$

Moreover $\mathrm{Re}\langle\ ,\ \rangle$: $\mathbf{H}^2 \times \mathbf{H}^2 \to \mathbf{R}$ is a positive definite inner product which gives $\mathbf{H}^2$ the structure of $\mathbf{E}^8$, Euclidean eight-space.

Let $\mathfrak{F}$ denote the space of pairs $f = (e_1, e_2)$ with $e_a \in \mathbf{H}^2$ for $a = 1, 2$ satisfying

$$(1.5) \qquad \langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1, \quad \langle e_1, e_2\rangle = 0.$$

We regard the components $e_a(f)$ of f as functions with values in $\mathbf{H}^2$. Clearly $e_1(\mathfrak{F}) = S^7 \subseteq \mathbf{E}^8 = \mathbf{H}^2$, and, for each $v \in S^7$,

$$e_1^{-1}(v) = (v\mathbf{H})^\perp \cap S^7 = S^3,$$

where $(v\mathbf{H})^\perp = \{w \in \mathbf{H}^2 \mid \langle v, w\rangle = 0\}$ is a quaternion line in $\mathbf{H}^2$ by (1.4). Thus we have a fibration

$$\begin{array}{ccc} S^3 & \longrightarrow & \mathfrak{F} \\ & & \downarrow e_1 \\ & & S^7. \end{array}$$

We may conclude from this that $\mathfrak{F}$ is a simply-connected compact manifold of real dimension 10, with

$$H^k(\mathfrak{F}, \mathbf{R}) = \begin{cases} \mathbf{R} & \text{if } k = 0, 3, 7, 10, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, it is well known that $\mathfrak{F}$ may be canonically identified with $\mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$ up to a left translation in $\mathrm{Sp}(2)$. Regarding each e_a as a vector-valued function on $\mathfrak{F}$, we see that there are unique quaternion-valued 1-forms $\{\phi^a_b \mid a, b = 1, 2\}$ so that

$$(1.6) \qquad de_a = e_b\phi^b_a.$$

Differentiating (1.6) we get

$$(1.7) \qquad d\phi^a_b = -\phi^a_c \wedge \phi^c_b.$$

Differentiating $\langle e_a, e_b\rangle = \delta_{ab}$, we get

$$\langle e_c\phi^c_a, e_b\rangle + \langle e_a, e_c\phi^c_b\rangle = 0,$$

or

$$(1.8) \qquad \overline{\phi^a_b} + \phi^b_a = 0, \quad a, b \in \{1, 2\}.$$

(1.6)–(1.8) will be referred to as the *structure equations* of $\mathfrak{F}$. From (1.8) we see that at most 10 of the real components of the ϕ^a_b can be independent. On the other hand (e_1, e_2): $\mathfrak{F} \to \mathbf{H}^2 \times \mathbf{H}^2$ is an imbedding by definition, so the real components of the ϕ^a_b must span $T^*_f\mathfrak{F}$ at every f. Since $\dim_{\mathbf{R}} T^*_f\mathfrak{F} = 10$, we see that these 10 components actually yield a co-framing of $\mathfrak{F}$.

We have a canonical map $C_a\colon \mathfrak{F} \to \mathbf{CP}^3$ for each $a = 1, 2$ defined as follows: let $C_a(f) = e_a(f)\mathbf{C} \in \mathbf{CP}^3$. For simplicity, we will concentrate on C_1, although, as we will see in §2, C_2 is also important. We immediately see that C_1 gives $\mathfrak{F}$ the structure of an $S^1 \times S^3$ bundle over $\mathbf{CP}^3$, where we have identified S^1 with the unit complex numbers and S^3 with the unit quaternions. The action is given by

$$f \cdot (z, q) = (e_1, e_2) \cdot (z, q) = (e_1 z, e_2 q),$$

where $z \in \mathbf{C}$ and $q \in \mathbf{H}$ satisfy $z\bar{z} = q\bar{q} = 1$. If we set

$$\begin{bmatrix} \phi_1^1 & \phi_2^1 \\ \phi_1^2 & \phi_2^2 \end{bmatrix} = \begin{bmatrix} i\rho_1 + j\phi_1 & -\bar{\omega}_1 + j\omega_2 \\ \omega_1 + j\omega_2 & i\rho_2 + j\phi_2 \end{bmatrix}, \tag{1.9}$$

where ρ_1 and ρ_2 are real 1-forms while ω_1, ω_2, ϕ_1, and ϕ_2 are complex valued, then the formula

$$de_1 \equiv (e_1 j)\phi_1 + e_2\omega_1 + (e_2 j)\omega_2, \quad \operatorname{mod} e_1\mathbf{C},$$

shows that $C_1^*(\Omega^{1,0}\mathbf{CP}^3) \equiv 0$, $\operatorname{mod}\{\phi_1, \omega_1, \omega_2\}$. We may rewrite part of the structure equations relative to this $S^1 \times S^3$-structure on $\mathbf{CP}^3$ as

$$d\begin{pmatrix} \phi_1 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 2i\rho_1 & -\omega_2 & \omega_1 \\ \bar{\omega}_2 & i(\rho_1 - \rho_2) & \bar{\phi}_2 \\ -\bar{\omega}_1 & -\phi_2 & i(\rho_1 + \rho_2) \end{pmatrix} \wedge \begin{pmatrix} \phi_1 \\ \omega_1 \\ \omega_2 \end{pmatrix}, \tag{1.10}$$

$$d\rho_1 = i\big(\phi_1 \wedge \bar{\phi}_1 + \omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2\big). \tag{1.11}$$

[In an exactly analogous fashion, $C_2^*(\Omega^{1,0}\mathbf{CP}^3) \equiv 0$, $\operatorname{mod}\{\phi_2, \omega_1, \omega_2\}$, and we have the formulas

$$d\begin{pmatrix} \phi_2 \\ \bar{\omega}_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 2i\rho_2 & \omega_2 & -\bar{\omega}_1 \\ -\bar{\omega}_2 & i(\rho_2 - \rho_1) & -\bar{\phi}_1 \\ \omega_1 & \phi_1 & i(\rho_1 + \rho_2) \end{pmatrix} \wedge \begin{pmatrix} \phi_2 \\ \bar{\omega}_1 \\ \omega_2 \end{pmatrix}, \tag{1.10$'$}$$

$$d\rho_2 = i\big(\phi_2 \wedge \bar{\phi}_2 + \bar{\omega}_1 \wedge \omega_1 + \omega_2 \wedge \bar{\omega}_2\big).] \tag{1.11$'$}$$

The map $H_a\colon \mathfrak{F} \to \mathbf{HP}^1$ is defined for $a = 1, 2$ by $H_a(f) = e_a(f)\mathbf{H} \in \mathbf{HP}^1$. Again, although the two maps are different, we will concentrate on H_1; in fact, once we have realized that $\mathbf{HP}^1 \simeq S^4$, then we will see that $H_1 = A \circ H_2$ where $A\colon S^4 \to S^4$ is the antipodal map. The formula

$$de_1 \equiv e_2\omega_1 + (e_2 j)\omega_2 \operatorname{mod} e_1\mathbf{H}$$

shows that $H_1^*(\Omega^1\mathbf{HP}^1) \equiv 0$, $\operatorname{mod}\{\omega_1, \omega_2, \bar{\omega}_1, \bar{\omega}_2\}$. In fact, $H_1\colon \mathfrak{F} \to \mathbf{HP}^1$ gives $\mathfrak{F}$ the structure of an $S^3 \times S^3$-bundle over $\mathbf{HP}^1$ where the action is given by

$$(e_1, e_2) \cdot (q_1, q_2) = (e_1 q_1, e_2 q_2),$$

where $q_a \in \mathbf{H}$ satisfies $q_a\bar{q}_a = 1$. The structure equations of $\mathbf{HP}^1$ relative to this $S^3 \times S^3$-structure on $\mathbf{HP}^1$ are

$$d\begin{pmatrix}\omega_1\\ \bar{\omega}_1\\ \omega_2\\ \bar{\omega}_2\end{pmatrix} = \begin{pmatrix} i(\rho_1-\rho_2) & 0 & \bar{\phi}_2 & -\phi_1 \\ 0 & i(\rho_2-\rho_1) & -\bar{\phi}_1 & \phi_2 \\ -\phi_2 & \phi_1 & i(\rho_1+\rho_2) & 0 \\ \bar{\phi}_1 & -\bar{\phi}_2 & 0 & -i(\rho_1+\rho_2)\end{pmatrix} \wedge \begin{pmatrix}\omega_1\\ \bar{\omega}_1\\ \omega_2\\ \bar{\omega}_2\end{pmatrix}$$

(1.12)

$$= -\Psi \wedge \begin{pmatrix}\omega_1\\ \bar{\omega}_1\\ \omega_2\\ \bar{\omega}_2\end{pmatrix},$$

(1.13) $$d\Psi + \Psi \wedge \Psi = 2\begin{pmatrix}\omega_1\\ \bar{\omega}_1\\ \omega_2\\ \bar{\omega}_2\end{pmatrix} \wedge (\bar{\omega}_1\, \omega_1\, \bar{\omega}_2\, \omega_2).$$

It follows that $\mathbf{HP}^1$, when endowed with the invariant metric $ds^2 = 4(\omega_1 \circ \bar{\omega}_1 + \omega_2 \circ \bar{\omega}_2)$, is a space of constant curvature $+1$. The fibration

$$\begin{array}{ccc} S^3 \times S^3 & \longrightarrow & \mathfrak{F} \\ & & \downarrow \\ & & \mathbf{HP}^1 \end{array}$$

shows that $\mathbf{HP}^1$ is compact and simply connected. It must therefore be isometric to the unit 4-sphere S^4. We may go further and see that $\mathfrak{F}$ is merely the spin double cover of the oriented orthonormal frame bundle of $\mathbf{HP}^1$ under this metric (we give $\mathbf{HP}^1$ the orientation determined by $-\omega_1 \wedge \bar{\omega}_1 \wedge \omega_2 \wedge \bar{\omega}_2$).

For this reason, we will, from now on, speak interchangably of $\mathbf{HP}^1$ and S^4, even though we have given no explicit isometry between them. For the sake of explicitness and to simplify an argument in §2, we may write (1.12) and (1.13) in real form by defining

(1.14) $$\tfrac{1}{2}(\eta^1 + i\eta^2) = \omega_1, \quad \tfrac{1}{2}(\eta^3 + i\eta^4) = \omega_2,$$

(1.15) $$\begin{aligned} i\mu_1 + j(\mu_2 + i\mu_3) &= i\rho_1 + j\phi_1, \\ i\nu_1 + j(\nu_2 + i\nu_3) &= i\rho_2 + j\phi_2, \end{aligned}$$

and we get

$$ds^2 = \sum_{\varepsilon=1}^{4} (\eta^\varepsilon)^2,$$

$$d(\eta^\varepsilon) = d\begin{pmatrix}\eta^1\\ \eta^2\\ \eta^3\\ \eta^4\end{pmatrix}$$

$$(1.16)\qquad = -\begin{bmatrix} 0 & \mu_1 - \nu_1 & \mu_2 - \nu_2 & \mu_3 - \nu_3 \\ -\mu_1 + \nu_1 & 0 & \mu_3 + \nu_3 & -\mu_2 - \nu_2 \\ -\mu_2 + \nu_2 & -\mu_3 - \nu_3 & 0 & \mu_1 + \nu_1 \\ -\mu_3 + \nu_3 & \mu_2 + \nu_2 & -\mu_1 - \nu_1 & 0 \end{bmatrix} \wedge \begin{bmatrix}\eta^1\\ \eta^2\\ \eta^3\\ \eta^4\end{bmatrix}$$

$$= -\theta^\varepsilon_\delta \wedge \eta^\delta \ \left(\text{this defines } \theta^\varepsilon_\delta = -\theta^\delta_\varepsilon\right),$$

$$(1.17)\qquad d\theta^\varepsilon_\delta = -\theta^\varepsilon_\gamma \wedge \theta^\gamma_\delta + \eta^\varepsilon \wedge \eta^\delta.$$

The verification of these formulas is routine and is left to the reader.

We now study how these maps are related to the fibration T: $\mathbf{CP}^3 \to \mathbf{HP}^1$. Clearly $T \circ C_a = H_a$ for $a = 1, 2$. Note that the fibres of T are complex submanifolds of $\mathbf{CP}^3$. It follows that the bundle $\mathfrak{T}$ of complex-linear 1-forms on $\mathbf{CP}^3$ splits as a direct sum $\mathfrak{T} = \mathcal{V} \oplus \mathcal{H}$ where $\mathcal{H} \subseteq T^*(\Lambda^1\mathbf{HP}^1)$ is the rank-2 complex bundle of semi-basic (for T) complex linear 1-forms and $\mathcal{V} = \mathcal{H}^\perp$ is the rank-one complex bundle dual to the vertical tangent bundle (for T) via the standard metric on $\mathbf{CP}^3$.

Theorem A. *The subbundle $\mathcal{V} \subseteq \mathfrak{T}$ is a holomorphic line bundle over $\mathbf{CP}^3$. Moreover, it induces a holomorphic contact structure on $\mathbf{CP}^3$, and $\mathcal{V} = L^2$, where L is the universal line bundle over $\mathbf{CP}^3$.*

Proof. Comparing the equations

$$C_1^*(\Omega^{1,0}\mathbf{CP}^3) \equiv 0, \quad \mathrm{mod}\{\phi_1, \omega_1, \omega_2\}$$

and

$$H_1^*(\Omega^1\mathbf{HP}^1) \equiv 0, \quad \mathrm{mod}\{\omega_1, \omega_2, \overline{\omega}_1, \overline{\omega}_2\},$$

we see that if σ: $\mathbf{CP}^3 \to \mathcal{V}$ is any section of $\mathcal{V}$, then

$$C_1^*(\sigma) = F\phi_1$$

for some function F on $\mathfrak{F}$. The structure equation

$$d\phi_1 = 2i\rho_1 \wedge \phi_1 + 2\omega_1 \wedge \omega_2$$

gives

$$d(F\phi_1) = (dF + 2i\rho_1 F) \wedge \phi_1 + 2F\omega_1 \wedge \omega_2,$$

in particular,

$$\left(\bar{\partial} F\phi_1\right) \wedge F^2\phi_1 = 0,$$

so $F\phi_1$ is locally a C^∞-multiple of a holomorphic form. Thus $\mathcal{V}$ is holomorphic. The formula

$$d(F\phi_1) \wedge F\phi_1 = 2F\phi_1 \wedge \omega_1 \wedge \omega_2$$

shows that $d\sigma \wedge \sigma \neq 0$ if $\sigma \neq 0$, so a holomorphic nonzero local section of $\mathcal{V}$ is a local contact form.

Finally, for each $f \in \mathcal{F}$, let $e_1^*(f) \in (e_1(f)\mathbf{C})^*$ be the vector dual to $e_1(f) \in e_1(f)\mathbf{C}$. One easily computes that

$$\sigma(e_1(f)\mathbf{C}) = \phi_1 \otimes e_1^*(f) \otimes e_1^*(f)$$

is a holomorphic nonzero well-defined section of $\mathcal{V} \otimes L^* \otimes L^*$, where L is the universal line bundle over $\mathbf{CP}^3$. Thus $\mathcal{V} \simeq (L)^2$ as holomorphic bundles.

Remark. We will verify this calculation in another way in §3, where we will explicitly construct a meromorphic section of $\mathcal{V}$ with a double pole along a $\mathbf{CP}^2 \subseteq \mathbf{CP}^3$ and with no zeros.

2. Surface theory in $\mathbf{HP}^1$

Let X: $M^2 \to S^4 = \mathbf{HP}^1$ be an immersion of an *oriented* surface M^2. (If M^2 is not orientable, we pass to the orientation double cover and choose the canonical orientation.) We let $\mathcal{F}_X^0 \subseteq M^2 \times \mathcal{F}$ be the set of pairs (p, f) which satisfy $X(p) = H_1(f)$. We have the diagram

$$\begin{array}{ccc} \mathcal{F}_X^0 & \xrightarrow{f} & \mathcal{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle H_1} \\ M & \xrightarrow{X} & S^4 \end{array}$$

and $\mathcal{F}_X^0$ is just the $S^3 \times S^3$-bundle over M induced by the immersion X. Since we will be working on $\mathcal{F}_X^0$ and its subspaces for some time now, we will omit references to f^* and write ϕ_b^a instead of $f^*(\phi_b^a)$ to denote forms on $\mathcal{F}_X^0$.

Because $X \circ p\colon \mathcal{F}_X^0 \to S^4$ has rank 2 at every point, and $H_1 \circ f = X \circ p$ by definition, we see that the four real components of ϕ_1^2 must satisfy two relations. Moreover, since ϕ_1^2 is semi-basic for H_1, it must be semi-basic for p, i.e., $\phi_1^2 \equiv 0$, mod $p^*(\Omega^1 M)$. Now we can compute in the standard way that

$$(2.1)\qquad R^*_{(q_1,q_2)}\left(\phi_1^2\right) = \bar{q}_1 \phi_1^2 q_2,$$

where $(q_1, q_2) \in S^3 \times S^3$, and $R_{(q_1,q_2)}\colon \mathcal{F}_X^0 \to \mathcal{F}_X^0$ is the bundle right action.

The induced metric on M, namely $X^*(ds^2)$, satisfies

$$p^*\left(X^*(ds^2)\right) = 4(\omega_1 \circ \bar{\omega}_1 + \omega_2 \circ \bar{\omega}_2).$$

We use this metric together with our chosen orientation to induce a compatible complex structure on M. Thus a 1-form α on M (with values in $\mathbf{C}$) is of type $(1,0)$ if and only if $\frac{1}{2} i\alpha \wedge \bar{\alpha}$ determines the correct orientation (when nonzero) and $\alpha \circ \bar{\alpha} = \lambda X^*(ds^2)$ for some λ.

It is not difficult to see that we may define a subbundle

$$(2.3)\qquad \mathcal{F}_X^1 = \left\{(p, f) \in \mathcal{F}_X^0 \middle| \omega^2 = 0 \text{ and } \omega^1 \in p^*(\Omega^{1,0} M)\right\}.$$

We leave the details to the reader. (Skeptics may check this most easily by using the structure equations (1.16) and (1.17) in real form. Our adaptation is just $\eta^3 = \eta^4 = 0$, and $\eta^1 \wedge \eta^2$ forms an oriented basis of $\Lambda^2 T^* M$.) The fiber of $p\colon \mathcal{F}_X^1 \to M^2$ is an $S^1 \times S^1$. The action is just

$$(2.4)\qquad (p, (e_1, e_2)) \cdot (z_1, z_2) = (p, (e_1 z_1, e_2 z_2)),$$

where $(p, (e_1, e_2)) \in \mathcal{F}_X^1$ and $z_1 \bar{z}_1 = z_2 \bar{z}_2 = 1$ with $z_a \in \mathbf{C}$.

With this in mind, we may define two "Gauss maps;" for $a = 1, 2$, $\Gamma_X^a\colon M^2 \to \mathbf{CP}^3$ is defined by

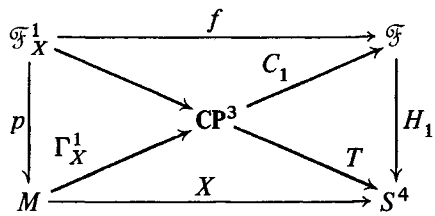
$$(2.5)\qquad \Gamma_X^a(p) = e_a(f)\mathbf{C} \in \mathbf{CP}^3,$$

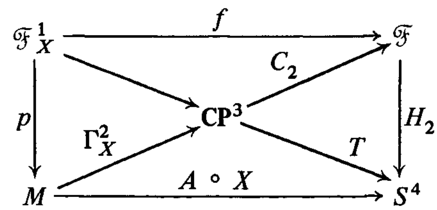
where $(p, f) \in \mathcal{F}_X^1$. The action (2.4) shows that this is well-defined. In fact, one easily verifies that $\Gamma_X^a\colon M^2 \to \mathbf{CP}^3$ is smooth and immersive since we have the formulas

$$(2.6)\qquad T \circ \Gamma_X^1 = X,$$

$$(2.7)\qquad A \circ T \circ \Gamma_X^2 = X,$$

where $A: S^4 \to S^4$ is the antipodal map. For example, we have the following commutative diagrams:





Let L be the universal line bundle over $\mathbf{CP}^3$, and let $L^a_X = \Gamma^{a*}_X(L)$ as an Hermitian bundle with connection. If $\sigma\colon M^2 \to L^a_X$ is a section, then

$$p^*(\sigma) = e_a(f)s(p, f) = e_a s \tag{2.8}$$

for some function s well defined on $\mathfrak{F}^1_X$. We give L^a_X the induced connection, so that

$$p^*(\nabla\sigma) = e_a \otimes (ds + is\rho_a). \tag{2.9}$$

While there is no possibility of confusion, we will simply write L^a instead of L^a_X. Since we are working over a Riemann surface M^2, we know that there is a unique holomorphic structure on L^a so that the above connection is compatible (see [8]). Henceforth, when we speak of holomorphic bundles L^a, this is the holomorphic structure we will mean. Thus a local section $\sigma\colon V \subseteq M^2 \to L^a$ with $p^*(\sigma) = e_a s$ will be holomorphic if and only if $(ds + is\rho_a) \wedge \omega_1 = 0$.

Now considering the structure equations (1.10) we see that $X^*(ds^2) = 2\omega_1 \circ 2\bar{\omega}_1$, and

$$d(2\omega_1) = i(\rho_1 - \rho_2) \wedge 2\omega_1. \tag{2.10}$$

Thus regarding the metric $X^*(ds^2)$ as defining an Hermitian structure on $\tau(M)$, the bundle of complex linear 1-forms on M, the covariant derivative of a 1-form α satisfying $p^*(\alpha) = \omega_1 A$, where $A \in C^\infty\mathfrak{F}^1_X$, satisfies

$$p^*(\nabla\alpha) = \omega_1 \otimes (\alpha A + i(\rho_1 - \rho_2)A). \tag{2.11}$$

Proposition 2.1. *$\tau(M) \simeq L^1 \otimes (L^2)^*$ as holomorphic bundles over M.*

Proof. The quantity $\sigma(p) = \omega_1(p, f) \otimes e_1^*(f) \otimes e_2(f)$ is seen to be independent of f and therefore defines a section $\sigma\colon M \to \tau(M) \otimes (L^1)^* \otimes L^2$. Not only does σ never vanish, but one clearly has $\nabla\sigma = 0$, so σ is covariant constant and, in particular, holomorphic. Thus $\tau(M) \otimes (L^1)^* \otimes L^2 \simeq \mathbf{C}$ as holomorphic bundles. q.e.d.

Now let us consider the equation $\omega_2 = 0$ on $\mathcal{F}_X^1$. This forces

$$d\omega_2 = -\bar{\omega}_1 \wedge \phi_1 - \phi_2 \wedge \omega_1 = 0.$$

It follows that there exist functions A, B_1, B_2 on $\mathcal{F}_X^1$ satisfying

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} A & \bar{B}_1 \\ B_2 & -A \end{pmatrix} \begin{pmatrix} \omega_1 \\ \bar{\omega}_1 \end{pmatrix}, \tag{2.12}$$

(by Cartan's Lemma, if one likes).

Proposition 2.2. *The immersion X: $M^2 \to S^4$ is minimal iff $A \equiv 0$.*

Proof. Inspecting the structure equations (1.16) and (1.17) one sees that X is minimal iff the two quadratic forms

$$II^3 = (-\mu_2 + \nu_2) \circ \eta^1 - (\mu_3 + \nu_3) \circ \eta^2 = \theta_a^3 \circ \eta^a,$$
$$II^4 = (-\mu_3 + \nu_3) \circ \eta^1 + (\mu_3 + \nu_3) \circ \eta^2 = \theta_a^4 \circ \eta^1$$

have zero trace with respect to $I = (\eta^1)^2 + (\eta^2)^2$. Inspecting (1.14) and (1.15) together with (2.12), we see that the condition $A = 0$ is exactly the pair of equations $\mathrm{tr}_I II^3 = \mathrm{tr}_I II^4 = 0$. q.e.d.

From now until we say otherwise, we shall assume that the immersion X: $M^2 \to S^4$ is minimal. Then equations (2.12) become

$$\phi_1 = \bar{B}_1\bar{\omega}_1, \quad \phi_2 = B_2\omega_1. \tag{2.13}$$

Proposition 2.3. *The quantities $\sigma_1 = e_1 \otimes e_1 \otimes \bar{\phi}_1$ and $e_2^* \otimes e_2^* \otimes \phi_2 = \sigma_2$ represent holomorphic sections of $(L^1)^2 \otimes \tau$ and $(L^{2*})^2 \otimes \tau$ respectively. Moreover, the quantity $Q = \bar{\phi}_1 \circ \phi_2 \circ \omega_1 \circ \omega_1$ is a holomorphic quartic form on M.*

Proof. The fact that these quantities are well-defined on M is left to the reader. The holomorphicity follows immediately from the structure equations. For example, from (1.10) we get $d\bar{\phi}_1 = -2i\rho_1 \wedge \bar{\phi}_1$, so $(dB_1 + i(3\rho_1 - \rho_2)B_1) \wedge \omega_1 = 0$. Thus $dB_1 = -i(3\rho_1 - \rho_2)B_1 + B_1'\omega_1$. This immediately implies $\bar{\partial}(e_1 \otimes e_1 \otimes \omega_1 B_1) = \pi^{0,1}(e_1 \otimes e_1 \otimes \omega_1 \otimes B_1'\omega_1) = 0$. The remaining equations follow similarly.

Theorem B. *The following are equivalent for a minimal immersion X: $M^2 \to S^4$:*

(i) $\sigma_1 = 0$,

(ii) Γ_X^1: $M \to \mathbf{CP}^3$ *is holomorphic,*

(iii) Γ_X^1: $M \to \mathbf{CP}^3$ *is an integral of* $\mathcal{V}$.

Proof. All of these follow immediately from the structure equation

$$de_1 \equiv (e_1 j)\phi_1 + e_2\omega_1 + e_2 j\omega_2 \mod e_1\mathbf{C}$$
$$\equiv (e_1 j)\bar{B}_1\bar{\omega}_2 + e_2\omega_1 \mod e_1\mathbf{C},$$

valid on $\mathcal{F}_X^1$ (see the proof of Theorem A). q.e.d.

We also have an analogous theorem for Γ_X^2.

Theorem B′. *The following are equivalent for a minimal immersion X: $M^2 \to S^4$:*

(i) $\sigma_2 = 0$,

(ii) Γ_X^2: $M \to \mathbf{CP}^3$ *is anti-holomorphic,*

(iii) Γ_X^2: $M \to \mathbf{CP}^3$ *is an integral of* $\mathcal{V}$.

Proof. Omitted.

It follows that those minimal immersions satisfying $Q \equiv 0$ fall into two classes. These classes are not as distinct as they appear at first glance. If X: $M^2 \to S^4$ is minimal and satisfies $\sigma_2 = 0$, then one can show that $A \circ X$: $M^2 \to S^4$ satisfies $\sigma_1 = 0$ (and conversely, of course). Thus it is reasonable to concentrate on those immersions satisfying $\sigma_1 = 0$ when one wishes to study those satisfying $Q \equiv 0$.

We will say that an immersion satisfying $Q \equiv 0$ is *superminimal*, and that the immersion has *positive* (resp. *negative*) *Spin* if it satisfies $\sigma_1 = 0$ (resp. $\sigma_2 = 0$).

Theorem C. *If $M \simeq \mathbf{P}^1$ as Riemann surfaces, then any minimal immersion X: $M \to S^4$ is superminimal. If X: $M \to S^4$ is superminimal with both positive and negative spin, then $X(M) \subseteq S^4$ is a geodesic 2-sphere.*

Proof. If $M \simeq \mathbf{P}^1$, then every holomorphic quartic form is identically 0. If X: $M \to S^4$ is superminimal with both positive and negative spin, then both ϕ_1 and ϕ_2 vanish identically, so $II^3 \equiv II^4 \equiv 0$. It is well-known that this implies that $X(M) \subseteq S^4$ is a geodesic 2-sphere.

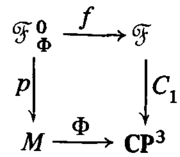
Remark. This theorem was essentially known to Calabi [2] and Chern [5].

We now sum up our results in the main theorem of this section.

Theorem D. *If X: $M^2 \to S^4$ is superminimal with positive spin, then Γ_X^1: $M^2 \to \mathbf{CP}^3$ is a holomorphic immersion which is an integral of the contact structure $\mathcal{V}$. Conversely, if Φ: $M^2 \to \mathbf{CP}^3$ is a holomorphic curve which is an integral of $\mathcal{V}$, and if $R \subseteq M^2$ is the ramification divisor of Φ, then $T \circ \Phi$: $M - \{R\} \to S^4$ is a superminimal immersion with positive spin satisfying $\Gamma_{T\circ\Phi}^1 = \Phi$. Moreover, the metrics induced on M^2 by Φ and by $T \circ \Phi$ are the same.*

Proof. One direction has already been done. Thus we may assume that Φ: $M^2 \to \mathbf{CP}^3$ is a holomorphic curve which is an integral of $\mathcal{V}$, and that $R = \phi$ by deleting points if necessary.

Let $\mathcal{F}_\Phi^0 \subseteq M \times \mathcal{F}$ be the set of pairs (p, f) satisfying $\Phi(p) = C_1(f)$. We have the diagram:



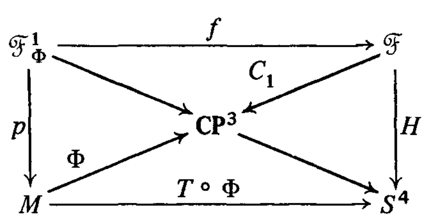
The map p: $\mathcal{F}_\Phi^0 \to M$ makes $\mathcal{F}_\Phi^0$ into an $S^1 \times SU(2)$ bundle over M. The assumption that Φ is an integral of $\mathcal{V}$ implies that $\phi_1 = 0$ on $\mathcal{F}_\Phi^0$. In the usual way, we may adapt frames to produce $\mathcal{F}_\Phi^1 \subseteq \mathcal{F}_\Phi^0$ consisting of those pairs $(p, f) \in \mathcal{F}_\Phi^0$ for which $\omega_2 = 0$. Then (1.10) implies

$$d\omega_2 = -\phi_2 \wedge \omega_1 = 0,$$

so

$$\phi_2 = B_2\omega_1$$

for some $B_2 \in C^\infty \mathcal{F}_\Phi^1$. Now p: $\mathcal{F}_\Phi^1 \to M$ makes $\mathcal{F}_\Phi^1$ into an $S^1 \times S^1$ bundle over M. We now have the diagram



with the equations

$$\phi_1 = \omega_2 = 0, \quad \phi_2 = B_2\omega_1.$$

The structure equation $de_1 \equiv e_2\omega_1$, mod $e\mathbf{C}$ (valid on $\mathcal{F}_\Phi^1$) together with the fact that we have assumed Φ holomorphic implies that $p^*(\Omega^{1,0}M) \equiv 0$, mod$\{\omega^1\}$. It is now immediate that $\mathcal{F}_\Phi^1 = \mathcal{F}_{T\circ\Phi}^1$ and that $T \circ \Phi$: $M^2 \to S^4$ is a superminimal immersion with positive spin. Moreover, the Kähler-form on $\mathbf{CP}^3$ pulls back to $\mathcal{F}$ as $2i(\phi_1 \wedge \bar{\phi}_1 + \omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2)$ which restricts to $2i\omega_1 \wedge \bar{\omega}_1$ on $\mathcal{F}_\Phi^1$, but this is clearly the Kähler form associated to the metric $p^*((T \circ \Phi)^*(ds^2)) = 2\omega_1 \circ 2\bar{\omega}_1$. Further details are left to the reader. q.e.d.

In closing this section, we would like to remark that if Φ: $M \to \mathbf{CP}^3$ is an arbitrary holomorphic integral of $\mathcal{V}$, then $T \circ \Phi$ will be a "generalized" superminimal immersion with positive spin in the sense that it will have "branch points" where Φ ramifies. It would be interesting to know if all "branch singularities" of superminimal varieties (in the appropriate sense of the word "varieties," "currents" perhaps would do) arise in this way.

We conclude with the following theorem whose proof is obvious from the fact that every holomorphic curve Φ: $M^2 \to \mathbf{CP}^3$ with M^2 compact has algebraic image (see [6]).

Theorem E. *If M^2 is a compact Riemann surface with a superminimal immersion X: $M^2 \to S^4$, then $X(M^2)$ is an algebraic surface in S^4.*

Proof. Omitted.

Calabi [2] has noted that every minimal 2-sphere in S^n has total area an integral multiple of 4π. By essentially the same reasoning, we establish the following proposition:

Proposition 2.4. *Let X: $M^2 \to S^4$ be a superminimal immersion (with positive spin, say), then* $\operatorname{vol}(M^2) = 4\pi d$ *where* $\operatorname{vol}(M^2)$ *is the volume of M^2 in the induced metric, and d is the degree of the algebraic curve $\Gamma_X^1(M^2) \subseteq \mathbf{CP}^3$.*

Proof. This follows immediately from Theorem D and the Wirtinger theorem stating that, up to a universal constant, the volume of an algebraic curve in $\mathbf{CP}^3$ is equal to its degree. One checks that the constant 4π is correct simply by noting that if X: $M^2 \to S^4$ is the inclusion of a geodesic 2-sphere into S^4, then $\operatorname{vol}(M^2) = 4\pi$ and, since Γ_X^1: $S^2 \to \mathbf{CP}^3$ is clearly a $\mathbf{P}^1 \subseteq \mathbf{CP}^3$, we must have $d = 1$.

3. Integral curves of the contact structure

In this section, we prove our main theorem by making a thorough study of the holomorphic integrals of the system $\mathcal{V}$ on $\mathbf{CP}^3$.

First we prove a formula for a meromorphic section of $\mathcal{V}$. Let $\mathbf{A}^3 \subseteq \mathbf{CP}^3$ be the subset of $[z_0, z_1, z_2, z_3] \in \mathbf{CP}^3$ where $z_0 \neq 0$. We may uniquely coordinatize $\mathbf{A}^3$ by meromorphic functions z_1, z_2, z_3 on $\mathbf{CP}^3$ with a simple pole along $\mathbf{CP}^2 = \mathbf{CP}^3 - \mathbf{A}^3$ so that (z_1, z_2, z_3): $\mathbf{A}^3 \to \mathbf{C}^3$ is a bi-holomorphism and $[1, z_1, z_2, z_3]$: $\mathbf{A}^3 \to \mathbf{CP}^3$ is just the inclusion. Moreover, the Fubini-Study metric form on $\mathbf{CP}^3$ restricts to $\mathbf{A}^3$ to be of the form $i\partial\bar{\partial} \log |Z|^2$ where $|Z|^2 = 1 + z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3$ (see [6]), up to a constant factor.

Proposition 3.1. *Let $\tilde{T}$: $\mathbf{A}^3 \to \mathbf{HP}^1$ be the restriction of T to $\mathbf{A}^3$. Then the fiber of $\tilde{T}$ through the point $[1, w_1, w_2, w_3]$ is the line*

$$\zeta \to [1, w_1 + (1 + w_1\bar{w}_1)\zeta, w_2 + (w_2\bar{w}_1 - \bar{w}_3)\zeta, w_3 + (w_3\bar{w}_1 + \bar{w}_2)\zeta].$$

It follows that the vector field

$$X = (1 + z_1\bar{z}_1)\frac{\partial}{\partial z_1} + (z_2\bar{z}_1 - \bar{z}_3)\frac{\partial}{\partial z_2} + (z_3\bar{z}_1 + \bar{z}_2)\frac{\partial}{\partial z_3}$$

is tangent to these fibers. The $(1,0)$*-form dual to* X *under the Fubini-Study metric is* $\tilde{\omega} = |Z|^{-2}(dz_1 - z_3dz_2 + z_2dz_3)$.

Proof. By definition

$$\tilde{T}[1, w_1, w_2, w_3] = [1 + jw_1, w_2 + jw_3] = [(1 + jw_1)q, (w_2 + jw_3)q]$$

for all $q \in \mathbf{H}^*$. Consider $q = (1 + \bar{w}_1\zeta) + j\zeta$ where ζ varies over $\mathbf{C}$. We compute

$$(1 + jw_1)q = 1 + j(w_1 + (1 + w_1\bar{w}_1)\zeta),$$
$$(w_2 + jw_3)q = (w_2 + (w_2\bar{w}_1 - \bar{w}_3)\zeta) + j(w_3 + (w_3\bar{w}_1 + \bar{w}_2)\zeta).$$

Since the fibers of $\tilde{T}$ are complex lines, this establishes the first claim. The second follows by differentiation with respect to ζ. In turn, the final claim follows by elementary calculation using $i\partial\bar{\partial}\log|Z|^2$ as the Kähler form on $\mathbf{A}^3$. Details are left to the reader. q.e.d.

By the above proposition, the form

$$\omega = dz_1 - z_3dz_2 + z_2dz_3$$

is a meromorphic section of $\mathcal{V}$. This form has a double pole along $\mathbf{CP}^2 = \mathbf{CP}^3 - \mathbf{A}^3$ and is nowhere zero on $\mathbf{A}^3$. This gives an alternate proof that $\mathcal{V}$ is the square of the universal line bundle over $\mathbf{CP}^3$. Note also that $\omega \wedge d\omega$ does not vanish on $\mathbf{A}^3$, so ω is a contact form on $\mathbf{A}^3$.

Theorem F. *Let* M^2 *be a connected Riemann surface, and let* f *and* g *be meromorphic functions on* M^2 *with* g *nonconstant. Let* $\Phi(f, g)$: $M^2 \to \mathbf{CP}^3$ *be defined by*

$$\Phi(f, g) = [1, f - \tfrac{1}{2}g(df/dg), g, \tfrac{1}{2}(df/dg)],$$

then $\Phi(f, g)$: $M^2 \to \mathbf{CP}^3$ *is a holomorphic integral of* $\mathcal{V}$. *Conversely, any nonconstant holomorphic integral of* $\mathcal{V}$, Φ: $M^2 \to \mathbf{CP}^3$, *is either of the form* $\Phi(f, g)$ *for some unique meromorphic functions* f *and* g *on* M *or* Φ *has image in some* $\mathbf{CP}^1 \subseteq \mathbf{CP}^3$.

Proof. First, write $\omega = d(z_1 + z_2z_3) - 2z_3dz_2$. It is now obvious that $\Phi(f, g)$: $M^2 \to \mathbf{CP}^3$ is a holomorphic integral of $\mathcal{V}$ for f and g as in the hypotheses.

Conversely, suppose Φ: $M^2 \to \mathbf{CP}^3$ is a holomorphic integral of $\mathcal{V}$ and write $\Phi = [f_0, f_1, f_2, f_3]$, where f_0, f_1, f_2, f_3 are meromorphic functions on M^2. First, assume $f_0 \not\equiv 0$, then we may divide all the f_ε by f_0 so as to reduce to the case $f_0 \equiv 1$. If f_2 is a constant, say $f_2 \equiv C$, then the condition that Φ be an integral reduces to $d(f_1 + f_2f_3) = 0$. Thus $\Phi = [1, a - cf_3, c, f_3]$ for some constant a, so $\Phi(M)$ lies in a $\mathbf{CP}^1$. If f_2 is not a constant, then set $f = f_1 + f_2f_3$ and $g = f_2$.

Then the condition $d(f_1 + f_2 f_3) - 2f_3 df_2 = 0$ on M forces $f_3 = \frac{1}{2}(df/dg)$ and then $f_1 = f - \frac{1}{2}g(df/dg)$ so the claim is established. Now assume $f_0 \equiv 0$. If we also had $f_1 \equiv 0$, then we would have $\Phi(M) \subseteq \mathbf{CP}^1$ again, so we may assume $f_1 \not\equiv 0$ and divide by it to reduce to the case $f_0 \equiv 0, f_1 \equiv 1$. Let w_0, w_2, w_3 be meromorphic functions on $\mathbf{CP}^3$ which satisfy

$$[w_0, 1, w_2, w_3] = [1, z_1, z_2, z_3]$$

on $\mathbf{A}^3 - \{z_1^{-1}(0)\}$. We compute

$$\omega = -w_0^{-2}(dw_0 + w_3 dw_2 - w_2 dw_3),$$

so $dw_0 + w_3 dw_2 - w_2 dw_3$ represents $\mathcal{V}$ on the affine chart where the second coordinate is nonzero. Since $\Phi = [0, 1, f_2, f_3]$ is an integral of this form, we must have

$$0 + f_3 df_2 - f_2 df_3 = 0.$$

In other words $f_2/f_3 \equiv C$ for some constant C. Again, we conclude that $\Phi(M) \subseteq \mathbf{CP}^1$.

Remark. The lines $\mathbf{P}^1 \subseteq \mathbf{CP}^3$ which are integrals of $\mathcal{V}$ represent the "Gauss maps" of geodesic two-spheres in S^4. They form a complex manifold isomorphic to $Q_3 \subseteq \mathbf{CP}^4$, the complex hyperquadric.

We now prove our main theorem and its corollary.

Theorem G. *Let M be a compact Riemann surface. There always exists a (holomorphic) embedding Φ: $M \to \mathbf{CP}^3$ which is an integral of $\mathcal{V}$.*

Corollary H. *Let M be a compact Riemann surface. There always exists a conformal minimal (in fact, superminimal) generically* 1-1 *immersion X: $M^2 \to S^4$ whose image $X(M)$ is, in addition, an algebraic surface in S^4.*

Proof of Corollary H. Let Φ: $M \to \mathbf{CP}^3$ be a holomorphic embedding of M as an integral curve of $\mathcal{V}$: this exists by Theorem G. By Theorem D, $T \circ \Phi$: $M \to S^4$ is a conformal minimal immersion of M into S^4 as a superminimal immersion with positive spin. If $T \circ \Phi$ were not generically 1-1, then $T \circ \Phi$ would be a covering map from M to $T \circ \Phi(M)$. However, since $\Phi = \Gamma^1_{T \circ \Phi}$, this would imply that Φ: $M \to \mathbf{CP}^3$ factors through this covering map and therefore could not be 1-1. Finally, Theorem E shows that $T \circ \Phi(M) \subseteq S^4$ is an algebraic surface.

Proof of Theorem G. Fix a generically 1-1 immersion Ψ: $M \to \mathbf{CP}^2$ so that Ψ is holomorphic and $\mathcal{C} = \Psi(M)$ is an algebraic curve whose only singularities are ordinary double points. By the Riemann-Roch theorem, this is always possible (see [6]). Let $Q_0 \in \mathbf{CP}^2$ be a point which does not lie on $\mathcal{C}$, any flex tangent or bitangent to $\mathcal{C}$, or the tangent cone to any double point of $\mathcal{C}$. Let L be a line through Q_0 which is not tangent to $\mathcal{C}$ nor does it pass through any double point of $\mathcal{C}$. Let $Q_1 \in L$ be a point distinct from Q_0 and not on $\mathcal{C}$. Let $\{p_1, \ldots, p_a\}$ be the set of points in M so that $\Psi(p_\alpha) \in L$ for all $1 \leqslant \alpha \leqslant a$. Let $\{q_1, \ldots, q_b\}$ be the set of points in M so that the tangent to $\mathcal{C}$ at $\Psi(q_\beta)$ passes through Q_0 for $1 \leqslant \beta \leqslant b$. Finally, choose homogeneous coordinates on $\mathbf{CP}^2$ so that $Q_0 = [0,1,0]$ and $Q_1 = [1,0,0]$, and write $\Psi(p) = [g(p), f(p), 1]$ for unique meromorphic functions g and f on M. We list the following consequences of our assumptions:

(i) If $p \notin \{p_\alpha\} \cup \{q_\beta\}$, then f and g are holomorphic at p, and dg does not vanish at p.

(ii) Both f and g have a simple pole at p_α. In fact, if we let $z(p) = 1/g(p)$, then z is a holomorphic coordinate near each p_α, and $z(p_\alpha) = 0$. There exist holomorphic functions $F_\alpha(z)$ for z near 0 so that

$$f(p) = F_\alpha(z(p))/z(p)$$

for p near p_α. Because Q_0, $Q_1 \notin \{p_\alpha\}$ and L intersects C in distinct points, the values $\{F_\alpha(0)\}$ are nonzero, finite, and distinct.

(iii) Near each q_β, there is a holomorphic coordinate $z(p)$, uniquely defined up to sign, so that

$$g(p) = A_\beta + (a(p))^2/2,$$
$$f(p) = F_\beta(z(p))$$

for unique constants A_β and holomorphic functions $F_\beta(z)$ near $z = 0$. This follows because dg only vanishes to first order (since Q_0 lies on no flex tangent). Because Ψ is an immersion we have $F_\beta'(0) \neq 0$. Since L is not tangent to $\mathcal{C}$ and Q_0 lies on no bitangent, the A_β are finite and distinct. We now claim that $\Phi(f, g)\colon M \to \mathbf{CP}^3$ is an embedding. We show this in three cases.

Case i. From the formula for $\Phi(f, g)$, it follows that $\Phi(f, g)(p) \in \mathbf{A}^3$ if and only if $p \notin \{p_\alpha\} \cup \{q_\beta\}$, and that $d\Phi(f, g) \neq 0$ at p (since $dg \neq 0$ at p). If $\Phi(f, g)(p) = \Phi(f, g)(p')$ for $p, p' \notin \{p_\alpha\} \cup \{q_\beta\}$, then we obviously have

$$g(p) = g(p'), \quad f(p) = f(p'), \quad \frac{df}{dg}(p) = \frac{df}{dg}(p').$$

Thus $\Psi(p) = \Psi(p')$, and the tangents at p and p' are the same. Since $\mathcal{C}$ has only ordinary double points, it follows that $p = p'$.

Case ii. In terms of the holomorphic coordinate $z = 1/g$ near p_α, we get

$$\Phi(f, g) = \left[z, \tfrac{1}{2}(F + zF_\alpha'), 1, \tfrac{1}{2}\left(zF_\alpha - z^2F_\alpha'\right)\right].$$

thus $d\Phi(f, g) \neq 0$ at p_α and $\Phi(f, g)(p_\alpha) = [0, F_\alpha(0)/2, 1, 0]$.

Case iii. In terms of the holomorphic coordinate z discussed in (iii) above, we have for p near q_β

$$\Phi(f, g) = \left[z, zF_\beta - \tfrac{1}{2}gF'_\beta, zg, F'_\beta/2\right].$$

Since $F'_\beta(0) \neq 0$, we see that $d\Phi(f, g) \neq 0$ at $p = q_\beta$. Furthermore $\Phi(f, g)(q_\beta) = [0, -A_\beta, 0, 1]$.

By Cases ii and iii, we see that the points $\{p_\alpha\} \cup \{q_\beta\}$ are sent to distinct points in $\mathbf{CP}^2 = \mathbf{CP}^3 - \mathbf{A}^3$. Combining this with Case i, we see that $\Phi(f, g)$: $M \to \mathbf{CP}^3$ is one-to-one. Since we have also shown that it is an immersion, we are done.

Remark. The formula appearing in Theorem F may be thought of as a sort of "Weierstrass formula" establishing the equivalence of algebraic plane curves $\mathcal{C} \subseteq \mathbf{CP}$ and holomorphic integrals of $\mathcal{V}$ in $\mathbf{CP}^3$. As a consequence, it is not difficult to prove Plücker formulas for holomorphic integrals of $\mathcal{V}$ completely analogous to the classical Plücker formulas for plane algebraic curves. We leave as an interesting problem the task of writing down these formulas for a superminimal immersion in S^4 with "traditional singularities" (see [6]). Somewhat more interesting is the fact that, for the purpose of finding examples, it is not necessary to start with an abstract Riemann surface M. One may just as well start with an implicitly defined curve $\mathcal{C} \subseteq \mathbf{CP}^2$ given as the set of points $[X, Y, 1]$ satisfying $F(X, Y) = 0$. The "Weierstrass" formula shows that the set of points $[Z_0, Z_1, Z_2, Z_3]$ satisfying

$$Z_0 = 2\frac{\partial F}{\partial Y}, \qquad Z_1 = 2Y\frac{\partial F}{\partial Y} + X\frac{\partial F}{\partial X}, \quad Z_2 = 2X\frac{\partial F}{\partial Y}, Z_3 = -\frac{\partial F}{\partial X},$$
$$F(X, Y) = 0,$$

in $\mathbf{CP}^3$ is an integral of $\mathcal{V}$. This set of points is a curve in $\mathbf{CP}^3$ satisfying polynomial relations which are computable from F by elimination theory. Thus in principle one could start with $F(X, Y) = 0$ in $\mathbf{CP}^2$ and derive the algebraic equations defining a surface in S^4 (implicitly, of course) which is superminimal and conformally equivalent to the given curve in $\mathbf{CP}^2$.

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