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Conformal Blocks on Elliptic Curves and the Knizhnik–Zamolodchikov–Bernard Equations

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Abstract: We give an explicit description of the vector bundle of WZW conformal blocks on elliptic curves with marked points as a subbundle of a vector bundle of Weyl group invariant vector valued theta functions on a Cartan subalgebra. We give a partly conjectural characterization of this subbundle in terms of certain vanishing conditions on affine hyperplanes. In some cases, explicit calculations are possible and confirm the conjecture. The Friedan–Shenker flat connection is calculated, and it is shown that horizontal sections are solutions of Bernard's generalization of the Knizhnik–Zamolodchikov equation.

1. Introduction

The aim of this work is to give a description of conformal blocks of the Wess–Zumino–Witten model on genus one curves as explicit as on the Riemann sphere.

Let us recall the well-known situation on the sphere. One fixes a simple finite dimensional complex Lie algebra $\mathfrak{g}$, with invariant bilinear form $(\,,)$ normalized so that the longest roots have length squared 2, and a positive integer k called level. One then considers the corresponding affine Kac–Moody Lie algebra, the one dimensional central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}((t))$ associated to the 2-cocycle $c(X \otimes f, Y \otimes g) = (X, Y)$ res $df g$. Its irreducible highest weight integrable representations of level (= value of central generator) k are in one to one correspondence with a certain finite set I_k of finite dimensional irreducible representations of $\mathfrak{g}$. These representations extend, by the Sugawara construction, to representations of the affine algebra to which an element L_{-1} is adjoined, such that $[L_{-1}, X \otimes f] = -X \otimes \frac{d}{dt} f$. Then to each n-tuple of distinct points $z_1, \dots, z_n$ on the complex plane, and of representations $V_1, \dots, V_n$ in I_k one associates the space of conformal blocks $E(z_1, \dots, z_n)$. It is the space of linear forms on the tensor product $\bigotimes_1^n V_i^\wedge$ of the corresponding level k representations of the affine algebra, which are annihilated by the Lie algebra $\mathscr{L}(z_1, \dots, z_n)$ of $\mathfrak{g}$-valued meromorphic functions with poles in $\{z_1, \dots, z_n\}$ and regular at infinity. The latter algebra acts on $\otimes V_i^\wedge$ by

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viewing $\mathscr{L}(z_1,\ldots,z_n)$ as a Lie subalgebra of the direct sum of n copies of the loop algebra via Laurent expansion at the poles. The central extension does not cause problems as the corresponding cocycle vanishes on $\mathscr{L}(z_1,\ldots,z_n)$ by virtue of the residue theorem.

It turns out that the spaces $E(z_1,\ldots,z_n)$ are finite dimensional and are fibers of a holomorphic vector bundle over the configuration space $\mathbb{C}^n$-diagonals, carrying the flat connection $d-\sum_i dz_i L_{-1}^{(i)}$ ($L_{-1}^{(i)}$ acts on the right of a linear form) given in terms of the Sugawara construction. We use the notation $X^{(i)}=\cdots\otimes \mathrm{Id}\otimes X\otimes \mathrm{Id}\cdots$ to denote the action on the i^{th} factor of a tensor product.

This part of the construction generalizes to surfaces of arbitrary genus (see [18]). What is new is that one has to also specify local coordinates around the points z_i to give a meaning to the Laurent expansion, and that the connection is in general only projectively flat (i.e., the curvature is a multiple of the identity).

To give a more explicit description of the vector bundle of conformal blocks on the sphere, and in particular to compute the holonomy of the connection, one observes that the map $E(z_1,\ldots,z_n)\to(\bigotimes_i V_i)^*$ given by restriction to $V_i\subset V_i^\wedge$ is injective. Thus we can view E as a subbundle of a trivial vector bundle of finite rank. This subbundle can be described by an explicit algebraic condition [10]. After this identification the connection can be given in explicit terms and the equation for horizontal sections reduces to the famous Knizhnik–Zamolodchikov equations

$$(k+h^\vee)\partial_{z_i}\omega(z_1,\ldots,z_n)=\sum_{j:j\neq i}\sum_a\frac{T_a^{(i)}T_a^{(j)}}{z_i-z_j}\omega(z_1,\ldots,z_n)\,.$$

In this equation, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and T_a, $a=1,\ldots,\dim(\mathfrak{g})$ is any orthonormal basis of $\mathfrak{g}$. We view here the dual spaces V_i^* as contragradient representations.

Let us now consider the situation on genus one curves, which we view as $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ for τ in the upper half plane. Let us denote by $E(z_1,\ldots,z_n,\tau)$ the space of conformal blocks. Again, by [18], this is the finite dimensional fiber of a holomorphic vector bundle with flat connection on the elliptic configuration space $C^{[n]}$ of $n+1$-tuples $(z_1,\ldots,z_n,\tau)$ with $\mathrm{Im}(\tau)>0$ and $z_i\neq z_j \mod \mathbb{Z}+\tau\mathbb{Z}$ if $i\neq j$.

The trouble is that the restriction to $\bigotimes_i V_i$ is no longer injective, the reason being that there are no meromorphic functions on elliptic curves with one simple pole only. The way out is the following construction which brings the moduli space of flat G-bundles into the game. Consider the Lie algebras $\mathscr{L}(z_1,\ldots,z_n,\tau,\lambda)$, parametrized by λ in a Cartan subalgebra $\mathfrak{h}$, of $\mathbb{Z}$-periodic meromorphic functions $X:C\to\mathfrak{g}$ with poles at $z_1,\ldots,z_n$ modulo $\mathbb{Z}+\tau\mathbb{Z}$, such that $X(t+\tau)=\exp(2\pi i\,\mathrm{ad}\,\lambda)X(t)$.

These algebras act on $\bigotimes_i V_i^\wedge$ and we can define a space of (twisted) conformal blocks $E_{\mathfrak{h}}(z,\tau,\lambda)$ as a space of invariant linear forms (see 2.3). The original space of conformal blocks is recovered by setting $\lambda=0$.

It turns out that $E_{\mathfrak{h}}(z,\tau,\lambda)$ is again the fiber over (z,τ,λ) of a holomorphic vector bundle $E_{\mathfrak{h}}$ over $C^{[n]}\times\mathfrak{h}$ with flat connection, whose restriction to $C^{[n]}\times\{0\}$ is E. Thus we can by parallel transport in the direction of $\mathfrak{h}$ identify the space of sections $E(U)$ of E over an open set $U\subset C^{[n]}$ with the space of sections of $E_{\mathfrak{h}}$ which are horizontal in the direction of $\mathfrak{h}$:

$$E(U)\simeq E_{\mathfrak{h}}(U\times\mathfrak{h})^{\text{hor}}\,.$$

The point is now that the restriction map

$$E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}} \to \left(\bigotimes_i V_i\right)^* \otimes \mathcal{O}(U \times \mathfrak{h}),$$

to $V^{[n]}$ is injective (Proposition 3.6). Composing these two maps we may view the vector bundle of conformal blocks as a subbundle of an explicitly given vector bundle on $C^{[n]}$ of finite rank. Indeed we show (Theorems 3.7, 3.8) that the image is contained in the space of functions on $U \times \mathfrak{h}$ which have definite transformation properties (of theta function type) under translations of λ by $Q^\vee + \tau Q^\vee$, where $Q^\vee$ denotes the coroot lattice. Moreover the theta functions in the image are invariant under a natural action of the Weyl group, and obey a certain vanishing condition as the argument approaches affine root hyperplanes. We conjecture that these conditions characterize completely the image. This conjecture is confirmed in some cases, including a special case which arises [6] in the theory of quantum integrable many body problems (see 4.1): we describe explicitly the space of conformal blocks in the case of sl_N, $n = 1$, where the representation is any symmetric power of the defining N-dimensional representation.

The characterization of conformal blocks in terms of invariant theta functions obeying vanishing conditions was first given (in the sl_2 case) by Falceto and Gawędzki [8], who define conformal blocks as Chern–Simons states in geometric quantization.

After the identification of conformal blocks as subbundle of the "invariant theta function" bundle, we describe the connection in explicit terms (Theorem 4.1), and get a generalization of the Knizhnik–Zamolodchikov equations. These equations were essentially written by Bernard [1, 2] in a slightly different context, and were recently reconsidered from a more geometrical point of view in [8]. They have the form (see Sect. 4)

$$\kappa \partial_{z_j} \tilde{\omega} = -\sum_\nu h_\nu^{(j)} \partial_{\lambda_\nu} \tilde{\omega} + \sum_{l: l \neq j} \Omega^{(j,l)}(z_j - z_l, \tau, \lambda)\tilde{\omega},$$

$$4\pi i \kappa \partial_\tau \tilde{\omega} = \sum_\nu \partial_{\lambda_\nu}^2 \tilde{\omega} + \sum_{j,l} H^{(j,l)}(z_j - z_l, \tau, \lambda)\tilde{\omega},$$

for some tensors Ω, $H \in \mathfrak{g} \otimes \mathfrak{g}$, given in terms of Jacobi theta functions. Here $\tilde{\omega}$ is related to ω by multiplication by the Weyl–Kac denominator. Thus, the right way to look at these equations is to view u as a section of a subbundle of the vector bundle over the elliptic configuration space of $n+1$-tuples $(z_1, \ldots, z_n, \tau)$, whose fiber is a finite dimensional space of invariant theta functions.

In this paper we do not discuss an alternative approach to conformal blocks on elliptic curves, which is in terms of traces of products of vertex operators. Bernard [1] showed that such traces obey his differential equations. Using this formulation, integral representation of solutions were given in the sl_2 case in [3]. To complete the picture, one should show that those solutions are indeed theta functions with vanishing condition.

Let us also point out the recent paper [7] that shows that the same space of invariant theta functions with vanishing condition can be identified with a space of equivariant functions on the corresponding Kac–Moody group.

Some of the results presented here were announced in [12].

2. Conformal Blocks on Elliptic Curves

2.1. Elliptic Configuration Spaces. Let $H_+ = \{\tau \in \mathbb{C} | \operatorname{Im} \tau > 0\}$ be the upper half plane and for $\tau \in H_+$ denote by $L(\tau)$ the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$. Let n be a positive integer. We define the elliptic configuration space to be the subset of $\mathbb{C}^n \times H_+$ consisting of points $(z_1, \dots, z_n, \tau)$ so that $z_i \neq z_j \bmod L(\tau)$ if $i \neq j$.

The space of points $(z, \tau) \in C^{[n]}$ with fixed τ is a covering of the configuration space of n ordered points on the elliptic curve $\mathbb{C}/L(\tau)$.

2.2. Meromorphic Lie Algebras. Let $\mathfrak{g}$ be a complex simple Lie algebra with dual Coxeter number $h^\vee$ and k be a positive integer. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ be the corresponding root space decomposition. The invariant bilinear form is normalized in such a way that $(\alpha^\vee, \alpha^\vee) = 2$ for long roots α (see [14]). We choose an orthonormal basis (h_ν) of $\mathfrak{h}$. The symmetric invariant tensor $C \in \mathfrak{g} \otimes \mathfrak{g}$ dual to $(\,,\,)$ admits then a decomposition $C = \sum_{\alpha \in \Delta \cup \{0\}} C_\alpha$, with $C_0 = \Sigma h_\nu \otimes h_\nu \in \mathfrak{h} \otimes \mathfrak{h}$ and $C_\alpha \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$, if $\alpha \in \Delta$.

We define a family of Lie algebras of meromorphic functions with values in $\mathfrak{g}$ parametrized by $C^{[n]} \times \mathfrak{h}$.

Definition. *For $(z, \tau) = (z_1, \dots, z_n, \tau) \in C^{[n]}$ and $\lambda \in \mathfrak{h}$, let $\mathscr{L}(z, \tau, \lambda)$ be the Lie algebra of meromorphic functions $t \mapsto X(t)$ on the complex plane with values in $\mathfrak{g}$ such that*

$$X(t+1) = X(t), \qquad X(t+\tau) = \exp(2\pi i \operatorname{ad} \lambda) X(t),$$

and whose poles belong to $\bigcup_{i=1}^n z_i + L(\tau)$. More generally, for any open set $U \subset C^{[n]} \times \mathfrak{h}$ let $\mathscr{L}_\mathfrak{h}(U)$ be the Lie algebra of meromorphic functions $(t, z, \tau, \lambda) \mapsto X(t, z, \tau, \lambda)$ on $\mathbb{C} \times U$ with values in $\mathfrak{g}$, whose poles are on the hyperplanes $t = z_i + r + s\tau$, $1 \leqq i \leqq n, r, s \in \mathbb{Z}$, and such that for all $(z, \tau, \lambda) \in U$, the function $t \to X(t, z, \tau, \lambda)$ belongs to $\mathscr{L}(z, \tau, \lambda)$. Similarly, define $\mathscr{L}(U)$ for an open subset U of $C^{[n]}$ to be the Lie algebra of meromorphic functions $(t, z, \tau) \mapsto X(t, z, \tau)$ on $\mathbb{C} \times U$ with values in $\mathfrak{g}$, whose poles are on the same hyperplanes, and such that for all $(z, \tau) \in U$, the function $t \to X(t, z, \tau)$ belongs to $\mathscr{L}(z, \tau, 0)$.

An explicit description of these Lie algebras is given in Appendix A. An important property is that they have a filtration by locally free finitely generated sheaves: Let $\mathcal{O}(U)$ be the algebra of holomorphic functions on an open set $U \subset C^{[n]} \times \mathfrak{h}$, and for any non-negative integer j let $\mathscr{L}_\mathfrak{h}^{\leqq j}(U)$ be the $\mathcal{O}(U)$-submodule of $\mathscr{L}_\mathfrak{h}(U)$ consisting of functions whose poles have order at most j. Similarly we define $\mathscr{L}^{\leqq j}(U)$ for open sets $U \in C^{[n]}$. The assignments $U \to \mathscr{L}^{\leqq j}(U)$, $U \to \mathscr{L}_\mathfrak{h}^{\leqq j}(U)$ are sheaves of $\mathcal{O}$-modules.

Proposition 2.1. *$\mathscr{L}_\mathfrak{h}^{\leqq j}$ is a locally free, locally finitely generated sheaf of $\mathcal{O}$-modules. In other words, every point in $C^{[n]} \times \mathfrak{h}$ has a neighborhood U such that $\mathscr{L}_\mathfrak{h}^{\leqq j}(U) \simeq \mathbb{C}^{n_j} \otimes \mathcal{O}(U)$ as an $\mathcal{O}(U)$-module, for some n_j. Moreover for each $x \in C^{[n]} \times \mathfrak{h}$, every $X \in \mathscr{L}(x)$ extends to a function in $\mathscr{L}_\mathfrak{h}^{\leqq j}(U)$ for some j and $U \ni x$. The same results hold for $\mathscr{L}^{\leqq j}$.*

The proof is contained in Appendix A (see Corollary A.3).

2.3. Tensor Product of Affine Kac–Moody Algebra Modules. Let $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$ be the loop algebra of $\mathfrak{g}$. Fix a positive integer $k \in \mathbb{N}$. Let $L\mathfrak{g}^{\wedge} = L\mathfrak{g} \oplus \mathbb{C}K$ be the central extension of $L\mathfrak{g}$ associated with the 2-cocycle

$$c(X \otimes f, Y \otimes g) = (X, Y)\,\mathrm{res}\,(f' g \mathrm{d}t)\,,$$

where the residue of a formal Laurent series is given by $\mathrm{res}(\sum_n a_n t^n \mathrm{d}t) = a_{-1}$. Thus the Lie bracket in $L\mathfrak{g}^{\wedge}$ has the form

$$[X \otimes f \oplus \zeta K, Y \otimes g \oplus \xi K] = [X, Y] \otimes fg \oplus c(X, Y)K\,.$$

With every irreducible highest weight $\mathfrak{g}$-module V is associated an irreducible highest weight $L\mathfrak{g}^{\wedge}$-module $V^{\wedge}$ of level k. Its construction goes as follows. The action of $\mathfrak{g}$ is first extended to the Lie subalgebra $\mathfrak{b}_+ = \mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}K$ of $L\mathfrak{g}^{\wedge}$, by letting $\mathfrak{g} \otimes t\mathbb{C}[[t]]$ act by zero and the central element K by k. Then a generalized Verma module $\tilde{V} = U(L\mathfrak{g}^{\wedge}) \bigotimes_{U(\mathfrak{b}_+)} V$ is induced. It is freely generated by (any basis of) V as a $\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$-module. The polynomial subalgebra $L\mathfrak{g}_P = \mathfrak{g} \otimes \mathbb{C}(t, t^{-1}] \oplus \mathbb{C}$ of $L\mathfrak{g}^{\wedge}$ is $\mathbb{Z}$-graded with $\deg(X \otimes t^j) = -j$. Since $\tilde{V} \simeq U(L\mathfrak{g}_P^{\wedge}) \bigotimes_{U(\mathfrak{b}_+ \cap L\mathfrak{g}_P^{\wedge})} V$, the generalized Verma module is naturally graded. By definition the irreducible module $V^{\wedge}$ is the quotient of the generalized Verma module by its maximal proper graded submodule.

We will consider integrable modules, $V^{\wedge}$ of fixed level $k = 0, 1, 2, \ldots$. If we fix a set of simple roots $\alpha_1, \ldots, \alpha_l \in \Delta$, and denote by θ the corresponding highest root, $V^{\wedge}$ is integrable of level k if the irreducible $\mathfrak{g}$-module V has highest weight μ in the subset

$$I_k = \{\mu \in P | (\mu, \alpha_i) \geqq 0, \qquad i = 1, \ldots, l, \qquad (\mu, \theta) \leqq k\}\,, \tag{1}$$

of the weight lattice P. Let v be the highest weight vector of V and e_θ a generator of $\mathfrak{g}_\theta$. Then the maximal proper submodule is generated by $(e_\theta \otimes t^{-1})^{k-(\mu,\theta)+1} v$.

The grading extends to $V^{\wedge[n]}$ by setting $\deg(v_1 \otimes \cdots \otimes v_n) = \Sigma \deg(v_i)$. With our convention all homogeneous vectors have non-negative degree.

Fix n highest weight $\mathfrak{g}$-modules V_j, $1 \leqq j \leqq n$ such that the corresponding $L\mathfrak{g}^{\wedge}$-modules $V_j^{\wedge}$ are integrable of level k, and let $\tau \in H_+$ and $z_1, \ldots, z_n$ complex numbers with $z_i \not\equiv z_j \mod L(\tau)$ if $i \neq j$. We think of $V_j^{\wedge}$ as an $L\mathfrak{g}^{\wedge}$-module which is attached to the point z_j.

In the following we will use the abbreviations $V^{[n]} = V_1 \otimes \cdots \otimes V_n$ and $V^{\wedge[n]} = V_1^{\wedge} \otimes \cdots \otimes V_n^{\wedge}$.

We now construct an action of $\mathscr{L}(z, \tau, \lambda)$ on $V^{\wedge[n]}$. For $X \in \mathscr{L}(z, \tau, \lambda)$ let $\delta_j(X) = X(z_j + t) \in \mathfrak{g} \otimes \mathbb{C}((t))$ be the Laurent expansion of X at z_j viewed as a formal Laurent series in t. Then

$$\delta(X) = \delta_1(X) \oplus \cdots \oplus \delta_n(X)\,, \tag{2}$$

defines a Lie algebra embedding of $\mathscr{L}(z, \tau, \lambda)$ into $L\mathfrak{g} \oplus \cdots \oplus L\mathfrak{g}$. As a vector space $L\mathfrak{g} \oplus \cdots \oplus L\mathfrak{g}$ is embedded in $L\mathfrak{g}^{\wedge} \oplus \cdots \oplus L\mathfrak{g}^{\wedge}$. The embedding is of course not a Lie algebra homomorphism. Since $L\mathfrak{g}^{\wedge} \oplus \cdots \oplus L\mathfrak{g}^{\wedge}$ acts on $V^{\wedge[n]}$ we obtain a map from $\mathscr{L}(z, \tau, \lambda)$ to $\mathrm{End}_{\mathbb{C}}(V^{\wedge[n]})$. This map will also be denoted by δ. Thanks to the residue theorem it turns out to be a Lie algebra homomorphism.

Proposition 2.2. *For X, $Y \in \mathscr{L}(z,\tau,\lambda)$,*

$$\delta([X,Y]) = [\delta(X),\delta(Y)]\,.$$

Proof. In $\mathrm{End}_{\mathbb{C}}(V^{\wedge[n]})$ we have the equation

$$\delta([X,Y]) = [\delta(X),\delta(Y)] + k\sum_{j=1}^{n}\mathrm{res}_{t=z_j}\,((X'(t),Y(t))\,dt)\,.$$

But $(X'(t),Y(t))$ is doubly periodic (by ad-invariance of $(\,,)$) so that the sum of residues vanishes. □

2.4. Vector Fields. The Lie algebra $\mathrm{Vect}(S^1) = \mathbb{C}((t))\frac{d}{dt}$ of formal vector fields on the circle acts by derivations on $L\mathfrak{g}$. Let us denote this action simply by $(\xi(t)\frac{d}{dt},X(t)) \mapsto \xi(t)\frac{d}{dt}X(t)$. It extends to an action on $L\mathfrak{g}^{\wedge}$ by letting vector fields act trivially on the center. The Sugawara construction yields a projective representation of $\mathrm{Vect}(S^1)$ on $V^{\wedge}$, for any finite dimensional $\mathfrak{g}$-module V. The Sugawara operators $L_n \in \mathrm{End}(V^{\wedge})$ are defined by choosing any basis $\{B_1,\dots,B_d\}$ of $\mathfrak{g}$, with dual basis $\{B^1,\dots,B^d\}$ of $\mathfrak{g}$ so that $(B^a,B_b) = \delta_{ab}$, and setting

$$L_n = \frac{1}{2(k+h^{\vee})}\sum_{m\in\mathbb{Z}}\sum_{a=1}^{d}(B^a\otimes t^{n-m})(B_a\otimes t^m), \quad n \neq 0\,,$$

$$L_0 = \frac{1}{2}[L_1,L_{-1}]\,.$$

These operators are independent of the choice of basis and obey the commutations relations of the Virasoro algebra $[L_n,L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)$ with central charge $c = k\dim(\mathfrak{g})/(k+h^{\vee})$. Then

$$\sum_n \xi_n t^{n+1}\frac{d}{dt} \mapsto -\sum_n \xi_n L_n \in \mathrm{End}(V^{\wedge})\,,$$

defines a projective representation of $\mathrm{Vect}(S^1)$ on $V^{\wedge}$, with the intertwining property $[\xi(t)\frac{d}{dt},X(t)] = \xi(t)\frac{d}{dt}X(t)$, for any $X(t) \in L\mathfrak{g}^{\wedge}$. Note that all infinite sums are actually finite when acting on any vector in $V^{\wedge}$.

2.5. Conformal Blocks. For a Lie algebra module V we denote by V^* the dual vector space with natural (right) action of the Lie algebra. The notation $\langle\omega,v\rangle$ will be used to denote the evaluation of a linear form ω on a vector v.

Definition. *The space of* **twisted conformal blocks** *associated to data* $\mathfrak{g}$, k, $V_1,\dots,V_n$ *as above, is the space* $E_{\mathfrak{h}}(z,\tau,\lambda)$ *of linear functionals on* $V^{\wedge[n]}$ *annihilated by* $\mathscr{L}(z_1,\dots,z_n,\tau,\lambda)$*. If* $\lambda = 0$*, then* $E(z,\tau) = E_{\mathfrak{h}}(z,\tau,0)$ *is called the space of* **conformal blocks** *at* (z,τ).

Let us vary the parameters: let U be an open subset of $C^{[n]}\times\mathfrak{h}$. Then the space of holomorphic functions $\omega : U \to V^{\wedge[n]*}$, (i.e., of functions ω whose evaluation $\langle\omega,u\rangle$ on any fixed vector $u \in V^{\wedge[n]}$ is holomorphic on U), is a right $\mathscr{L}_{\mathfrak{h}}(U)$-module.

Definition. *The space* $E_{\mathfrak{h}}(U)$ *of* **holomorphic twisted conformal blocks on** $U \subset C^{[n]}\times\mathfrak{h}$ *is the space of* $V^{\wedge[n]*}$*-valued holomorphic functions* ω*, so that for all open* *subsets U' of U, the restriction of ω to U' is annihilated by $\mathscr{L}_{\mathfrak{h}}(U')$. We also define the space $E(U)$ of* **holomorphic conformal blocks on** *$U \subset C^{[n]}$ by replacing $\mathscr{L}_{\mathfrak{h}}$ by $\mathscr{L}$.*

With this definition, the assignments $U \mapsto E_{\mathfrak{h}}(U)$, $U \mapsto E(U)$ are sheaves of $\mathcal{O}$-modules.

Lemma 2.3. *Let U be an open subset of $C^{[n]} \times \mathfrak{h}$ (resp. of $C^{[n]}$). Then $\omega \in E_{\mathfrak{h}}(U)$ (resp. $E(U)$) if and only if ω is holomorphic on U and $\omega(x) \in E_{\mathfrak{h}}(x)$ (resp. $E(x)$) for all $x \in U$.*

Proof. It is obvious that if ω is holomorphic and if $\omega(x) \in E_{\mathfrak{h}}(x)$ for all $x \in U$, then $\omega \in E_{\mathfrak{h}}(U)$. Let $\omega \in E_{\mathfrak{h}}(U)$, and $x \in U$. To show that $w(x) \in E_{\mathfrak{h}}(x)$, we have to show that every element X of $\mathscr{L}(x)$ is the restriction of an element of $\mathscr{L}_{\mathfrak{h}}(U')$ for some neighborhood U' of x. But his follows from Prop. 2.1. The same applies in the untwisted case. □

3. Flat Connections, Theta Functions

3.1. The Flat Connection. For each open subset U of $C^{[n]} \times \mathfrak{h}$ we have defined a Lie algebra $\mathscr{L}_{\mathfrak{h}}(U)$ acting on $V^{\wedge[n]}(U)$, the space of holomorphic functions on U with values in $V^{\wedge[n]}$. It is convenient to extend this definition. Let G be the simply connected complex Lie group whose Lie algebra is $\mathfrak{g}$, and for $(z,\tau,g) \in C^{[n]} \times G$, let $\mathscr{L}(z,\tau,g)$ be the Lie algebra of meromorphic $\mathfrak{g}$-valued functions $X(t)$, on the complex plane whose poles modulo L_τ belong to $\{z_1,\dots,z_n\}$, and with multipliers

$$X(t+1) = X(t)\,,$$

$$X(t+\tau) = \mathrm{Ad}(g)X(t)\,.$$

If U is an open subset of $C^{[n]} \times G$, define $\mathscr{L}_G(U)$ to be the Lie algebra of meromorphic functions on $U \times \mathbb{C} \ni (z,\tau,g,t)$ whose poles are on the hyperplanes $t = z_i + n + m\tau, n, m \in \mathbb{Z}$, and restricting to functions in $\mathscr{L}(z,\tau,g)$ for fixed $(z,\tau,g) \in U$. As above, we introduce the space $E_G(z,\tau,g)$ of $\mathscr{L}(z,\tau,g)$-invariant linear forms on $V^{\wedge[n]}$, and the sheaf $U \to E_G(U)$ of $\mathscr{L}_G(U)$ invariant holomorphic $V^{\wedge[n]*}$-valued functions.

Let $\eta(z,\tau,t)$ be a meromorphic function on $C^{[n]} \times \mathbb{C}$ whose poles belong to the hyperplanes $t = z_i + n + m\tau$ and such that, as function of $t \in \mathbb{C}$,

$$\eta(z,\tau,t+1) = \eta(z,\tau,t), \quad \eta(z,\tau,t+\tau) = \eta(z,\tau,t) - 2\pi i\,. \tag{3}$$

Although the construction does not depend on which η we choose, we will always set

$$\eta(z,\tau,t) = \rho(t - z_1,\tau)\,,$$

$$\rho(t,\tau) = \frac{\partial}{\partial t}\log\theta_1(t|\tau)\,,$$

$$\theta_1(t|\tau) = -\sum_{j=-\infty}^{\infty} e^{\pi i(j+\frac{1}{2})^2\tau + 2\pi i(j+\frac{1}{2})(t+\frac{1}{2})}\,,$$

for definiteness.

Let $A_Y(z,\tau,g,t)$ to a meromorphic function on $C^{[n]} \times G \times \mathbb{C}$, depending linearly on $Y \in \mathfrak{g}$, whose poles as a function of t belong to $\{z_1,\dots,z_n\}$, and such that

$$A_Y(z,\tau,g,t+1) = A_Y(z,\tau,g,t)\,,$$
$$A_Y(z,\tau,g,t+\tau) = \mathrm{Ad}(g)(A_Y(z,\tau,g,t) - Y)\,. \tag{4}$$

If $\psi(A) = \sum_{j\in\mathbb{Z}} e^{i\pi(j+\frac{1}{2})^2\tau}(-A)^j$, we may take A_Y to be

$$A_Y(z,\tau,g,t) = \frac{1}{1-\mathrm{Ad}(g^{-1})}\left(1 - \frac{\psi(e^{2\pi i(t-z_1)}\mathrm{Ad}(g^{-1}))}{\psi(e^{2\pi i(t-z_1)})}\right)Y\,. \tag{5}$$

Note that for fixed z,τ,t and Y, A_Y extends to a regular function of $g \in G$.

Denote by ∂_Y the derivative in the direction of the left invariant vector field on G associated on $Y \in \mathfrak{g} : \partial_Y f(g) = \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}f(g\exp\varepsilon Y)$, and by ∂_{z_i}, ∂_τ, ∂_t the partial derivatives with respect to the coordinates z_i,τ,t of $C^{[n]}\times\mathbb{C}$.

The properties (3), (4) imply the

Proposition 3.1. *Let U be an open subset of $C^{[n]}\times G$. The differential operators*

$$D_{z_i}X(x,t) = \partial_{z_i}X(x,t)\,,$$
$$D_\tau X(x,t) = \partial_\tau X(x,t) - \frac{1}{2\pi i}\eta(x,t)\partial_t X(x,t)\,,$$
$$D_Y X(x,t) = \partial_Y X(x,t) + [A_Y(x,t), X(x,t)],\quad x=(z,\tau,g)\in U\times\mathbb{C}\,,$$

map $\mathscr{L}_G(U)$ to itself.

Therefore, we have a connection $D : \mathscr{L}_G(U) \to \Omega^1(U)\otimes\mathscr{L}_G(U)$, defined by $D = \sum dz_j\otimes D_{z_j} + d\tau\otimes D_\tau + \sum\theta_a\otimes D_{\theta^a}$, for any basis of left invariant vector fields θ^a on G with dual basis θ_a.

We proceed to define a connection on E_G. Consider first the following differential operators on the space $V^{\wedge[n]}(U)$ of $V^{\wedge[n]}$-valued holomorphic functions on the open set $U \subset C^{[n]}\times G$,

$$\nabla_{z_i}v(x) = \partial_{z_i}v(x) + L_{-1}^{(i)}v(x)\,,$$
$$\nabla_\tau v(x) = \partial_\tau v(x) - \frac{1}{2\pi i}\delta(\eta(x)\partial_t)v(x)\,,$$
$$\nabla_Y v(x) = \partial_Y v(x) + \delta(A_Y(x))v(x)\,,\quad x\in U\,.$$

In this formula the definition of the operator δ taking the Laurent expansion at the points z_i (see (2)) is extended to general meromorphic $\mathfrak{g}$-valued functions and vector fields considered as a function of the variable $t \in \mathbb{C}$. For a meromorphic vector field $\xi = \xi(t)\frac{d}{dt}$ on the complex plane we set $\delta(\xi) = \Sigma\delta_i(\xi)$, with $\delta_i(\xi) = \xi(z_i+t)\frac{d}{dt} \in \mathbb{C}((t))\frac{d}{dt}$. Let $\nabla : V^{\wedge[n]}(U) \to \Omega^1(U)\otimes V^{\wedge[n]}(U)$ denote the connection $\sum dz_j\otimes\nabla_{z_j} + d\tau\otimes\nabla_\tau + \sum\theta_a\otimes\nabla_{\theta^a}$,

Proposition 3.2. *The connections D,∇ obey the compatibility condition*

$$\nabla(Xv) = (DX)v + X\nabla v,\quad X\in\mathscr{L}_G(U),\quad v\in V^{\wedge[n]}\,.$$

Proof. This is verified by explicit calculation. □

This has the following consequence. Define ∇ on holomorphic functions ω on U with values in the dual $V^{\wedge[n]*}$ (i.e., such that $\langle\omega(x), v\rangle$ is holomorphic on U for all $v \in V^{\wedge[n]}$) by the formula $\langle\nabla\omega(x), v(x)\rangle = d\langle\omega(x), v(x)\rangle - \langle\omega(x), \nabla v(x)\rangle$.

Corollary 3.3. *The connection ∇ preserves twisted holomorphic conformal blocks, i.e., it maps $E_G(U)$ to $\Omega^1(U) \otimes E_G(U)$.*

Proposition 3.4. *The connection ∇ on $E_G(U)$ is flat.*

Proof. For $X, Y \in \mathfrak{g}$, the curvature $F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is given by the expression

$$F(X,Y) = \partial_X\delta(A_Y) - \partial_Y\delta(A_X) + [\delta(A_X), \delta(A_Y)] - \delta(A_{[X,Y]}).$$

Note that the cocycle

$$\int_\gamma \left(\frac{d}{dt}A_X, A_Y\right) dt,$$

vanishes: indeed, the integrand $I(t)$ is $\mathbb{Z}$-periodic and obeys $I(t+\tau) = I(t) + \frac{d}{dt}g(t)$ for some $\mathbb{Z}$-periodic function $g(t)$ and the integration cycle γ can be decomposed into a sum of contours bounding some fundamental domains. The contributions of the four edges cancel by periodicity, except for a term $\int_x^{x+1} g'(t)dt = 0$.

Thus we can write F as

$$F(X,Y) = \delta(\tilde{F}(X,Y)),$$

$$\tilde{F}(X,Y) = \partial_X A_Y - \partial_Y A_X + [A_X, A_Y] - A_{[X,Y]}.$$

Now, as a simple calculation shows, $\tilde{F}(X,Y)$, viewed as a function of $t \in \mathbb{C}$ with values is $\mathfrak{g}$, is $\mathbb{Z}$-periodic, and obeys $\tilde{F}(X,Y)(t+\tau) = \mathrm{Ad}(g)\tilde{F}(X,Y)(t)$, as a consequence of (4). Thus $F(X,Y)$ is in the image of $\mathscr{L}_G(U)$, and vanishes on invariant linear forms.

A similar reasoning applies to the commutators $[\nabla_{z_i}, \nabla_X], [\nabla_\tau, \nabla_X], X \in \mathfrak{g}$. These commutators are also in the image of $\mathscr{L}_G(U)$ and thus vanish on invariant forms. We are left with the commutator $[\nabla_\tau, \nabla_{z_i}]$, which vanishes except possibly for $i = 1$. The proof that it vanishes also for $i = 1$ will be given later on (see 4.1). □

The group G acts on $V^{\wedge[n]}$, since the cocycle vanishes on $\mathfrak{g} \subset L\mathfrak{g}^\wedge$. Denote this action simply $G \times V^{\wedge[n]} \ni (h, v) \mapsto hv$.

Proposition 3.5. *For all $h \in G$, $X \to \mathrm{Ad}(h)X$ is a Lie algebra isomorphism from $\mathscr{L}(z,\tau,g)$ to $\mathscr{L}(z,\tau,hgh^{-1})$. Thus the map $X \mapsto \phi_h X$ with $\phi_h X(z,\tau,g) = \mathrm{Ad}(h)X(z,\tau,h^{-1}gh)$ is an isomorphism from $\mathscr{L}_G(U)$ to $\mathscr{L}_G(U')$ for any open $U \subset C^{[n]} \times G$, where $U' = \{(z,\tau,hgh^{-1}) | (z,\tau,g) \in U\}$. Moreover, for any $X \in \mathscr{L}_G(U)$, $\delta(\mathrm{Ad}(h)X) = h\delta(X)h^{-1}$, and thus $\rho_h\omega(z,\tau,g) = \omega(z,\tau,hgh^{-1})h$ defines an isomorphism $\rho_h : E_G(U') \to E_G(U)$. This isomorphism maps horizontal sections to horizontal sections.*

Proof. The first statement follows immediately from the definitions. The fact that $\delta(\mathrm{Ad}(h)X) = h\delta(X)h^{-1}$ is also clear, once one notices that the 2-cocycle defining the central extension vanishes if one of the arguments is a constant Lie algebra element. Finally h commutes with ∇_{z_i} and ∇_τ, and we have $\nabla_X \rho_h = \rho_h \nabla_{\mathrm{Ad}(h)X}$, $X \in \mathfrak{g}$. The latter identity follows from the equality (see (5))

$$\mathrm{Ad}(h) A_X(z, \tau, g, t) = A_{\mathrm{Ad}(h)X}(z, \tau, hgh^{-1}, t).$$

Thus ρ_h preserves horizontality. □

The existence of a connection implies, as in [18], that the sheaf $U \mapsto E_G(U)$ is (the sheaf of holomorphic sections of) a holomorphic vector bundle whose fiber over x is $E_G(x)$. This follows once one notices that $E_G(U)$ is actually a subsheaf of a locally free finitely generated sheaf carrying a connection whose restriction to E_G is ∇. Details on this point are in Appendix B.

To make connection with the previous sections, consider the pull back of E_G by the map $\lambda \mapsto \exp(2\pi i \lambda)$, from $\mathfrak{h}$ to G. It is the vector bundle $E_{\mathfrak{h}}$ on $C^{[n]} \times \mathfrak{h}$. Let us introduce coordinates λ_ν on $\mathfrak{h}$ with respect to some orthonormal basis (h_ν). Then the pull-back connection on $E_{\mathfrak{h}}$ is given by (3.1), (3.1), and, in the direction of λ,

$$\nabla_{\lambda_\nu} = \partial_{\lambda_\nu} - \delta(h_\nu \rho(\,\cdot\, - z_1, \tau)).$$

Moreover, we can use the connection of identify by parallel translation the space of conformal blocks $E(U)$ with the space of twisted conformal blocks ω in $E_G(U \times G)$ (or in $E_{\mathfrak{h}}(U \times \mathfrak{h})$) such that $\nabla_X \omega = 0, X \in \mathfrak{g}$ (or $\nabla_{\lambda_\nu}\omega = 0$, respectively). Here we use the fact that G and $\mathfrak{h}$ are simply connected.

The point of this construction is given by the following result. Let $V^{[n]*}(U)$ be the space of holomorphic functions on an open set U with values in the finite dimensional space $V^{[n]*}$. We also set, for any open subset U of $C^{[n]} \times G$, or $C^{[n]} \times \mathfrak{h}$, respectively,

$$E_G(U)^{\mathrm{hor}} = \{\omega \in E_G(U) | \nabla_X \omega = 0, \quad \forall X \in \mathfrak{g}\},$$

$$E_{\mathfrak{h}}(U)^{\mathrm{hor}} = \{\omega \in E_{\mathfrak{h}}(U) | \nabla_X \omega = 0, \quad \forall X \in \mathfrak{h}\}.$$

Proposition 3.6. *The compositions*

$$\iota_G : E(U) \to E_G(U \times G)^{\mathrm{hor}} \to V^{[n]*}(U \times G),$$

$$\iota_{\mathfrak{h}} : E(U) \to E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}} \to V^{[n]*}(U \times \mathfrak{h}),$$

where the first map sends a holomorphic conformal block ω to the unique twisted holomorphic conformal block horizontal in the G (resp. $\mathfrak{h}$) direction, which coincides with ω on $U \times \{1\}$ (resp. $U \times \{0\}$), and the second map is the restriction to $V^{[n]}$, are injective.

Proof. The first map is an isomorphism to the space of twisted holomorphic conformal blocks horizontal in the G (resp. $\mathfrak{h}$) direction. The fact that the second map is injective follows from the fact that using the invariance and the equation $\nabla_X \omega = 0$ (resp. $\nabla_{\lambda_\nu}\omega = 0$), one can express $\langle \omega, v \rangle$ for any $v \in V^{\wedge[n]}$ linearly in terms of the restriction of ω to $V^{[n]}$. □

We may (and will) thus view the sheaf E of sections of the vector bundle on $C^{[n]}$ of conformal blocks as a subsheaf of $V^{[n]}(U \times \mathfrak{h})$. The next steps are a characterization of this subsheaf and a formula for the connection after this identification.

3.2. Theta Functions. Let $Q^\vee = \{q \in \mathfrak{h} | \exp(2\pi i q) = 1 \in G\}$ be the coroot lattice of $\mathfrak{g}$.

Definition. *Let* $(z,\tau) \in C^{[n]}$, *and* $V_1, \dots, V_n$ *be finite dimensional* $\mathfrak{g}$*-modules, and* k *a non-negative integer. The space* $\Theta_k(z,\lambda)$ *of theta functions of level* k *is the space of holomorphic functions* $f : \mathfrak{h} \to V^{[n]*}$ *such that*

(i) $\sum_{i=1}^{n} f(\lambda) h^{(i)} = 0$.

(ii) *One has the following transformation properties under the lattice* $Q^\vee + \tau Q^\vee \subset \mathfrak{h}$:

$$f(\lambda + q) = f(\lambda),$$
$$f(\lambda + q\tau) = f(\lambda) \exp\left(-\pi i k (q,q)\tau - 2\pi i k (q,\lambda) - 2\pi i \sum_{j=1}^{n} z_j q^{(j)} \right).$$

The space of such theta functions is finite dimensional, as can be easily seen by Fourier series theory. Denote by W the Weyl group of $\mathfrak{g}$, generated by reflection with respect to root hyperplanes. It is known that this group is isomorphic to $N(H)/H$, $N(H) \subset G$ being the normalizer of $H = \exp(\mathfrak{h})$. For $w \in W$, let $\hat{w} \in N(H)$ be any representative of the class of w in $N(H)/H$. The Weyl group acts on the space of theta functions. Indeed, if $f \in \Theta_k(z,\tau)$, then $(wf)(\lambda) = f(w^{-1}\lambda)\hat{w}^{-1}$ also in $\Theta_k(z,\tau)$, (the coroot lattice and the invariant bilinear form are preserved by the Weyl group), and is independent of the choice of representative $\hat{w}$ by (i). Let $\Theta_k(z,\tau)^W$ denote the space of W-invariant theta functions.

Theorem 3.7. *Let* $\mathfrak{g} = A_l, l \geqq 2, D_l, l \geqq 4, E_6, E_7, E_8, F_4$, *or* G_2. *Then the image of* $\iota_\mathfrak{h}$ *is contained in the space of holomorphic functions* $w \in V^{[n]*}(U \times \mathfrak{h})$ *such that for all* $(z,\tau) \in C^{[n]}, \omega(z,\tau,\cdot)$ *belongs to* $\Theta_k(z,\tau)^W$, *and such that for all roots* $\alpha, X \in \mathfrak{g}_\alpha$ *and nonnegative integers* p,

$$\omega(z,\tau,\lambda) X^p = O(\alpha(\lambda)^p),$$

as $\alpha(\lambda) \to 0$.

In the remaining cases, we have

Theorem 3.8. *Let* $\mathfrak{g} = A_1, B_l$ *or* $C_l, l \geqq 2$. *Then the image of* $\iota_\mathfrak{h}$ *is contained in the space of holomorphic functions* $\omega \in V^{[n]*}(U \times \mathfrak{h})$ *such that for all* $(z,\tau) \in C^{[n]}, \omega(z,\tau,\cdot)$ *belongs to* $\Theta_k(z,\tau)^W$, *and such that for all* $\alpha \in \Delta, r,s \in \{0,1\}, X \in \mathfrak{g}_\alpha$ *and nonnegative integers* p,

$$\omega(z,\tau,\lambda) \exp\left(2\pi i c_{r,s} \sum_j z_j \lambda^{(j)} \right) X^p = O((\alpha(\lambda) - r - s\tau)^p),$$

as $\alpha(\lambda) \to r + s\tau$, *with* $c_{r,0} = 0, c_{r,1} = (r+\tau)^{-1}$.

The proof of these theorems will be completed in 3.7. We conjecture that the space of functions described in Theorems 3.7, 3.8 actually coincides with the image of $\iota_\mathfrak{h}$. This conjecture is verified in a simple class of examples in 3.8 below.

The fact that the formulation of the result is simpler for certain Lie algebras is due to the following property shared by the Lie algebras of Theorem 3.7: for each root α and integer m there exist an element q in the coroot lattice with $\alpha(q) = m$. For the other simple Lie algebras this is true only if m is assumed to be even. More on this in 3.6.

3.3. Affine Weyl Group. Since H acts trivially on $E_{\mathfrak{h}}$, the Weyl group acts (on the right) on the values of $E_{\mathfrak{h}}$: w acts as $\hat{w}$, and this is independent of the choice of representative.

Proposition 3.9. *Let $\omega \in E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}}$. Then, for all $q \in Q^{\vee}$,*

$$\omega(z, \tau, \lambda + q) = \omega(z, \tau, \lambda) .$$

For all $w \in W$,

$$\omega(z, \tau, w\lambda) = \omega(z, \tau, \lambda)\hat{w}^{-1} .$$

Proof. The value of ω at $(z, \tau, \lambda + q)$ is obtained from $\omega(z, \tau, \lambda)$ by parallel transport along some path from λ to $\lambda + q$. Recall that ω is the pull back of a section of $E(U \times G)^{\mathrm{hor}}$ to $U \times \mathfrak{h}$. The image of the path in G is closed, and contractible (G is simply connected), which proves the first claim.

From Prop. 3.5 and the fact that $w \cdot \lambda = \mathrm{Ad}(\hat{w})\lambda$, we see that if ω is horizontal then also $\rho_{\hat{w}}\omega$ is horizontal. But these horizontal sections coincide at $\lambda = 0$, and thus everywhere. □

3.4. Modular Transformations. The group $\mathrm{SL}(2, \mathbb{Z})$ acts as follows on $C^{[n]} \times \mathfrak{h}$: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$,

$$A \cdot (z, \tau, \lambda) = \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}, \frac{\lambda}{c\tau + d} \right) .$$

Lemma 3.10. *Introduce the linear functions $\ell_\lambda(t) = 2\pi i \lambda t$ for $\lambda \in \mathfrak{h}$. For all $x \in C^{[n]} \times \mathfrak{h}$, $A \in \mathrm{SL}(2, \mathbb{Z})$, the map $X \to \phi_A(x)X$ with*

$$(\phi_A(x)X)(t) = \exp(-\mathrm{ad}\, c\ell_\lambda(t))X((c\tau + d)t) , \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

is a Lie algebra isomorphism from $\mathscr{L}(x)$ to $\mathscr{L}(Ax)$.

Proof. This follows directly from the definitions. □

We have defined an action $\mathscr{L}(x) \otimes V^{\wedge[n]} \to V^{\wedge[n]}$, for all $x \in C^{[n]} \times \mathfrak{h}$. Let us denote it as $X \otimes v \mapsto \delta_x(X)v$ (see (2)) to emphasize the x-dependence.

Lemma 3.11. *Define a linear isomorphism $v \to \rho_A(x)v$ from $V^{\wedge[n]}$ viewed as $\mathscr{L}(x)$-module to $V^{\wedge[n]}$ viewed as $\mathscr{L}(Ax)$-module: if $x = (z, \tau, \lambda)$,*

$$\rho_A(x) = \eta_A(x)(c\tau + d)^{-\sum_{j=1}^{n} L_0^{(j)}} \exp\left(-\frac{c\delta_x(\ell_\lambda)}{c\tau + d} \right), \qquad \eta_A(x) = \exp\left(-\frac{\pi i c k(\lambda, \lambda)}{c\tau + d} \right) .$$

This map has the intertwining property

$$\rho_A(x)\delta_x(X) = \delta_{Ax}(\phi_A(x)X)\rho_A(x) ,$$

for all $X \in \mathscr{L}(x)$.

Note that the choice of coefficient η_A is irrelevant for the validity of the lemma. However, it is important for compatibility with the connection, see below.

We should also add a remark about the power of $(c\tau + d)$. The exponent $\Sigma L_0^{(i)}$ is diagonalizable with finite dimensional eigenspaces. However the eigenvalues are fractional in general, and the power is defined for a choice of branch of the logarithm for each $A \in \mathrm{SL}(2,\mathbb{Z})$. This is made more systematic in the next subsection.

Proof. This is again straightforward. The only subtlety is that, a priori, there could be a contribution from the central extension in the computation of the intertwining property. However the central term appearing in this computation is proportional to the sum of the residues of the component of $\partial_t X$ along λ, which is doubly periodic. By the residue theorem, this sum vanishes. □

We can thus define linear maps $\omega \mapsto \rho_A^*\omega$ by $\rho_A^*\omega(x) = \omega(Ax)\rho_A(x)$. Lemma 3.11 implies then that ρ_A^* is an isomorphism from $E_{\mathfrak{h}}(U)$ to $E_{\mathfrak{h}}(A^{-1}(U))$.

Lemma 3.12. *Let A^* be the pull back on one-forms of the map $x \mapsto Ax$ defined on some open $U \subset C^{[n]} \times \mathfrak{h}$, and $\nabla : E_{\mathfrak{h}}(U) \to \Omega^1(U) \otimes E_{\mathfrak{h}}(U)$ be the connection defined in* 3.1. *We have*

$$\nabla \rho_A^* = A^* \otimes \rho_A^* \nabla .$$

This fact can be derived from a straightforward but unfortunately lengthy calculation. The main identity one uses is

$$\rho\left(\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)\rho(t, \tau) + 2\pi i c t .$$

Lemma 3.12 ensures that ρ_A^* maps horizontal sections to horizontal sections. Moreover, since $A_*\partial_{\lambda_\nu} = (c\tau + d)^{-1}\partial_{\lambda_\nu}$ does not have components in z or τ direction, ρ_A^* maps sections which are horizontal in the $\mathfrak{h}$ direction to sections with the same property:

$$\rho_A^* : E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}} \to E_{\mathfrak{h}}(A^{-1}U \times \mathfrak{h})^{\mathrm{hor}} .$$

Let us apply this in the special case $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proposition 3.13. *Let $\omega \in V^{[n]*}(U)$ be in the image of $\iota_{\mathfrak{h}}$. Then for all $q \in Q^\vee$,*

$$\omega(z, \tau, \lambda + \tau q) = \omega(z, \tau, \lambda) \exp\left(- 2\pi i(q, \lambda)k - \pi i(q, q)k\tau - 2\pi i \sum_{j=1}^{n} z_j q^{(j)}\right) .$$

Proof. We have $\rho_A^*\omega \in E_{\mathfrak{h}}(A^{-1}U \times \mathfrak{h})^{\mathrm{hor}}$. Thus,

$$\rho_A^*\omega\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\lambda}{\tau} + q\right) = \rho_A^*\omega\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\lambda}{\tau}\right) ,$$

for all coroots q, by Lemma 3.9, and $(z, \tau) \in U$. Explicitly,

$$\omega(z, \tau, \lambda + q\tau)\rho_A\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\lambda}{\tau} + q\right) = \omega(z, \tau, \lambda)\rho_A\left(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\lambda}{\tau}\right) .$$

Inserting the formula for ρ_A we obtain

$$\omega(z, \tau, \lambda + q\tau)\tau^{\Sigma_j L_0^{(j)}} = \omega(z, \tau, \lambda)\tau^{\Sigma_j L_0^{(j)}} e^{-2\pi i(q,\lambda)k - \pi i(q,q)k\tau} \exp(-\tau\delta_x(\ell_q)) ,$$

with $x = (z/\tau, -1/\tau, \lambda/\tau)$. Now we use the fact that, on $V^{[n]}, \delta_x(\ell_q)$ acts as Σ_j $2\pi i(z_j/\tau)q^{(j)}$, and that $L_0^{(j)}$ acts as a multiple of the identity, to conclude the proof. □

3.5. Monodromy (projective) Representations of $SL(2,\mathbb{Z})$. In this subsection, we assume that $n = 1$, set $z_1 = 0$, and show that a central extension of $\mathrm{SL}(2,\mathbb{Z})$ acts on the space of horizontal sections of the bundle of conformal blocks.

The fact that we have a central extension comes from the necessity to choose a branch of the logarithm to define the expression $(c\tau + d)^{L_0}$. In fact L_0 is diagonalizable with finite dimensional eigenspaces, and any two eigenvalues differ by an integer. Moreover, L_0 acts by a non-negative rational multiple $(\mathrm{Cas}(V)/(k + h^\vee))$ of the identity on $V \subset V^\wedge$ for any integrable $V^\wedge$ of level k. Let $L_0|_V = \frac{r}{s}\mathrm{id}_V, r, s \in \mathbb{N}$. We introduce a central extension

$$0 \to \mathbb{Z}/s\mathbb{Z} \to \Gamma_s \to \mathrm{SL}(2,\mathbb{Z}) \to 1\,,$$

of $\mathrm{SL}(2,\mathbb{Z})$ by the cyclic group of order s. The group Γ_s consists of pairs of (A, ϕ), where $A \in \mathrm{SL}(2,\mathbb{Z})$ has matrix elements a, b, c, d and ϕ is a holomorphic function on the upper half plane such that $\phi(\tau)^s = c\tau + d$. The product is $(A, \phi)(B, \psi) = (AB, \phi \circ A \cdot \psi)$. Then this group acts on V^* valued functions on $H_+ \times \mathfrak{h}$ as above, but keeping track of the choice of branch:

$$(A, \phi)^{-1}\omega(\lambda, \tau) = \omega(A \cdot (\lambda, \tau))\eta_A(\lambda, \tau)\phi(\tau)^{-r}\,.$$

This action preserves the connection. (The inversion here is to correct for the "wrong" order $\rho_A^* \rho_B^* = \rho_{BA}^*$, up to ambiguity in the choice of branch). Thus we conclude that Γ_s acts on the space of global horizontal sections on $H_+ \times \mathfrak{h}$ of $E_\mathfrak{h}$. This monodromy representation restricts to the character $[m] \mapsto \exp(2\pi i m r/s)$ of $\mathbb{Z}/s\mathbb{Z}$.

In the case of $V =$ trivial representation, this monodromy representation is just the representation of $\mathrm{SL}(2,\mathbb{Z})$ on characters of affine Lie algebras (see [15]). It would be interesting to calculate this monodromy representation explicitly for general V. Some progress in the sl_2 case was made in [5], where a connection with the adjoint representation of the corresponding quantum group was established.

3.6. The Vanishing Condition. Let G be a simply connected complex Lie group with Lie algebra $\mathfrak{g}, \mathfrak{h}$ a Cartan subalgebra, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ a Cartan decomposition and $H = \exp \mathfrak{h}$. Suppose that $\rho : G \to \mathrm{End}(V)$ is a finite dimensional representation of G. Thus V is also a $\mathfrak{g}$-module. For $K = G$ or H let $I(K, V)$ be the space of holomorphic functions on K with values in V such that $\forall g, h \in K, u(ghg^{-1}) = \rho(g)u(h)$. The Weyl group W acts on $I(H, V)$: let $\hat{w}$ be a representative of $w \in W$ in $N(H)$. Then $(wf)(h) - \rho(\hat{w})f(w^{-1} \cdot h)$ is well defined for $f \in I(H, V)$, since H acts trivially on the image of functions in $I(H, V)$. We denote by $I(H, V)^W$ the space of Weyl-invariant functions in $I(H, V)$.

Lemma 3.14. *The restriction map $I(G, V) \to I(H, V)$ is injective. Its image is the space $I_0(H, V)^W$ of functions u in $I(H, V)^W$ such that for all positive roots $\alpha, X \in \mathfrak{g}_\alpha p \in \mathbb{N}$, and $m \in \mathbb{Z}$,*

$$X^p u(\exp 2\pi i\lambda) = O((\alpha(\lambda) - m)^p)\,, \tag{6}$$

as $\alpha(\lambda) \to m$.

Proof. The behavior of functions in $I(G, V)$ under conjugation by $N(H)$ implies Weyl invariance

Let $X \in \mathfrak{g}_\alpha$ and $\lambda \in \mathfrak{h}$. Then

$$\mathrm{Ad}(\exp(2\pi i\lambda))X = e^{2\pi i\alpha(\lambda)}X\,. \tag{7}$$

If $u \in I(G,V), u(\exp(X)\exp(2\pi i\lambda))$ is a holomorphic $Q^\vee$-periodic function of $\lambda \in \mathfrak{h}$, (thus a holomorphic function on H). On the other hand, by (7),

$$\begin{aligned} u(\exp(X)\exp(2\pi i\lambda)) &= u\left(\exp\left(\frac{X}{1-e^{2\pi i\alpha(\lambda)}}\right)\exp(2\pi i\lambda)\exp\left(-\frac{X}{1-e^{2\pi i\alpha(\lambda)}}\right)\right) \\ &= \rho(\exp((1-e^{2\pi i\alpha(\lambda)})^{-1}X))u(\exp(2\pi i\lambda)) \\ &= \sum_{p=0}^{M}\frac{1}{p!}(1-e^{2\pi i\alpha(\lambda)})^{-p}X^p u(\exp(2\pi i\lambda)), \end{aligned}$$

for some M. We see that the latter expression is holomorphic on the affine hyperplanes $\alpha(\lambda)=m$, if and only if, for all $p, X^p u$ vanishes there to order at least p.

To conclude the proof, we use some facts about conjugacy classes in algebraic groups (see, e.g., [16], Chapter 3). Let, for each root α and integer $m, H_{\alpha,m} \subset H$ be the set of elements of the form $\exp(2\pi i\lambda)$ such that $\alpha(\lambda)=m$. The conjugacy classes containing elements in $H_{ss} = H - \cup H_{\alpha,m}$ form the dense open subset G_{ss} of regular semisimple elements in G. Its complement contains the set H_1 consisting of conjugacy classes of elements of the form $\exp(X)\exp(2\pi i\lambda)$, where λ lies on precisely one of the distinct $H_{\alpha,m}$. These elements are regular, as they are regular in the identity component of the stabilizer of $\exp(2\pi i\lambda)$, (see [16], 3.5), which is the direct product of a torus of dimension rank-1 times the SL(2) subgroup associated with α. By the above reasoning, a Weyl invariant function on H extends uniquely to an equivariant holomorphic function on G_{ss}, and the vanishing conditions imply that it extends to a holomorphic function on $G_{ss} \cup H_1$. The complement of $G_{ss} \cup H_1$ consists of higher codimension classes whose closure intersects H_1, and of classes whose closure do not intersect $H_{ss} \cup H_1$. Counting dimensions shows that this complement is of codimension at least two, so by Hartogs' theorem, our vanishing conditions are sufficient to have an extension to all of G. □

By Weyl invariance, we may replace the set of positive roots in the formulation of the lemma to a subset of roots consisting of one root for each Weyl group orbit. Also, we may restrict the values of m, by $Q^\vee$ periodicity of $u(\exp 2\pi i\lambda)$. Indeed, if the vanishing condition holds at $\alpha(\lambda)=m$, it also holds at $\alpha(\lambda)=m-\alpha(q)$ for all $q \in Q^\vee$

We thus have the following result. The action of the affine Weyl group $W^\wedge = W \tilde{\times} Q^\vee$ on $\Delta \times \mathbb{Z}$ is defined by

$$(w,q)(\alpha,m) = (w\alpha, m-\alpha(q)).$$

Lemma 3.15. *The subspace $I_0(H,V)^W \subset I(H,V)^W$ is characterized by the vanishing condition* (6), *for* (α,m) *in any fundamental domain for the action of $W^\wedge$ on $\Delta \times \mathbb{Z}$.*

From [4], we see that in the cases $A_l, l \geqq 2, D_l, l \geqq 4, E_6, E_7, E_8, F_4, G_2$ a fundamental domain is $\{(\alpha,0), \alpha \in F\}$, where α runs over a fundamental domain F (consisting of one or two elements) of W. If $\mathfrak{g} = A_1, B_l, C_l$, then we have to add $(\alpha,1)$, where α is a long root.

As a corollary we obtain a more precise characterization of the image of $i_{\mathfrak{h}}$. Let us identify functions on H with $Q^\vee$-periodic functions on $\mathfrak{h}$ via the map $\lambda \mapsto \exp(2\pi i\lambda)$, and view $V^{[n]*}$ as a representation of G by $\langle\rho(g)u,v\rangle = \langle u, g^{-1}v\rangle$.

Corollary 3.16. *The image of* $E(U)$ *by* $\iota_{\mathfrak{h}}$ *is contained in the space of functions* $\omega \in V^{[n]*}(U \times \mathfrak{h})$ *such that for all* $(z,\tau) \in C^{[n]}, \omega(z,\tau,\ \cdot\ )$ *belongs to* $I_0(H, V^{[n]*})^W$.

Moreover, if $\omega \in E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}}$, then $\rho_A^* \omega \in E_{\mathfrak{h}}(A^{-1}U \times \mathfrak{h})^{\mathrm{hor}}$, implying further vanishing conditions: let $x = (z,\tau,\lambda)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. On $V^{[n]} L_0^{(j)}$ acts by a scalar Δ_j. The restriction of $\rho_A^*\omega$ to $V^{[n]}$ is

$$(\rho_A^*\omega)(x) = \omega(Ax)(c\tau + d)^{-\Sigma_j \Delta_j} \eta_A(x) e^{-\frac{c}{c\tau+d}\Sigma_j z_j \lambda^{(j)}} .$$

It follows that, for all p,

$$\omega(Ax)\exp\left(-\frac{c}{c\tau+d}\sum_j z_j\lambda^{(j)}\right)X^p = O((\alpha(\lambda)-m)^p),$$

if $\alpha(\lambda) \to m$. Changing variables, this implies that

$$\omega(z,\tau,\lambda)\exp\left(-2\pi i\frac{c}{a-c\tau}\sum_j z_j\lambda^{(j)}\right)X^p = O((\alpha(\lambda)-m(a-c\tau))^p).$$

Since any pair a, c of relatively prime integers appear in the first column of some $\mathrm{SL}(2,\mathbb{Z})$ matrix, we obtain the result.

Corollary 3.17. *The image of* $E(U)$ *by* $\iota_{\mathfrak{h}}$ *is contained in the space of functions* $\omega \in V^{[n]*}(U \times \mathfrak{h})$ *such that for all* $(z,\tau,\lambda) \in C^{[n]}, r, s, p \in \mathbb{Z}, p \geqq 1, (r,s) \neq (0,0)$,

$$\omega(z,\tau,\lambda)\exp\left(2\pi i\frac{s}{r+s\tau}\sum_j z_j\lambda^{(j)}\right)X^p = O((\alpha(\lambda)-r-s\tau)^p), \qquad (8)$$

as $\alpha(\lambda) \to r + s\tau$.

3.7. Proof of Theorems 3.7, 3.8. Theorems 3.7, 3.8 follow from Propositions 3.9, 3.13, and Corollaries 3.16, 3.17 together with the fact that twisted conformal blocks are annihilated by $\mathfrak{h} \subset \mathscr{L}_{\mathfrak{h}}(U)$.

3.8. Examples. Here we give an explicit description of the space of conformal blocks in some special cases. The discussion parallels the constructions in [8], where Chern–Simons states in the case of sl_2 are studied. First of all consider the case of one point z_1 with the trivial representation. Then the vanishing condition is vacuous, and we are left to classify scalar Weyl invariant theta functions of level k. This space coincides with the space spanned by characters of irreducible highest weight $L\mathfrak{g}^\wedge$-modules, in accordance with the Verlinde formula.

Next, we consider the case of one point z_1, with a symmetric tensor power of the defining representation $\mathbb{C}^N$ of sl_N.

If $N \geqq 3$, the problem is reduced to describing the space of Weyl invariant theta functions ω of level k, with the property that

$$e_\alpha^p u(\alpha(\lambda)) = O(\alpha(\lambda)^p), \quad \alpha(\lambda) \to 0,$$

for all $p = 1, 2, \ldots$ and root vectors $e_\alpha \in \mathfrak{g}_\alpha$. Actually it is sufficient to consider one root α, since the Weyl group acts transitively on the set of roots of sl_N.

The symmetric power $S^j\mathbb{C}^N$ has a non-zero weight space if and only if j is a multiple of N. Let us set $j = lN$, and denote by ε_i the elements of the standard basis of $\mathbb{C}^N$. Then the weight zero subspace of $S^{lN}\mathbb{C}^N = (\mathbb{C}^N)^{\otimes l}/S_N$ is one-dimensional and is spanned by the class of $v = \varepsilon_1^{\otimes l} \otimes \cdots \otimes \varepsilon_N^{\otimes l}$. The following considerations apply also to the case $l = 0$, if we agree that $S^0\mathbb{C}^N$ is the trivial representation.

The Weyl group of sl_N is the symmetric group S_N and is generated by adjacent transpositions $s_j, j = 1, \ldots, N-1$. If we identify the Weyl group with $N(H)/H$, then a representative in $N(H)$ of s_j is given by $\hat{s}_j\varepsilon_r = \varepsilon_r$, if $r \neq j, j+1, \hat{s}_j\varepsilon_j = \varepsilon_{j+1}$, $\hat{s}_j\varepsilon_{j+1} = -\varepsilon_j$. It follows that S_N acts on the weight zero space by the l^{th} power of the alternating representation: $\hat{w}v = \varepsilon(w)^l v$.

The next remark is that $e_\alpha^{l+1}v = 0$ but $e_\alpha^l v \neq 0$. We thus see that $\omega(\alpha(\lambda)) = O(\alpha(\lambda)^l)$ as λ approaches the hyperplane $\alpha(\lambda) = 0$. If ω is a Weyl-invariant theta function, it then also vanishes to order l on all hyperplanes $\alpha(\lambda) = n + m\tau, n, m \in \mathbb{Z}$. Therefore the quotient of ω by the l^{th} power of the Weyl–Kac denominator $\Pi(\lambda, \tau)$ (see 4.1) is an entire function, as Π has simple zeroes on those hyperplanes. Moreover, Π is a (scalar) theta function of level N (the dual Coxeter number of sl_N), and $\Pi(w\lambda, \tau) = \varepsilon(w)\Pi(\lambda, \tau)$.

We conclude that the space of conformal blocks at fixed τ is contained in the space of functions of the form

$$\omega(\lambda) = \Pi(\lambda, \tau)^l u(\lambda) v \,, \tag{9}$$

where u is an entire $Q^\vee$-periodic scalar function on $\mathfrak{h}$, such that $u(w\lambda) = u(\lambda)$, for all $w \in S_N$ and

$$u(\lambda + q\tau) = \alpha(q, \lambda, \tau)^{k-Nl} u(\lambda) \,,$$
$$\alpha(q, \lambda, \tau) = \exp(-2\pi i(q, \lambda) - \pi i(q, q)\tau), \quad q \in Q^\vee \,. \tag{10}$$

We have assumed here that $N \geqq 3$. In the $_2$ case, where the vanishing condition must be satisfied also at 3 other points on $\mathfrak{h}$, one can proceed in the same way, noticing that the Weyl denominator vanishes there too.

A basis of $Q^\vee$-periodic functions with multipliers (10) is easily given using Fourier series. The basis elements θ_μ are labeled by $\mu \in P/(k - Nl)Q^\vee$, where the weight lattice P is dual to $Q^\vee$ (if $k < N$ there are no non-zero conformal blocks). The Weyl group acts as $\theta_\mu(w^{-1}\lambda) = \theta_{w\mu}(\lambda)$.

Therefore the dimension of our space is the number of orbits of the Weyl group in $P/(k - Nl)Q^\vee$. This number is well-known: a fundamental domain in P for the action of the semidirect product of the Weyl group by the group of translations by $(k - Nl)Q^\vee$ is the set of weights in the (dilated) Weyl alcôve I_{k-Nl}, see (1).

More explicitly, if α_i are simple roots, w_i fundamental weights with $(w_i, \alpha_j) = \delta_{ij}$, and $\mu = \Sigma_i n_i w_i$, then $\mu \in I_{k-Nl}$ if and only if the integers n_i satisfy the inequalities

$$n_i \geqq 0, \quad i = 1, \ldots, N-1, \quad \sum_{i=1}^{N-1} n_i \leqq k - Nl \,.$$

The number of $N-1$-tuples of integers with these properties is calculated to be

$$\binom{k - N(l-1) - 1}{N-1} \,.$$

This is the formula for the dimension of the space of Weyl-invariant theta functions of level k extending to holomorphic functions on SL_N, with values in the

$(l \cdot N)^{\text{th}}$ symmetric power of the defining representation of sl_N. We now show[1] that this coincides with the Verlinde formula [19], which according to [18, 9] give the dimension of the space of conformal blocks.

Let I_k be the set of integrable highest weights of level k. It consists of dominant integral weights μ with $(\mu, \theta) \leq k$. The dimension of the space of conformal blocks with one point, to which an irreducible representation of highest weight $\mu \in I_k$ is attached, is given by the formula

$$d_\mu = \sum_{\nu \in I_k} N_{\nu\mu}^{\nu} ,$$

in terms of the structure constants N_{bc}^a of Verlinde's fusion ring. A convenient formula for these constants in terms of the classical fusion coefficients m_{bc}^a (= the multiplicity of a in the decomposition of the tensor product of b with c) was given in [13, and 15], Exercise 13.35.

Let $W_k^\wedge \simeq W^\wedge$ be the group of affine transformations of $\mathfrak{h}^*$ generated by the Weyl group W and the reflection s_0 at the hyperplane $\{\lambda \in \mathfrak{h}^* | (\theta, \lambda) = k + h^\vee\}$ (θ is the highest root and $h^\vee$ the dual Coxeter number). Let ρ be half the sum of the positive roots of $\mathfrak{g}$ and define another action of $W_k^\wedge$ on $\mathfrak{h}^*$ by $w * \lambda = w(\lambda + \rho) - \rho$. Let $\varepsilon : W_k^\wedge \to \{1, -1\}$ be the homomorphism taking reflections to -1.

Then, for all $a, b, c \in I_k$,

$$N_{bc}^a = \sum_{w \in W_k^\wedge} m_{bc}^{w*a} . \tag{11}$$

Actually, in Verlinde's formula the coefficients N_{bc}^a are given in terms of modular transformation properties of characters. They are uniquely determined by the equation

$$\frac{S_{db}}{S_{d0}} \frac{S_{dc}}{S_{d0}} = \sum_a N_{bc}^a \frac{S_{da}}{S_{d0}} ,$$

where, according to [15], (13.8.9),

$$\frac{S_{ab}}{S_{a0}} = \chi_b \left(\exp \left(-2\pi i \frac{a + \rho}{k + h^\vee} \right) \right) .$$

Here, χ_a is the character of the representation of G with highest weight a.

Let us check that the two formulas agree (this is essentially the solution to Exercise 13.35 of [15]). Let $w \in W$ and $q \in (k + h^\vee)Q^\vee$ and suppose that both a and $w * a + q$ are dominant integral weights. Then it is easy to see from the Weyl character formula (see [14]) that if $\lambda \in (k + h^\vee)^{-1} P$,

$$\chi_{w*a+q}(\exp(2\pi i \lambda)) = \varepsilon(w) \chi_a(\exp(2\pi i \lambda)) .$$

There is a unique element in each affine Weyl group orbit in the shifted Weyl alcôve $I_k + \rho$. Using these facts and the formula for the multiplicities in the decomposition of tensor products $\chi_b \chi_c = \Sigma n_{bc}^a \chi_a$, we deduce (11).

Let us apply this to our example. Identify $\mathfrak{h}^*$ with $\mathbb{C}^N / \mathbb{C}(1, 1, \ldots, 1)$. Then integral weights are classes $a = [a_1, \ldots, a_n]$ of n-tuples of integers defined modulo $\mathbb{Z}(1, \ldots, 1)$. The Weyl group S_N acts in the obvious way, and a weight is dominant if $a_j \geqq a_{j+1}$. The affine reflection s_0 is

$$s_0[a_1, \ldots, a_N] = [a_N + k + N, a_2, \ldots, a_{N-1}, a_1 - k - N] , \tag{12}$$

[1] We learned how to do this computation from H. Wenzl.

and $\rho = [N-1, N-2, \dots, 0]$. Let $c = [r, 0, \dots, 0]$ be the highest weight of $S^r\mathbb{C}^N$. Then the decomposition rules of tensor products say that $m^a_{bc} = 1$ if $a_j = b_j + l_j$ $(1 \leqq j \leqq N)$ for some integers l_j such that $0 \leqq l_j \leqq a_{j-1} - a_j, 2 \leqq j \leqq N$ and $\Sigma_j l_j = r$. Otherwise, $m^a_{bc} = 0$. As $\theta = [1, 0, 0, \dots, 0, -1]$, a dominant weight a belongs to I_k if and only if $a_1 - a_N \leqq k$.

We need two properties (see [13]) of the coefficients N^a_{bc}, valid for any $a, b, c \in I_k$: (i) $0 \leqq N^a_{bc} \leqq m^a_{bc}$, for all $a, b, c, \in I_k$, and (ii) $N^{\sigma(a)}_{\sigma(b),c} = N^a_{bc}$, where $\sigma([a_1, \dots, a_N]) = [k + a_N, a_1, \dots, a_{N-1}]$. We will also use: (iii) Each orbit of $W^\wedge_k$, acting via $*$ on $\mathfrak{h}^*$, contains at most one point in I_k.

Let us now fix $c = [Nl, 0, \dots, 0]$, and do the classical calculation first.

Lemma 3.18. *Let $c = [Nl, 0, \dots, 0]$. Then $m^a_{ac} = 1$ iff $a_j - a_{j+1} \geqq 1$ for all $j \in \{1, \dots, N-1\}$.*

Proof. The coefficient m^a_{ac} is non-zero if and only if there exist non-negative integers $l_1, \dots, l_N$, summing up to Nl, such that $l_j \leqq a_{j-1} - a_j$ if $j \geqq 2$ and $[a_1 + l_1, \dots, a_N + l_N] = a$. It follows that $l_j = l$ for all j, and this solution obeys the inequality iff $a_{j-1} - a_j \geqq l$ for all $j \geqq 2$. □

Lemma 3.19. *Let $c = [Nl, 0, \dots, 0]$, with $Nl \leqq k$ and suppose $a \in I_k$. Then $N^a_{ac} = 0$ if $a_1 - a_N > k - l$.*

Proof. In this case $\sigma(a) = [k + a_N, a_1, \dots]$, and since $(k + a_N) - a_1 < l$, $m^{\sigma(a)}_{\sigma(a),c} = 0$, by Lemma 3.18. Therefore $N^a_{ac} = 0$, by properties (i), (ii). □

Lemma 3.20. *Let $c = [Nl, 0, \dots, 0]$, with $Nl \leqq k$. Then $N^a_{ac} = 1$ if and only if $a_j - a_{j+1} \geqq l$, $1 \leqq j \leqq N-1$, and $a_1 - a_N \leqq k - l$.*

Proof. We need to prove only the "if" part. We do this by showing that only the first term in the sum (11) is non-zero. Let us suppose that a obeys the hypothesis of the lemma, and that $m^a_{ac} = m^{w*a}_{ac} = 1$, with $w \neq 1$ and derive a contradiction. Since $b = w * a$ is dominant, and is not in I_k by (iii), we have $b_1 - b_N \geqq k + 1$. Let us choose the representative in a with $a_N = 0$, and identify $a_1, \dots, a_{N-1}$ with the row lengths of a Young diagram. Then b is obtained by adding Nl boxes to this Young diagram, in such a way that $a_i \leqq b_i \leqq a_{i-1}$. Then w^{-1} with $w^r - 1 * b = a$ is the unique element mapping b to I_k. This element is constructed as follows: (i) Add, for all $j, N - j$ boxes to the j^{th} row of b (this adds ρ). (ii) Draw a vertical line at distance $k + N$ from the end of the N^{th} row to the right of it; the only boxes to the right of this line are in the first row, and their number is at most $Nl \leqq k$. (iii) Take these boxes and add them to the N^{th} row (i.e., act by s_0, see (12)); permute the rows to get a Young diagram (i.e. act by an element of W). (iv) Subtract $N - j$ boxes from the j^{th} row, $j = 1, \dots, n$.

We obtain in this way a diagram which has Nl boxes more than the original diagram with row lengths a_i and whose first row has $b_N + k + 1 \geqq k + 1$ boxes. The two diagrams are equivalent, meaning that the latter is obtained from the former by adding the same number of boxes to each row. This number is at least $k + 1 - a_1$ which by hypothesis is strictly larger than l. We need thus more than Nl boxes, and this is a contradiction. □

The dimension of the space of conformal blocks can be now computed: note that $a \mapsto a - l\rho$ (i.e. $a_j \mapsto a_j - l(N - j)$) maps bijectively the set of weights obeying the conditions of Lemma 3.20 onto I_{k-lN} whose cardinality coincides with the dimension of the space of invariant theta functions with vanishing condition.

We conclude that the space of invariant theta functions satisfying our vanishing condition coincides with the space of conformal blocks, in accordance with our conjecture.

4. The Knizhnik–Zamolodchikov–Bernard Equations, and Generalized Classical Yang–Baxter Equation

4.1. The KZB Equations. The Knizhnik–Zamolodchikov–Bernard (KZB) equations, first written in [2], for a holomorphic conformal block $\omega \in E(U)$ are the horizontality conditions $\nabla_\omega = 0$, where ω is identified with its image by $\iota_{\mathfrak{h}}$. To write these equations explicitly, let us compute the expression of the connection ∇ on $E(U)$ viewed as a subsheaf of $V^{[n]}(U \times \mathfrak{h})$ via $\iota_{\mathfrak{h}}$.

It is convenient to introduce functions $\rho, \sigma_w, w \in \mathbb{C}$ expressed in terms of the function θ_1:

$$\rho(t,\tau) = \partial_t \log \theta_1(t|\tau)\,,$$

$$\sigma_w(t,\tau) = \frac{\theta_1(w-t|\tau)\partial_t\theta_1(0|\tau)}{\theta_1(w|\tau)\theta_1(t|\tau)}\,.$$

See Appendix A for details on these functions. We use the notation

$$\kappa = k + h^\vee\,,$$

and the abbreviation X_m for $X \otimes t^m$. We also identify $\mathfrak{g}$ as a Lie subalgebra of $L\mathfrak{g}^\wedge$: $X_0 = X \in \mathfrak{g}$. Let $C_\alpha = e_\alpha \otimes e_{-\alpha}$ (see (2.2)). Then we can write L_{-1} as

$$L_{-1} = \frac{1}{\kappa}\sum_{n=0}^{\infty}\left(\sum_{\alpha\in\Delta} e_{\alpha,-n-1}e_{-\alpha,n} + \sum_{\nu} h_{\nu,-n-1}h_{\nu,n}\right)\,.$$

Now let $U \subset C^{[n]}$, and $\omega \in E(U)$, which we identify via $\iota_{\mathfrak{h}}$ with a function on U with values in $V^{[n]}$. We then have, for fixed $u \in V^{[n]}$,

$$\kappa\langle\nabla_{z_j}\omega, u\rangle = \kappa\frac{\partial}{\partial z_j}\langle\omega,u\rangle - \left\langle\omega, \left(\sum_\nu h_{\nu,-1}h_\nu + \sum_\alpha e_{\alpha,-1}e_{-\alpha}\right)u\right\rangle\,.$$

Recall that vectors in V are annihilated by X_n, with $X \in \mathfrak{g}$, $n > 0$. We now use the invariance of ω under the action of $\mathscr{L}$. The functions $t \mapsto e_\alpha\sigma_{\alpha(\lambda)}(t-z_j)$ are elements of $\mathscr{L}(z,\tau,\lambda)$. They have simple poles at $t = z_j$ with residue e_α. As a consequence of the invariance of ω, we have

$$\langle\omega, e^{(j)}_{\alpha,-1}u\rangle = \rho(\alpha(\lambda))\langle\omega, e^{(j)}_\alpha u\rangle - \sum_{k:k\neq j}\sigma_{\alpha(\lambda)}(z_k - z_j)\langle\omega, e^{(k)}_\alpha u\rangle\,,$$

for all $u \in \bigotimes_j V_j$. We can use this identity to compute the value of ω on vectors $e^{(j)}_{-1}u$. The flatness condition $\nabla_{\lambda_\nu}\omega = 0$ translates to

$$\langle\omega, h^{(j)}_{\nu,-1}u\rangle = \frac{\partial}{\partial\lambda_\nu}\langle\omega,u\rangle - \sum_{k:k\neq j}\rho(z_k - z_j)\langle\omega, h^{(k)}_\nu u\rangle\,.$$

To compute further we need the commutation relation $[e_\alpha, e_{-\alpha}] = \sum_\nu \alpha(h_\nu)h_\nu$, that follows from $([e_\alpha, e_{-\alpha}], h_\nu) = (e_\alpha, [e_{-\alpha}, h_\nu])$. We therefore obtain the formula

$$\kappa\langle\nabla_{z_j}\omega, u\rangle = \kappa\frac{\partial}{\partial z_j}\langle\Pi\omega, u\rangle - \sum_\nu \frac{\partial}{\partial\lambda_\nu}\langle\Pi\omega, h_\nu^{(j)}u\rangle - \sum_{k\neq j}\langle\Pi\omega, \Omega^{(k,j)}(z_k - z_j, \tau, \lambda)u\rangle\ , \tag{13}$$

where $\Pi = \Pi(\lambda,\tau)$ is (essentially) the "Weyl–Kac denominator" (for any choice of positive roots Δ_+)

$$q^{\frac{\dim \mathfrak{g}}{24}}\prod_{\alpha\in\Delta_+}(e^{i\pi\alpha(\lambda)} - e^{-\pi i\alpha(\lambda)})\prod_{n=1}^{\infty}\left[(1-q^n)^{\operatorname{rank}\mathfrak{g}}\prod_{\alpha\in\Delta}(1-q^n e^{2\pi i\alpha(\lambda)})\right]\ ,$$

($q = e^{2\pi i\tau}$), and with the abbreviation

$$\Omega(t,\tau,\lambda) = \rho(t)C_0 + \sum_{\alpha\in\Delta}\sigma_{\alpha(\lambda)}(t)C_\alpha\ . \tag{14}$$

We also use the standard notation $\Omega^{(i,j)}$ to denote $\sum_s X_s^{(i)}Y_s^{(j)}$, if $\Omega = \sum_s X_s\otimes Y_s$. This notation will be used below also in the case $i = j$. The λ independent factors in Π do not play a role here, but will provide some simplifications later. In deriving (13), we have used that, by the classical product formula for Jacobi theta functions, Π is, up to a λ independent fact, the product $\prod_{\alpha\in\Delta_+}\theta_1(\alpha(\lambda))$. Before continuing, we can use the formula (13) to complete the proof of Prop. 3.4.

End of the proof of Prop. 3.4. What is left to prove is that $[\nabla_\tau, \nabla_{z_1}]$ on $E(U)$. But from the above formula for ∇_{z_j} it follows that $\Sigma_j\nabla_{z_j}$ vanishes. Indeed we have $\omega\Sigma_j h_\nu^{(j)} = 0$ by $\mathfrak{h}$-invariance, and the other terms cancel by antisymmetry. As ∇_τ preserves conformal blocks, we have $[\nabla_\tau, \Sigma_j\nabla_{z_j}] = 0$, and claim follows from the fact that ∇_τ commutes with ∇_{z_j} with $j\neq 1$. □

A more involved but similar calculation gives a formula for ∇_τ, also essentially due to Bernard, which will be given here without full derivation,

One of the ingredients is Macdonald's (or denominator) identity (see [15])

$$\Pi(\lambda,\tau) = \sum_{q\in Q^\vee} e^{i\pi\tau\frac{1}{2h^\vee}(\rho+h^\vee q, \rho+h^\vee q)}\sum_{w\in W}\varepsilon(w)e^{2\pi i(\rho+h^\vee q, w\lambda)}\ ,$$

implying (one form of) Fegan's heat kernel identity

$$4\pi i h^\vee\partial_\tau\Pi(\lambda,\tau) = \sum_\nu \partial^2_{\lambda_\nu}\Pi(\lambda,\tau)\ .$$

Here, ρ is half the sum of all positive roots of $\mathfrak{g}$, W is the Weyl group, and $\varepsilon(w)$ is the sign of $w\in W$. The (complex) dimension of $\mathfrak{g}$ enters the game through the Freudenthal-de Vries strange formula $(\rho,\rho)/2h^\vee = \dim\mathfrak{g}/24$.

Let us summarize the results. We switch to the more familiar left action notation, by setting $\langle X\omega, v\rangle = -\langle\omega, Xv\rangle$ if X is in a Lie algebra and ω is in the dual space to $\mathfrak{g}$-module. We also need the following special functions of $t \in \mathbb{C}$, expressed in terms of $\sigma_w(t)$, $\rho(t)$ and Weierstrass' elliptic function $\wp$ with periods $1, \tau$:

$$I(t) = \frac{1}{2}(\rho(t)^2 - \wp(t)),$$

$$J_w(t) = \partial_t \sigma_w(t) + (\rho(t) + \rho(w))\sigma_w(t).$$

These functions are regular at $t = 0$. Introduce the tensor

$$H(t, \tau, \lambda) = I(t)C_0 + \sum_{\alpha \in \Delta} J_{\alpha(\lambda)}(t) C_\alpha . \tag{15}$$

Theorem 4.1. *The image* ω *by* $\iota_{\mathfrak{h}}$ *of a horizontal section of* $E(U)$ *obeys the KZB equations*

$$\kappa \partial_{z_j} \tilde{\omega} = -\sum_\nu h_\nu^{(j)} \partial_{\lambda_\nu} \tilde{\omega} + \sum_{l: l \neq j} \Omega^{(j,l)}(z_j - z_l, \tau, \lambda)\tilde{\omega},$$
$$4\pi i \kappa \partial_\tau \tilde{\omega} = \sum_\nu \partial_{\lambda_\nu}^2 \tilde{\omega} + \sum_{j,l} H^{(j,l)}(z_j - z_l, \tau, \lambda)\tilde{\omega},$$

where $\tilde{\omega}(z, \tau, \lambda) = \Pi(\tau, \lambda)\omega(z, \tau, \lambda)$, *and* Ω, H *are the tensors* (14), (15), *respectively.*

Remark. For $n = 1$, these equations reduce to $\partial_{z_1}\tilde{\omega} = 1$, thus $\tilde{\omega}$ is a V^*-valued function of τ and λ only, and

$$4\pi i \kappa \frac{\partial}{\partial \tau}\tilde{\omega} = \sum_\nu \frac{\partial^2}{\partial \lambda_\nu^2}\tilde{\omega} - \eta_1(\tau)\,\mathrm{Cas}(V)\tilde{\omega} - \sum_{\alpha \in \Delta} \wp(\alpha(\lambda)) e_\alpha e_{-\alpha}\tilde{\omega},$$

where $\rho(z) = z^{-1} - \eta_1 z + O(z^2)$, and $\mathrm{Cas}(V)$ is the value of the quadratic Casimir element $C^{(1,1)}$ in the representation V. This equation was considered recently by Etingof and Kirillov [6], who noticed that if $\mathfrak{g} = sl_N$ and V^* is the symmetric tensor product $S^{lN}\mathbb{C}^N$, $e_\alpha e_{-\alpha} = l(l+1)$ Id on the one dimensional weight zero space of V^*, and the equation reduces to the heat equation associated to the elliptic Calogero–Moser–Sutherland–Olshanetsky–Perelomov integrable N-body system:

$$4\pi i \kappa \frac{\partial}{\partial \tau}\tilde{\omega} = \sum_\nu \frac{\partial^2}{\partial \lambda_\nu^2}\tilde{\omega} - \eta_1(\tau) l(l+1)N(N-1)\tilde{\omega} - l(l+1)\sum_{i \neq j} \wp(\lambda_i - \lambda_j)\tilde{\omega}.$$

See also 3.8 for a description of the space of conformal blocks in this case.

4.2. The Classical Yang–Baxter Equation. The tensor $\Omega^{(1,2)} = \Omega^{(1,2)}(z_1 - z_2, \tau, \lambda) \in \mathfrak{g} \otimes \mathfrak{g}$ obeys the "unitarity" condition

$$\Omega^{(1,2)} + \Omega^{(2,1)} = 0.$$

Let us remark that the fact that the connection is flat is then equivalent to the identity

$$\sum_\nu \partial_{\lambda_\nu}\Omega^{(1,2)} h_\nu^{(3)} + \sum_\nu \partial_{\lambda_\nu}\Omega^{(2,3)} h_\nu^{(1)} + \sum_\nu \partial_{\lambda_\nu}\Omega^{(3,1)} h_\nu^{(2)}$$
$$- [\Omega^{(1,2)}, \Omega^{(1,3)}] - [\Omega^{(1,2)}, \Omega^{(2,3)}] - [\Omega^{(1,3)}, \Omega^{(2,3)}] = 0$$

in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. This identity may be thought of as the genus one generalization of the classical Yang–Baxter equation. It admits an interesting "quantization" [11].

Appendix A. Lie Algebras of Meromorphic Functions

We have the following explicit description of $\mathscr{L}(z_1,\dots,z_n,\tau,\lambda)$. Let $\rho,\sigma_w,(w\in\mathbb{C})$ be meromorphic $\mathbb{Z}$-periodic functions on the complex plane, whose poles are simple and belong to $L(\tau)$, and such that

$$\rho(t+\tau)=\rho(t)-2\pi i\,,$$

$$\sigma_w(t+\tau)=e^{2\pi i w}\sigma_w(t)\,,$$

$$\sigma_w(t)\sim\frac{1}{t},\quad t\to 0\,.$$

Such functions exist (for $w\in\mathbb{C}-L(\tau)$) and are unique, if we require that $\rho(-t)=-\rho(t)$. They can be expressed in terms of the Jacobi theta function θ_1:

$$\rho(t)=\frac{\partial}{\partial t}\log\theta_1(t|\tau)\,,$$

$$\sigma_w(t)=\frac{\theta_1(t-w|\tau)\theta_1'(0|\tau)}{\theta_1(t|\tau)\theta_1(-w|\tau)}\,,$$

$$\theta_1(t|\tau)=-\sum_{n=-\infty}^{\infty}e^{2\pi i(t+\frac{1}{2})(n+\frac{1}{2})+\pi i\tau(n+\frac{1}{2})^2}\,.$$

Here a prime denotes a derivative with respect to the first argument.

Proposition A.1. *For $\alpha\in\Delta\cup\{0\}$, $\lambda\in\mathfrak{h}$, and $(z,\tau)\in C^{[n]}$, the meromorphic functions of t (defined as limits at the removable singularities $\alpha(\lambda)\in\mathbb{Z}$)*

$$X(e^{2\pi i\alpha(\lambda)}-1)\sigma_{\alpha(\lambda)}(t-z_1)\,,$$

$$X(\sigma_{\alpha(\lambda)}(t-z_l)-\sigma_{\alpha(\lambda)}(t-z_1)),\quad 2\leqq l\leqq n\,,$$

$$X\frac{\partial^j}{\partial t^j}\sigma_{\alpha(\lambda)}(t-z_l),\quad j\geqq 1,\quad 2\leqq l\leqq n\,,$$

$X\in\mathfrak{g}_\alpha$ or $\mathfrak{h}$ if $\alpha=0$, are well defined provided $|\operatorname{Im}\alpha(\lambda)|<\operatorname{Im}\tau$ and belong to $\mathscr{L}(z_1,\dots,z_n,\lambda,\tau)$. If α runs over $\Delta\cup\{0\}$ and X runs over a basis of $\mathfrak{g}_\alpha$ ($\mathfrak{h}$ if $\alpha=0$), then these functions form a basis of $\mathscr{L}(z,\tau,\lambda)$.

Proof. It is easy to check that these functions belong to $\mathscr{L}(z,\tau,\lambda)$. Let $\mathscr{L}^{\leqq j}(z,\tau,\lambda)$ be given by the functions in $\mathscr{L}(z,\tau,\lambda)$ whose pole orders do not exceed j. By the Riemann–Roch theorem,

$$d(j):=\dim(\mathscr{L}^{\leqq j}(z,\tau,\lambda))=\dim(\mathfrak{g})\,jn\,,$$

if $j\geqq 1$. Indeed $\mathscr{L}^{\leqq j}(z,\tau,\lambda)$ is the space of holomorphic sections of the tensor product of a flat vector bundle on the elliptic curve by the line bundle associated to jD, where D is the positive divisor Σz_i.

The functions given here are linear independent, as can be easily checked by looking at their poles, and have the property that for $j\geqq 1$, the first $d(j)$ functions belong to $\mathscr{L}^{\leqq j}(z,\tau,\lambda)$. □

To obtain a basis outside the strip $|\operatorname{Im}\alpha(\lambda)|<\operatorname{Im}\tau$ we can transport our basis using the following isomorphisms.

Proposition A.2. *Let $(z,\tau) \in C^{[n]}$, and $q, q' \in P^\vee$. Then the map sending $X \in \mathscr{L}(z,\tau,\lambda)$ to the function*

$$t \mapsto \exp(2\pi i t \operatorname{ad} q')X(t),$$

is a Lie algebra isomorphism from $\mathscr{L}(z,\tau,\lambda)$ to $\mathscr{L}(z,\tau,\lambda + q + q'\tau)$.

For any open subset U of $C^{[n]}$, $C^{[n]} \times \mathfrak{h}$ or $C^{[n]} \times G$ define $\mathscr{L}^{\leqq j}(U), \mathscr{L}_{\mathfrak{h}}^{\leqq j}(U)$, $\mathscr{L}_G^{\leqq j}(U)$ to be the space of functions in $\mathscr{L}(U), \mathscr{L}_{\mathfrak{h}}(U), \mathscr{L}_G(U)$, respectively, whose pole orders do not exceed j.

Corollary A.3. *The sheaves $\mathscr{L}^{\leqq j}$, $\mathscr{L}_{\mathfrak{h}}^{\leqq j}$ are locally free, finitely generated for all $j \geqq 1$. Moreover for each $x \in C^{[n]} \times \mathfrak{h}$, every $X \in \mathscr{L}_{\mathfrak{h}}(x)$ extends to a function in $\mathscr{L}_{\mathfrak{h}}^{\leqq j}(U)$ for some j and $U \ni x$.*

The proof in the case of $\mathscr{L}^{\leqq j}$ is obtained by setting simply $\lambda = 0$.

We wish to extend this result to $\mathscr{L}_G$. Let us first notice that the function $\sigma_w(t)$ is actually a meromorphic function of $e^{2\pi i w}$. Thus if $g = \exp(2\pi i\lambda)$, the functions in Prop. A.1 can be written as $f(\operatorname{Ad}(g), t, z, \tau)X$, where the meromorphic function f is regular as a function of the first argument in the range corresponding to $|\operatorname{Im}\alpha(\lambda)| < \operatorname{Im}(\tau)$. Therefore we may extend the definition of the basis to give a basis of $\mathscr{L}_G(z,\tau,g)$ for g in some neighborhood of $g = \exp(2\pi i\lambda)u$, with $\operatorname{Ad}(u)$ unipotent commuting with $\operatorname{Ad}(g)$. (It is clear that the multipliers are correct if g is on some Cartan subalgebra, but such g's form a dense set in G.) The pole structure does not change if the neighborhood is sufficiently small. In this way by choosing properly the Cartan subalgebra, we find local bases of $\mathscr{L}_G$ in the neighborhood of all points in G whose semisimple parts are of the form $\exp(2\pi i\lambda)$ with λ in some Cartan subalgebra and $|\operatorname{Im}\alpha(\lambda)| < \operatorname{Im}(\tau)$, for all $\alpha \in \Delta$.

Proposition A.4. *Let $(z,\tau) \in C^{[n]}$, and $q, q' \in P^\vee$. Then the map sending $X \in \mathscr{L}_G(z,\tau,g)$ to the function*

$$t \mapsto \exp(2\pi i t \operatorname{ad} q')X(t),$$

is a Lie algebra isomorphism from $\mathscr{L}_G(z,\tau,g)$ to $\mathscr{L}_G(z,\tau,\exp(2\pi i(q+\tau q'))g)$.

With the Jordan decomposition theorem, we get a local basis around all points of G, and we obtain:

Proposition A.5. *The sheaf $\mathscr{L}_G^{\leqq j}$, is locally free, finitely generated for all $j \geqq 1$. Moreover for each $x \in C^{[n]} \times G$, every $X \in \mathscr{L}(x)$ extends to a function in $\mathscr{L}_G^{\leqq j}(U)$ for some j and $U \ni x$.*

Appendix B. Connections on Filtered Sheaves

Let S be a complex manifold, and denote by $\mathcal{O}$ the sheaf of germs of holomorphic sections on S. A sheaf of Lie algebras over S is a sheaf of $\mathcal{O}$-modules $\mathscr{L}$ with Lie bracket $\mathscr{L} \otimes_{\mathcal{O}} \mathscr{L} \to \mathscr{L}$ a homomorphism of sheaves of $\mathcal{O}$-modules, obeying anti-symmetry and Jacobi axioms. A sheaf of Lie algebras $\mathscr{L}$ over S is said to be locally free if it is locally free as an $\mathcal{O}$-module, i.e., if every $x \in S$ has a neighborhood U such that, as an $O(U)$-module, $\mathscr{L}(U) \simeq W \otimes \mathcal{O}(U)$ for some complex vector space W. In this case, $\mathscr{L}(U)$ is freely generated over $\mathscr{O}(U)$ by a basis $e_1, e_2, \ldots$ with Lie brackets $[e_i, e_j] = \Sigma f_{ij}^k e_k$, (with finitely many non-zero summands) for some holomorphic functions f_{ij}^k on U.

We will consider the case in which the sheaf $\mathscr{L}$ of Lie algebra is filtered by locally free sheaves of $\mathscr{O}$-modules of finite type. In other words, $\mathscr{L}$ admits a filtration,

$$\mathscr{L}^{\leqq 0} \subset \cdots \subset \mathscr{L}^{\leqq j} \subset \mathscr{L}^{\leqq j+1} \subset \cdots \subset \mathscr{L} = \bigcup_{j=0}^{\infty} \mathscr{L}^{\leqq j},$$

with $\mathscr{L}^{\leqq j}$ locally isomorphic to some $\mathbb{C}^{n_j} \otimes \mathscr{O}$, inclusions induced from inclusions $\mathbb{C}^{n_j} \subset \mathbb{C}^{n_{j+1}}$, and such that $[\mathscr{L}^{\leqq j}, \mathscr{L}^{\leqq l}] \subset \mathscr{L}^{\leqq j+l}$. In particular $\mathscr{L}$ is locally free.

A sheaf of $\mathscr{L}$-modules is a sheaf V of $\mathscr{O}$-modules with an action $\mathscr{L} \bigotimes_{\mathscr{O}} V \to V$ which is assumed to be a homomorphism of $\mathscr{O}$-modules. The image sheaf of this homomorphism is denoted by $\mathscr{L}V$. In the filtered situation it is assumed further that V is filtered by locally free, finitely generated $\mathscr{O}$-modules:

$$V^{\leqq 0} \subset \cdots \subset V^{\leqq j} \subset V^{\leqq j+1} \subset \cdots \subset V = \bigcup_{j=0}^{\infty} V^{\leqq j},$$

and that the action is compatible with the filtration, i.e., $\mathscr{L}^{\leqq j} V^{\leqq l} \subset V^{\leqq j+l}$. In particular V is locally free, and we can define a dual sheaf V^* locally as $V^*(U) = \mathrm{Hom}_{\mathscr{O}(U)}(V(U), \mathscr{O}(U))$. If $V(U)$ is of the form $\bar{V} \otimes \mathscr{O}(U)$ for some vector space $\bar{V}$, then $V^*(U)$ is the space of functions u on U with values in the dual $\bar{V}^*$ such that $\langle u, w \rangle$ is holomorphic for all $w \in \bar{V}$. The dual sheaf V^* has a natural structure of a sheaf of right $\mathscr{L}$-modules and we have a natural pairing $\langle , \rangle : V^* \times V \to \mathscr{O}$.

We can define the associated graded objects

$$\mathrm{Gr}\, \mathscr{L} = \bigoplus_{j=0}^{\infty} \mathscr{L}^{\leqq j} / \mathscr{L}^{\leqq j-1},$$

$$\mathrm{Gr}\, V = \bigoplus_{j=0}^{\infty} V^{\leqq j} / V^{\leqq j-1},$$

with the understanding that $V^{\leqq -1} = 0 = \mathscr{L}^{\leqq -1}$.

Then $\mathrm{Gr}\, \mathscr{L}$ is a graded sheaf of Lie algebras acting on the graded sheaf $\mathrm{Gr}\, V$ of $\mathscr{O}$-modules, and homogeneous components are locally free and finitely generated.

The sheaf of coinvariants is $V/\mathscr{L}V$, and the sheaf of invariant forms E is locally given by

$$U \mapsto E(U) = \{\omega \in V^*(U) | \omega X = 0 \,\forall X \in \mathscr{L}(U)\}.$$

In the filtered situation, $\mathscr{L}V$ is filtered, with $(\mathscr{L}V)^{\leqq j} = \Sigma_{r+s=j} \mathscr{L}_r V_s$ and we have induced homomorphisms

$$(V/\mathscr{L}V)^{\leqq 0} \to \cdots \to (V/\mathscr{L}V)^{\leqq j} \to (V/\mathscr{L}V)^{\leqq j+1} \to \cdots \to V/\mathscr{L}V$$
$$= \varinjlim (V/\mathscr{L}V)^{\leqq j}.$$

Locally, $(V/\mathscr{L}V)^{\leqq j}(U)$ is the quotient $V^{\leqq j}(U)$ by the submodule of linear combinations of elements of the form Xv, $X \in \mathscr{L}^{\leqq r}$, $v \in V^{\leqq s}$ with $r + s \leqq j$.

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[2] i.e., every point of S has a neighborhood such that the statement holds for the restriction of the sheaf to this neighborhood.

A connection ∇ on a sheaf of $\mathcal{O}$-modules V is a $\mathbb{C}$ linear map $V \to \Omega^1 \bigotimes_{\mathcal{O}} V$, where Ω^1 is the sheaf of holomorphic $(1,0)$-differential forms on S, such that for all open sets $U \subset S$,

$$\nabla(fv) = f\nabla v + df \otimes v,$$

for any $f \in \mathcal{O}(U)$, $v \in V(U)$. The notation ∇_ξ is used to denote the covariant derivative in the direction of a local holomorphic vector field ξ : if $\nabla v = \Sigma_i \alpha_i \otimes v_i$, $\nabla_\xi v = \Sigma \alpha_i(\xi) v_i$. A connection D on a sheaf of Lie algebras is furthermore assumed to have covariant derivatives being derivations for all local vector field ξ:

$$D_\xi[X,Y] = [D_\xi X, Y] + [X, D_\xi Y], \quad X, Y \in \mathcal{L}(U),$$

and a connection ∇ on a sheaf of $\mathcal{L}$-modules with connection D is assumed to be compatible with the action, i.e.,

$$\nabla_\xi(Xv) = (D_\xi X)v + X\nabla_\xi v, \quad X \in \mathcal{L}(U), \quad v \in V(U).$$

Such a connection induces a unique connection, also called ∇ on V^* such that for all open $U \subset S$, $u \in V^*(U)$, $v \in V(U)$,

$$d\langle u, v\rangle = \langle \nabla u, v\rangle + \langle u, \nabla v\rangle.$$

Let ∇ be a connection on a sheaf V of $\mathcal{O}$-modules. If V is filtered by free, finitely generated $\mathcal{O}$-modules $V^{\leqq j}$, we say that ∇ is *of finite depth* if there exists an integer d such that $\nabla V^{\leqq j} \subset \Omega^1 \otimes V^{\leqq j+d}$. The smallest non-negative such integer will be called depth of the connection.

Theorem B.1. *Let $\mathcal{L}$ be a sheaf of Lie algebras and V a sheaf of $\mathcal{L}$-modules over a complex manifold S. Suppose that $\mathcal{L}$ and V have a filtration by locally free finitely generated $\mathcal{O}$-modules, and compatible connections D and ∇ of finite depth. If* $\mathrm{Gr}\, V/\mathrm{Gr}\, \mathcal{L}\mathrm{Gr}\, V$ *has only finitely many non-zero homogeneous summands, then the sheaf of invariant forms E is locally free and finitely generated.*

Proof. Let $z_0 \in S$ and U be a neighborhood of z_0, such that the restriction of V to U is free. Thus there exist vector spaces $\bar{V}^{\leqq j}, \bar{V}$, such that

$$V^{\leqq j}(U) \simeq \bar{V}^{\leqq j} \otimes \mathcal{O}(U) \qquad V(U) \simeq \bar{V} \otimes \mathcal{O}(U).$$

The assumption that $\mathrm{Gr}\, V/\mathrm{Gr}\, \mathcal{L}\mathrm{Gr}\, V$ has vanishing components of degree $\geqq N$ means that if $j \geqq N$ and $v \in V^{\leqq j}(U)$ we have a decomposition (not necessarily unique)

$$v = v' + Xv'', \tag{16}$$

for some $v' \in V^{\leqq j-1}$ and $X \in \mathcal{L}(U)$. By iterating this we see that we can take $v' \in V^{\leqq N}$.

The first consequence of this is that the restriction map $E(U) \to V^{\leqq l*}(U)$ is injective for all sufficiently large l.

The second consequence is that we can replace the connection by a connection which preserves $V^{\leqq l*}(U)$ for some large l, and coincides with the given one on the image of invariant forms. The construction goes as follows.

Let us choose a basis $e_1, e_2, \ldots$ of $\bar{V}$ with the property that, for all j, a basis of $\bar{V}^{\leqq j}$ is obtained by taking the first $\dim(\bar{V}^{\leqq j})$ elements of this sequence. View $\bar{V}$ as the subspace of constant functions in $V(U)$, and choose a decomposition (16) for all e_i:

$$e_i = e_i' + X_i e_i'' ,$$

with $e_i' \in V^{\leqq N}(U)$. Define a new connection $\tilde{\nabla}$ by

$$\tilde{\nabla} e_i = \nabla e_i' .$$

This formula uniquely determines a connection $\tilde{\nabla}$ on the restriction of E to U. The dual connection on $V^*(U)$, also denoted $\tilde{\nabla}$ is defined as usual by $\langle \tilde{\nabla}\alpha, e_i\rangle = d\langle \alpha, e_i\rangle - \langle \alpha, \tilde{\nabla} e_i\rangle$. By construction, this dual connection coincides with ∇ on invariant forms, and, if d denotes the depth of the connection ∇, it maps $V^{\leqq N+d*}(U)$ to itself.

If we introduce local coordinates $t_1, \dots, t_n$ around z_0, with z_0 at the origin, we see that we have to solve the following problem: given a subsheaf E of a finitely generated free sheaf F on an open neighborhood U of the origin in $\mathbb{C}^n$, with connection $\tilde{\nabla}$ on F preserving E, show that there exists an open set $U' \subset U$ containing z_0, such that $E(U')$ is a free $\mathcal{O}(U')$-module. Write F as $F_0 \otimes \mathcal{O}$, for a vector space F_0. We may assume that U is a ball centered at the origin.

Lemma B.2. *Let ξ be the vector field $\sum_i t_i \partial_{t_i}$ on $\mathbb{C}^n$, and $\tilde{\nabla}$ be a connection on a free, finitely generated sheaf of $\mathcal{O}$-modules $F = F_0 \otimes \mathcal{O}$ on a ball U centered at the origin of $\mathbb{C}^n$. For each $\phi \in F_0$ there is a unique $\hat{\phi} \in F(U)$ such that $\hat{\phi}(0) = \phi$, and $\nabla_\xi \phi = 0$.*

The proof is more or less standard: the F_0-valued holomorphic function $\hat{\phi}$ on U is a solution of the system of linear differential equations

$$\sum_{i=1}^{n} t_i \frac{\partial}{\partial t_i} \hat{\phi}(t) = \sum_{i=1}^{n} t_i A_i(t) \hat{\phi}(t) .$$

for some holomorphic matrix-valued functions A_i, with initial condition $\hat{\phi}(0) = \phi$. It is convenient to rewrite this equation in the form

$$\frac{d}{dx} \hat{\phi}(xt) = B(x,t)\hat{\phi}(xt), \quad B(x,t) = \sum t_i A_i(xt) ,$$

In this form we can apply the standard existence and uniqueness theorem: the unique solution with initial condition ϕ is given by the absolutely convergent Dyson series

$$\hat{\phi}(t) = \phi + \sum_{m=1}^{\infty} \int_{\Delta_m} B(x_1,t) \cdots B(x_m,t)\phi dx_1 \cdots dx_m .$$

The domain Δ_m of integration is the simplex $0 < x_1 < \cdots < x_m < 1$. It is clear from this formula that $\hat{\phi}$ is holomorphic on U. This concludes the proof of the lemma.

Let E_0 be the subspace of F_0 consisting of all values at 0 of sections of $E(U')$, where U' runs over all open balls contained in U and centered at the origin.

Let $e_1, \dots, e_r$ be a basis of F_0 such that the first $s e_i$ build a basis of E_0. The homomorphism of $\mathcal{O}(U')$-modules

$$\tau : E_0 \otimes \mathcal{O}(U') \to F(U'),$$
$$\phi \otimes h \mapsto \hat{\phi} h,$$

is injective since $\hat{\phi}$ vanishes if and only if ϕ vanishes. We claim that the image of τ is precisely $E(U')$, if U' is small enough. Let $\psi \in E(U')$. Then, we can write ψ as

$$\psi(t) = \sum_{j=1}^{r} a_j(t) \hat{e}_j(t),$$

for some holomorphic functions $a_j(t)$. By assumption, $a_j(0) = 0$ if $j > s$. Since $\tilde{\nabla}_\xi \hat{e}_i = 0$, we have

$$\tilde{\nabla}_\xi \psi(t) = \sum_{j=1}^{r} \sum_{i=1}^{n} t_i \partial_{t_i} a_j(t) \hat{e}_j(t).$$

But $\tilde{\nabla}_\xi$ preserves E and, therefore, $\Sigma_i t_i \partial_{t_i} a_j(t) = 0$ if $j > s$. It follows that $a_j(t) = a_j(0) = 0$ if $j > s$. We have shown that $E(U')$ is contained in the image of the homomorphism τ. Now let, for $j = 1, \dots, s, \psi_j(t)$ be sections of $E(U')$ such that $\psi_j(0) = e_j$. Such sections exist, by definition of E_0, for some neighborhood U'. Then, the construction above gives

$$\psi_j(t) = \sum_{l=1}^{s} a_{jl}(t) \hat{e}_l(t).$$

The holomorphic matrix-valued function $(a_{ij}(t))$ is the unit matrix at $t = 0$ and is thus invertible for $t \in U'$, if the ball U' is small enough. We conclude that $\hat{e}_j \in E(U')$, which completes the proof. $\square$

Let us see how this applies to our situation, following [18]. For us S is either of $C^{[n]}, C^{[n]} \times \mathfrak{h}, C^{[n]} \times G$, and $\mathscr{L}$ is the corresponding sheaf of Lie algebras, which we denoted $\mathscr{L}, \mathscr{L}_\mathfrak{h}, \mathscr{L}_G$, respectively. The module V is the free graded $\mathcal{O}$-module $V^{\wedge[n]} \otimes \mathcal{O}$. The key observation is that $\mathrm{Gr}(\mathscr{L})_j$ consists of the degree j part of $(\mathfrak{g} \otimes \mathbb{C}[t^{-1}])^n \otimes \mathcal{O}$ for all sufficiently large j. Moreover $\mathrm{Gr}(V) = V$ canonically since V is graded, and the action of elements of sufficiently high degrees in $\mathrm{Gr}(\mathscr{L})$ on $\mathrm{Gr}(V)$ comes from the action of $\mathfrak{g} \otimes \mathbb{C}[t^{-1}]$ on the factors $V_j^\wedge$.

The fact that $\mathrm{Gr}\, V / \mathrm{Gr}\, \mathscr{L}\, \mathrm{Gr}\, V$ has only finitely many non-trivial homogeneous components follows then from the fact that $V^\wedge / t^{-N} \mathfrak{g} \otimes \mathbb{C}[t^{-1}]$ is finite dimensional for all positive integers N, which is proved in [18] using Gabber's theorem.

Note Added in Proof. We recently realized that in fact the conjecture we stated in the Introduction follows from the work of Etingof, Frenkel and Kirillov [7], and the Tsuchyia–Ueno–Yamada factorization theorem [18]. Indeed, the authors of [7] show that the space of Weyl invariant theta functions obeying the vanishing condition of Theorems 3.7, 3.8 can be mapped isomorphically to a suitable direct sum of spaces of conformal blocks on the sphere, which by [18] has the dimension of the space of conformal blocks on elliptic curves. It thus follows that our inclusion ι_h of conformal blocks into Weyl-invariant theta functions with vanishing condition is actually an isomorphism, as conjectured.

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