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Conformal Blocks and Generalized Theta Functions

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Abstract: Let $\mathcal{SU}_X(r)$ be the moduli space of rank r vector bundles with trivial determinant on a Riemann surface X. This space carries a natural line bundle, the determinant line bundle $\mathcal{L}$. We describe a canonical isomorphism of the space of global sections of $\mathcal{L}^k$ with the space of conformal blocks defined in terms of representations of the Lie algebra $\mathfrak{sl}_r(\mathbf{C}((z)))$. It follows in particular that the dimension of $H^0(\mathcal{SU}_X(r), \mathcal{L}^k)$ is given by the Verlinde formula.

Introduction

The aim of this paper is to construct a canonical isomorphism between two vector spaces associated to a Riemann surface X. The first of these spaces is the space of *conformal blocks* $B_c(r)$ (also called the space of vacua), which plays an important role in conformal field theory. It is defined as follows: choose a point $p \in X$, and let A_X be the ring of algebraic functions on $X - p$. To each integer $c \geq 0$ is associated a representation V_c of the Lie algebra $\mathfrak{sl}_r(\mathbf{C}(z))$, the *basic representation* of level c (more correctly it is a representation of the universal extension of $\mathfrak{sl}_r(\mathbf{C}((z)))$ – see Sect. 7 for details). The ring A_X embeds into $\mathbf{C}((z))$ by associating to a function its Laurent development at p; then $B_c(r)$ is the space of linear forms on V_c which vanish on the elements $A(z)v$ for $A(z) \in \mathfrak{sl}_r(A_X)$, $v \in V_c$.

The second space comes from algebraic geometry, and is defined as follows. Let $\mathcal{SU}_X(r)$ be the moduli space of semi-stable rank r vector bundles on X with trivial determinant. One can define a theta divisor on $\mathcal{SU}_X(r)$ in the same way one does in the rank 1 case: one chooses a line bundle L on X of degree $g-1$, and considers the locus of vector bundles $E \in \mathcal{SU}_X(r)$ such that $E \otimes L$ has a nonzero section. The associated line bundle $\mathcal{L}$ is called the *determinant bundle*; the space we are interested in is $H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$. This space can be considered as a non-Abelian version of the

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space of c^{th}-order theta functions on the Jacobian of X, and is sometimes called the space of generalized theta functions. We will prove that it is canonically isomorphic to $B_c(r)$. By [T-U-Y] this implies that the space $H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$ satisfies the so-called *fusion rules*, which allow to compute its dimension in a purely combinatorial way, giving the famous Verlinde formula ([V], see Corollary 8.6).

The isomorphism $B_c(r) \xrightarrow{\sim} H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$ is certainly known to the physicists – see e.g. [W]. Our point it that this can be proved in a purely mathematical way. In fact we hope to convince the reader that even in an infinite-dimensional context, the methods of algebraic geometry provide a flexible and efficient language (though a little frightening at first glance!).

Our strategy is as follows. First, by trivializing vector bundles on $X - p$ and on a neighborhood of p, we construct a bijective correspondence between the moduli space and the double coset space $SL_r(A_X)\backslash SL_r(\mathbf{C}((z)))/SL_r(\mathbf{C}[[z]])$ (this is a quite classical idea which goes back to Weil). Sections 1 to 3 are devoted to make sense of this as an isomorphism between geometric objects. We show that the quotient $\mathcal{Q} := SL_r(\mathbf{C}((z)))/SL_r(\mathbf{C}[[z]])$ as well as the group $SL_r(A_X)$ is an *ind-variety*, that is a direct limit of an increasing sequence of algebraic varieties. The quotient $SL_r(A_X)\backslash\mathcal{Q}$ makes sense as a *stack* (not far from what topologists call an orbifold), and this stack is canonically isomorphic to the moduli stack $\mathcal{SL}_X(r)$ of vector bundles on X with trivial determinant.

The determinant line bundle $\mathcal{L}$ lives naturally on the moduli stack, and the next step is to identify its pull back to $\mathcal{Q}$. In order to do this we first construct the central $\mathbf{C}^*$-extension $\widehat{SL}_r(\mathbf{C}((z)))$ and the τ function on this group, and show that the τ function defines a section of a line bundle $\mathcal{L}_\chi$ on $\mathcal{Q}$ (Sect. 4). We then prove that the pull back of $\mathcal{L}$ to $\mathcal{Q}$ is isomorphic to $\mathcal{L}_\chi$ (Sect. 5). A theorem of Kumar and Mathieu identifies the space $H^0(\mathcal{Q}, \mathcal{L}_\chi^c)$ with the dual V_c^* of the basic representation V_c; it follows, almost by definition of a stack, that $H^0(\mathcal{SL}_X(r), \mathcal{L}^c)$ can be identified with the elements of V_c^* which are invariant under the group $SL_r(A_X)$ (Sect. 7). This turns out to be the same as the linear forms annihilated by the Lie algebra: the key point is that the group $SL_r(A_X)$ is *reduced* (Sect. 6) – a highly non-trivial property in our infinite-dimensional set-up. The final step is to prove that the sections of $\mathcal{L}^c$ on the moduli stack and on the moduli space are the same (Sect. 8) – this is essentially Hartog's theorem, since the substack of non-stable bundles is of codimension ≥ 2.

In the last section we state and prove the corresponding result for the moduli space of vector bundles of rank r and determinant L for any line bundle L on X.

The methods of this paper should extend to the general case of principal bundles under a semi-simple algebraic group G. We have chosen to work in the context of vector bundles (i.e. $G = SL_r(\mathbf{C})$) because it is by far the most important case for algebraic geometers, and it is easier to explain in so far as it appeals very little to the rather technical machinery of Kac-Moody groups. Also the general case can be to a large extent reduced to this one.

Most of this work was done in the Spring of 1992, and we have lectured in various places about it. In July 1992 we heard of G. Faltings beautiful ideas, which should prove at the same time both our result and that of [T-U-Y] (in the more general case of principal bundles). These ideas are sketched in [F], but (certainly due to our own incompetence) we were unable to understand some of the key points in the proof. We have therefore decided after some time to write a complete version of our proof, if only to provide an introduction to Faltings' ideas.

Part of our results have been obtained independently (also in the context of principal bundles) by Kumar, Narasimhan and Ramanathan [K-N-R].

1. The Ind-Groups $\mathbf{GL}_r(K)$ and $\mathbf{SL}_r(K)$

K-Spaces and Ind-Schemes

(1.1) Throughout this paper we'll work over an algebraically closed field k of characteristic 0. A k-algebra will always be assumed to be associative, commutative and unitary. Our basic objects will be *k-spaces* in the sense of [L-MB]: by definition, a k-space (resp. a k-group) is a functor F from the category of k-algebras to the category of sets (resp. of groups) which is a sheaf for the faithfully flat topology. Recall [1] that this means that for any faithfully flat homomorphism $R \to R'$, the diagram

$$F(R) \to F(R') \rightrightarrows F(R' \otimes_R R')$$

is exact; in most cases the verification that this is indeed the case is quite easy, and will be left to the reader. Any scheme X over k provides such a functor (by associating to a k-algebra R the set $X(R)$ of morphisms of Spec(R) into X); in this way we will consider the category of schemes over k as a full subcategory of the category of k-spaces. A scheme will always be assumed to be quasi-compact and quasi-separated.

Direct limits exist in the category of k-spaces; we'll say that a k-space (resp. a k-group) is an *ind-scheme* (resp. an *ind-group*) if it is the direct limit of a directed system of schemes. Let $(X_\alpha)_{\alpha \in I}$ be a directed system of schemes, X its limit in the category of k-spaces, and S a k-scheme. The set Mor(S, X) of morphisms of S into X is the direct limit of the sets Mor(S, X_α), while the set Mor(X, S) is the inverse limit of the sets Mor(X_α, S).

The Groups $\mathbf{GL}_r(K)$ and $\mathbf{GL}_r(\mathscr{O})$

(1.2) Let z be an indeterminate. We will denote by $\mathscr{O}$ the formal series ring $k[[z]]$ and by K the field $k((z))$ of meromorphic formal series in z. We let $\mathbf{GL}_r(\mathscr{O})$ (or $\mathbf{GL}_r(k[[z]])$) be the k-group $R \mapsto GL_r(R[[z]])$, and $\mathbf{GL}_r(K)$ (or $\mathbf{GL}_r(k((z)))$) be the k-group $R \mapsto GL_r(R((z)))$. We define in the same way the k-groups $\mathbf{SL}_r(\mathscr{O})$ and $\mathbf{SL}_r(K)$. For $N \geq 0$, we denote by $G^{(N)}(R)$ (resp. $S^{(N)}(R)$) the set of matrices $A(z)$ in $GL_r(R((z)))$ (resp. in $SL_r(R((z)))$) such that both $A(z)$ and $A(z)^{-1}$ have a pole of order $\leq N$. This defines subfunctors $G^{(N)}$ and $S^{(N)}$ of $\mathbf{GL}_r(K)$ and $\mathbf{SL}_r(K)$ respectively.

Proposition 1.2. *The k-group $\mathbf{GL}_r(\mathscr{O})$ (resp. $\mathbf{SL}_r(\mathscr{O})$) is an affine group scheme. The k-group $\mathbf{GL}_r(K)$ (resp. $\mathbf{SL}_r(K)$) is an ind-group, direct limit of the sequence of schemes $(G^{(N)})_{N \geq 0}$ (resp. $(S^{(N)})_{N \geq 0}$).*

For any k-algebra R, let us denote by $M_r(R)$ the vector space of r-by-r matrices with entries in R. The set $GL_r(R[[z]])$ consists of matrices $A(z) = \sum_{n \geq 0} A_n z^n$, with

[1] An accessible introduction to Grothendieck topologies and descent theory can be found in the first pages of [SGA4$\frac{1}{2}$].

$A_0 \in GL_r(R)$ and $A_n \in M_r(R)$ for $n \geq 1$; therefore the group $\mathbf{GL}_r(\mathcal{O})$ is represented by the affine scheme $GL_r(k) \times \prod_1^{\infty} M_r(k)$.

Let $M^{(N)}(R)$ be the space of r-by-r matrices $A(z) = \sum_{n \geq -N} A_n z^n$, with $A_n \in M_r(R)$. The functor $M^{(N)}$ is represented by the affine scheme $\prod_{n \geq -N} M_r(k)$, and the functor $G^{(N)}$ is represented by a closed (affine) subscheme of $M^{(N)} \times M^{(N)}$ (identify $G^{(N)}(R)$ with the subset of $M^{(N)}(R) \times M^{(N)}(R)$ consisting of couples $(A(z), B(z))$ such that $A(z)B(z) = I$). One has $GL_r(R((z))) = \bigcup_{N \geq 0} G^{(N)}(R)$, hence the k-group $\mathbf{GL}_r(K)$ is the direct limit of the sequence of schemes $(G^{(N)})_{N \geq 0}$.

Let N be a non-negative integer. There exist universal polynomials $P_m^{(N)}((A_n)_{n \geq -N})$ $(m \geq -rN)$ on the affine space $\prod_{n \geq -N} M_r(k)$ such that the determinant of an element $A(z) = \sum_{n \geq -N} A_n z^n$ of $GL_r(R((z)))$ is given by

$$\det A(z) = \sum_{m \geq -rN} P_m^{(N)}((A_n)_{n \geq -N}) z^m .$$

It follows that the functor $S^{(N)}$ is representable by a closed affine subscheme of $G^{(N)}$. In particular, $S^{(0)} = \mathbf{SL}_r(\mathcal{O})$ is an affine scheme, and $\mathbf{SL}_r(K)$ is an ind-scheme, direct limit of the sequence $(S^{(N)})_{N \geq 0}$. □

$\mathbf{GL}_r(K)$ *and Vector Bundles*

(1.3) We now start the geometric side of this paper; we fix once and for all a smooth (connected) projective curve X over k, and a closed point p of X. We put $X^* = X - p$. We denote by $\mathcal{O}$ the completion of the local ring of X at p, and by K its field of fractions. We will choose a local coordinate z at p and identify $\mathcal{O}$ with $k[[z]]$ and K with $k((z))$. Let R be a k-algebra. We put $X_R = X \times_k \operatorname{Spec}(R)$, $X_R^* = X^* \times_k \operatorname{Spec}(R)$, $D_R = \operatorname{Spec}(R[[z]])$ and $D_R^* = \operatorname{Spec}(R((z)))$ [2]. We consider the cartesian diagram

$$\begin{array}{ccc} D_R^* & \hookrightarrow & D_R \\ \downarrow & & \downarrow f \\ X_R^* & \overset{j}{\hookrightarrow} & X_R . \end{array} \tag{1.3}$$

When $R = k$, we may think of $f(D)$ as a small disk in X around p, and of $f(D^*)$ as the punctured disk $f(D) - p$. We want to say that the ind-group $\mathbf{GL}_r(K)$ parametrizes bundles which are trivialized on X^* and on D.

We consider triples (E, ϱ, σ), where E is a vector bundle on X_R, $\varrho : \mathcal{O}_{X_R^*}^r \to E_{|X_R^*}$ a trivialization of E over X_R^*, $\sigma : \mathcal{O}_{D_R}^r \to E_{|D_R}$ a trivialization of E over D_R. We let $T(R)$ be the set of isomorphism classes of triples (E, ϱ, σ) (with the obvious notion of isomorphism).

[2] The R-algebras $R[[z]]$ and $R((z))$ do not acutally depend on the choice of a local coordinate z at p: $R[[z]]$ is the completion of the tensor product $R \otimes_k \mathcal{O}$ with respect to the $(R \otimes \mathfrak{m})$-adic topology, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$, and $R((z))$ is $R[[z]] \otimes_{\mathcal{O}} K$.

Proposition 1.4. *The ind-Group* $\mathbf{GL}_r(K)$ *Represents the Functor* T.

Let (E, ϱ, σ) be an element of $T(R)$. Pulling back the trivializations ϱ and σ to D_R^* provides two trivializations ϱ^* and σ^* of the pull back of E over D_R^*: these trivializations differ by an element $\gamma = \varrho^{*-1} \circ \sigma^*$ of $GL_r(R((z)))$.

Let us now drop the suffix R to simplify the notation. Let $\mathscr{K}_D$ be the quasi-coherent sheaf on D associated to the $R[[z]]$-module $R((z))$. We have an exact sequence

$$0 \to \mathscr{O}_X \to j_*\mathscr{O}_{X^*} \to f_*(\mathscr{K}_D/\mathscr{O}_D) \to 0 .$$

Tensoring with E and using the trivializations ϱ and σ, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & j_*(E_{|X^*}) & \longrightarrow & f_*(E_{|D} \otimes \mathscr{K}_D/\mathscr{O}_D) & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \varrho} & & \uparrow{\scriptstyle \sigma} & & \\ 0 & \longrightarrow & E & \longrightarrow & j_*\mathscr{O}_{X^*}^r & \xrightarrow{\bar{\gamma}} & f_*(\mathscr{K}_D/\mathscr{O}_D)^r & \longrightarrow & 0 , \end{array}$$

where $\bar{\gamma}$ is the composition of the natural map $j_*\mathscr{O}_{X^*}^r \to f_*(\mathscr{K}_D)^r$, the automorphism γ^{-1} of $f_*(\mathscr{K}_D)^r$, and the canonical projection $f_*(\mathscr{K}_D)^r \to f_*(\mathscr{K}_D/\mathscr{O}_D)^r$.

Conversely, let us start from an element γ of $GL_r(R((z)))$. We claim that the homomorphism $\bar{\gamma}: j_*\mathscr{O}_{X^*}^r \to f_*(\mathscr{K}_D/\mathscr{O}_D)^r$ defined by the above recipe is surjective, and that its kernel E_γ is locally free of rank r. By descent theory it is enough to check these assertions after pull back to X^* and to D. They are clear over X^*, since the exact sequence reduces to an isomorphism $\varrho^{-1}: E_\gamma \to \mathscr{O}_{X^*}^r$. Over D, we observe that the canonical map $f^*f_*(\mathscr{K}_D/\mathscr{O}_D) \to \mathscr{K}_D/\mathscr{O}_D$ is an isomorphism (express for instance $\mathscr{K}_D/\mathscr{O}_D$ as the limit of the direct system

$$(\ldots \to \mathscr{O}_D/(z^n) \xrightarrow{z} \mathscr{O}_D/(z^{n+1}) \to \ldots)) .$$

Therefore we get an exact sequence

$$0 \to f^*E \to \mathscr{K}_D^r \xrightarrow{p \circ \gamma^{-1}} (\mathscr{K}_D/\mathscr{O}_D))^r \to 0 ,$$

where $p: \mathscr{K}_D^r \to (\mathscr{K}_D/\mathscr{O}_D))^r$ is the canonical map. In other words, γ induces an isomorphism $\sigma: \mathscr{O}_D^r \to f^*E$. Thus E_γ is a vector bundle, so we have associated to γ a triple $(E_\gamma, \varrho, \sigma)$ in $T(R)$. The two constructions are clearly inverse of each other, hence the proposition. □

From this proposition we get immediately

Proposition 1.5. *The ind-group* $\mathbf{SL}_r(K)$ *represents the subfunctor* T_0 *of* T *which associates to a* k*-algebra* R *the set of isomorphism classes of triples* (E, ϱ, σ), *where* E *is a vector bundle on* X_R, $\varrho: \mathscr{O}_{X_R^*}^r \to E_{|X_R^*}$ *and* $\sigma: \mathscr{O}_{D_R}^r \to E_{|D_R}$ *are isomorphisms such that* $\wedge^r \varrho$ *and* $\wedge^r \sigma$ *coincide over* D_R^*. □

Remarks. (1.6) The condition that the trivializations $\wedge^r \varrho$ and $\wedge^r \sigma$ coincide over D_R^* means that they come from a global trivialization of $\bigwedge^r E$. So we can rephrase Proposition 1.5 by saying that $T_0(R)$ is the set of isomorphism classes of data $(E, \varrho, \sigma, \delta)$, where δ is a trivialization of $\bigwedge^r E$, ϱ and σ are trivializations of $E_{|X_R^*}$ and $E_{|D_R}$ respectively, such that $\wedge^r \varrho$ coincide with $\delta_{|X_R^*}$ and $\wedge^r \sigma$ with $\delta_{|D_R}$.

(1.7) There is an obvious extension of Proposition 1.5 which will be useful to deal with vector bundles with arbitrary determinant. For $d \in \mathbf{Z}$, let us denote by $\mathbf{SL}_r(K)^{(d)}$

the sub-ind-scheme of $\mathbf{GL}_r(K)$ parametrizing matrices with determinant z^{-d}. Then $\mathbf{SL}_r(K)^{(d)}$ *represents the subfunctor of* T *which associates to a* k*-algebra* R *the set of isomorphism classes of triples* (E, ϱ, σ) *over* X_R *such that* $\wedge^r_\varrho$ *and* $z^d \wedge^r_\sigma$ *coincide over* D^*_R (thus defining as above an isomorphism $\delta: \mathscr{O}_{X_R}(dp) \to \bigwedge^r E$). We could clearly replace z^d by any element of K in this statement. It follows in particular that the determinant of a vector bundle corresponding to an element γ of $GL_r(K)$ is $\mathscr{O}_X(dp)$, where d is the order of the Laurent series $\det\gamma$.

Let us specialize Propositions 1.4 and 1.5 to the case $R = k$:

Corollary 1.8. *Let us denote by* A_X *the affine algebra* $\Gamma(X - p, \mathscr{O}_X)$. *There is a canonical bijective correspondence between the set of isomorphism classes of rank* r *vector bundles on* X *with trivial determinant* (resp. *with determinant of the form* $\mathscr{O}_X(np)$ *for some integer* n) *and the set of double classes* $SL_r(A_X)\backslash SL_r(K)/SL_r(\mathscr{O})$ (resp. $GL_r(A_X)\backslash GL_r(K)/GL_r(\mathscr{O})$).

Since two trivializations of $E_{|D}$ differ by an element of $GL_r(\mathscr{O})$, and two trivializations of $E_{|X_*}$ by an element of $GL_r(A_X)$, we deduce from Proposition 1.4 a bijection between $GL_r(A_X)\backslash GL_r(K)/GL_r(\mathscr{O})$ and the set of isomorphism classes of rank r vector bundles on X which are trivial on X^*. But a projective module over a Dedekind ring is free if and only if its determinant is free ([B], Chap. 7, Sect. 4, Proposition 24), hence our assertion for GL_r. The same proof applies for SL_r. □

(1.9) Our first goal in the following sections will be to show that the bijection defined in Corollary 1.8 comes actually from an isomorphism between algebro-geometric objects. Let us observe here that the functor $R \mapsto SL_r(A_{X_R})$ is a k-group, which will play an important rôle in our story; we denote it by $\mathbf{SL}_r(A_X)$. It is actually an ind-variety, limit of the affine varieties $\Gamma^{(N)}$ parametrizing matrices $A = (a_{ij})$ with $\det A = 1$ and $a_{ij} \in H^0(X, \mathscr{O}_X(Np))$ for all i, j. We shall study this group in more detail in Sect. 7.

Application: The Birkhoff Decomposition

Let us apply Corollary 1.8 when $X = \mathbf{P}^1$, and $p = 0$. The vector bundles on $\mathbf{P}^1$ with rank r and trivial determinant are parametrized by sequences of integers $\mathbf{d} = (d_1, \dots, d_r)$ with $d_1 \le \dots \le d_r$ and $\sum d_i = 0$: to such a sequence corresponds the vector bundle $\mathscr{O}_{\mathbf{P}^1}(d_1) \oplus \dots \oplus \mathscr{O}_{\mathbf{P}^1}(d_r)$, which is defined by the diagonal matrix $z^{\mathbf{d}} := \mathrm{diag}(z^{d_1}, \dots, z^{d_r})$. The k-algebra $A_{\mathbf{P}^1}$ is simply $k[z^{-1}]$. We obtain the *Birkhoff decomposition*

$$SL_r(K) = \bigcup_{\mathbf{d}} SL_r(k[z^{-1}])\, z^{\mathbf{d}} SL_r(\mathscr{O}) \,. \tag{1.10}$$

We shall see that the *big cell* $\mathbf{SL}_r(K)^0 := \mathbf{SL}_r(k[z^{-1}])\mathbf{SL}_r(\mathscr{O})$ is open in $\mathbf{SL}_r(K)$. More precisely, let us denote (abusively) by $\mathbf{SL}_r(\mathscr{O}_-)$ the (closed) sub-ind-group of $\mathbf{SL}_r(k[z^{-1}])$ parametrizing matrices of the form $A(z) = I + \sum_{n\ge 1} A_n z^{-n}$.

Proposition 1.11. *The multiplication map* $\mu: \mathbf{SL}_r(\mathscr{O}_-) \times \mathbf{SL}_r(\mathscr{O}) \to \mathbf{SL}_r(K)$ *is an open immersion.*

Let first S be a scheme and $\mathscr{E}$ a vector bundle over $S \times \mathbf{P}^1$; we denote by $q: S \times \mathbf{P}^1 \to S$ the projection map. Let S^0 be the biggest open subset of S over which the canonical map $q^*q_*\mathscr{E} \to \mathscr{E}$ is an isomorphism; this is the locus of points s in S such that $\mathscr{E}_{|\{s\}\times\mathbf{P}^1}$ is trivial [3]. If moreover the bundle $\mathscr{E}_{|S\times\{0\}}$ is trivial, so is the restriction of $\mathscr{E}$ to S^0.

We apply these remarks to $S = \mathbf{SL}_r(K)$. Let R be any k-algebra. Clearly the map $\mathbf{SL}_r(z^{-1}R[z^{-1}]) \times \mathbf{SL}_r(R[\![z]\!]) \to \mathbf{SL}_r(R((z)))$ is injective, and its image corresponds to triples (E, ϱ, σ) over $\mathbf{P}^1_R$, where the vector bundle E is trivial. Therefore μ induces an isomorphism from $\mathbf{SL}_r(z^{-1}k[z^{-1}]) \times \mathbf{SL}_r(\mathscr{O})$ onto the open sub-ind-scheme $\mathbf{SL}_r(K)^0$. □

2. The Homogeneous Space $\mathbf{SL}_r(K)/\mathbf{SL}_r(\mathscr{O})$

In the preceding section we have described (Corollary 1.8) a bijection between the set of isomorphism classes of rank r vector bundles on X with trivial determinant and the double coset space $SL_r(A_X)\backslash SL_r(K)/SL_r(\mathscr{O})$. Our aim in this section and the following is to show that this gives in fact a description of the moduli space – actually of the moduli stack. We therefore need to understand the algebraic structure of the set $SL_r(A_X)\backslash SL_r(K)/SL_r(\mathscr{O})$. We'll start with the quotient $\mathbf{SL}_r(K)/\mathbf{SL}_r(\mathscr{O})$, which will turn out to be as nice as we can reasonably hope, namely a direct limit of projective varieties (Theorem 2.5 below). Let us first recall that such a quotient always exists as a k-space – it is simply the sheaf (for the faithfully flat topology) associated to the presheaf $R \mapsto SL_r(R((z)))/SL_r(R[\![z]\!])$.

Proposition 2.1. *The k-space $\mathscr{Q} := \mathbf{SL}_r(K)/\mathbf{SL}_r(\mathscr{O})$ represents the functor which associates to a k-algebra R the set of isomorphism classes of pairs (E, ϱ), where E is a vector bundle over X_R and ϱ a trivialization of E over X_R^* such that $\wedge^r\varrho$ extends to a trivialization of $\bigwedge^r E$.*

Let R be a k-algebra and q an element of $\mathscr{Q}(R)$. By definition there exists a faithfully flat homomorphism $R \to R'$ and an element γ of $SL_r(R'((z)))$ such that the image of q in $\mathscr{Q}(R')$ is the class of γ. To γ corresponds by Proposition 1.5 a triple (E', ϱ', σ') over $X_{R'}$. Let $R'' = R' \otimes_R R'$, and let (E''_1, ϱ''_1), (E''_2, ϱ''_2) denote the pull-backs of (E', ϱ') by the two projections of $X_{R''}$ onto $X_{R'}$. Since the two images of γ in $SL_r(R''((z)))$ differ by an element of $SL_r(R''[\![z]\!])$, these pairs are isomorphic; this means that the isomorphism $\varrho''_2\varrho''^{-1}_1$ over $X^*_{R''}$ extends to an isomorphism $u\colon E''_1 \to E''_2$ over $X_{R''}$. This isomorphism satisfies the usual cocycle condition, because it is enough to check it over X^*, where it is obvious. Therefore (E', ϱ') descends to a pair (E, ϱ) on X_R as in the statement of the proposition.

Conversely, given a pair (E, ϱ) as above over X_R, we can find a faithfully flat homomorphism $R \to R'$ and a trivialization σ' of the pull back of E over $D_{R'}$ such that $\wedge^r\sigma'$ coincides with $\wedge^r\varrho$ over $D^*_{R'}$ (in fact Spec(R) is covered by open subsets Spec(R_α) such that E is trivial over D_{R_α}, and we can take $R' = \prod R_\alpha$). By Proposition 1.5 we get an element γ' of $SL_r(R'((z)))$ such that the two images of γ' in $SL_r(R''((z)))$ (with $R'' = R' \otimes_R R'$) differ by an element of $SL_r(R''[\![z]\!])$; this gives an element of $\mathscr{Q}(R)$. The two constructions are clearly inverse one of each other. □

[3] We are using here (and will use in the sequel) the fact that the standard base change theorems for coherent cohomology are valid without any noetherian hypothesis for projective morphisms (see [SGA6, Exp. III]).

Remark 2.2. Instead of the k-group $\mathbf{SL}_r(K)$ we might as well consider the k-space $\mathbf{SL}_r(K)^{(d)}$ which parametrizes matrices with determinant z^{-d} (1.6); we obtain exactly in the same way that *the k-space $\mathcal{Q}_d := \mathbf{SL}_r(K)^{(d)}/\mathbf{SL}_r(\mathcal{O})$ represents the functor which associates to a k-algebra R the set of isomorphism classes of pairs (E, ϱ), where E is a vector bundle over X_R and ϱ a trivialization of E over X_R^* such that $\wedge^r \varrho$ extends to an isomorphism of $\mathcal{O}_{X_R}(dp)$ onto $\bigwedge^r E$.* If γ_d is an element of $GL_r(K)$ with determinant z^{-d}, left multiplication by γ_d defines an isomorphism of $\mathcal{Q}$ onto $\mathcal{Q}_d$.

The same construction applies also to the case of the group GL_r, giving a k-space $\mathbf{GL}_r(K)/\mathbf{GL}_r(\mathcal{O})$ which represents the functor associating to a k-algebra R the set of isomorphism classes of pairs (E, ϱ), where E is a vector bundle over X_R and ϱ a trivialization of E over X_R^*. This k-space is a disjoint union of the k-spaces $(\mathcal{Q}'_d)_{d\in\mathbf{Z}}$ which parametrize those pairs (E, ϱ) for which $\deg(E_{|X\times\{t\}}) = d$ for all $t \in \operatorname{Spec}(R)$. In group-theoretic terms, $\mathcal{Q}'_d$ is the quotient $\mathbf{GL}_r(K)^{(d)}/\mathbf{GL}_r(\mathcal{O})$, where $\mathbf{GL}_r(K)^{(d)}$ is the open and closed sub-ind-scheme of $\mathbf{GL}_r(K)$ parametrizing matrices $A(z)$ such that the Laurent series[4] $\det A(z)$ has order $-d$ (1.7). One sees easily that the natural map $\mathcal{Q}_d(R) \longrightarrow \mathcal{Q}'_d(R)$ is bijective when the ring R is reduced, but not in general – we'll see this phenomenon in (2.4) below in another guise. This means that the ind-variety $\mathbf{GL}_r(K)/\mathbf{GL}_r(\mathcal{O})$ is not reduced (6.3). We will now concentrate on the quotient $\mathcal{Q} = \mathbf{SL}_r(K)/\mathbf{SL}_r(\mathcal{O})$, which will turn out to be a much nicer object.

$\mathcal{Q}$ as a Grassmannian

The quotient space $\mathcal{Q}$ is related to the infinite Grassmannian used by the Japanese school (see [S-W]) in the following way. For any k-algebra R, define a *lattice* in $R((z))^r$ as a sub-$R[[z]]$-module W of $R((z))^r$ which is projective of rank r, and such that $\bigcup z^{-n}W = R((z))^r$. It is an exercise in algebra to show that this amounts to say that W is a sub-$R[[z]]$-module of $R((z))^r$ such that

$$z^N R[[z]]^r \subset W \subset z^{-N} R[[z]]^r$$

for some integer N, and such that the R-module $z^{-N}R[[z]]^r/W$ is projective. Let us say moreover that the lattice W is *special* if the lattice $\bigwedge^r W \subset \bigwedge^r R((z))^r = R((z))$ is trivial, i.e. equal to $R[[z]] \subset R((z))$.

Proposition 2.3. *The k-space $\mathcal{Q}$* (resp. $\mathbf{GL}_r(K)/\mathbf{GL}_r(\mathcal{O})$) *represents the functor which associates to a k-algebra R the set of special lattices* (resp. *of lattices) $W \subset R((z))^r$. The group $\mathbf{SL}_r(K)$ acts on $\mathcal{Q}$ by $(\gamma, W) \mapsto \gamma W$ (for $\gamma \in GL_r(R((z)))$, $W \subset R((z))^r$).*

Let us fix the k-algebra R, and consider our diagram (1.3)

$$\begin{array}{ccc} D^* & \hookrightarrow & D \\ \big\downarrow & & \big\downarrow \\ X^* & \hookrightarrow & X \end{array}$$

where for simplicity we have dropped the suffix R. Let us start with a pair (E, ϱ) over X. The trivialization ϱ gives an isomorphism $R((z))^r \longrightarrow H^0(D^*, E_{|D^*})$; the

[4] A Laurent series $\varphi \in R((z))$ is said to be of order d if its image in $F((z))$ has order d for each homomorphism of R into a field F.

inverse image W of $H^0(D, E_{|D})$ is a lattice in $R((z))^r$, and it is a special lattice if $\wedge^r \varrho$ extends to a trivialization of $\bigwedge^r E$ over X.

Conversely, given a lattice W in $R((z))^r$, we define a vector bundle E_W on X by glueing the trivial bundle over X^* with the bundle on D associated to the $R[[z]]$-module W; the glueing isomorphism is the map $W \otimes_{R[[z]]} R((z)) \to R((z))^r$ induced by the embedding $W \hookrightarrow R((z))^r$. By definition E_W has a natural trivialization ϱ_W over X^*, and if W is a special lattice $\wedge^r \varrho$ extends to a trivialization of $\bigwedge^r E$ over X. It is easy to check that these two constructions are inverse one of each other.

Let γ be an element of $GL_r(R((z)))$, corresponding to a triple (E, ϱ, σ) (1.4). By construction the corresponding lattice is $\varrho^{-1}\sigma(R[[z]]^r) = \gamma(R[[z]]^r)$. This proves the last assertion of the proposition. □

Recall that we have denoted by $S^{(N)}$ the subscheme of $\mathbf{SL}_r(K)$ parametrizing matrices $A(z)$ such that $A(z)$ and $A(z)^{-1}$ have a pole of order $\leq N$; it is stable under right multiplication by $S^{(0)} = \mathbf{SL}_r(\mathscr{O})$. We will denote by $\mathscr{Q}^{(N)}$ its image in $\mathscr{Q}$, i.e. the quotient k-space $S^{(N)}/S^{(0)}$.

Proposition 2.4. *Let F_N be a free module of rank r over the ring $k[z]/(z^{2N})$ (so that F_N is a k-vector space of dimension $2rN$), and let $\mathbf{G}_z(rN, F_N)$ be the subvariety of the Grassmannian parametrizing z-stable rN-dimensional subspaces of F_N. The k-space $\mathscr{Q}^{(N)} = S^{(N)}/S^{(0)}$ is isomorphic to a closed subvariety of $\mathbf{G}_z(rN, F_N)$ with the same underlying topological space.*

It was pointed out to us by Genestier that the variety $\mathbf{G}_z(rN, F_N)$ is *not* reduced, even in the case $r = N = 1$. The variety $\mathscr{Q}^{(N)}$ turned out to be reduced in the examples we worked out, but we do not know whether this is true in general; this will cause us some trouble in the sequel.

Let R be a k-algebra. An element γ of $SL_r(R((z)))$ belongs to $S^{(N)}$ if and only if the lattice $W = \gamma R[[z]]^r$ satisfies $z^N R[[z]]^r \subset W \subset z^{-N} R[[z]]^r$. Therefore $\mathscr{Q}^{(N)}(R)$ is the subset of $\mathscr{Q}(R)$ consisting of special lattices with the above property. Let us associate to such a lattice its image $\overline{W}$ in $R \otimes_k F_N = z^{-N} R[[z]]^r / z^N R[[z]]^r$. We first observe that when R is a field, the lattice W is special if and only if $\dim \overline{W} = rN$: by the elementary divisors theorem the $R[[z]]$-module W has a basis $(z^{d_1} e_1, \ldots, z^{d_r} e_r)$, where $(e_1, \ldots, e_r)$ is a basis of $R[[z]]^r$, and $-N \leq d_i \leq N$; both conditons are then equivalent to $\sum d_i = 0$.

In general, let $W \subset R((z))^r$ be a lattice such that $z^N R[[z]]^r \subset W \subset z^{-N} R[[z]]^r$. Then $\overline{W}$ is a direct sub-R-module of F_N, stable by z; if W is special, $\overline{W}$ is of rank rN (because $\dim_F F \otimes_R \overline{W} = rN$ for every homomorphism of R into a field F). Conversely, assume that $\overline{W}$ is of rank rN. Locally over Spec(R), one has $\bigwedge^r W = z^{-rN} \varphi R[[z]]$ for some element $\varphi = a_0 + a_1 z + \ldots$ of $R[[z]]$. We know that for each homomorphism of R into a field F, the image of φ in $F[[z]]$ can be written $z^{rN} u$, where u is a unit of $F[[z]]$. It follows that the coefficients $a_0, \ldots, a_{rN-1}$ of φ are nilpotent, while a_{rN} is invertible. It is immediate that the nilpotent ideal I_W of R spanned by $a_0, \ldots, a_{rN-1}$ does not change when φ is multiplied by a unit, and therefore is defined globally over Spec(R). For any ring homomorphism $u: R \to R'$, the lattice $R' \otimes_R W \subset R'((z))^r$ is special if and only if $u(I_W) = 0$.

This means that the functor which associates to a k-algebra R the set of direct z-stable sub-R-modules $\overline{W}$ of $R \otimes_k F_N$ of rank rN such that the corresponding lattice W is special is represented by a closed subvariety of $\mathbf{G}_z(rN, F_N)$, defined by a nilpotent ideal. Associating to a special lattice $W \subset R((z))^r$ its image $\overline{W}$ defines a functorial isomorphism of $\mathcal{Q}^{(N)}$ onto this functor, hence the proposition. □

Recall that we have denoted by $\mathbf{SL}_r(\mathcal{O}_-)$ the subgroup of $\mathbf{SL}_r(k[z^{-1}])$ parametrizing matrices $\sum_{n\geq 0} A_n z^{-n}$ with $A_0 = I$. It is an ind-variety.

Theorem 2.5. *The k-space $\mathcal{Q} = \mathbf{SL}_r(K)/\mathbf{SL}_r(\mathcal{O})$ is an ind-variety, direct limit of the system of projective varieties $(\mathcal{Q}^{(N)})_{N\geq 0}$. It is covered by open subsets which are isomorphic to $\mathbf{SL}_r(\mathcal{O}_-)$, and over which the fibration $p:\mathbf{SL}_r(K) \to \mathcal{Q}$ is trivial.*

The first assertion follows from Proposition 2.4. Recall from Proposition 1.11 that the ind-group $\mathbf{SL}_r(K)$ contains an open subset $\mathbf{SL}_r(K)^0$ which is isomorphic to $\mathbf{SL}_r(\mathcal{O}_-)\times\mathbf{SL}_r(\mathcal{O})$, the isomorphism being equivariant with respect to the right action of $\mathbf{SL}_r(\mathcal{O})$. Since $\mathbf{SL}_r(K)$ is covered by the open subsets $g\mathbf{SL}_r(K)^0$ for $g \in SL_r(K)$, the second assertion follows. □

The (left) action of $\mathbf{SL}_r(K)$ on $\mathcal{Q}$ restricts to an action on the variety $\mathcal{Q}^{(N)}$ of the group scheme $\mathbf{SL}_r(\mathcal{O})$ (which actually acts through its finite dimensional quotient $\mathbf{SL}_r(k[z]/(z^{2N}))$). We are going to study the orbits of this action. Let us denote by ω the class of I in $\mathcal{Q}(k)$, and by $z^{\mathbf{d}}$ the matrix $\mathrm{diag}(z^{d_1}, \ldots, z^{d_r})$.

Proposition 2.6. a) *The orbits of $SL_r(\mathcal{O})$ in $\mathcal{Q}(k)$ are the orbits of the points $z^{\mathbf{d}}\omega$, where $\mathbf{d}$ runs through the sequences $d_1 \leq \ldots \leq d_r$ with $\sum d_i = 0$.*

b) *The orbit of $z^{\mathbf{d}'}\omega$ lies in the closure of the orbit of $z^{\mathbf{d}}\omega$ if and only if one has $d'_1 + \ldots + d'_p \geq d_1 + \ldots + d_p$ for $1 \leq p \leq r$.*

c) *The subset $\mathcal{Q}^{(N)}(k)$ is the union of the orbits of the points $z^{\mathbf{d}}\omega$, where $\mathbf{d}$ runs through the sequences with $-N \leq d_1 \leq \ldots \leq d_r \leq N$ and $\sum d_i = 0$.*

d) *Let $\mathbf{d}(N)$ denote the sequence $d_1 \leq \ldots \leq d_r$ with $d_i = -N$ for $i < \frac{r+1}{2}$, $d_i = N$ for $i > \frac{r+1}{2}$, and $d_{\frac{r+1}{2}} = 0$ when r is odd. Then the orbit of $z^{\mathbf{d}(N)}\omega$ is dense in $\mathcal{Q}^{(N)}$.*

e) *The variety $\mathcal{Q}^{(N)}$ is irreducible.*

Let W be a lattice in $k((z))^r$. Since $k[[z]]$ is a principal ring, there exists a uniquely determined sequence of integers $d_1 \leq \ldots \leq d_r$ and a basis $(e_1, \ldots, e_r)$ of $k[[z]]^r$ such that W is the lattice spanned by $z^{d_1}e_1, \ldots, z^{d_r}e_r$; this lattice is special if and only if $\sum d_i = 0$. This means that the point W of $\mathcal{Q}(k)$ belongs to the orbit of $z^{\mathbf{d}}\omega$, which proves a). Since the condition $W \in \mathcal{Q}^{(N)}$ is equivalent to $-N \leq d_1 \leq \ldots \leq d_r \leq N$, c) follows.

The formula

$$\begin{pmatrix} t^{-1}z & t^{-1} \\ -t & 0 \end{pmatrix}\begin{pmatrix} z^{d_1} & 0 \\ 0 & z^{d_2} \end{pmatrix}\begin{pmatrix} t & -t^{-1}z^{d_2-d_1-1} \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} z^{d_1+1} & 0 \\ -t^2z^{d_2} & z^{d_2-1} \end{pmatrix}$$

for $t \in k^*$, shows that the point $z^{\mathbf{d}'}\omega$ belongs to the closure of the orbit of $z^{\mathbf{d}}\omega$ whenever $\mathbf{d}'$ is obtained from $\mathbf{d}$ by replacing a pair of indices (d_i, d_j) with $d_i < d_j$ by the pair $(d_i + 1, d_j - 1)$. An easy combinatorial argument then shows that every sequence $\mathbf{d}'$ with $d'_1 + \ldots + d'_p \geq d_1 + \ldots + d_p$ for $1 \leq p \leq r$ can be obtained from $\mathbf{d}$ by iterating this operation, which proves the "if" part of b). To prove the converse, observe that any matrix $A(z)$ in $SL_r(\mathcal{O})z^{\mathbf{d}}SL_r(\mathcal{O})$ has the property that $z^{d_1}\ldots z^{d_p}$ divides the coefficients of $\wedge^p A(z)$ for $1 \leq p \leq r$. So if $z^{\mathbf{d}'}$ is a specialization of such a matrix one must have $d_1 + \ldots + d_p \leq d'_1 + \ldots + d'_p$ for all p, which gives b).

The assertion d) is an easy consequence of b) and c). To prove e), it remains to show that the group scheme $\mathbf{SL}_r(\mathscr{O})$ is irreducible; one way is to observe that the group scheme $\mathbf{GL}_r(\mathscr{O})$ is irreducible $\Big($it is isomorphic to $GL_r(k) \times \prod_1^\infty M_r(k)$, see (1.2)$\Big)$, and maps onto $\mathbf{SL}_r(\mathscr{O})$ by the morphism $A \mapsto A\delta(A)^{-1}$, where $\delta(A)$ is the diagonal matrix $\operatorname{diag}(\det A, 1, \ldots, 1)$. □

Remark 2.7. One can refine the above decomposition of $\mathcal{Q}$ as follows. Let U be the subgroup of $SL_r(\mathscr{O})$ consisting of matrices $A(z)$ such that $A(0)$ is upper-triangular with diagonal coefficients equal to 1. Using the Bruhat decomposition of $SL_r(k)$ one sees easily that the $SL_r(\mathscr{O})$-orbit of $z^{\mathbf{d}}\omega$ is the disjoint union of the sets $Uz^{\mathbf{d}_\sigma}\omega$, where $\mathbf{d}_\sigma$ runs over all permutations of the sequence $(d_1, \ldots, d_r)$. This is the *parabolic Bruhat decomposition* of the Kac-Moody groups theory [Ku, Sl].

3. The Stack $\mathbf{SL}_r(\mathscr{O})\backslash\mathbf{SL}_r(K)/\mathbf{SL}_r(A_X)$

Stacks

We will need a few properties of stacks. Rather than giving formal definitions (for which we refer to [D-M] and especially [L-MB]), we will try here to give a rough idea of what stacks are and what they are good for. Many geometric objects (like vector bundles on a fixed variety, or varieties of a given type) have no fine moduli space because of automorphisms. The remedy is to consider, instead of the set of isomorphism classes, the *groupoid* of such objects (recall that a groupoid is a category where every arrow is an isomorphism).

A *stack* over k associates to any k-algebra R a groupoid $F(R)$, and to any homomorphism $u: R \to S$ a functor $F(u): F(R) \to F(S)$; these data should satisfy some natural compatibility conditions as well as some localization properties.

By considering a set as a groupoid (with the identity of each object as only arrows), a k-space can be viewed in a natural way as a stack over k. Conversely, a stack over k with the property that any object has the identity as only automorphism is a k-space.

Examples. (3.1) The *moduli stack* $\mathscr{GL}_X(r)$ of rank r vector bundles on X is defined by associating to a k-algebra R the groupoid of rank r vector bundles over X_R. Similarly, one defines a stack $\mathscr{SL}_X(r)$ by associating to R the groupoid of pairs (E, δ), where E is a vector bundle over X_R and $\delta: \mathscr{O}_{X_R} \to \bigwedge^r E$ an isomorphism; this is the fibre over the trivial bundle of the morphism of stacks $\det: \mathscr{GL}_X(r) \to \mathscr{GL}_X(1)$.

(3.2) Let be Γ a k-group (1.1). Recall that a *Γ-torsor* (or principal Γ-bundle) over a k-scheme S is a k-space over S with an action of Γ, which after a faithfully flat extension $S' \to S$ becomes trivial, that is isomorphic to $\Gamma \times S'$ with the action of Γ by multiplication. Let Q be a k-space with an action of Γ. The quotient stack $F = \Gamma\backslash Q$ is defined in the following way: an object of $F(R)$ is a Γ-torsor P together with a Γ-equivariant morphism $\alpha: P \to Q$; arrows in $F(R)$ are defined in the obvious way, and so are the functors $F(u)$. The stack $\Gamma\backslash Q$ is indeed the quotient of Q by Γ in the category of stacks, in the sense that any Γ-invariant morphism from Q into a stack factors through $\Gamma\backslash Q$ in a unique way. If Γ acts freely on Q (i.e. $\Gamma(R)$ acts freely on $Q(R)$ for each k-algebra R), then the stack $\Gamma\backslash Q$ is a k-space.

When $Q = \mathrm{Spec}(k)$ (with the trivial action), $\Gamma\backslash Q$ is the *classifying stack* $B\Gamma$: for each k-algebra R, $B\Gamma(R)$ is the groupoid of Γ-torsors over $\mathrm{Spec}(R)$.

Proposition 3.4. *The quotient stack* $\mathbf{SL}_r(A_X)\backslash\mathbf{SL}_r(K)/\mathbf{SL}_r(\mathcal{O})$ *is canonically isomorphic to the algebraic stack* $\mathcal{SL}_X(r)$ *of vector bundles on* X *with trivial determinant. The projection map* $\pi:\mathbf{SL}_r(K)/\mathbf{SL}_r(\mathcal{O}) \to \mathcal{SL}_r(X)$ *is locally trivial for the Zariski topology.*

Let us denote as before by $\mathcal{Q}$ the ind-variety $\mathbf{SL}_r(K)/\mathbf{SL}_r(\mathcal{O})$, and by Γ the group $\mathbf{SL}_r(A_X)$. The universal vector bundle $\mathcal{E}$ over $X \times \mathcal{Q}$, together with the trivialization of $\bigwedge^r \mathcal{E}$ given by ϱ (Proposition 2.1), gives rise to a map $\pi:\mathcal{Q} \to \mathcal{SL}_X(r)$. This map is Γ-invariant, hence induces a morphism of stacks $\bar{\pi}:\Gamma\backslash\mathcal{Q} \to \mathcal{SL}_X(r)$.

On the other hand we can define a map $\mathcal{SL}_X(r) \to \Gamma\backslash\mathcal{Q}$ as follows. Let R be a k-algebra, E a vector bundle over X_R and δ a trivialization of $\bigwedge^r E$. For any R-algebra S, let $P(S)$ be the set of trivializations ϱ of E_S over X_S^* such that $\wedge^r\varrho$ coincides with the pull back of δ. This defines a R-space P on which the group Γ acts; by the lemma below, it is a torsor under Γ (it is in fact an ind-scheme, but we need not worry about that). To any element of $P(S)$ corresponds a pair (E_S, ϱ), hence by Proposition 2.1 an element of $\mathcal{Q}(S)$. In this way we associate functorially to an object (E,δ) of $\mathcal{SL}_X(r)(R)$ a Γ-equivariant map $\alpha:P \to \mathcal{Q}$. This defines a morphism of stacks $\mathcal{SL}_X(r) \to \Gamma\backslash\mathcal{Q}$ which is the inverse of $\bar{\pi}$.

The second assertion means that for any scheme T and morphism $f:T \to \mathcal{SL}_X(r)$, the pull back to T of the fibration π is (Zariski) locally trivial, i.e. admits local sections. Now f corresponds to a pair (E,δ), where E is a vector bundle over $X \times T$ and δ a trivialization of $\bigwedge^r E$. Let $t \in T$. By the lemma below, we can find an open neighborhood U of t in T and a trivialization ϱ of $E_{|X^* \times U}$; modifying ϱ by an automorphism of $\mathcal{O}^r_{X^* \times U}$ if necessary, we can moreover assume $\wedge^r\varrho = \delta_{|X^* \times U}$. The pair (E,ϱ) defines a morphism $g:U \to \mathcal{Q}$ (Proposition 2.1) such that $\pi \circ g = f$, that is a section over U of the pull back of the fibration π. □

Lemma 3.5. *Let* T *be a scheme, and* E *a vector bundle over* $X \times T$ *with trivial determinant. Then there exists an open covering* (U_α) *of* T *such that the restriction of* E *to* $X^* \times U_\alpha$ *is trivial.*

We proceed by induction on the rank r of E – the case $r = 1$ being trivial. Suppose $r \geq 2$. Let us denote simply by p the divisor $\{p\} \times T$ in $X \times T$. There exists an integer n such that $E(np)$ is spanned by its global sections and has no H^1. Let t be a point of T. An easy count of constants provides a section s of $E(np)_{|X\times\{t\}}$ which does not vanish at any point of X. Shrinking T if necessary, we may assume that s is the restriction of a global section of $E(np)$ which vanishes nowhere on $X \times T$. By restriction to $X^* \times T$ we get an exact sequence

$$0 \to \mathcal{O}_{X^*\times T} \to E_{|X^*\times T} \to F \to 0\,,$$

where F is a vector bundle of rank $r - 1$ over $X^* \times T$. Again by shrinking T if necessary, we may assume that this sequence is split and (thanks to the induction hypothesis) that F is trivial, so E is trivial over $X^* \times T$. □

Remark 3.6. The proof of the proposition applies without any modification to the case of vector bundles with determinant $\mathcal{O}_X(dp)$, $d \in \mathbf{Z}$: the ind-group $\mathbf{SL}_r(A_X)$ acts by left multiplication on $\mathcal{Q}_d = \mathbf{SL}_r(K)^{(d)}/\mathbf{SL}_r(\mathcal{O})$ (2.2), and the quotient stack $\mathbf{SL}_r(A_X)\backslash\mathcal{Q}_d$ is canonically isomorphic to the moduli stack $\mathcal{SL}_X(r,d)$ parametrizing vector bundles on E on X_R together with an isomorphism $\mathcal{O}_{X_R}(dp) \xrightarrow{\sim} \bigwedge^r E$.

Let γ_d be an element of $GL_r(K)$ with $\det(\gamma_d) = z^{-d}$. Since left multiplication by γ_d induces an isomorphism of $\mathcal{Q}$ onto $\mathcal{Q}_d$ (2.2), we can also describe $\mathcal{SL}_X(r,d)$ as the quotient stack $(\gamma_d^{-1}\mathbf{SL}_r(A_X)\gamma_d)\backslash\mathcal{Q}$.

Line Bundles over k-Spaces and Stacks

(3.7) Let Q be a k-space. A *line bundle* (or a vector bundle, or a coherent sheaf) $\mathcal{L}$ on Q can be defined as the data of a line bundle (resp. a vector bundle, resp. a coherent sheaf) $\mathcal{L}_\mu$ on T for each morphism μ of a scheme T into Q, and of isomorphisms $g_{\mu,f} : f^*\mathcal{L}_\mu \xrightarrow{\sim} \mathcal{L}_{\mu\circ f}$ for each morphism of schemes $f : T' \to T$; these data must satisfy the obvious compatibility conditions. Morphisms of line bundles (resp. ...) are defined in an analogous way; in particular, a section of $\mathcal{L}$ is a compatible family of sections $s_\mu \in H^0(T, \mathcal{L}_\mu)$, which means $g_{\mu,f}(f^*s_\mu) = s_{\mu\circ f}$ for all $f : T' \to T$. We leave to the reader to check that all the standard constructions for line bundles on schemes extend naturally to this situation.

Of course these definitions coincide with the usual ones when Q is scheme. Suppose Q is an ind-scheme, limit of an increasing sequence Q_n of schemes; then a line bundle $\mathcal{L}$ on Q is determined by the data of a line bundle (L_n) on Q_n for each n, and isomorphisms $L_q|_{Q_p} \xrightarrow{\sim} L_p$ for $q \geq p$, again with the obvious compatibility conditions. The space $H^0(Q, \mathcal{L})$ is then the inverse limit of the system $(H^0(Q_n, L_n))_{n\geq 1}$.

These definitions can be easily generalized to the case of stacks. A line bundle $\mathcal{L}$ on a stack $\mathcal{S}$ is defined as the data of a line bundle $\mathcal{L}_\mu$ on T for each scheme T and object μ of the groupoid $\mathcal{S}(T)$, and of an isomorphism $g_\alpha : f^*L_\mu \to L_\nu$ for each morphism $f : T' \to T$ and each arrow $\alpha : f^*\mu \to \nu$ in $\mathcal{S}(T')$ – these data should satisfy some standard compatibility conditions. A section of $\mathcal{L}$ is again given by a family of sections $s_\mu \in H^0(T, \mathcal{L}_\mu)$ such that $g_\alpha(f^*s_\mu) = s_\nu$ for each arrow $\alpha : f^*\mu \to \nu$ in $\mathcal{S}(T')$.

(3.8) *Example: the determinant bundle.* Let T be a scheme and E a vector bundle on $X \times T$. The derived direct image $R(pr_T)_*(E)$ is given by a complex of vector bundles $L^0 \to L^1$. The line bundle $\det(L^1) \otimes \det(L^0)^{-1}$ is independent of the choice of this complex, hence canonically defined on T; this is the "determinant of the cohomology" $\det R\Gamma_T(E)$. Associating to each bundle E on $X \times T$ the line bundle $\det R\Gamma_T(E)$ defines a line bundle $\mathcal{L}$ on the stack $\mathcal{SL}_X(r)$ (or $\mathcal{GL}_X(r)$), the *determinant line bundle.*

There is a useful way to produce sections of the line bundle $\det R\Gamma_T(E)$ and of its multiples. Suppose for simplicity that T is integral, and that the line bundle $\bigwedge^r E$ is the pull back of some line bundle on X. Let F be a vector bundle of rank s on X; let us use the same notation to denote its pull back to $X \times T$. Then *the line bundle* $\det R\Gamma_T(E \otimes F)$ *is isomorphic to* $(\det R\Gamma_T(E))^{\otimes s}$ (write F as an extension to reduce to the case $s = 1$, then use repeatedly the exact sequence $0 \to E \to E(q) \to E \otimes (\mathcal{O}(q)/\mathcal{O}) \to 0$ to prove that the line bundle $\det R\Gamma_T(E(D))$ is isomorphic to $\det R\Gamma_T(E)$ for any divisor D on X). Put $E_t := E_{|X\times\{t\}}$ for $t \in T$. Choose F such that the vector bundle $E_t \otimes F$ has trivial cohomology for some t in T. Let $L^0 \xrightarrow{u} L^1$ be a complex of vector bundles isomorphic to $R(pr_T)_*(E \otimes F)$. Then $\det u$ is a nonzero section of $\det R\Gamma_T(E\otimes F) \cong (\det R\Gamma_T(E))^{\otimes s}$, which is well defined up to an invertible function on T. In particular, its divisor Θ_F is canonically defined on T (the support of Θ_F is the set of points $t \in T$ such that $H^0(X, E_t \otimes F) \neq 0$).

Example 3.9. Let G be a k-group, H a k-subgroup of G, and $\chi : H \to \mathbf{G}_m$ a character of H. As usual we associate to this situation a line bundle $\mathscr{L}_\chi$ on G/H: the group Hacts freely on the trivial bundle $G \times \mathbf{A}^1$ by $h \cdot (g,t) = (gh, \chi(h^{-1})t)$ and we define $\mathscr{L}_\chi$ as the quotient k-space $(G \times \mathbf{A}^1)/H$. It is an easy exercise in descent theory to prove that the pull back of the fibration $\mathscr{L}_\chi \to G/H$ by any morphism $\mu : T \to G/H$ is indeed a line bundle (use the fact that μ lifts locally to G, and that the pull back of $\mathscr{L}_\chi$ to G is the trivial bundle). Again by descent, sections of $\mathscr{L}_\chi$ corresponds in a one-to-one way to sections of the trivial bundle $G \times \mathbf{A}^1$ over G which are H-invariant, that is to *functions f on G such that* $f(gh) = \chi(h^{-1})f(g)$ for any k-algebra R and elements $g \in G(R)$, $h \in H(R)$.

There is a more fancy way to describe the line bundle $\mathscr{L}_\chi$. Consider the classifying stack $B\mathbf{G}_m$ over k (3.2). A morphism $\mu : T \to B\mathbf{G}_m$ is given by a $\mathbf{G}_m$-torsor over T, which defines a line bundle $\mathscr{U}_\mu$ over T: this defines the universal line bundle $\mathscr{U}$ over $B\mathbf{G}_m$. As a stack over $B\mathbf{G}_m$, it is simply the quotient $\mathbf{A}^1/\mathbf{G}_m$. The character $\chi : H \to \mathbf{G}_m$ induces a morphism $B\chi : BH \to B\mathbf{G}_m$, hence a line bundle $(B\chi)^*\mathscr{U}$ on BH. The structural map $G \to \operatorname{Spec}(k)$ induces a morphism of stacks $G/H \to BH$, hence by pull back we get a line bundle on G/H, which is (almost by definition) $\mathscr{L}_\chi$.

Let G' be another k-group, H' a k-subgroup of G' and $f : G' \to G$ a morphism of k-groups which maps H' into H. It follows from either of these definitions that the pull back of $\mathscr{L}_\chi$ by the morphism $G'/H' \to G/H$ induced by f is the line bundle $\mathscr{L}_{\chi'}$ associated to the character $\chi' := \chi \circ f$ of H'.

4. The Central Extension

Let $\pi : \mathscr{Q} \to \mathscr{SL}_X(r)$ be the canonical morphism of stacks defined in the preceding section, and $\mathscr{L}$ the determinant line bundle on $\mathscr{SL}_X(r)$ (3.8). We want to identify the line bundle $\pi^*\mathscr{L}$ on $\mathscr{Q}$.

It will turn out that, though it is invariant under the action of $\mathbf{SL}_r(K)$, this line bundle does *not* admit an action of $\mathbf{SL}_r(K)$. But it does admit an action of a canonical extension $\widehat{\mathbf{SL}}_r(K)$ of $\mathbf{SL}_r(K)$, which we are now going to describe paraphrasing [S-W].

The Canonical Extension of the Fredholm Group

(4.1) Let V be an infinite-dimensional vector space over k. Denote by $\operatorname{End}^f(V)$ the two-sided ideal of $\operatorname{End}(V)$ formed by the endomorphisms of finite rank and by $\mathscr{F}(V)$ the group of units of the quotient algebra $\operatorname{End}(V)/\operatorname{End}^f(V)$. The elements of $\mathscr{F}(V)$ are classes of equivalence of endomorphisms with finite-dimensional kernel and cokernel. We let $\mathscr{F}(V)^0$ be the subgroup of classes of index 0 endomorphisms, i.e. endomorphisms with $\dim \operatorname{Ker} u = \dim \operatorname{Coker} u$. It is an easy exercise to show that the image of the canonical homomorphism $\operatorname{Aut}(V) \to \mathscr{F}(V)$ is $\mathscr{F}(V)^0$; its kernel consists of the automorphisms u of V such that $u \equiv I$ (mod. $\operatorname{End}^f(V)$). The determinant of such an endomorphism is naturally defined, e.g. by the formula

$\det(I + v) = \sum_{n\geq 0} \operatorname{Tr} \wedge^n v$. Let us denote by $(I + \operatorname{End}^f(V))_1$ the subgroup of automorphisms of the form $I + v$ with $v \in \operatorname{End}^f(V)$ and $\det(I + v) = 1$; we get an exact sequence

$$1 \to k^* \to \operatorname{Aut}(V)/(I + \operatorname{End}^f(V))_1 \to \mathscr{F}(V)^0 \to 1 .$$

For $v \in \operatorname{End}^f(V)$, $u \in \operatorname{Aut}(V)$, one has $\det(I + uvu^{-1}) = \det(I + v)$; this means that the element $I + v$ belongs to the *center* of the group $\operatorname{Aut}(V)/(I + \operatorname{End}^f(V))_1$. We have thus defined a canonical central extension of the group $\mathscr{F}(V)^0$ by k^*.

(4.2) We want to make sense of this in an algebraic setting, at least at the level of k-groups. We define the k-space $\operatorname{End}(V)$ in an obvious way, as the functor $R \mapsto \operatorname{End}_R(V \otimes_k R)$, and the k-group $\operatorname{Aut}(V)$ as its group of units. We'll say that an endomorphism of $V \otimes R$ has finite rank if its image is contained in a finitely generated submodule; we define $\operatorname{End}^f(V)(R)$ as the ideal formed by these endomorphisms, and take for $\mathscr{F}(V)$ the group of units of the algebra $\operatorname{End}(V)/\operatorname{End}^f(V)$. We don't know a good definition for the subgroup $\mathscr{F}(V)^0$, so we just define it as the image of $\operatorname{Aut}(V)$ in $\mathscr{F}(V)$. We then get again a central extension of k-groups

$$(\mathscr{F}) \qquad 1 \to \mathbf{G}_m \to \operatorname{Aut}(V)/(I + \operatorname{End}^f(V))_1 \to \mathscr{F}(V)^0 \to 1 .$$

The Central Extension of $\mathbf{SL}_r(K)$

Let us go back to the ind-group $\mathbf{GL}_r(K)$. We choose a supplement V of $\mathscr{O}^r$ in K^r. For any k-algebra, we get a direct sum decomposition (over k)

$$R((z))^r = V_R \oplus R[[z]]^r ,$$

with $V_R := V \otimes_k R$. Let γ be an element of $GL_r(R((z)))$, and let

$$\gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix} \tag{4.2}$$

be its matrix with respect to the above decomposition. Let $\bar{a}(\gamma)$ denote the class of $a(\gamma)$ in $\operatorname{End}(V_R)/\operatorname{End}^f(V_R)$.

Proposition 4.3. a) *The map $\gamma \mapsto \bar{a}(\gamma)$ is a group homomorphism from $GL_r(R((z)))$ into $\mathscr{F}(V_R)$; it defines a morphism of k-groups:*

$$\bar{a} : \mathbf{GL}_r(K) \to \mathscr{F}(V) .$$

b) *Let V' be another supplement of $\mathscr{O}^r$ in K^r and let $\bar{a}' : \mathbf{GL}_r(K) \to \mathscr{F}(V')$ be the morphism associated to V'. Let $\varphi : V \to V'$ be the isomorphism obtained by restricting to V the projector onto V'. Then $\bar{a}'$ is equal to $\varphi \bar{a} \varphi^{-1}$.*

Since γ maps $R[[z]]^r$ into $z^{-N} R[[z]]^r$ for some N, the map $b(\gamma) : R[[z]]^r \to V_R$ has finite rank. From this and the formula for the product of two matrices follows first that the endomorphism a of V_R is invertible modulo finite rank endomorphisms, then that the map $GL(R((z))) \to \mathscr{F}(V)(R)$ which associates to γ the class of $a(\gamma)$ is a group homomorphism. This proves a).

Let p, q be the projectors of $R((z))^r$ onto V'_R and $R[[z]]^r$ relative to the decomposition $R((z))^r = V'_R \oplus R[[z]]^r$, and let $\begin{pmatrix} a'(\gamma) & b'(\gamma) \\ c'(\gamma) & d'(\gamma) \end{pmatrix}$ be the matrix of an element $\gamma \in GL_r(R((z)))$ relative to this decomposition. An easy computation gives $pa(\gamma) = a'(\gamma)p + b'(\gamma)q$. Since $b'(\gamma)$ has finite rank, we get the equality $a'(\gamma) \equiv \varphi a(\gamma)\varphi^{-1}$ (mod. $\mathrm{End}^f(V_R)$). □

Proposition 4.4. *Let R be a k-algebra and γ an element of $SL_r(R((z)))$; locally on* Spec(R) *(for the Zariski topology), the endomorphism $a(\gamma)$ of V_R is equivalent* mod. $\mathrm{End}^f(V_R)$ *to an automorphism.*

By Proposition 4.3b), it is enough to prove the result for one particular choice of V; we'll take $V = (z^{-1}k[z^{-1}])^r$. The assertion is clear when γ belongs to $SL_r(R[[z]])$ or to $SL_r(R[z^{-1}])$: in those cases the matrix (4.2) is triangular, so that $a(\gamma)$ itself is an isomorphism. The result then follows when R is a field, since any matrix $\gamma \in SL_r(R((z)))$ can be written as a product of elementary matrices $I + \lambda E_{ij}$, where λ can be taken either in $R[[z]]$ or in $R[z^{-1}]$. The general case is a consequence of the following lemma:

Lemma 4.5. *Locally over* Spec(R)*, any element γ of $SL_r(R((z)))$ can be written $\gamma_0\gamma^-\gamma^+$, with $\gamma_0 \in SL_r(K)$, $\gamma^- \in SL_r(R[z^{-1}])$, $\gamma^+ \in SL_r(R[[z]])$.*

Let us assume first that the k-algebra R is finitely generated. Let t be a closed point of Spec(R); put $\gamma_0 = \gamma(t)$. By (1.11) $\gamma_0^{-1}\gamma$ can be written in a neighborhood of t as $\gamma^-\gamma^+$, hence the result in this case.

In the general case, R is the union of its finitely generated subalgebras R_α. Let $p: \mathbf{SL}_r(K) \to \mathcal{Q} = \mathbf{SL}_r(K)/\mathbf{SL}_r(\mathcal{O})$ be the quotient map. Since $\mathcal{Q}$ is an ind-variety, the morphism $p\circ\gamma$: Spec$(R) \to \mathcal{Q}$ factors through Spec(R_α) for some α. Locally over Spec(R_α), this morphism can be written $p\circ\gamma_\alpha$ for some element γ_α of $SL_r(R_\alpha((z)))$, which differ from γ by an element of $SL_r(R[[z]])$ (Theorem 2.5). Since R_α is of finite type, the lemma holds for γ_α, hence also for γ. □

Corollary 4.6. *The image of* $\mathbf{SL}_r(K)$ *by $\bar{a}$ is contained in the subgroup $\mathcal{F}(V)^0$.* □

We will denote by $\widehat{\mathbf{SL}}_r(K) \to \mathbf{SL}_r(K)$ the pull back of the central extension $(\mathcal{F})$ by $\bar{a}$, so that we get a central extension of k-groups

$$(\mathcal{E}) \qquad 0 \to \mathbf{G}_m \to \widehat{\mathbf{SL}}_r(K) \xrightarrow{\psi} \mathbf{SL}_r(K) \to 0\,.$$

By descent theory any $\mathbf{G}_m$-torsor over a scheme is representable, so the k-group $\widehat{\mathbf{SL}}_r(K)$ is also an ind-group.

(4.7) Let R be a k-algebra; an element of $\widehat{\mathbf{SL}}_r(K)(R)$ is given, locally on Spec R, by a pair (γ, u) with γ in $SL_r(R((z)))$, u in Aut(V_R), and $u \equiv a(\gamma)$ (mod. $\mathrm{End}^f(V_R)$); two pairs (γ, u) and (γ, v) give the same element if $u^{-1}v$ (which belongs to $I + \mathrm{End}^f(V_R)$) has determinant 1. In particular, the kernel of ψ consists of the pairs (I, u) with $u \in I + \mathrm{End}^f(V)$, modulo the pairs (I, u) with $\det u = 1$; the map $u \mapsto \det u$ provides an isomorphism from Ker ψ onto $\mathbf{G}_m$.

Because of Proposition 4.3b), the extension $(\mathcal{E})$ is independent of the choice of the supplement V of $\mathcal{O}^r$ in K^r. More precisely, given two such supplements V and V', there is a canonical isomorphism from the group $\widehat{\mathbf{SL}}_r(K)$ defined using V onto the group $\widehat{\mathbf{SL}}'_r(K)$ defined using V': it associates to a pair (γ, u) as above the pair $(\gamma, \varphi u\varphi^{-1})$, where φ is the natural isomorphism from V onto V' (loc. cit.). One can then define in the usual way a *canonical* central extension of $\mathbf{SL}_r(K)$ by taking the projective (or inductive) limit, over the set of all supplements of $\mathscr{O}^r$ in K^r, of the extensions we have constructed.

(4.8) Let H be a sub-k-group of $\mathbf{SL}_r(K)$, such that $\mathscr{O}^r$ (resp. V) is stable under H. Then *the extension* ($\mathscr{E}$) *is canonically split over* H. For any element γ of $H(R)$ satisfies $b(\gamma) = 0$ (resp. $c(\gamma) = 0$), so that the map $\gamma \mapsto a(\gamma)$ is a homomorphism from $H(R)$ into $\mathrm{Aut}(V_R)$. Then the map $\gamma \mapsto (\gamma, a(\gamma))$ defines a section of ψ over H. In particular, we see that *the pull back* $\widehat{\mathbf{SL}}_r(\mathscr{O})$ *of* $\mathbf{SL}_r(\mathscr{O})$ *is canonically isomorphic to* $\mathbf{SL}_r(\mathscr{O}) \times \mathbf{G}_m$. We will denote by $\chi_0 : \widehat{\mathbf{SL}}_r(\mathscr{O}) \to \mathbf{G}_m$ the second projection; if the element $\tilde{\delta}$ of $\widehat{\mathbf{SL}}_r(\mathscr{O})$ is represented by a pair (δ, v), one has $\chi_0(\tilde{\delta}) = \det(a(\delta)^{-1}v)$.

More generally, suppose that there exists an element $\lambda \in SL_r(K)$ such that the subgroup H preserves the subspace $\lambda(\mathscr{O}^r)$ (resp. $\lambda(V)$). We choose an automorphism u of V such that $u \equiv a(\lambda) \pmod{\mathrm{End}^f(V)}$, and define a section of ψ over H by $\gamma \mapsto ua(\lambda^{-1}\gamma\lambda)u^{-1}$. This section is independent of the choice of u, so once again the group H embeds canonically into $\widehat{\mathbf{SL}}_r(K)$.

The Lie Algebra of the Central Extension and the Tate Residue

(4.9) We want to show that at the level of Lie algebras, the extension ($\mathscr{E}$) is the universal central extension which appears in the theory of Kac-Moody algebras [K]. This is essentially known (see e.g. [A-D-K], where very similar computations appear). We have included the computation because it is extremely simple and gives a nice generalization of the residue defined by Tate in [T].

Let us start from the central extension ($\mathscr{F}$). Since $\mathscr{F}(V)$ is the group of invertible elements of the associative algebra $\mathrm{End}(V)/\mathrm{End}^f(V)$, its Lie algebra is simply the quotient of the Lie algebra $\mathrm{End}(V)$ by the ideal $\mathrm{End}^f(V)$. The Lie algebra of $(I + \mathrm{End}^f(V))_1$ is the sub-Lie algebra $\mathrm{End}^f(V)_0$ of $\mathrm{End}^f(V)$ consisting of traceless endomorphisms. Therefore the Lie algebras extension corresponding to ($\mathscr{F}$) is

$$(\mathfrak{F}) \qquad 0 \to k \to \mathrm{End}(V)/\mathrm{End}^f(V)_0 \to \mathrm{End}(V)/\mathrm{End}^f(V) \to 0\,.$$

Let α be an element of $\mathfrak{sl}_r(k((z)))$; it corresponds to the element $I + \varepsilon\alpha$ of $SL_r(k[\varepsilon]((z)))$. Since $a(I + \varepsilon\alpha) = I + \varepsilon a(\alpha)$, the tangent map $L(\bar{a})$ at I to $\bar{a} : \mathbf{SL}_r(K) \to \mathscr{F}(V)$ associates to α the class of $a(\alpha)$ in $\mathrm{End}(V)/\mathrm{End}^f(V)$. By construction the extension of $\mathfrak{sl}_r(K)$ we are looking for is the pull back of $\mathfrak{F}$ by $L(\bar{a})$. This means that the Lie algebra $\widehat{\mathfrak{sl}}_r(K)$ of $\widehat{\mathbf{SL}}_r(K)$ consists of pairs (α, u) with $\alpha \in \mathfrak{sl}_r(K)$, $u \in \mathrm{End}(V)$, $a(\alpha) \equiv u \pmod{\mathrm{End}^f(V)}$; two pairs (α, u) and (α, v) give the same element if $\mathrm{Tr}(u - v) = 0$. We get a central extension

$$(\mathfrak{E}) \qquad 0 \to k \to \widehat{\mathfrak{sl}}_r(K) \xrightarrow{L(\psi)} \mathfrak{sl}_r(K) \to 0$$

with $L(\psi)(\alpha, u) = \alpha$, the kernel of $L(\psi)$ being identified with k by $(0, u) \mapsto \mathrm{Tr}\, u$. As before, this extension does not depend on the choice of the supplement V of $\mathscr{O}^r$ in K^r. We claim that it is the well-known *universal central extension* of $\mathfrak{sl}_r(K)$. Recall that this extension is obtained by defining a Lie algebra structure on $\mathfrak{sl}_r(K) \oplus k$ by the formula

$$[(\alpha, s), (\beta, t)] = \left([\alpha, \beta], \mathrm{Res}_0 \mathrm{Tr}\left(\frac{d\alpha}{dz}\beta\right)\right);$$

the projection p onto the first summand and the injection i of the second one define the universal central extension

$$(\mathfrak{U}) \qquad 0 \to k \xrightarrow{i} \mathfrak{sl}_r(K) \oplus k \xrightarrow{p} \mathfrak{sl}_r(K) \to 0 .$$

Proposition 4.10. *There exists a Lie algebra isomorphism* $\widehat{\mathfrak{sl}}_r(K) \xrightarrow{\sim} \mathfrak{sl}_r(K) \oplus k$ *inducing an isomorphism of the extension* $\mathfrak{E}$ *onto the universal central extension of* $\mathfrak{sl}_r(K)$.

It is enough to prove the proposition for one particular choice of V; it will be convenient to choose for V the subspace $\mathcal{O}_-^r$, where $\mathcal{O}_-$ denotes the subspace $z^{-1}k[z^{-1}]$ of K.

Let us define a map $\varphi : \widehat{\mathfrak{sl}}_r(K) \to \mathfrak{sl}_r(K) \oplus k$ by $\varphi(\alpha, u) = (\alpha, \mathrm{Tr}(u - a(\alpha)))$. One has $p \circ \varphi = L(\psi)$ and φ induces an isomorphism of $\mathrm{Ker}\, L(\psi)$ onto $i(k)$; this implies that φ is bijective. It remains to check that φ is a Lie algebra homomorphism. Since φ maps $\mathrm{Ker}\, L(\psi)$ into the center $i(k)$, it is enough to prove the equality $\varphi([\tilde{\alpha}, \tilde{\beta}]) = [\varphi(\tilde{\alpha}), \varphi(\tilde{\beta})]$ for $\tilde{\alpha} = (\alpha, a(\alpha))$, $\tilde{\beta} = (\beta, a(\beta))$. This amounts to the following formula:

Lemma 4.11. *Let* α, β *be two matrices in* $M_r(K)$. *One has*

$$\mathrm{Tr}([a(\alpha), a(\beta)] - a([\alpha, \beta])) = \mathrm{Res}_0 \mathrm{Tr} \left(\frac{d\alpha}{dz} \beta \right) .$$

This is precisely Tate's definition of the residue [T] in the case $r = 1$; we will actually reduce the proof to the rank 1 case.

Assume first that for some integer N one has $\alpha \in z^{N+1} M_r(\mathcal{O})$, $\beta \in z^{-N} M_r(\mathcal{O})$. For $p \geq 0$, let us denote by V_p the subspace $V \cap z^{-p}\mathcal{O}^r$ of V. Then $(V_p)_{p\geq 0}$ is an increasing filtration of V, and for each $p \geq 0$, the endomorphism $[a(\alpha), a(\beta)] - a([\alpha, \beta])$ maps V_p into V_{p-1}. This implies that its trace is zero, which gives the formula in this case.

By bilinearity, we can therefore assume that α and β are polynomial in z, and even of the form $z^p A$ for some integer p and some matrix $A \in M_r(k)$. Let us identify K^r with $K \otimes_k k^r$. The direct sum decomposition $K^r = V \oplus \mathcal{O}^r$ is induced by tensor product from the decomposition $K = \mathcal{O}_- \oplus \mathcal{O}$. It follows that $a(z^p \otimes A)$ is $a_1(z^p) \otimes A$, where $a_1(z^p)$ is the endomorphism of $\mathcal{O}_-$ associated to z^p. Since the trace of $u \otimes M$, for $u \in \mathrm{End}^f(\mathcal{O}_-)$ and $M \in M_r(k)$, is $(\mathrm{Tr}\, u)(\mathrm{Tr}\, M)$, and since the endomorphisms z^p and z^q of K commute, we obtain

$$\mathrm{Tr}([a(z^p A), a(z^q B)] - a([z^p A, z^q B]) = \mathrm{Tr}\, AB\, \mathrm{Tr}[a_1(z^p), a_1(z^q)] .$$

It remains to compute the trace of the (finite rank) endomorphism $u = [a_1(z^p), a_1(z^q)]$ of $\mathcal{O}_-$, which we do using the basis $(z^{-n})_{n\geq 1}$ of $\mathcal{O}_-$. One has $u(z^{-n}) = \varepsilon z^{p+q-n}$, with $\varepsilon \in \{-1, 0, 1\}$; therefore $\mathrm{Tr}\, u$ is zero except when $p + q = 0$. Assume $q = -p$ and, say, $p \geq 0$; then we find $u(z^{-n}) = 0$ for $n > p$, and $u(z^{-n}) = z^{-n}$ for $1 \leq n \leq p$. We conclude that $\mathrm{Tr}[a_1(z^p), a_1(z^q)] = \delta_{p,-q} p = \mathrm{Res}_0(p z^{p-1} z^q)$, from which the proposition follows. □

(4.12) The above computations extend in a straightforward way when the base field k is replaced by an arbitrary k-algebra R. In particular, the kernel of the homomorphism $\widehat{\mathbf{SL}}_r(K)(R[\varepsilon]) \to \widehat{\mathbf{SL}}_r(K)(R)$ is the Lie algebra $\widehat{\mathfrak{sl}}_r(R((z))) = \mathfrak{sl}_r(R((z))) \oplus R$, where the Lie bracket is defined by formula (4.10). This defines an adjoint action of the group $SL_r(R((z)))$ onto $\widehat{\mathfrak{sl}}_r(R((z)))$, which is trivial on the center and induces on the quotient $\mathfrak{sl}_r(R((z)))$ the usual action by conjugation. We claim that it is given by the following formula:

$$\operatorname{Ad}(\gamma)(\alpha, s) = \left(\gamma\alpha\gamma^{-1}, s + \operatorname{Res}_0 \operatorname{Tr}\left(\gamma^{-1}\frac{d\gamma}{dz}\alpha\right)\right). \tag{4.12}$$

In fact, let $\gamma \in SL_r(R((z)))$, and let l be a R-linear form on $\mathfrak{sl}_r(R((z)))$. The condition for the map $(\alpha, s) \mapsto (\gamma\alpha\gamma^{-1}, s + l(\alpha))$ to be a Lie algebra homomorhism is $l([\alpha,\beta]) = \operatorname{Res}_0 \operatorname{Tr}\left(\frac{d(\gamma\alpha\gamma^{-1})}{dz}\gamma\beta\gamma^{-1} - \frac{d\alpha}{dz}\beta\right)$. Since the Lie algebra $\mathfrak{sl}_r(S)$, for any ring S, is equal to its commutator algebra, this condition determines l uniquely. On the other hand, it is checked readily that the linear form $\alpha \mapsto \operatorname{Res}_0 \operatorname{Tr}\left(\gamma^{-1}\frac{d\gamma}{dz}\alpha\right)$ has the required property.

The τ Function

Let R be a k-algebra, and $\tilde{\gamma}$ an element of $\widehat{\mathbf{SL}}_r(K)(R)$. Locally on $\operatorname{Spec}(R)$ we can write $\tilde{\gamma} = (\gamma, u)$ with γ in $SL_r(R((z)))$, u in $\operatorname{Aut}(V_R)$ and $u \equiv a(\gamma)$ (mod. $\operatorname{End}^f(V_R)$). We associate to this pair the element $\tau_V(\gamma, u) := \det(ua(\gamma^{-1}))$ of R. This is clearly well-defined, so we get an algebraic function τ_V on $\widehat{\mathbf{SL}}_r(K)$.

Proposition 4.13. *Let R be a k-algebra, $\tilde{\gamma}$ an element of $\widehat{\mathbf{SL}}_r(K)(R)$. One has $\tau_V(\tilde{\gamma}\tilde{\delta}) = \chi_0(\tilde{\delta})\tau_V(\tilde{\gamma})$ for all $\tilde{\delta}$ in $\widehat{\mathbf{SL}}_r(\mathscr{O})(R)$.*

Let us choose representatives (γ, u) of $\tilde{\gamma}$ and (δ, v) of $\tilde{\delta}$. Since $b(\delta^{-1}) = 0$, one has $a(\delta^{-1}\gamma^{-1}) = a(\delta^{-1})a(\gamma^{-1})$, and

$$\begin{aligned}\tau_V(\tilde{\gamma}\tilde{\delta}) &= \det(uva(\delta^{-1})a(\gamma^{-1})) \\ &= \det(va(\delta^{-1}))\det(ua(\gamma^{-1})) = \chi_0(\tilde{\delta})\tau_V(\tilde{\gamma}).\end{aligned}$$ □

(4.14) Let us denote by χ the character χ_0^{-1} of $\widehat{\mathbf{SL}}_r(\mathscr{O})$. The function τ_V thus defines a section of the line bundle $\mathscr{L}_\chi$ on the ind-variety $\mathscr{Q} = \widehat{\mathbf{SL}}_r(K)/\widehat{\mathbf{SL}}_r(\mathscr{O})$ (3.8). More generally, let $\delta \in SL_r(K)$, and let $\tilde{\delta}$ be a lift of δ in $\widehat{SL}_r(K)$; the function $\tilde{\gamma} \mapsto \tau_V(\tilde{\delta}^{-1}\tilde{\gamma})$ still defines an element of $H^0(\mathscr{Q}, \mathscr{L}_\chi)$, whose divisor is $\delta(\operatorname{div}(\tau_V))$.

To conclude this section, let us mention that one gets slightly more natural conventions by having the group $SL_r(\mathscr{O})$ acting *on the left* on $SL_r(K)$: in particular the twist $\gamma \mapsto \gamma^{-1}$ in the definition of the τ function disappears, and the τ function becomes a section of $\mathscr{L}_{\chi_0}$. We have chosen instead to follow the standard conventions of Kac-Moody theory.

5. The Determinant Bundle

We will now compare the pull back over $\mathscr{Q}$ of the determinant line bundle $\mathscr{L}$ on the moduli stack with the line bundle $\mathscr{L}_\chi$ we have just described.

Proposition 5.1. *Let R be a k-algebra, γ an element of $GL_r(R)$, and (E, ϱ, σ) the corresponding triple over X_R (1.4). There is a canonical exact sequence*

$$0 \to H^0(X_R, E) \to A_X^r \otimes_k R \xrightarrow{\bar{\gamma}} (R((z))/R[[z]])^r \to H^1(X_R, E) \to 0,$$

where $\bar\gamma$ is the composition of the injection $A_X^r \otimes_k R \hookrightarrow R((z))^r$ deduced from the restriction map $A_X \to k((z))$, the automorphism $\gamma^{-1}: R((z))^r \to R((z))^r$, and the canonical surjection $R((z))^r \to (R((z))/R[[z]])^r$.

In fact this is the cohomology exact sequence associated to the exact sequence

$$0 \to E \to j_*\mathcal{O}_{X^*}^r \xrightarrow{\bar\gamma} f_*(\mathcal{K}_D/\mathcal{O}_D)^r \to 0$$

defined in (1.4). □

(5.2) Let us choose an element γ_0 in $GL_r(K)$ such that the associated bundle E_{γ_0} has trivial cohomology. According to what we have just seen, this means that the map $\bar\gamma_0: A_X^r \to (K/\mathcal{O})^r$ is an isomorphism, or in other words that *the subspace $V := \gamma_0^{-1}(A_X^r)$ is a supplement of $\mathcal{O}^r$ in K^r*. Let us identify A_X^r to V with the help of γ_0, and the quotient map $K^r \to (K/\mathcal{O})^r$ to the projection of K^r onto V; we obtain that $\bar\gamma$ is the composition of the mappings $V \hookrightarrow K^r \xrightarrow{\gamma^{-1}\gamma_0} K^r \to V$. In other words, $\bar\gamma$ is the coefficient $a(\gamma^{-1}\gamma_0)$ of the matrix of $\gamma^{-1}\gamma_0$ with respect to the decomposition $K^r = V \oplus \mathcal{O}^r$ (Sect. 4). We have thus obtained:

Proposition 5.2. *Let γ be an element of $GL_r(R((z)))$, and let E be the associated vector bundle over X_R. There is a canonical exact sequence*

$$0 \to H^0(X_R, E) \to V_R \xrightarrow{a(\gamma^{-1}\gamma_0)} V_R \to H^1(X_R, E) \to 0\,. \quad \square \tag{5.2}$$

Corollary 5.3. *Assume that there exists an automorphism u of V_R such that $u \equiv a(\gamma_0^{-1}\gamma)$ (mod. $\mathrm{End}^f(V_R)$). Then there is an exact sequence*

$$0 \to H^0(X_R, E) \to V_0 \xrightarrow{v_0} V_0 \to H^1(X_R, E) \to 0\,, \tag{5.3}$$

where V_0 is a free finitely generated R-module, and $\det(v_0) = \tau_V(\gamma_0^{-1}\gamma, u)$.

Let $v = ua(\gamma^{-1}\gamma_0) \in I + \mathrm{End}^f(V_R)$, and let V_0 be a free finitely generated direct factor of V_R containing $\mathrm{Im}(v - I)$. We denote by v_0 the restriction of v to V_0. The matrix of v relative to a direct sum decomposition $V_R = V_0 \oplus V_1$ is of the form $\begin{pmatrix} v_0 & * \\ 0 & I \end{pmatrix}$, from which one gets at once $\det v_0 = \det v = \tau_V(\gamma_0^{-1}\gamma, u)$. It also follows that $\mathrm{Ker}\, v_0 = \mathrm{Ker}\, v$ and that the inclusion $V_0 \hookrightarrow V_R$ induces an isomorphism $\mathrm{Coker}\, v_0 \xrightarrow{\sim} \mathrm{Coker}\, v$, so we deduce from (5.2) the exact sequence (5.3). □

The order of $\det\gamma_0$ is $r(g-1)$ (1.7), so we can choose γ_0 so that $\delta = z^{-(g-1)}\gamma_0$ belongs to $SL_r(K)$.

Proposition 5.4. *Let T be an integral scheme, and E a vector bundle on $X \times T$, with a trivialization ϱ over $X^* \times T$, such that $\wedge^r\varrho$ extends to a trivialization of $\bigwedge^r E$. Let $\mu: T \to \mathcal{Q}$ be the corresponding morphism (2.1). Assume that for some $t \in T$, the bundle $E_{|X\times\{t\}}((g-1)p)$ has trivial cohomology. Then the determinant bundle* $\det R\Gamma_T(E)$ (3.8) *is isomorphic to the line bundle $\mu^*\mathcal{L}_X$, and the theta divisor $\Theta_{(g-1)p}$ is the pull back of the divisor $\delta(\mathrm{div}\,\tau_V)$.*

Since $\mathcal{L}_X = \mathcal{O}_{\mathcal{Q}}(\delta(\mathrm{div}\,\tau_V))$ (4.13), the first assertion follows from the second one, which is local over T. Therefore we may assume that $T = \mathrm{Spec}(R)$, and that μ is defined by an element γ of $SL_r(R((z)))$ (2.5). The vector bundle $E((g-1)p)$ is defined by the element $z^{g-1}\gamma$ of $GL_r(R((z)))$. By shrinking Spec(R) if necessary, we may assume that there exists an automorphism u of V_R such that $u \equiv a(z^{g-1}\gamma_0^{-1}\gamma)$ (mod. $\mathrm{End}^f(V_R)$) (4.4); the result then follows from Corollary 5.3. □

Corollary 5.5. *The pull back $\pi^*\mathscr{L}$ is the line bundle $\mathscr{L}_\chi$ on $\mathscr{Q}$ associated to the character χ.*

By Proposition 6.4 below we can write $\mathscr{Q}$ as a direct limit of integral subvarieties Q_n. For n large enough, some points of Q_n will correspond to vector bundles E on X such that $E((g-1)p)$ has trivial cohomology. Therefore by (5.4) the line bundles $\pi^*\mathscr{L}$ and $\mathscr{L}_\chi$ have isomorphic restrictions to Q_n for n large enough, hence they are isomorphic. □

6. The Group $\mathbf{SL}_r(A_X)$

The next sections will be devoted to descend from the ind-variety $\mathscr{Q}$ to its quotient $\mathbf{SL}_r(A_X)\backslash\mathscr{Q}$, which is isomorphic to the moduli stack $\mathscr{S}\mathscr{L}_X(r)$. In order to do this we will need an important technical property of the ind-varieties $\mathscr{Q}$ and $\mathbf{SL}_r(A_X)$, namely that they are *integral*. We first study a particular case (from which we will deduce the general case): the group $\mathbf{SL}_r(A_X)$ when $X = \mathbf{P}^1$ and $p = 0$. This is simply the k-group $\mathbf{SL}_r(k[t])$ with $t = z^{-1}$.

Proposition 6.1. *The k-group $\mathbf{SL}_r(k[t])$ is the direct limit of an increasing sequence $(\Gamma^{(N)})_{N\geq 1}$ of subvarieties which are integral, normal, and locally complete intersections.*

For any k-algebra R, define $\Gamma^{(N)}(R)$ as the set of matrices of degree $\leq N$ in $SL_r(R[t])$. The k-space $\Gamma^{(N)}$ is represented by a closed subvariety of $M_r(k)^{N+1}$, defined by the equation $\det\left(\sum_{n=0}^{N} A_n t^n\right) = 1$. In other words, $\Gamma^{(N)}$ is the fibre over 1 of the map $\det : M_r(k)^{N+1} \to S_{rN}$ (we denote by S_d the space of polynomials in t of degree $\leq d$).

Let $\Gamma_0^{(N)}$ be the open subset of $\Gamma^{(N)}$ consisting of matrices $A(t) = \sum_{n=0}^{N} A_n t^n$ with $\mathrm{rk}(A_N) = N-1$ (the equality $\det A(t) = 1$ forces $\mathrm{rk}(A_N) \leq N-1$). Let us first prove that *the map* $\det : M_r(k)^{N+1} \to S_{rN}$ *is smooth along* $\Gamma_0^{(N)}$. Let $A(t) \in \Gamma_0^{(N)}$, and let $M(t) = A(t)^{-1}$. The differential of det at $A(t)$ is the map $B(t) \mapsto \mathrm{Tr}\, M(t)B(t)$. The minor $M_{ij}(t)$ is of degree $\leq (r-1)N$, and its highest degree coefficient is the corresponding minor for the matrix A_N. Since $A(t)$ belongs to $\Gamma_0^{(N)}$ there exist indices i, j such that $M_{ij}(t)$ has degree exactly $(r-1)N$. Then the minors $M_{i1}(t), \ldots, M_{ir}(t)$, viewed as elements of $H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}((r-1)N))$, have no common zeros: this is clear at infinity, because $M_{ij}(t)$ does not vanish, and on the affine line it follows from the formula $\sum_k M_{ik}(t)A_{ki}(t) = 1$. Therefore the usual resolution for the ideal spanned by the maximal minors of a matrix of type $(r, r-1)$ gives an exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}^1}^{r-1} \xrightarrow{A[i]} \mathscr{O}_{\mathbf{P}^1}(N)^r \xrightarrow{(M_{i1},\ldots,M_{ir})} \mathscr{O}_{\mathbf{P}^1}(rN) \to 0\,,$$

where $A[i]$ is obtained by deleting the i^{th} column of $A(t)$ (see e.g. [P-S], Lemma 3.1). Taking cohomology we see that the map $(P_j) \mapsto \sum_k P_k M_{ik}$ of S_M^r into S_{rN} is surjective, which implies the surjectivity of $T_{A(t)}(\det)$. Therefore $\Gamma_0^{(N)}$ is smooth, with the expected dimension $r^2(N+1)-(rN+1)$.

We will now prove that $\Gamma_0^{(N)}$ is *irreducible*, and that $\Gamma^{(N)}-\Gamma_0^{(N)}$ *is of codimension* ≥ 2. An element $A(t)$ of $\Gamma^{(N)}$ can be viewed as an homomorphism $A(t):\mathcal{O}_{\mathbf{P}^1}^r \to \mathcal{O}_{\mathbf{P}^1}(N)^r$, which is bijective over $\mathbf{P}^1-\{\infty\}$. Let us denote as before by $\mathcal{O}=k[[z]]$ the complete local ring of $\mathbf{P}^1$ at ∞. Then the cokernel of $A(t)$ is of the form $\mathcal{O}/(z^{d_1})\oplus\ldots\oplus\mathcal{O}/(z^{d_r})$, with $0\leq d_1\leq\ldots\leq d_r$ and $\sum d_i=rN$. The elements $z^{d_1},\ldots,z^{d_r}$ are the invariant factors of the matrix $A(t)$ at ∞ (i.e. of the matrix $z^N A(z^{-1})$ over the ring $k[[z]]$). In particular the case $(0,\ldots,0,rN)$ corresponds exactly to $\Gamma_0^{(N)}$.

Let $\mathbf{d}=(d_1,\ldots,d_r)$ be a sequence of integers satisfying the above properties. Let us denote by $C_{\mathbf{d}}$ the $\mathcal{O}$-module $\oplus\mathcal{O}/(z^{d_i})$. Using the local coordinate z we can identify the k-vector space $\mathrm{Hom}(\mathcal{O}_{\mathbf{P}^1}(N)^r, C_{\mathbf{d}})$ with $C_{\mathbf{d}}^r$. Let $H_{\mathbf{d}}$ be the open subset of this vector space consisting of those homomorphisms φ such that $\varphi(-1):H^0(\mathbf{P}^1,\mathcal{O}_{\mathbf{P}^1}(N-1)^r)\to C_{\mathbf{d}}$ is bijective. This means that the vector bundle $\mathrm{Ker}\,\varphi$ is trivial; it admits a unique trivialization τ such that the composite map $A_\varphi(t):\mathcal{O}_{\mathbf{P}^1}^r \xrightarrow{\tau} \mathrm{Ker}\,\varphi \hookrightarrow \mathcal{O}_{\mathbf{P}^1}(N)^r$ is the identity at 0. Let us consider the map $p_{\mathbf{d}}:H_{\mathbf{d}}\times SL_r(k)\to\Gamma^{(N)}$ defined by $p_{\mathbf{d}}(\varphi,B)=A_\varphi(t)B$. The image of $p_{\mathbf{d}}$ is the locally closed subvariety of $\Gamma^{(N)}$ consisting of matrices $A(t)$ with invariant factors at infinity $(z^{d_1},\ldots,z^{d_r})$. We see that these subvarieties are irreducible; in particular, *the open subset* $\Gamma_0^{(N)}$ *is irreducible.*

The automorphism group $G_{\mathbf{d}}:=\mathrm{Aut}_{\mathcal{O}}(C_{\mathbf{d}})$ acts freely on $H_{\mathbf{d}}$, and p clearly factors through this action. The group $G_{\mathbf{d}}$ is an affine algebraic group, which can be realized as an open subvariety of the space $\bigoplus_{i,j}\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}/(z^{d_i}),\mathcal{O}(z^{d_j}))$. An easy computation gives $\dim G_{\mathbf{d}}=(2r-1)d_1+(2r-3)d_2+\ldots+d_r\geq rN+2$ if $d_{r-1}\neq 0$. Since $\dim H_{\mathbf{d}}=r^2N$, we conclude that

$$\dim\Gamma^{(N)}-\Gamma_0^{(N)}\leq(r^2N+r^2-1)-(rN+2)=\dim\Gamma_0^{(N)}-2\,.$$

Since $\Gamma^{(N)}$ is defined by $rN+1$ equations in $k^{r^2(N+1)}$, every component of $\Gamma^{(N)}$ has dimension $\geq r^2(N+1)-(rN+1)=\dim\Gamma_0^{(N)}$. We conclude that $\Gamma^{(N)}$ is irreducible, and is a (global) complete intersection in $M_r(k)^{N+1}$. In particular it is locally complete intersection, hence Cohen-Macaulay, and normal by Serre's criterion. □

Remark 6.2. Let $\Gamma_{\mathbf{d}}^{(N)}$ be the image of $p_{\mathbf{d}}$; it follows easily from the proof that $p_{\mathbf{d}}$ induces an isomorphism of $(H_{\mathbf{d}}/G_{\mathbf{d}})\times SL_r(k)$ onto $\Gamma_{\mathbf{d}}^{(N)}$. So we get a stratification of $\Gamma^{(N)}$ by the smooth subvarieties $\Gamma_{\mathbf{d}}^{(N)}$, which admit a very explicit description. One sees easily for instance that the variety $\Gamma^{(N)}$ is rational.

(6.3) We now come back to the general case. Let us say that an ind-scheme is *reduced* (resp. *irreducible*, resp. *integral*) if it is the direct limit of an increasing sequence of reduced (resp. irreducible, resp. integral) schemes.

Lemma 6.3. *Let X be an ind-scheme, limit of an increasing sequence of schemes.*

a) *If X is reduced, and is the direct limit of an increasing sequence (X_n) of schemes, then $X=\varinjlim(X_n)_{\mathrm{red}}$.*

b) *If X is covered by reduced open sub-ind-schemes, X is reduced.*

c) *X is integral if and only if it is reduced and irreducible.*

d) *Let V be a scheme. If $V \times X$ is integral, X is integral.*

Let us prove a). Let (Y_n) be an increasing sequence of reduced schemes such that $X = \varinjlim Y_n$. We have to prove that any morphism f from an affine scheme into X factors through some $(X_n)_{\mathrm{red}}$. But f factors through some Y_p, and the inclusion $Y_p \hookrightarrow X$ factors through some X_q. Since Y_p is reduced, f factors through $(X_q)_{\mathrm{red}}$.

Let us prove b). Write $X = \varinjlim X_n$; we want to show that given $p \in \mathbf{N}$, the inclusion $X_p \hookrightarrow X_n$ factors through $(X_n)_{\mathrm{red}}$ for n large enough. Since X_p is quasi-compact it is enough to prove this statement locally over X_p, so we are reduced to the case where X is reduced; then it follows from a).

The assertion c) follows from a). Let us prove d). Let (T_n) be an increasing sequence of reduced schemes such that $V \times X = \varinjlim T_n$. Let $p: V \times X \to X$ denote the second projection. Choose a point $v \in V(k)$, and let $s_v: X \to V \times X$ be the section of p defined by $s_v(v) = (v, y)$. Since X is and ind-variety, the induced morphism $p: T_n \to X$ factors through a subvariety T'_n of X, which we may assume to be reduced (resp. irreducible) if T_n is reduced (resp. irreducible). Let S be an affine scheme, and $f: S \to X$ a morphism; writing $f = p \circ s_v \circ f$ we see that f factors through T'_n for some n. Therefore X is the direct limit of the varieties T'_n. □

Proposition 6.4. *The ind-varieties $\mathcal{Q}$ and $\mathbf{SL}_r(A_X)$ are integral.*

The ind-variety $\mathcal{Q}$ is reduced by Theorem 2.5, Proposition 6.1 and Lemma 6.3b), and irreducible by Proposition 2.6e). To prove the result for $\mathbf{SL}_r(A_X)$, we'll use the well-known fact that the open substack $\mathcal{SL}_X(r)^s$ of $\mathcal{SL}_X(r)$ parametrizing stable bundles is the quotient of a smooth variety H by a group $GL_M(k)$ (see Sect. 8 for an explicit construction). Consider the cartesian diagram

$$\begin{array}{ccc} H' & \xrightarrow{f'} & \mathcal{Q} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ H & \xrightarrow{f} & \mathcal{SL}_X(r) \end{array}$$

with $H' = H \times_{\mathcal{SL}_X(r)} \mathcal{Q}$. Reducing H if necessary we may assume that H' is isomorphic to $\mathbf{SL}_r(A_X) \times H$ (Proposition 3.4). The ind-variety $\mathcal{Q}$ is integral and the morphism f' is smooth with connected fibres (it makes H' a $GL_M(k)$-torsor over $\mathcal{Q}$); therefore H' is integral, and so is $\mathbf{SL}_r(A_X)$ by Lemma 6.3d). □

Corollary 6.5. *$\mathcal{Q}$ is the direct limit of the integral projective varieties $\mathcal{Q}^{(N)}_{\mathrm{red}}$.*

This follows from Proposition 6.4 and Lemma 6.3a). □

Corollary 6.6. *Every character $\chi: \mathbf{SL}_r(S_X) \to \mathbf{G}_m$ is trivial.*

We claim that the derivative of χ (considered as a function on $\mathbf{SL}_r(A_X)$) is everywhere 0. In fact, since χ is a homomorphism, this is equivalent to saying that the Lie algebra homomorphism $L(\chi): \mathfrak{sl}_r(A_X) \to k$ is zero. But for any commutative ring R the Lie algebra $\mathfrak{sl}(R)$ is equal to its commutator algebra, so any Lie algebra homomorphism of $\mathfrak{sl}(A_X)$ into k is trivial.

Let us write $\mathbf{SL}_r(A_X)$ as the limit of a sequence V_n of integral varieties. The restriction of χ to V_n has again zero derivative, hence is constant. Since 1 belongs to V_n for n large enough, one has $\chi_{|V_n} = 1$ for all n, that is $\chi = 1$. □

This has the following interesting consequence:

Proposition 6.7. *There is a unique embedding of* $\mathbf{SL}_r(A_X)$ *in* $\widehat{\mathbf{SL}}_r(K)$ *lifting the inclusion* $\mathbf{SL}_r(A_X) \subset \mathbf{SL}_r(K)$. *The corresponding embedding* $i : \mathfrak{sl}_r(A_X) \hookrightarrow \widehat{\mathfrak{sl}}_r(K)$ *is given in terms of the decomposition* $\widehat{\mathfrak{sl}}_r(K) = \mathfrak{sl}_r(K) \oplus k$ *(4.10) by* $i(\alpha) = (\alpha, 0)$.

The unicity of the lifting follows from 6.6. To prove the existence, choose an element δ in $SL_r(K)$ such that the bundle $E_\delta((g-1)p)$ has trivial cohomology. The subspace $V := z^{-(g-1)}\delta^{-1}(A_X^r)$ is a supplement of $\mathscr{O}^r$ in K^r (5.2), and the elements of $\mathbf{SL}_r(A_X)$ preserve $\delta(V)$, so the universal extension splits over $\mathbf{SL}_r(A_X)$ (4.7). To prove the assertion about the Lie algebras, the simplest way is to notice that the embedding of $\mathfrak{sl}_r(A_X)$ in $\widehat{\mathfrak{sl}}_r(K)$ will also be unique, so it is enough to check that i is a Lie homomorhism. By Proposition 4.10, we must prove $\operatorname{Res}_0 \operatorname{Tr}\left(\frac{d\alpha}{dz}\beta\right) = 0$ for α, β in $\mathfrak{sl}_r(A_X)$; but this is a consequence of the residue theorem. □

7. The Space of Sections of the Determinant Bundle

(7.1) The aim of this section is to identify the space of sections of the (powers of) the determinant bundle over the moduli stack $\mathscr{SL}_X(r)$ in group-theoretic terms. We first need some general formalism about descent. We will consider a k-space Q and a k-group Γ acting on Q (on the left). This means that we are given a morphism $m : \Gamma \times Q \to Q$ satisfying the usual conditions of a group action. Let $\Gamma\backslash Q$ be the quotient stack (3.2), and $\pi : Q \to \Gamma\backslash Q$ be the canonical map.

We suppose given a line bundle $\mathscr{M}_0$ on $\Gamma\backslash Q$ (3.7). Its pull back $\mathscr{M} = \pi^*\mathscr{M}_0$ to Q has a canonical Γ*-linearization*, that is an isomorphism $\varphi : m^*\mathscr{M} \xrightarrow{\sim} pr_1^*\mathscr{M}$ satisfying the usual cocycle condition. Though we will not need it, let us observe that conversely, any line bundle on Q with a Γ-linearization comes by descent theory from a uniquely determined line bundle on $\Gamma\backslash Q$.

We'll say that a section $s \in H^0(Q, \mathscr{M})$ is Γ*-invariant* if $\varphi(m^*s) = pr_1^*s$. We will need the following formal lemma about quotient stacks:

Lemma 7.2. *The map* $\pi^* : H^0(\Gamma\backslash Q, \mathscr{M}_0) \to H^0(Q, \mathscr{M})$ *induces a isomorphism of* $H^0(\Gamma\backslash Q, \mathscr{M}_0)$ *onto the space of* Γ*-invariant sections of* $\mathscr{M}$.

Since $\pi \circ m = \pi \circ q$, the pull back of a section of $\mathscr{M}_0$ is Γ-invariant. Conversely, let s be a Γ-invariant section in $H^0(Q, \mathscr{M})$, and let μ be a morphism of a scheme S into $\Gamma\backslash Q$. Recall (3.2) that μ corresponds to a diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\mu}} & Q \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mu} & \Gamma\backslash Q \end{array}$$

where P is a Γ-torsor over S and the map $\tilde{\mu}$ is Γ-equivariant. By construction the section $\tilde{\mu}^*s$ over P is Γ-invariant (in the preceding sense); we want to show that it is the pull back of a unique section s_μ over S. By standard descent theory, it is enough to check this locally for the faithfully flat topology, so we can suppose $P = \Gamma \times S$. Saying that $\tilde{\mu}^*s$ is Γ-invariant means that for any map $\nu : T \to \Gamma$, where T is a scheme, the section $(\nu \times 1_S)^*\tilde{\mu}^*s$ on $T \times S$ satisfies the usual descent condition with respect to the projection $T \times S \to S$. Therefore this section descends to a unique section $s_\mu \in H^0(S, \mu^* \mathscr{M}_0)$, which is clearly independent of T, and satisfies the required property. □

(7.3) In this situation, each element of $\Gamma(R)$ gives an automorphism of the k-space $Q_R := Q \times \mathrm{Spec}(R)$, hence acts on the space $H^0(Q_R, \mathscr{M}_R)$; we get in this way a representation of the (abstract) group $\Gamma(k)$ in the space $V := H^0(Q, \mathscr{M})$. If Q is a scheme, the space $H^0(Q_R, \mathscr{M}_R)$ is canonically isomorphic to V_R, so the above representation is algebraic in the sense that it is given by a morphism of k-groups $\Gamma \to \mathrm{Aut}(V)$. This is no longer true when Q is only an ind-scheme, because inverse limits do not commute with tensor products. They do however commute with tensor products by finite-dimensional algebras over k, so what we get is a morphism of Γ into $\mathrm{Aut}(V)$ viewed as functors on the category of finite-dimensional k-algebras. In particular the homomorphism $\Gamma(k[\varepsilon]) \to \mathrm{Aut}_{k[\varepsilon]}(V \otimes_k k[\varepsilon])$ defines in the usual way a representation of the Lie algebra $\mathrm{Lie}(\Gamma)$ on V.

Proposition 7.4. *Suppose Γ and Q are integral ind-varieties* (6.3). *Let $s \in V = H^0(Q, \mathscr{M})$. The following properties are equivalent:*

(i) *The section s is Γ-invariant;*

(ii) *The element s of V is invariant under the action of $\Gamma(k)$;*

(iii) *s is annihilated by* $\mathrm{Lie}(\Gamma)$.

(i) implies that for every k-algebra R the image of s in $H^0(Q_R, \mathscr{M}_R)$ is invariant under $\Gamma(R)$; taking $R = k$ (resp. $R = k[\varepsilon]$) gives (ii) (resp. (iii)).

Suppose (ii) holds. Then the section $\sigma = \varphi(m^* s) - pr_1^* s$ on $\Gamma \times Q$ vanishes by restriction to $\{\gamma\} \times Q$ for all $\gamma \in \Gamma(k)$; in particular, its value at any k-point of $\Gamma \times Q$ is zero. Since $\Gamma \times Q$ is reduced, this implies $\sigma = 0$, hence (i).

Suppose (iii) holds. Let $q \in Q(k)$, and let $i_q : \Gamma \hookrightarrow \Gamma \times Q$ denote the injection $\gamma \mapsto (\gamma, q)$. The line bundle $i_q^* \mathscr{M}$ is trivialized once we choose a generator of $\mathscr{M}$ at q, so we may consider $i_q^* \sigma$ as a function over Γ: its value at a point $\gamma \in \Gamma(R)$ is obtained by evaluating the section $\gamma^* s - s$ at q. The hypothesis (iii) means that the derivative of this function is identically zero. As in the proof of Corollary 6.6 this implies that σ vanishes on $\Gamma \times \{q\}$ for all q in $Q(k)$, which implies as before $\sigma = 0$. □

(7.5) Let G be a k-group, H a subgroup of G, and Q the quotient G/H. The group G acts on Q by left multiplication. Recall that we have associated to each character χ of H a line bundle $\mathscr{L}_\chi$ on Q (3.9). We claim that this line bundle admits a *canonical G-linearization.* The easiest way to see that is to notice that the quotient stack $G \backslash Q$ can be canonically identified with the classifying space BH, with the morphism $Q \to BH$ induced by the structural map $G \to \mathrm{Spec}(k)$. We have seen in (3.9) that $\mathscr{L}_\chi$ is the pull back of a line bundle on BH, hence our assertion.

(7.6) Let us now go back to our situation, and consider the action of the ind-group $\widehat{\mathbf{SL}}_r(K)$ on the ind-variety $\mathscr{Q}$. According to (7.5) the line bundle $\mathscr{L}_\chi$ on $\mathscr{Q}$ admits a canonical $\widehat{\mathbf{SL}}_r(K)$-linearization. We therefore obtain an action of the Lie algebra $\widehat{\mathfrak{sl}}_r(K)$ on the space $H^0(\mathscr{Q}, \mathscr{L}_\chi)$, and similarly on the spaces $H^0(\mathscr{Q}, \mathscr{L}_\chi^c)$ for all $c \in \mathbf{N}$. The identification of this representation is an important result of Kumar and Mathieu [Ku, M]. Before stating it, we need to recall a few facts about representations of Kac-Moody algebras, for which we refer to [K].

Let us introduce the triangular decomposition $\mathfrak{sl}_r(K) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{h}$ is the Cartan algebra of diagonal matrices in $\mathfrak{sl}_r(k)$ and $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the Lie subalgebra of matrices $\sum\limits_{n\geq 0} A_n z^n$ $\Big($resp. $\sum\limits_{n\geq 0} A_n z^{-n}\Big)$ such that the matrix A_0 is strictly upper (resp. lower) triangular. The most interesting representations of the Lie algebra $\widehat{\mathfrak{sl}}_r(K)$ are the so-called *integrable highest weight representations* [5]; they are associated to a dominant weight λ of the simple Lie algebra $\mathfrak{sl}_r(k)$ and an integer $c \geq \langle\lambda, \tilde{\alpha}\rangle$, where $\tilde{\alpha}$ is the highest root of $\mathfrak{sl}_r(k)$. The highest weight representation $V_{\lambda,c}$ corresponding to the weight λ and the integer c is characterized by the following properties ([K], 9.10): it is irreducible, and element t of the central factor $k \subset \widehat{\mathfrak{sl}}_r(K)$ acts as the homothety of ratio ct, and there exists a vector $v \in V_{\lambda,c}$ which is annihilated by $\mathfrak{n}_+$ and satisfies $Hv = \lambda(H)v$ for all H in $\mathfrak{h}$. The vector v, which is uniquely determined up to a scalar, is called a highest weight vector of the representation.

We will be mainly interested in the case $\lambda = 0$; the corresponding representation V_c ($c \geq 0$) is sometimes called the *basic representation of level* (or charge) c. In this case the annihilator of a highest weight vector $v_c \in V_c$ is $\mathfrak{sl}_r(\mathcal{O})$.

Theorem 7.7 (S. Kumar, Mathieu). *The space $H^0(\mathcal{Q}, \mathcal{L}_X^c)$ is isomorphic (as a $\widehat{\mathfrak{sl}}_r(K)$-module) to the dual of the basic representation V_c of level c of $\widehat{\mathfrak{sl}}_r(K)$.*

This theorem is proved in [Ku] and [M], with one important difference. Both S. Kumar and Mathieu define the structure of ind-variety on $SL_r(K)/SL_r(\mathcal{O})$ in an *ad hoc* way, using representation theory of Kac-Moody algebras; we must show that it coincides with our functorial definition. For instance Kumar, following Slodowy [Sl], consider the representation V_c for a fixed c, and a highest weight vector v_c. The subgroup $SL_r(\mathcal{O})$ is the stabilizer of the line kv_c in $\mathbf{P}(V_c)$, so the map $g \mapsto gv_c$ induces an injection $i_c : SL_r(K)/SL_r(\mathcal{O}) \hookrightarrow \mathbf{P}(V_c)$. Let U be the subgroup of $SL_r(\mathcal{O})$ consisting of matrices $A(z)$ such that $A(0)$ is upper-triangular with diagonal coefficients equal to 1; to each element w of the Weyl group is associated a "Schubert variety" X_w which is a finite union of orbits of U. It turns out that the image under i_c of X_w is actually contained in some finite-dimensional projective subspace $\mathbf{P}_w$ of $\mathbf{P}(V_c)$, and is Zariski closed in $\mathbf{P}_w$. This defines on X_w a structure of reduced projective variety, and a structure of ind-variety on $SL_r(K)/SL_r(\mathcal{O}) = \varinjlim X_w$.

To check that this ind-variety coincides with $\mathcal{Q}$, we will use the fact that the map i_c is actually a morphism of ind-schemes of $\mathcal{Q}$ into $\mathbf{P}(V_c)$ (which is the direct limit of its finite-dimensional subspaces). In fact, we will prove in the Appendix below, following G. Faltings, that the integrable representation V_c of $\widehat{\mathfrak{sl}}_r(K)$ can be "integrated" to an *algebraic* projective representation of $\widehat{\mathbf{SL}}_r(K)$, that is a morphism of k-groups $\widehat{\mathbf{SL}}_r(K) \to PGL(V_c)$. We claim that i_c is an *embedding*. It is injective by what we said above; let us check that it induces an injective map on the tangent spaces. Since it is equivariant under the action of $\widehat{\mathbf{SL}}_r(K)$ it is enough to prove this at the origin ω of $\mathcal{Q}$; then it follows from the fact that the annihilator of v_c in the Lie algebra $\widehat{\mathfrak{sl}}_r(K)$ is $\mathfrak{sl}_r(\mathcal{O})$.

Therefore the restriction of i_c to each of the subvarieties $\mathcal{Q}^{(N)}$ is proper, injective, and injective on the tangent spaces, hence is an embedding (in some finite-dimensional projective subspace of $\mathbf{P}(V_c)$). Each X_w is contained in some $\mathcal{Q}^{(N)}$, and therefore is

[5] In [K] they are defined as representations of $\widehat{\mathfrak{sl}}_r(k[z, z^{-1}])$, but we will see in (A.1) below that they extend to Laurent series.

a closed subvariety of $\mathscr{Q}^{(N)}_{\mathrm{red}}$. Each orbit of U is contained in some X_w; since the X_w's define an ind-structure, each $\mathscr{Q}^{(N)}$ is contained in some X_w, so that $\mathscr{Q}^{(N)}_{\mathrm{red}}$ is a subvariety of X_w. Since $\mathscr{Q}$ is the direct limit of the $\mathscr{Q}^{(N)}_{\mathrm{red}}$, the two ind-structures coincide, and the theorem follows from the Kumar-Mathieu theorem. □

It remains to descend to the quotient $\Gamma\backslash\mathscr{Q}$, i.e. to apply Proposition 7.4 in the case where Q is the ind-variety $\mathscr{Q}$, and Γ the ind-group $\mathbf{SL}_r(A_X)$. The quotient stack $\Gamma\backslash\mathscr{Q}$ is the moduli stack $\mathscr{S}\mathscr{L}_X(r)$ (3.4); we take for $\mathscr{M}_0$ a power $\mathscr{L}^c$ ($c \in \mathbf{N}$) of the determinant bundle $\mathscr{L}$ on $\mathscr{S}\mathscr{L}_X(r)$. What corresponds to $\mathscr{M}$ is the line bundle $\mathscr{L}_X^c$ on $\mathscr{Q}$ (Corollary 5.5). Since $\mathscr{L}_X$ is the pull back of $\mathscr{L}$, it has a canonical $\mathbf{SL}_r(A_X)$-linearization; on the other hand, it has a natural $\widehat{\mathbf{SL}}_r(K)$-linearization (7.5), and we know that the inclusion of $\mathbf{SL}_r(A_X)$ in $\mathbf{SL}_r(K)$ lifts canonically to an embedding of $\mathbf{SL}_r(A_X)$ in $\widehat{\mathbf{SL}}_r(K)$ (6.7), which gives another $\mathbf{SL}_r(A_X)$-linearization of $\mathscr{L}_X$. We claim that *these two linearizations are the same.* Actually there is no choice:

Lemma 7.8. *The line bundle $\mathscr{L}_X$ admits a unique $\mathbf{SL}_r(A_X)$-linearization.*

Let us write $\Gamma = \mathbf{SL}_r(A_X)$. Two Γ-linearizations differ by an automorphism of $p^*\mathscr{L}_X$, i.e. by an invertible function on $\Gamma \times \mathscr{Q}$. Since $\mathscr{Q}$ is the direct limit of the integral projective varieties $\mathscr{Q}^{(N)}_{\mathrm{red}}$ (6.5), this function is the pull back of an invertible function f on Γ; the cocycle conditions on the linearization imply that f is a character, hence $f = 1$ by (6.6). □

Therefore the action of the Lie algebra $\mathfrak{sl}_r(A_X)$ on $H^0(\mathscr{Q},\mathscr{L}_X^c)$ is the restriction via the natural embedding (6.7) of the action of $\widehat{\mathfrak{sl}}_r(K)$ on $H^0(\mathscr{Q},\mathscr{L}_X^c) \cong V_c^*$ (Theorem 7.7). Since the ind-varieties $\mathscr{Q}$ and $\mathbf{SL}_r(A_X)$ are integral (Proposition 6.4), we can apply Lemma 7.2 and Proposition 7.4, and we get:

Theorem 7.9. *The space $H^0(\mathscr{S}\mathscr{L}_X(r),\mathscr{L}^c)$ is canonically isomorphic to the space of conformal blocks $B_c(r)$, that is the subspace of V_c^* annihilated by the Lie algebra $\mathfrak{sl}_r(A_X)$.* □

Example 7.10. The only case where a direct computation seems possible is the case $g = 0$. We take as before $X = \mathbf{P}^1$, $p = 0$, so that $A_{\mathbf{P}^1} = k[z^{-1}]$. The space $B_c(r)$ is the dual of $V_c/\mathscr{U}^+V_c$, where $\mathscr{U}^+$ is the augmentation ideal of the enveloping algebra $\mathscr{U}$ of $\mathfrak{sl}_r(k[z^{-1}])$. By definition of a highest weight module, V_c is generated as a $\mathscr{U}$-module by a highest weight vector v_c, and one has $V_c = kv_c \oplus \mathscr{U}^+v_c = kv_c \oplus \mathscr{U}^+V_c$. We conclude that *the space $H^0(\mathscr{S}\mathscr{L}_{\mathbf{P}^1}(r),\mathscr{L}^c)$ is one-dimensional for all c.*

Remark 7.11. One can deduce from the results of Mathieu that the Picard group of $\mathscr{Q}$ is generated by the line bundle $\mathscr{L}_X$ (see [M, Proposition 5]). It then follows from Lemma 7.8 that *the Picard group of $\mathscr{S}\mathscr{L}_X(r)$ is generated by the determinant bundle $\mathscr{L}$* (the corresponding statement for the moduli space $\mathscr{S}\mathscr{U}_X(r)$ is proved in [D-N]).

Appendix to Sect. 7: Integration of Integrable Highest Weight Modules (According to Faltings)

In this appendix we want to show that integrable highest weight representations of the Lie algebra $\widehat{\mathfrak{sl}}_r(K)$ can be integrated to algebraic representations of the group $\widehat{\mathbf{SL}}_r(K)$. We will actually content ourselves with a *projective* representation of this group, since this is sufficient for our purpose and that the complete result requires some more work.

(A.1) Let us denote by $\widehat{\mathfrak{sl}}_r(k[z,z^{-1}])$ the sub-Lie algebra $\mathfrak{sl}_r(k[z,z^{-1}]) \oplus k$ of $\widehat{\mathfrak{sl}}_r(K) = \mathfrak{sl}_r(k((z))) \oplus k$ (4.10). Let V be an integrable highest weight module for $\widehat{\mathfrak{sl}}_r(k[z,z^{-1}])$. The integrability property has the following consequence:

For each vector v in V there exists an integer p such that $A(z)v = 0$ for every element $A(z) = \sum_{n \geq p} A_n z^n$ of $\mathfrak{sl}_r(k[z,z^{-1}])$.

This means that the homomorphism $\pi : \widehat{\mathfrak{sl}}_r(k[z,z^{-1}]) \to \mathrm{End}(V)$ is continuous when $\widehat{\mathfrak{sl}}_r(k[z,z^{-1}])$ is endowed with the z-adic topology, V with the discrete topology, and $\mathrm{End}(V)$ with the topology of pointwise convergence. It implies that π extends to a continuous homomorphism – still denoted by π – from the z-adic completion $\widehat{\mathfrak{sl}}_r(K)$ of $\widehat{\mathfrak{sl}}_r(k[z,z^{-1}])$ to $\mathrm{End}(V)$: one has $\pi\Big(\sum_{n \geq -N} A_n z^n\Big) = \sum_{n \geq -N} \pi(A_n z^n)$, where the second sum is *locally finite*, i.e. on each vector, all but finitely many of the endomorphisms in the sum are zero. More generally, for any k-algebra R, one gets by tensor product a homomorphism $\pi_R : \widehat{\mathfrak{sl}}_r(k[z,z^{-1}]) \otimes_k R \to \mathrm{End}(V_R)$, which by continuity extends to $\widehat{\mathfrak{sl}}_r(R((z)))$ (4.12).

Suppose π is the derivative of an algebraic representation (i.e. a morphism of k-groups $\widehat{\mathbf{SL}}_r(K) \to \mathrm{Aut}(V)$), such that the center of $\widehat{\mathbf{SL}}_r(K)$ acts on V by homotheties. Then we get a *projective representation* of $\mathbf{SL}_r(K)$ in V, that is a homomorphism $\bar{\varrho}$ of $\mathbf{SL}_r(K)$ into the quotient k-group $PGL(V) := \mathrm{Aut}(V)/\mathbf{G}_m$, whose derivative $L(\bar{\varrho}) : \mathfrak{sl}_r(K) \to \mathrm{End}(V)/k1_V$ coincides with π up to homotheties. We claim that we can always find such a representation:

Proposition A.2. *Let $\pi : \widehat{\mathfrak{sl}}_r(K) \to \mathrm{End}(V)$ be an integrable highest weight representation. There exists a (unique) projective representation $\bar{\varrho} : \mathbf{SL}_r(K) \to PGL(V)$ whose derivative coincides with π up to homotheties.*

The proof which follows has been shown to us by G. Faltings.

Lemma A.3. *Let R be a k-algebra and γ an element of $SL_r(R((z)))$. Locally over* $\mathrm{Spec}(R)$, *there exists an automorphism u of V_R, uniquely determined up to an invertible element of R, satisfying*

$$u\pi_R(\alpha)u^{-1} = \pi_R(\mathrm{Ad}(\gamma)(\alpha)) \tag{A.4}$$

for any $\alpha \in \widehat{\mathfrak{sl}}_r(R((z)))$ (cf. (4.12) for the definition of the adjoint action).

We'll say for short that an automorphism u satisfying the above condition is *associated* to γ.

Let us show first that this lemma implies the proposition. Thanks to the unicity property, the automorphisms u associated locally to γ glue together to define a uniquely determined element $\bar{\varrho}(\gamma)$ in $PGL(V)(R)$. Still because of the unicity property, $\bar{\varrho}$ is a homomorphism of k-groups of $\mathbf{SL}_r(K)$ into $PGL(V)$. Let $\beta \in \widehat{\mathfrak{sl}}_r(K)$; the element $\bar{\varrho}(\exp \varepsilon\beta)$ of $PGL(V_{k[\varepsilon]})$ can be written as the class of an automorphism $I + \varepsilon u_\beta$ of $V_{k[\varepsilon]}$, where u_β is an endomorphism of V whose class in $\mathrm{End}(V)/k1_V$ is $L(\bar{\varrho})(\beta)$. Formula (A.4) applied to $R = k[\varepsilon]$ and $\gamma = \exp \varepsilon\beta$ gives $[u_\beta, \pi(\alpha)] = [\pi(\beta), \pi(\alpha)]$ for each α in $\widehat{\mathfrak{sl}}_r(K)$. Since π is irreducible this implies that u_β coincides with $\pi(\beta)$ up to homotheties [K, Lemma 9.3], hence the proposition. □

We will prove Lemma A.3 in several steps.

a) Let us prove first the unicity assertion. We just need to observe that an endomorphism u of V_R which commutes with $\pi_R(\widehat{\mathfrak{sl}}_r(R((z))))$ is a homothety: for each k-linear form $\varphi: R \to k$, the endomorphism $(1_V \otimes \varphi) \circ u$ of V commutes with $\pi(\widehat{\mathfrak{sl}}_r(K))$, hence is a homothety, from which it follows that u is a homothety.

b) Assume that γ is the exponential of a matrix $\nu \in \mathfrak{sl}_r(R((z)))$ which is either nilpotent, or of positive order (so that its exponential is well-defined). Then the automorphism $\mathrm{Ad}(\gamma)$ of $\widehat{\mathfrak{sl}}_r(R((z)))$ is the exponential of the derivation $\mathrm{ad}\,\nu$. Because of the continuity property of π_R (A.1), the series $\exp(\pi_R(\nu))$ is locally finite, hence defines an automorphism u of V_R; one has

$$\pi_R(\mathrm{Ad}(\gamma)\,(\alpha)) = \exp(\mathrm{ad}\,\pi_R(\nu))\,(\alpha) = u\pi_R(\alpha)u^{-1},$$

so u satisfies (A.4).

c) Let us observe that if two elements γ, δ of $SL_r(R((z)))$ have associated automorphisms u and v, then uv is associated to $\gamma\delta$. If R is a field, so is $R((z))$, hence any element of $SL_r(R((z)))$ is a product of elementary matrices $I + \lambda E_{ij} = \exp(\lambda E_{ij})$. The result then follows from b).

d) The exponential mapping is a bijection from the space of matrices $\sum_{n\geq 1} A_n z^n$ with zero trace onto the group of matrices $B(z) = I + \sum_{n\geq 1} B_n z^n$ with determinant 1 (the inverse bijection is given by the logarithm), so b) gives the result for the matrices $B(z)$. Let now $\gamma \in SL_r(R)$; locally over $\mathrm{Spec}(R)$ we can again write γ as a product of elementary matrices, so the result follows as in c). Finally we see that the result holds for γ in $SL_r(R[[z]])$.

e) Assume now that the ring R is local artinian; let $\mathfrak{m}$ be its maximal ideal and κ its residue field. The quotient map $R \to \kappa$ has a section, so the group $SL_r(R((z))$ is a semi-direct product of $SL_r(\kappa((z))$ by the kernel N of the map $SL_r(R((z))) \to SL_r(\kappa((z))$. The lemma holds for $\gamma \in SL_r(\kappa((z)))$ by c). The elements of N are of the form $I + A(z)$, where all the coefficients of $A(z)$ belong to $\mathfrak{m}R((z))$; since $\mathfrak{m}$ is nilpotent, $I + A(z)$ is the exponential of a nilpotent matrix, hence the lemma holds for the elements of N by c) and therefore for all elements of $SL_r(R((z)))$.

f) We now arrive at the heart of the proof, the case $\gamma \in SL_r(R[z^{-1}])$. Let us observe that in this case one can *normalize* the automorphism u of V_R in the following way. Let $\mathscr{U}(\mathfrak{n}_-)$ be the enveloping algebra of $\mathfrak{n}_-$ (7.6), and $\mathscr{U}^*(\mathfrak{n}_-)$ its augmentation ideal. The space V is spanned as a $\mathscr{U}(\mathfrak{n}_-)$-module by a highest weight vector v, and the quotient $V/\mathscr{U}^*(\mathfrak{n}_-)V$ is one-dimensional. Therefore the R-module $V_R/\mathscr{U}^*(\mathfrak{n}_-)V_R$ is free of rank 1. Since γ normalizes $\mathfrak{n}_- \otimes R$, u induces an automorphism of this R-module, so we can choose u (in a unique way) so that it induces the identity mod. $\mathscr{U}^*(\mathfrak{n}_-)V_R$.

Since the group $\mathbf{SL}_r(\mathscr{O}_-)$ is an ind-variety, we may assume that the k-algebra R is finitely generated. We will prove the existence of an associated automorphism u by induction on $\dim(R)$.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals of R, and $S = R - \cup\mathfrak{p}_i$. Over the arthinian ring $S^{-1}R$ (isomorphic to $\prod R_{\mathfrak{p}_i}$) we can construct by e) an automorphism u_S of $V_{S^{-1}R}$ associated to γ. Let us denote by $\mathscr{U}$ the enveloping algebra of $\widehat{\mathfrak{sl}}_r(K)$.

Observe that (A.4) is equivalent to saying that u is a *semi-linear* endomorphism of the $\mathscr{U}$-module V_R relative to the automorphism $\mathrm{Ad}(\gamma)$ of $\mathscr{U}$. Since the $\mathscr{U}$-module V is finitely presented ([K], 10.4.6), we can find an element f of S such that fu_S comes from a $(\mathscr{U} \otimes R, \mathrm{Ad}(\gamma))$-linear endomorphism u_f of V_R (chase denominators in generators and relations). Then the class of $u_f(v) - fv$ in $V_R/\mathscr{U}^*(\mathfrak{n}\_)V_R$ is annihilated by some element s of S, so repalacing f by fs we may assume $u_f(v) \equiv v\,(\mathrm{mod.}\ \mathscr{U}^*(\mathfrak{n}\_)V_R)$. Moreover we can modify f so that fu_S^{-1} also comes from an endomorphism u' of V_R, such that the endomorphisms $\dfrac{u_f}{f}$ and $\dfrac{u'}{f}$ of V_{R_f} are inverse of each other.

Since $f \in S$, one has $\dim(R/f^nR) < \dim(R)$ for each n, hence the induction hypothesis provides a normalized automorphism of V_{R/f^nR} satisfying (A.4). These automorphisms define an automorphism $\hat{u}$ of the f-adic completion $\hat{V}_R$ of V_R. On the other hand, u_f extends to an endomorphism $\hat{u}_f$ of $\hat{V}_R$; one has $\hat{u}_f \equiv f\hat{u}\,(\mathrm{mod.}\ f^n\hat{V}_R)$ for all n, hence $\hat{u}_f = f\hat{u}$.

Unfortunately $\hat{V}_R$ is bigger than $V_{\hat{R}} := V \otimes_k \hat{R}$: if we choose a basis $(e_\iota)_{\iota \in I}$ of V, the elements of $\hat{V}_R$ are formal sums $\sum r_\iota e_\iota$, where for every $n \geq 0$, f^n divides all but finitely many of the r_ι's. However, since $\hat{R}$ is noetherian, there exists an integer n such that $\mathrm{Ann}_{\hat{R}}(f^n) = \mathrm{Ann}_{\hat{R}}(f^{n+1})$, which implies $f^n\hat{R} \cap \mathrm{Ann}_{\hat{R}}(f) = 0$; therefore *an element x of $\hat{V}_R$ such that $fx \in V_{\hat{R}}$ belongs itself to $V_{\hat{R}}$*. Coming back to our situation, we deduce from the formula $\hat{u}_f = f\hat{u}$ that $\hat{u}$ induces an endomorphism $\tilde{u}$ of $V_{\hat{R}}$; using the same construction with u_S^{-1} shows that $\hat{u}^{-1}$ also preserves $V_{\hat{R}}$, so that $\tilde{u}$ is an automorphism. It acts trivially on $V_{\hat{R}}/\mathscr{U}^*(\mathfrak{n}\_)V_{\hat{R}}$, because it does mod. f^n for all n.

By the unicity property, the automorphisms $\dfrac{u_f}{f}$ of V_{R_f} and $\tilde{u}$ of $V_{\hat{R}}$ have the same image in $\mathrm{Aut}(V_{\hat{R}\otimes_R R_f})$. Since the homomorphism $R \to R_f \times \hat{R}$ is faithfully flat, they can be glued together to define an automorphism u of V_R, which satisfies (A.4) because both $\dfrac{u_f}{f}$ and $\tilde{u}$ do.

g) Finally the general case follows from Lemma (4.5) and cases c), d) and f). □

8. From the Moduli Stack to the Moduli Space

The last step is to compare the sections of the determinant bundle (or of its powers) over the moduli space and over the moduli stack. *Throughout this section we assume* $g \geq 1$ (by Example 7.10 there is essentially nothing to say in the case $g = 0$).

(8.1) We first review briefly the standard construction of the moduli space (or stack) of vector bundles. For each integer N, we will denote by $\mathscr{SL}_X(r)_N$ the open substack of $\mathscr{SL}_X(r)$ parametrizing vector bundles E on X such that $H^1(X, E(Np))$ is 0 and $H^0(X, E(Np))$ is generated by its global sections. Let $h(N) = \dim H^0(X, E(Np))$ $(= r(N + 1 - q))$. Choosing an isomorphism $k^{h(N)} \to H^0(X, E(Np))$ realizes E as a quotient of the bundle $\mathscr{O}_X(-Np)^{h(N)}$. The stack which parametrizes such quotients is represented by a smooth scheme K_N. Let $\mathscr{E}$ be the universal quotient bundle over $X \times K_N$, and let $q: X \times K_N \to K_N$ denote the second projection. The sheaf $q_* \bigwedge^r \mathscr{E}$ is the sheaf of sections of a line bundle on K_N: let H_N be the complement of the zero section in this line bundle. By construction H_N parametrizes quotients E of $\mathcal{O}_X(-Np)^{h(N)}$ together with a trivialization of $\bigwedge^r E$. The group $GL(h(N))$ acts on H_N, and the quotient stack is $\mathcal{SL}_X(r)_N$.

We'll denote by $\mathcal{SL}_X(r)^{ss}$ the open substack of $\mathcal{SL}_X(r)$ parametrizing semi-stable bundles, and by H_N^{ss} the corresponding open subset of H_N. We'll assume $N \geq 2g$, which implies that $\mathcal{SL}_X(r)^{ss}$ is contained in $\mathcal{SL}_X(r)_N$.

Lemma 8.2. *The codimension of $H_N - H_N^{ss}$ in H_N is at least 2.*

For each pair of integers (s,d) with $0 < s < r$ and $d > 0$, let us define a stack $\mathcal{SL}_X^{s,d}(r)$ by associating to a k-algebra R the groupoid of triples (E, F, δ) where E is a rank r vector bundle over X_R, F a rank s subbundle of degree d, and δ a trivialization of $\bigwedge^r E$. Forgetting F gives a morphism of stacks of $\mathcal{SL}_X^{s,d}(r)$ to $\mathcal{SL}_X(r)$; the (reduced) substack $\mathcal{SL}_X(r) - \mathcal{SL}_X(r)^{ss}$ is the union of these images (for variable s, d). According to [L, Corollary 2.10], the dimension of $\mathcal{SL}_X^{s,d}(r)$ is $(g-1)(r^2 - 1 + s^2 - rs) - rd$, so the codimension of its image is at least rd, which is ≥ 2. Since H_N is a torsor over $\mathcal{SL}_X(r)_N$ the lemma follows. □

Proposition 8.3. *For any integer c, the restriction map*

$$H^0(\mathcal{SL}_X(r), \mathcal{L}^c) \to H^0(\mathcal{SL}_X(r)^{ss}, \mathcal{L}^c)$$

is an isomorphism.

Let $N \geq 2g$. Consider the diagram

$$\begin{array}{ccc} H_N^{ss} & \hookrightarrow & H_N \\ \downarrow & & \downarrow \\ \mathcal{SL}_X(r) & \hookrightarrow & \mathcal{SL}_X(r)_N \end{array}$$

By Lemma 7.2, the sections of $\mathcal{L}^c$ over $\mathcal{SL}_X(r)^{ss}$ (resp. $\mathcal{SL}_X(r)_N$) are the invariant sections of the pull back of $\mathcal{L}^c$ over H_N^{ss} (resp. H_N). But any section over H_N^{ss} extends to H_N by (8.2), so the restriction map $H^0(\mathcal{SL}_X(r)_N, \mathcal{L}^c) \to H^0(\mathcal{SL}_X(r)^{ss}, \mathcal{L}^c)$ is an isomorphism for each N. Since any map from a scheme to $\mathcal{SL}_X(r)$ factors through $\mathcal{SL}_X(r)_N$ for some N, the proposition follows. □

Let $\mathcal{SU}_X(r)$ be the moduli space of semi-stable rank r vector bundles on X with trivial determinant (the notation is meant to remind that these correspond to unitary representations). It is usually constructed as the geometric invariant theory quotient of K_N^{ss} (8.1) by the group $PGL(h(N))$. We have a forgetful morphism $\varphi : \mathcal{SL}_X(r)^{ss} \to \mathcal{SU}_X(r)$. It is known that the determinant bundle $\mathcal{L}$ is the pull back of a line bundle (that we will still denote by $\mathcal{L}$) on $\mathcal{SU}_X(r)$ (see [D-N], and [Tu] for the case $g = 1$).

Proposition 8.4. *Let $c \in \mathbf{N}$. The map*

$$\varphi^* : H^0(\mathcal{SU}_X(r), \mathcal{L}^c) \to H^0(\mathcal{SL}_X(r)^{ss}, \mathcal{L}^c)$$

is an isomorphism.

Let us choose an integer $N \geq 2g$. We claim that both spaces can be identified with the space of $GL(h(N))$-invariant sections of the pull back of $\mathcal{L}^c$ to H_N^{ss}. For $H^0(\mathcal{SL}_X(r)^{ss}, \mathcal{L}^c)$, this follows from Lemma 7.2.

Let us consider the space $H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$. We will write simply H, K and $\mathcal{S}$ for H_N^{ss}, K_N^{ss} and $\mathcal{SU}_X(r)$. By definition of the GIT quotient, the quotient map $p: K \to \mathcal{S}$ is affine and the sheaf $\mathcal{O}_{\mathcal{S}}$ is the subsheaf of local $PGL(h(N))$-invariant sections in $p_*\mathcal{O}_K$. On the other hand, since the map $q: H \to K$ is a k^*-fibration, the subsheaf of local k^*-invariant sections of $q_*\mathcal{O}_H$ is $\mathcal{O}_K$. Putting things together we conclude that the sheaf of $GL(h(N))$-invariant sections of $p_*q_*\mathcal{O}_H$ is $\mathcal{O}_{\mathcal{S}}$. Therefore for any sheaf $\mathcal{F}$ on $\mathcal{S}$ the space $H^0(\mathcal{S}, \mathcal{F})$ is the space of $GL(h(N))$-invariant sections of the pull back of $\mathcal{F}$ to H. □

Putting together Proposition 8.3 and 8.4 and Theorem 7.9, we obtain:

Theorem 8.5. *For all $c \in \mathbf{N}$, the space $H^0(\mathcal{SU}_X(r), \mathcal{L}^c)$ is canonically isomorphic to the space of conformal blocks $B_c(r)$.* □

It follows from [T-U-Y] that the dimension of the space $B_c(r)$ can be computed in terms of the representation theory of $SL_r(k)$ – more precisely in terms of the *fusion algebra* associated to this group. In the case (of interest here) of $SL_r(k)$, this computation has been done in [G]; the reader will find a treatment valid for all classical groups (and possibly more accessible to mathematicians) in the Appendix of [F]. The outcome is the following formula[6]:

Corollary 8.6 (Verlinde formula). *One has*

$$\dim H^0(\mathcal{SU}_X(r), \mathcal{L}^c) = \left(\frac{r}{r+c}\right)^g \sum_{\substack{S \subset [1, r+c] \\ |S| = r}} \prod_{\substack{s \in S \\ t \notin S}} \left| 2 \sin \pi \frac{s-t}{r+c} \right|^{g-1}. \quad \square$$

9. Arbitrary Degree

In this last section we will extend our results to the case of vector bundles of arbitrary degree. We fix an integer d, and let $\mathcal{SL}_X(r,d)$ be the moduli stack parametrizing vector bundles E on X of rank r with an isomorphism $\delta: \mathcal{O}_X(dp) \xrightarrow{\sim} \bigwedge^r E$. This stack depends only on the class of $d \bmod. r$, so we'll loose no generality by assuming $0 < d < r$. We will still use the letter $\mathcal{L}$ to denote the determinant bundle on $\mathcal{SL}_X(r,d)$ (3.8).

Recall that the *fundamental weights* $\varpi_1, \dots, \varpi_{r-1}$ of $\mathfrak{sl}_r(k)$ are the linear forms on the Cartan algebra $\mathfrak{h} \subset \mathfrak{sl}_r(k)$ defined by $\langle \varpi_k, H \rangle = \sum_{i=1}^{k} H_{ii}$. Using the notation of (7.6), we can state the main result of this section:

Theorem 9.1. *Let $0 < d < r$. The space $H^0(\mathcal{SL}_X(r,d), \mathcal{L}^c)$ is canonically isomorphic to the subspace of $V^*_{c\varpi_{r-d}, c}$ annihilated by the Lie algebra $\mathfrak{sl}_r(A_X)$.*

The proof follows the same lines as in the degree zero case. We choose once and for all an element γ_d of $GL_r(K)$ with determinant z^{-d}. Then $\mathcal{SL}_X(r,d)$ can be described as the quotient stack $(\gamma_d^{-1} \mathbf{SL}_r(A_X) \gamma_d) \backslash \mathcal{Q}$ (3.6). Let $\pi_d: \mathcal{Q} \to \mathcal{SL}_X(r,d)$ be the canonical morphism.

Proposition 9.2. *One has $\pi_d^* \mathcal{L} \cong \mathcal{L}_\chi$.*

[6] This slightly exotic formulation has been shown to us by D. Zagier.

We will reduce this assertion to the case $d = 0$ by using the following trick. The line bundle $\mathscr{O}_X(-dp)$ has natural trivializations ϱ_0 over X^* and σ_0 over D such that the element $\varrho_0^{-1}\sigma_0$ of K^* is equal to z^d (1.5). Let R be a k-algebra, and E a vector bundle on X_R, of rank r and degree d, with trivializations ϱ over X_R^* and σ over D_R, corresponding to an element δ of $GL_r(R((z)))$ (Proposition 1.4). Then the triple $(E\oplus\mathscr{O}_X(-dp), \varrho\oplus\varrho_0, \sigma\oplus\sigma_0)$ corresponds to the matrix $\begin{pmatrix}\delta & 0\\ 0 & (\det\gamma_d)^{-1}\end{pmatrix}$. If $\delta = \gamma_d\gamma$, with $\gamma \in SL_r(R((z)))$, this matrix is the product of $\gamma_d' := \begin{pmatrix}\gamma_d & 0\\ 0 & (\det\gamma_d)^{-1}\end{pmatrix}$ with the matrix $t(\gamma) := \begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}$. We have therefore obtained a commutative diagram (we use a $'$ when we replace r by $r+1$ in the objects defined in Sects. 3 and 4):

$$\begin{array}{ccc} \mathscr{Q} & \xrightarrow{f} & \mathscr{Q}' \\ {\scriptstyle \pi_d}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathscr{SL}_X(r,d) & \xrightarrow{s} & \mathscr{SL}_X(r+1) \end{array}$$

where f is induced by the map $\gamma \mapsto \gamma_d' t(\gamma)$ from $\mathbf{SL}_r(K)$ into $\mathbf{SL}_{r+1}(K)$, and s associates to a vector bundle E on X_R the vector bundle $E\oplus\mathscr{O}_X(-dp)$.

Let us denote by E and E' the universal bundles on $X\times\mathscr{SL}_X(r,d)$ and $X\times\mathscr{SL}_X(r+1)$ respectively. By construction the pull back of E' by $1_X\times s$ is $E\oplus\mathscr{O}_X(-dp)$. Let p be the projection from $X\times\mathscr{SL}_X(r,d)$ onto $\mathscr{SL}_X(r,d)$. One has $Rp_*(E\oplus\mathscr{O}_X(-dp)) \cong Rp_*(E)\oplus Rp_*(\mathscr{O}_X(-dp))$ and the bundles $R^ip_*(\mathscr{O}_X(-dp))$ are trivial, so we get $s^*\mathscr{L}' \cong \det Rp_*(E\oplus\mathscr{O}_X(-dp)) \cong \det Rp_*(E) = \mathscr{L}$. Therefore our assertion is equivalent to $f^*\mathscr{L}_{\chi'} \cong \mathscr{L}_\chi$.

The group morphism $t:\mathbf{SL}_r(K)\to\mathbf{SL}_{r+1}(K)$ extends in a straightforward way to $\hat{t}:\widehat{\mathbf{SL}}_r(K)\to\widehat{\mathbf{SL}}_{r+1}(K)$: from the decomposition $K^r = V\oplus\mathscr{O}^r$ and an arbitrary decomposition $K = V_0\oplus\mathscr{O}$ one gets $K^{r+1} = (V\oplus V_0)\oplus\mathscr{O}^{r+1}$; then $\hat{t}$ is defined by $\hat{t}(\gamma,u) = \left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}u & 0\\ 0 & 1_{V_0}\end{pmatrix}\right)$. This morphism maps $\widehat{\mathbf{SL}}_r(\mathscr{O})$ into $\widehat{\mathbf{SL}}_{r+1}(\mathscr{O})$ and satisfies $\hat{t}(\gamma, a(\gamma)) = (t(\gamma), a(t(\gamma)))$ for $\gamma\in SL_r(R((z)))$, from which one deduces $\chi'\circ\hat{t} = \chi$. By (3.9) this implies that the pull back of $\mathscr{L}_{\chi'}$ by the morphism $\mathscr{Q}\to\mathscr{Q}'$ deduced from $\hat{t}$ is isomorphic to $\mathscr{L}_\chi$. Since $\mathscr{L}_{\chi'}$ is invariant under the action of $SL_{r+1}(K)$, we conclude that $f^*\mathscr{L}_{\chi'}$ is isomorphic to $\mathscr{L}_\chi$, which proves the proposition. □

It follows from the proposition that the line bundle $\mathscr{L}_\chi$ on $\mathscr{Q}$ has a $(\gamma_d^{-1}\mathbf{SL}_r(A_X)\gamma_d)$-linearization (7.1), which means that the map $\delta\mapsto\gamma_d^{-1}\delta\gamma_d$ of $\mathbf{SL}_r(A_X)$ into $\mathbf{SL}_r(K)$ lifts to $\widehat{\mathbf{SL}}_r(K)$. It is not difficult to describe explicitly this lifting (use the same trick as in the above proof), but we need only to know the corresponding Lie algebra map $i_d:\mathfrak{sl}_r(A_X)\to\widehat{\mathfrak{sl}}_r(K)$. As in (6.7) we just have to exhibit one homomorphism which coincides with $\alpha\mapsto\gamma_d^{-1}\alpha\gamma_d$ modulo the center of $\widehat{\mathfrak{sl}}_r(K)$.

Recall (4.12) that the adjoint action of $SL_r(K)$ on $\widehat{\mathfrak{sl}}_r(K)$ is given by

$$\mathrm{Ad}(\gamma)(\alpha,s) = \left(\gamma\alpha\gamma^{-1}, s + \mathrm{Res}_0\,\mathrm{Tr}\left(\gamma^{-1}\frac{d\gamma}{dz}\alpha\right)\right). \tag{9.3}$$

We observe that the formula makes sense for $\gamma \in GL_r(K)$, and defines a group homomorphism $GL_r(K) \to \mathrm{Aut}(\mathfrak{sl}_r(K))$, that we still denote by Ad. Then the homomorphism $\mathrm{Ad}(\gamma_d^{-1}) \circ i$, where i is the canonical embedding of $\mathfrak{sl}_r(A_X)$ into $\widehat{\mathfrak{sl}}_r(K)$ (6.7), satisfies the required conditions and is therefore equal to i_d.

Let us denote by π the homomorphism $\widehat{\mathfrak{sl}}_r(K) \to \mathrm{End}(V_c)$. By Proposition 7.4 and Theorem 7.7, the space $H^0(\mathscr{SL}_X(r,d), \mathscr{L}^c)$ is canonically isomorphic to the space of linear forms on V_c which vanish on the image of $\pi(i_d(\alpha))$ for every α in $\mathfrak{sl}_r(A_X)$, i.e. of linear forms killed by $\mathfrak{sl}_r(A_X)$ acting on V_c through the representation $\pi \circ \mathrm{Ad}(\gamma_d^{-1})$. Therefore Theorem 9.1 will be a consequence of the following lemma:

Lemma 9.3. *The representation $\pi \circ \mathrm{Ad}(\gamma_d^{-1})$ is isomorphic to the highest weight representation $V_{c\varpi_{r-d},c}$.*

By Lemma (A.3), the representations π and $\pi \circ \mathrm{Ad}(\gamma)$ are isomorphic for $\gamma \in SL_r(K)$, so the representation $\pi \circ \mathrm{Ad}(\gamma_d^{-1})$ doesn't depend (up to isomorphism) of the choice of the particular element γ_d; we choose for γ_d the matrix $\mathrm{diag}(1, \ldots, 1, z, \ldots, z)$, where z appears d times, and denote by π_d the representation $\pi \circ \mathrm{Ad}(\gamma_d^{-1})$. Let $\alpha = \sum A_n z^n \in \mathfrak{sl}_r(K)$; an easy computation [using (9.3)] gives

$$\mathrm{Ad}(\gamma_d^{-1})(\alpha, s) = (\gamma_d^{-1}\alpha\gamma_d, s + \langle \varpi_{r-d}, A_0 \rangle) .$$

This implies $\mathrm{Ad}(\gamma_d^{-1})(\mathfrak{n}_+) \subset \mathfrak{sl}_r(\mathscr{O})$, and therefore the highest weight vector v_c of V_c is annihilated by $\pi_d(\mathfrak{n}_+)$ (7.6). Let $H \in \mathfrak{h}$, and $s \in k$; the above formula gives $\pi_d((H,s))v_c = c(s + \langle \varpi_{r-d}, H \rangle)v_c$. Moreover the representation π_d is irreducible. Therefore π_d is isomorphic to the highest weight representation $V_{c\varpi_{r-d},c}$ (9.6). $\square$

Let $\mathscr{SU}_X(r,d)$ be the moduli space of semi-stable vector bundles on X of rank r and determinant $\mathscr{O}_X(dp)$. As in Sect. 8 we have a forgetful morphism $\varphi: \mathscr{SL}_X(r,d)^{ss} \to \mathscr{SU}_X(r,d)$. According to [D-N], the determinant bundle $\mathscr{L}$ itself does not descend in general to a line bundle on $\mathscr{SU}_X(r,d)$; the pull back of the ample generator $\mathscr{L}_{r,d}$ of $\mathrm{Pic}(\mathscr{SU}_X(r,d))$ is the line bundle $\det R\Gamma_{\mathscr{SL}_X(r,d)}(\mathscr{E} \otimes F)$ (3.8), where $\mathscr{E}$ is the universal bundle over $X \times \mathscr{SL}_X(r,d)$ and F a vector bundle on X of rank $s := \dfrac{r}{(r,d)}$. By (3.8) we get $\varphi^*\mathscr{L}_{r,d} \cong \mathscr{L}^s$. Now the proof of Theorem 8.5 applies almost without modification to this situation; the only point which requires some care is Lemma 8.2, where one gets $\mathrm{codim}(H_N - H_N^{ss}) = 1$ in the case $g = 1$, $(r,d) = 1$. Assuming $g \geq 2$ for simplicity, we obtain

Theorem 9.4. *Assume $0 < d < r$ and $g \geq 2$; let $s = \dfrac{r}{(r,d)}$. The space*

$$H^0(\mathscr{SU}_X(r,d), \mathscr{L}_{r,d}^c)$$

is canonically isomorphic to the subspace of $V^*_{cs\varpi_{r-d},cs}$ annihilated by the Lie algebra $\mathfrak{sl}_r(A_X)$. $\square$

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