Published for SISSA by Springer

Received: January 20, 2015
Accepted: February 7, 2015
Published: February 24, 2015

Conformal correlators of mixed-symmetry tensors

Miguel S. Costa[a,b] **and Tobias Hansen**[c]

[a] *Centro de Física do Porto, Departamento de Física e Astronomia, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169–007 Porto, Portugal*

[b] *Theory Division, Department of Physics, CERN, CH-1211 Genève 23, Switzerland*

[c] *II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany*

E-mail: miguelc@fc.up.pt, tobias.hansen@desy.de

Abstract: We generalize the embedding formalism for conformal field theories to the case of general operators with mixed symmetry. The index-free notation encoding symmetric tensors as polynomials in an auxiliary polarization vector is extended to mixed-symmetry tensors by introducing a new commuting or anticommuting polarization vector for each row or column in the Young diagram that describes the index symmetries of the tensor. We determine the tensor structures that are allowed in n-point conformal correlation functions and give an algorithm for counting them in terms of tensor product coefficients. A simple derivation of the unitarity bound for arbitrary mixed-symmetry tensors is obtained by considering the conservation condition in embedding space. We show, with an example, how the new formalism can be used to compute conformal blocks of arbitrary external fields for the exchange of any conformal primary and its descendants. The matching between the number of tensor structures in conformal field theory correlators of operators in d dimensions and massive scattering amplitudes in $d+1$ dimensions is also seen to carry over to mixed-symmetry tensors.

Keywords: Conformal and W Symmetry, Higher Spin Symmetry, Field Theories in Higher Dimensions

ArXiv ePrint: 1411.7351

doi:10.1007/JHEP02(2015)151

JHEP02(2015)151

Contents

JHEP02(2015)151

1 Introduction

The study of Conformal Field Theories (CFTs) is among the most important subjects in theoretical physics, with implications to critical phenomena, particle physics and, in the light of the AdS/CFT duality, quantum gravity. In past years we have witnessed a revival in the study of CFTs in dimensions higher than two. This study is considerably more difficult than the two-dimensional case, where the conformal group possesses an infinitely dimensional extension given by the Virasoro algebra, which leads to many known exactly solvable models. On one hand, the *conformal bootstrap* program [1, 2] applied to higher dimensional CFTs, revived in [3], has already shown its merits by providing the most accurate computation to date of 3D Ising model critical exponents [4–6]. On the other hand, the most studied case of the AdS/CFT duality [7] considers $\mathcal{N} = 4$ Super Yang-Mills, which is a four-dimensional CFT. In particular, this theory is believed to be integrable in the planar limit [8, 9], thus providing the first example of an exactly solvable 4D gauge theory.

To advance in the *conformal bootstrap* program, as well as our understanding of AdS/CFT, it is necessary to further develop analytic and computational techniques to deal with arbitrary tensor primary fields. In d dimensions, these fields are classified by the unitary irreducible representations of the conformal group $\mathrm{SO}(d+1,1)$, which are labeled by the conformal dimension Δ and by an irreducible representation (irrep) of $\mathrm{SO}(d)$. A first step in this direction was made in [10], where symmetric tensors of arbitrary spin were studied in detail. The goal of this paper is to extend this work by considering $\mathrm{SO}(d)$ tensors with mixed symmetry.

We shall start, in section 2, with the general classification of irreducible tensor representations of $\mathrm{SO}(d)$, which can be represented by Young diagrams. This is a well known subject, which we shall review in order to introduce the reader to the necessary formalism. We will then see how to encode, in general, mixed-symmetry tensors in terms of polynomials of polarization vectors. To encode their mixed symmetry it is necessary to employ a combination of Grassmann valued and ordinary commuting polarizations. Actual computations simplify considerably if fields that live in d-dimensional Euclidean space are embedded in an auxiliary $(d+2)$-dimensional Minkowski space, where the conformal group $\mathrm{SO}(d+1,1)$ acts linearly as the usual Lorentz transformations. We shall see that this formalism can be easily extended to include mixed-symmetry tensors by encoding them in polynomials of polarization vectors in the embedding space.

In section 3 we show how to construct CFT n-point correlation functions of arbitrary tensors. The formalism is presented in general terms, but we shall give a number of simple examples for two-, three- and four-point functions, so the reader can appreciate the simplicity and efficiency of the method. We will also describe the general case of n-point functions.

In section 4 we consider arbitrary conserved tensors. We see how to implement the conservation equation in the embedding formalism, and also how to derive the unitary bound for conserved tensors in arbitrary irreducible $\mathrm{SO}(d)$ representations.

As an application of the new formalism we consider, in section 5, the problem of computing conformal blocks for any desired external primary fields, describing the exchange

JHEP02(2015)151

of an arbitrary conformal primary and its descendants. With the help of shadow operators, these conformal blocks can be written as an integral of three-point functions, leading to an expression of the conformal blocks in terms of a finite number of integrals, which can be expressed in terms of hypergeometric functions for even dimensions [11]. To see the method at work, we shall consider explicitly the example of the four-point function of two scalars and two vectors, exchanging a mixed-symmetry tensor of rank three.

In section 6 we show that the number of tensor structures in CFT correlators of non-conserved mixed-symmetry tensors in d dimensions matches that of massive scattering amplitudes in $d+1$ dimensions, as expected. Section 7 presents final comments.

2 Mixed-symmetry tensors

2.1 Parametrizing Young diagrams

In this paper the traceless irreducible tensor representations of SO(d) are considered. These representations are enumerated mostly[1] by Young diagrams, which encode the (anti-) symmetry of the tensors under permutation of their indices.

There are two different ways to parametrize the shape of a Young diagram λ. The first is by giving a partition $l^\lambda = \left(l_1^\lambda, l_2^\lambda, \ldots\right)$ containing the lengths of the rows, l_i^λ being the length of the i-th row. The diagram that is obtained from λ by exchanging rows and columns is called the transpose λ^t. The partition h^λ describes the column heights of λ and is the conjugate partition to l^λ, $l^{\lambda^t} \equiv h^\lambda = \left(h_1^\lambda, h_2^\lambda, \ldots\right)$. A second way to describe the shape of a Young diagram is by its Dynkin label $\lambda = \left[\lambda_1, \lambda_2, \ldots, \lambda_{h_1^\lambda}\right]$, which lists the numbers λ_i of columns with i boxes. Apart from the exception mentioned in the footnote, the Young diagram λ labels an irrep of SO(d) if and only if its overall height h_1^λ does not exceed the rank of the Lie algebra corresponding to SO(d),

$$h_1^\lambda \leq \left\lfloor \frac{d}{2} \right\rfloor = \begin{cases} \frac{d}{2}\,, & d \text{ even}\,, \\ \frac{d-1}{2}\,, & d \text{ odd}\,. \end{cases} \tag{2.1}$$

The total number of boxes is denoted by $|\lambda|$,

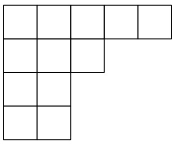
$$|\lambda| = \sum_i i\lambda_i = \sum_i l_i^\lambda = \sum_i h_i^\lambda\,. \tag{2.2}$$

It will be useful to label the number of rows with more than one box n_Z^λ and the number of columns with more than one box n_Θ^λ,

$$n_Z^\lambda = \sum_{i=2}^{l_1^\lambda} \lambda_i^t\,, \qquad n_\Theta^\lambda = \sum_{i=2}^{h_1^\lambda} \lambda_i\,. \tag{2.3}$$

[1] There is a one-to-one correspondence between traceless irreducible tensor representations of SO(d) and the Young diagrams satisfying (2.1) except for the case $d = 2n, h_1^\lambda = n$ [12]. In this case the representation with the symmetry corresponding to λ can be decomposed further using the Levi-Civita tensor and is therefore not irreducible. A well-known example is the decomposition of the two-form in four dimensions into self-dual and anti-self-dual parts.

All of this is best illustrated by the following example:



$$\lambda = [2,1,0,2]\,, \qquad |\lambda| = 12\,,$$
$$l^\lambda = (5,3,2,2)\,, \qquad n_Z^\lambda = 4\,,$$
$$h^\lambda = (4,4,2,1,1)\,, \qquad n_\Theta^\lambda = 3\,. \tag{2.4}$$

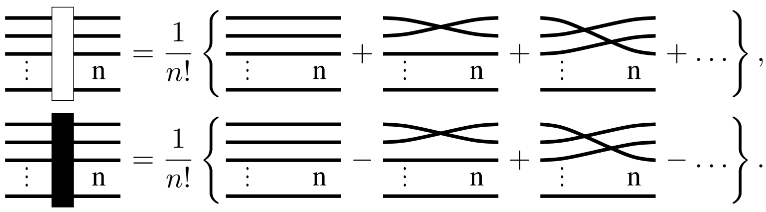
The λ on l^λ, h^λ, n_Z^λ and n_Θ^λ will frequently be omitted or replaced by i if the Young diagram is of shape λ_i.

2.2 Birdtracks and Grassmann variables

Probably the best way to think about mixed-symmetry tensors is in terms of birdtrack notation[2] where index contractions are simply drawn as lines

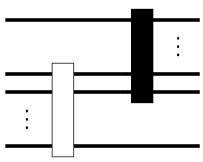
$$= \delta^{a_1 b_1} \delta^{a_2 b_2}\,. \tag{2.5}$$

Symmetrization and antisymmetrization are indicated by the symbols



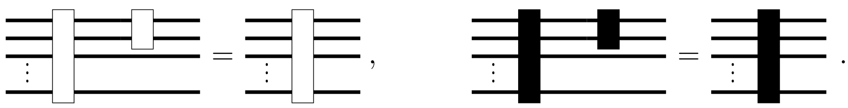
(2.6)

This notation has the advantage that it makes it immediately visible when terms are vanishing because two or more symmetric indices are antisymmetrized or vice versa



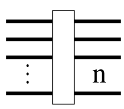
$$= 0\,. \tag{2.7}$$

Furthermore, birdtracks can be diagrammatically transformed, for example using that repeated (anti)symmetrizations of subsets of indices have no effect



(2.8)

A symmetrized contraction of n indices is generated by the n-th derivative of n components of an auxiliary vector z,

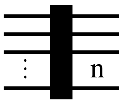


$$= \frac{1}{n!}\, \partial_{z_{a_1}} \ldots \partial_{z_{a_n}} z_{b_1} \ldots z_{b_n}\,. \tag{2.9}$$

[2]See [13] for a beautiful group theory book entirely in terms of birdtracks.

JHEP02(2015)151

Antisymmetrization works analogously with an auxiliary vector in Grassmann variables θ,



$$= \frac{1}{n!} \partial_{\theta_{a_1}} \dots \partial_{\theta_{a_n}} \theta_{b_1} \dots \theta_{b_n} \,. \tag{2.10}$$

The Grassmann variables are anticommuting in the sense that

$$\theta_a^{(p)} \theta_b^{(q)} = (-1)^{\delta^{pq}} \theta_b^{(q)} \theta_a^{(p)} \,. \tag{2.11}$$

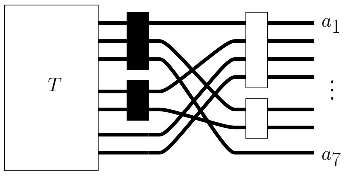
Here an additional label (p) was introduced to allow for several independent antisymmetrizations at the same time. Derivatives with respect to Grassmann variables are implied to be right derivatives,

$$\partial_{\theta_c^{(r)}} \theta_a^{(p)} \theta_b^{(q)} = \delta^{rq} \delta^{cb} \theta_a^{(p)} + (-1)^{\delta^{pq}} \delta^{rp} \delta^{ca} \theta_b^{(q)} \,. \tag{2.12}$$

2.3 Young symmetrization and antisymmetric basis

The symmetry of a Young diagram is imposed on a tensor via Young symmetrizers. Each row of the diagram corresponds to a symmetrization and each column corresponds to an antisymmetrization. This can be nicely illustrated by an example, following [13]. To actually write down components of mixed-symmetry tensors it is necessary to choose a basis for the irreducible representation at hand. This requires an assignment between the boxes of the Young diagram and the indices of the tensor. Therefore the bases of the irreps under consideration are labeled by Young tableaux. A symmetrizer given by the Young tableau $\mathbf{YT}$ creates the tensor $T^{\mathbf{YT}}$ with appropriate symmetry from a generic tensor T,

$$\mathbf{YT} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad \to \qquad T^{\mathbf{YT}} = \tag{2.13}$$

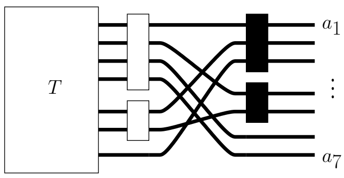


This tensor has the manifest symmetry properties

$$T^{\mathbf{YT}}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7} = T^{\mathbf{YT}}_{(a_1 a_2 a_3 a_4)(a_5 a_6) a_7} \,, \tag{2.14}$$

but there are also less obvious symmetries caused by the antisymmetrizations. Due to the manifest symmetries, $T^{\mathbf{YT}}$ is said to belong to the symmetric basis. The antisymmetric basis is obtained by changing the order of symmetrization and antisymmetrization

$$\mathbf{YT}' = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \qquad \to \qquad T^{\mathbf{YT}'} = \tag{2.15}$$



Here we have manifest antisymmetry

$$T^{\mathbf{YT}'}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7} = T^{\mathbf{YT}'}_{[a_1 a_2 a_3][a_4 a_5](a_6 a_7)} \,. \tag{2.16}$$

The only reason we used a different Young tableau for this second example is to spare us from having to cross lines on the right hand side of the birdtrack diagram. We will in this paper work only in antisymmetric bases with Young tableaux where the boxes are enumerated column by column, as in (2.15). The tensors corresponding to different bases (different tableaux) can be obtained simply by commutation of indices.

It may also be instructive to see how the non-explicit index symmetries manifest themselves on the components of the tensors, again in the antisymmetric basis with boxes labeled column by column. To this end assign different labels to each anticommuting group of indices

$$f_{a_1 \dots a_{h_1} b_1 \dots b_{h_2} c_1 \dots c_{h_3} \dots g_1 \dots g_{h_{l_1}}} = f_{[a_1 \dots a_{h_1}][b_1 \dots b_{h_2}][c_1 \dots c_{h_3}] \dots [g_1 \dots g_{h_{l_1}}]} \,. \tag{2.17}$$

Apart from the antisymmetry, the Young symmetrization implies that the antisymmetrization of any of the indices b with all the a vanishes, as well as the antisymmetrization of any of the c with all indices a or all b and so forth [14]. Explicitly this means that

$$\begin{aligned} & f_{[a_1 \dots a_{h_1}][b_1 \dots b_{h_2}][c_1 \dots c_{h_3}] \dots [g_1 \dots g_{h_{l_1}}]} \\ & \quad = f_{[b_1 a_2 \dots a_{h_1}][a_1 b_2 \dots b_{h_2}][c_1 \dots} + f_{[a_1 b_1 a_3 \dots a_{h_1}][a_2 b_2 \dots b_{h_2}][c_1 \dots} + \dots + f_{[a_1 \dots a_{h_1 - 1} b_1][a_{h_1} b_2 \dots b_{h_2}][c_1 \dots} \\ & \quad = f_{[c_1 a_2 \dots a_{h_1}][b_1 \dots b_{h_2}][a_1 c_2 \dots} + f_{[a_1 c_1 a_3 \dots a_{h_1}][b_1 \dots b_{h_2}][a_2 c_2 \dots} + \dots + f_{[a_1 \dots a_{h_1 - 1} c_1][b_1 \dots b_{h_2}][a_{h_1} c_2 \dots} \,. \end{aligned} \tag{2.18}$$

There are also more general relations that arise from exchanging k indices from one column with all possible k-element subsets of a column to its left. Here the order of the two sets of indices is kept, so that the right hand side of the general equation has $\binom{h_l}{k}$ terms if the left column has height h_l. As a special case of these relations the tensors are symmetric under exchange of complete groups of antisymmetric indices if the corresponding columns in the Young tableau are of equal height, e.g. for $h_2 = h_3$,

$$f_{[a_1 \dots a_{h_1}][b_1 \dots b_{h_2}][c_1 \dots c_{h_3}] \dots [g_1 \dots g_{h_{l_1}}]} = f_{[a_1 \dots a_{h_1}][c_1 \dots c_{h_3}][b_1 \dots b_{h_2}] \dots [g_1 \dots g_{h_{l_1}}]} \,. \tag{2.19}$$

Since it will be needed in section 4 we also state the equation analogous to (2.18) for a tensor

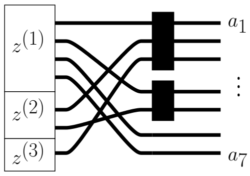
$$f_{a_1 \dots a_{l_1} b_1 \dots b_{l_2} c_1 \dots c_{l_3} \dots g_1 \dots g_{l_{h_1}}} = f_{(a_1 \dots a_{l_1})(b_1 \dots b_{l_2})(c_1 \dots c_{l_3}) \dots (g_1 \dots g_{l_{h_1}})} \,, \tag{2.20}$$

in the symmetric basis with boxes enumerated row by row as in (2.13),

$$\begin{aligned} & - f_{(a_1 \dots a_{l_1})(b_1 \dots b_{l_2})(c_1 \dots c_{l_3}) \dots (g_1 \dots g_{l_{h_1}})} \\ & \quad = f_{(b_1 a_2 \dots a_{l_1})(a_1 b_2 \dots b_{l_2})(c_1 \dots} + f_{(a_1 b_1 a_3 \dots a_{l_1})(a_2 b_2 \dots b_{l_2})(c_1 \dots} + \dots + f_{(a_1 \dots a_{l_1 - 1} b_1)(a_{l_1} b_2 \dots b_{l_2})(c_1 \dots} \\ & \quad = f_{(c_1 a_2 \dots a_{l_1})(b_1 \dots b_{l_2})(a_1 c_2 \dots} + f_{(a_1 c_1 a_3 \dots a_{l_1})(b_1 \dots b_{l_2})(a_2 c_2 \dots} + \dots + f_{(a_1 \dots a_{l_1 - 1} c_1)(b_1 \dots b_{l_2})(a_{l_1} c_2 \dots} \,. \end{aligned} \tag{2.21}$$

2.4 Encoding mixed-symmetry tensors by polynomials

In general, to encode a mixed-symmetry tensor by a polynomial, the strategy is to contract it with a tensor with the same mixed symmetry, which is built out of auxiliary polarizations. To construct a Young symmetrized tensor in the antisymmetric basis out of auxiliary vectors, one can start with a set of polarizations that is already symmetrized so that only the antisymmetrization is left to do. For the example (2.15), the following tensor depending on the auxiliary vectors $z^{(1)}$, $z^{(2)}$ and $z^{(3)}$ is appropriately symmetrized



. (2.22)

Using (2.10) to encode the antisymmetrization, (2.22) can be written as

$$\frac{1}{3!2!}\left(z^{(1)}\cdot\partial_{\theta^{(1)}}\right)\left(z^{(1)}\cdot\partial_{\theta^{(2)}}\right)\left(z^{(1)}\cdot\partial_{\theta^{(3)}}\right)\left(z^{(1)}\cdot\partial_{\theta^{(4)}}\right)$$
$$\left(z^{(2)}\cdot\partial_{\theta^{(1)}}\right)\left(z^{(2)}\cdot\partial_{\theta^{(2)}}\right)\left(z^{(3)}\cdot\partial_{\theta^{(1)}}\right)\theta^{(1)}_{a_1}\theta^{(1)}_{a_2}\theta^{(1)}_{a_3}\theta^{(2)}_{a_4}\theta^{(2)}_{a_5}\theta^{(3)}_{a_6}\theta^{(4)}_{a_7}\,. \tag{2.23}$$

This can be shortened by avoiding the introduction of polarizations that appear only once and hence do not cause any (anti-)symmetrization, i.e. doing explicitly the derivatives in the polarizations $\theta^{(3)}$ and $\theta^{(4)}$,

$$\left(z^{(1)}\cdot\partial_{\theta^{(3)}}\right)\left(z^{(1)}\cdot\partial_{\theta^{(4)}}\right)\theta^{(3)}_{a_6}\theta^{(4)}_{a_7}=z^{(1)}_{a_6}z^{(1)}_{a_7}\,. \tag{2.24}$$

After this step the symmetry in the indices a_6 and a_7 is manifest. Likewise, $z^{(3)}$ that appears only once in this example through the derivative $\left(z^{(3)}\cdot\partial_{\theta^{(1)}}\right)$, does not encode any symmetry. More generally, for diagrams with more than one row of length one, the action of such derivatives hides antisymmetry. We shall therefore omit these derivative terms, with the result that the encoding polynomial will depend not only on symmetric polarizations, but also on $\theta^{(1)}$, therefore making antisymmetrization explicit on the indices corresponding to all rows of length one.

Thus, the slightly less elegant, but more pragmatic Young symmetric polarization we use for the example at hand will be the polynomial in $\mathbf{z}\equiv\left(z^{(1)},z^{(2)},\theta^{(1)}\right)$ given by

$$\left(z^{(1)}\cdot\partial_{\theta^{(1)}}\right)\left(z^{(1)}\cdot\partial_{\theta^{(2)}}\right)\left(z^{(2)}\cdot\partial_{\theta^{(1)}}\right)\left(z^{(2)}\cdot\partial_{\theta^{(2)}}\right)\theta^{(1)}_{a_1}\theta^{(1)}_{a_2}\theta^{(1)}_{a_3}\theta^{(2)}_{a_4}\theta^{(2)}_{a_5}z^{(1)}_{a_6}z^{(1)}_{a_7}\,, \tag{2.25}$$

which is quartic in $z^{(1)}$, quadratic in $z^{(2)}$ and linear in $\theta^{(1)}$, as appropriate for a Young diagram with lengths of rows given by $l^\lambda=(4,2,1)$. This Young symmetric polarization is obtained by acting with derivatives of the type $\left(z^{(p)}\cdot\partial_{\theta^{(q)}}\right)$ on a polynomial in $\boldsymbol{\theta}\equiv\left(\theta^{(1)},\theta^{(2)},z^{(1)}\right)$, cubic in $\theta^{(1)}$, quadratic in $\theta^{(2)}$ and quadratic in $z^{(1)}$, as appropriate for a Young diagram with lengths of columns given by $h^\lambda=(3,2,1,1)$. A tensor with components $f^{a_1\ldots a_7}$ in the irrep of this example will then be encoded by the polynomial

$$f(\mathbf{z})\equiv\left(z^{(1)}\cdot\partial_{\theta^{(1)}}\right)\left(z^{(1)}\cdot\partial_{\theta^{(2)}}\right)\left(z^{(2)}\cdot\partial_{\theta^{(1)}}\right)\left(z^{(2)}\cdot\partial_{\theta^{(2)}}\right)\bar{f}(\boldsymbol{\theta})\,, \tag{2.26}$$

where

$$\bar{f}(\boldsymbol{\theta})\equiv\theta^{(1)}_{a_1}\theta^{(1)}_{a_2}\theta^{(1)}_{a_3}\theta^{(2)}_{a_4}\theta^{(2)}_{a_5}z^{(1)}_{a_6}z^{(1)}_{a_7}f^{a_1\ldots a_7}\,. \tag{2.27}$$

Notice that, in this example, the assignment of the polarization vectors in $\mathbf{z}$ and in $\boldsymbol{\theta}$ to the boxes of the Young diagram is done according to

$$\begin{array}{|c|c|c|c|}\hline z^{(1)} & z^{(1)} & z^{(1)} & z^{(1)} \\ \hline z^{(2)} & z^{(2)} \\ \cline{1-2} \theta^{(1)} \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|c|}\hline \theta^{(1)} & \theta^{(2)} & z^{(1)} & z^{(1)} \\ \hline \theta^{(1)} & \theta^{(2)} \\ \cline{1-2} \theta^{(1)} \\ \cline{1-1}\end{array} \quad , \tag{2.28}$$

respectively.

In general we shall consider n_Θ anticommuting and n_Z commuting polarization vectors for a given tensor operator. A convenient notation for the mostly anticommuting polarizations which are first contracted to the tensor is

$$\boldsymbol{\theta} \equiv \left(\theta^{(1)}, \theta^{(2)} \ldots, \theta^{(n_\Theta)}, z^{(1)}\right). \tag{2.29}$$

In cases where there are no columns with one box the last entry is absent and $\boldsymbol{\theta}$ contains only anti-commuting polarizations. Similarly, we will write for the mostly commuting polarizations on which the final encoding polynomial depends

$$\mathbf{z} \equiv \left(z^{(1)}, z^{(2)}, \ldots, z^{(n_Z)}, \theta^{(1)}\right). \tag{2.30}$$

Again, in cases where there are no rows with one box the last entry is absent and $\mathbf{z}$ contains only commuting polarizations. Generalizing the previous example, we have that a tensor $f^{a_1 \ldots a_{|\lambda|}}$ in the irrep λ is encoded by the polynomial

$$f(\mathbf{z}) \equiv \prod_{p=1}^{n_Z} \prod_{q=1}^{\min(l_p, n_\Theta)} \left(z^{(p)} \cdot \partial_{\theta^{(q)}}\right) \bar{f}(\boldsymbol{\theta}), \tag{2.31}$$

where

$$\begin{aligned} \bar{f}(\boldsymbol{\theta}) \equiv\; & \theta^{(1)}_{a_1} \ldots \theta^{(1)}_{a_{h_1}} \theta^{(2)}_{a_{h_1+1}} \ldots \theta^{(2)}_{a_{h_1+h_2}} \\ & \ldots \theta^{(n_\Theta)}_{a_{h_1+\ldots+h_{n_\Theta-1}+1}} \ldots \theta^{(n_\Theta)}_{a_{h_1+\ldots+h_{n_\Theta}}} z^{(1)}_{a_{|\lambda|-\lambda_1+1}} \ldots z^{(1)}_{a_{|\lambda|}} f^{a_1 \ldots a_{|\lambda|}}. \end{aligned} \tag{2.32}$$

When there are more than one row with one box, the dependence of $f(\mathbf{z})$ on $\theta^{(1)}$ makes manifest the antisymmetry of the indices corresponding to such boxes. Likewise, when there are more than one column with one box, the dependence of $\bar{f}(\boldsymbol{\theta})$ on $z^{(1)}$ makes manifest the symmetry of the indices corresponding to such boxes.

The condition that $f^{a_1 \ldots a_{|\lambda|}}$ is traceless can be used to choose the polarizations to have vanishing products

$$\begin{aligned} f^{a_1 \ldots a_{|\lambda|}} \text{ traceless } \;&\leftrightarrow\; \bar{f}(\boldsymbol{\theta})\big|_{\theta^{(p)} \cdot \theta^{(q)} = \theta^{(p)} \cdot z^{(1)} = z^{(1)2} = 0}\,, \\ &\leftrightarrow\; f\left(\mathbf{z}\right)\big|_{z^{(p)} \cdot z^{(q)} = z^{(p)} \cdot \theta^{(1)} = 0}\,. \end{aligned} \tag{2.33}$$

This means that all terms in the tensor proportional to Kronecker deltas $\delta^{a_i a_j}$ are discarded. They have to be restored by projection to traceless tensors if one wishes to extract the tensor from the polynomial.

To extract the tensor $f^{a_1 \ldots a_{|\lambda|}}$ back from the polynomials one can simply restore the indices by acting with $|\lambda|$ derivatives on the polarizations and then project to the irreducible representation λ with the projector $\pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}$,

$$f^{a_1\dots a_{|\lambda|}} = \pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}} \frac{1}{h_1!}\,\partial_{\theta^{(1)}_{b_1}}\dots\partial_{\theta^{(1)}_{b_{h_1}}}\frac{1}{h_2!}\,\partial_{\theta^{(2)}_{b_{h_1+1}}}\dots\partial_{\theta^{(2)}_{b_{h_1+h_2}}} \tag{2.34}$$

$$\dots\frac{1}{h_{n_\Theta}!}\,\partial_{\theta^{(n_\Theta)}_{b_{h_1+\dots+h_{n_\Theta-1}+1}}}\dots\partial_{\theta^{(n_\Theta)}_{b_{h_1+\dots+h_{n_\Theta}}}}\frac{1}{\lambda_1!}\,\partial_{z^{(1)}_{b_{|\lambda|-\lambda_1+1}}}\dots\partial_{z^{(1)}_{b_{|\lambda|}}}\bar f(\boldsymbol{\theta})$$

$$= \pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}\frac{1}{H(\lambda)}\,\partial_{z^{(1)}_{b_1}}\dots\partial_{z^{(1)}_{b_{l_1}}}\partial_{z^{(1)}_{b_{l_1+1}}}\dots\partial_{z^{(1)}_{b_{l_1+l_2}}} \tag{2.35}$$

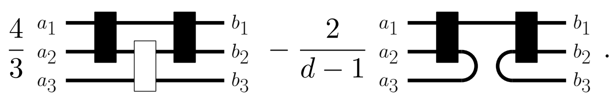
$$\dots\partial_{z^{(n_Z)}_{b_{l_1+\dots+l_{n_Z-1}+1}}}\dots\partial_{z^{(n_Z)}_{b_{l_1+\dots+l_{n_Z}}}}\;\partial_{\theta^{(1)}_{b_{|\lambda|-\lambda_1^t+1}}}\dots\partial_{\theta^{(1)}_{b_{|\lambda|}}}f(\mathbf{z})\,.$$

The normalizations can be explained as follows. When extracting the components $f^{a_1\dots a_{|\lambda|}}$ from the polynomial $\bar f(\boldsymbol{\theta})$ all that happens is the antisymmetrization of a tensor which is already in the antisymmetric basis. For each set of antisymmetric indices every generated term is the same and the normalization factor only has to cancel the number of terms. Going from $f(\mathbf{z})$ to $f^{a_1\dots a_{|\lambda|}}$ involves a Young projection of a tensor that is already Young symmetrized. Therefore the normalization $H(\lambda)$ is that of the Young projectors, which are given in [13]. It is computed from the shape of λ by a hook rule. Write into each box of a Young diagram the number of boxes to its right and below, including the box itself. The product of all numbers is $H(\lambda)$. For example,

$$H\left(\yng(4,2,1)\right) = H\left(\young(6421,31,1)\right) = 6\cdot4\cdot3\cdot2\,. \tag{2.36}$$

As far as we are aware an explicit general formula for the projector $\pi_\lambda^{a_1\dots a_{|\lambda|},b_1\dots b_{|\lambda|}}$ is only known for symmetric tensors [15]. For the simplest mixed-symmetry tensor $\yng(2,1)$ the projector is [13]

$$\pi_{\yng(2,1)}^{a_1a_2a_3,b_1b_2b_3} = \tag{2.37}$$



Let $f^{a_1a_2a_3}$ and $g^{b_1b_2b_3}$ be two tensors in the irrep $\yng(2,1)$ and

$$\begin{aligned} f(\mathbf{z}) = f(z,\theta) &= (z\cdot\partial_\theta)\,\theta_{a_1}\theta_{a_2}z_{a_3}f^{a_1a_2a_3} = (\theta_{a_1}z_{a_2}z_{a_3}-\theta_{a_2}z_{a_1}z_{a_3})\,f^{a_1a_2a_3}\big|_{z^2=z\cdot\theta=0}\,,\\ g(\mathbf{z}) = g(z,\theta) &= (z\cdot\partial_\theta)\,\theta_{a_1}\theta_{a_2}z_{a_3}g^{a_1a_2a_3} = (\theta_{a_1}z_{a_2}z_{a_3}-\theta_{a_2}z_{a_1}z_{a_3})\,g^{a_1a_2a_3}\big|_{z^2=z\cdot\theta=0}\,, \end{aligned} \tag{2.38}$$

their encoding polynomials. We would like to know how to contract these tensors using directly the polynomials. The antisymmetrization in the projector (2.37) is already done in the construction of the polynomials, only the symmetrization and subtraction of the trace is left to do. This can be done by introducing a differential operator D_z^a that satisfies

$$D_z^{a_1}D_z^{a_2}z^{b_1}z^{b_2} = \frac{1}{4}\left(\frac{2}{3}\left(\delta^{a_1b_1}\delta^{a_2b_2}+\delta^{a_1b_2}\delta^{a_2b_1}\right)-\frac{2}{d-1}\,\delta^{a_1a_2}\delta^{b_1b_2}\right), \tag{2.39}$$

where the factor $\frac{1}{4}$ normalizes the antisymmetrizations. D_z^a can be found to be

$$D_z^a = \frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial z_a}-\frac{3}{2(d-1)}\,z^a\frac{\partial^2}{\partial z\cdot\partial z}\right). \tag{2.40}$$

The contraction of the two traceless tensors can then be expressed in terms of the encoding polynomials as

$$f^{a_1a_2a_3}g_{a_1a_2a_3} = f(D_z, \partial_\theta)\, g(z,\theta)\,. \tag{2.41}$$

This is entirely analogous to the situation of symmetric traceless tensors, but now the explicit form of the projector and corresponding differential operator acting on the polarization vectors is not known in general. We will assume that there exists for every irrep λ a set of differential operators

$$\mathbf{D_z} = \left(D^{(1)}_{z^{(1)}}, \ldots, D^{(n_Z)}_{z^{(n_Z)}}, D^{(n_Z+1)}_{\theta^{(1)}}\right), \tag{2.42}$$

that reproduces the projector in this way. We have no proof that every projector can be expressed like this. If nothing else it is a notation that allows us to write any contraction as

$$f^{a_1\ldots a_{|\lambda|}}g_{a_1\ldots a_{|\lambda|}} = f(\mathbf{D_z})g(\mathbf{z})\,. \tag{2.43}$$

We postpone a more general treatment of the projectors to traceless mixed-symmetry tensors to a subsequent paper.

2.5 Tensors in embedding space

To work out the constrains conformal symmetry imposes on correlation functions of tensor operators, it is convenient to use the embedding formalism. The idea, which dates back at least to Dirac [16], is to lift the problem to the embedding space $\mathbb{M}^{d+2}$ where the conformal group $\mathrm{SO}(d+1,1)$ acts linearly as standard Lorentz transformations in $(d+2)$-dimensional Minkowski space. Let $P \in \mathbb{M}^{d+2}$ be a point in this embedding space. Points in physical space are identified with light-rays, i.e. with null vectors in $\mathbb{M}^{d+2}$ up to rescalings,

$$P^2 = 0\,, \qquad P \sim \alpha P \quad (\alpha > 0)\,. \tag{2.44}$$

Then, a specific choice of conformal frame corresponds to a specific section of the light cone. In particular, for a CFT on d-dimensional Euclidean space $\mathbb{R}^d$, we consider the Poincaré section of the light-cone

$$P^A = \left(P^+, P^-, P^a\right) = \left(1, x^2, x^a\right), \tag{2.45}$$

where we are using light-cone coordinates with metric

$$P_1 \cdot P_2 = \eta_{AB}P_1^A P_2^B = -\frac{1}{2}\left(P_1^+P_2^- + P_1^-P_2^+\right) + \delta_{ab}P_1^aP_2^b\,. \tag{2.46}$$

For example, it is simple to see that the Euclidean distance between two points in $\mathbb{R}^d$ is written in the embedding space as $-2P_1\cdot P_2 = (x_1-x_2)^2$. It will later be abbreviated by $P_{ij} \equiv -2P_i\cdot P_j$. In general, $\mathrm{SO}(d+1,1)$ Lorentz transformations map the light-cone into itself and, by the identification (2.45), define the action of the conformal group in physical space. A more thorough discussion of the embedding formalism can be seen in [10, 17], whose notation we follow here.

Let us now consider a mixed-symmetry tensor primary field of dimension Δ. This field will have components $f^{a_1\dots a_{|\lambda|}}(x)$ with symmetries given by the Young diagram λ. We wish to express it in terms of a field on the embedding space. This new tensor field will have components $F^{A_1\dots A_{|\lambda|}}(P)$ with the same symmetries as the physical tensor, it should be defined on the light cone $P^2=0$ and it should be homogeneous of degree $-\Delta$,

$$F_{A_1\dots A_{|\lambda|}}(\alpha P)=\alpha^{-\Delta}F_{A_1\dots A_{|\lambda|}}(P)\,,\quad \alpha>0\,. \tag{2.47}$$

It should also obey the transversality condition

$$P^{A_i}F_{A_1\dots A_i\dots A_{|\lambda|}}=0\,. \tag{2.48}$$

Components of the physical tensor are then obtained by projecting into physical space by

$$f_{a_1\dots a_{|\lambda|}}=\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|}}}{\partial x^{a_{|\lambda|}}}F_{A_1\dots A_{|\lambda|}}\,. \tag{2.49}$$

Next we wish to encode the tensor in the embedding space $F^{A_1\dots A_{|\lambda|}}(P)$ by a polynomial. The discussion is entirely analogous to that of the previous section, only that now the tensor will be a polynomial $F(P,\mathbf{Z})$ in the embedding space polarization vectors

$$\mathbf{Z}\equiv\left(Z^{(1)},Z^{(2)},\dots,Z^{(n_Z)},\Theta^{(1)}\right). \tag{2.50}$$

Explicitly, the polynomial $F(P,\mathbf{Z})$ is given by

$$F(P,\mathbf{Z})\equiv\prod_{p=1}^{n_Z}\prod_{q=1}^{\min(l_p,n_\Theta)}\left(Z^{(p)}\cdot\partial_{\Theta^{(q)}}\right)\bar{F}(P,\mathbf{\Theta})\,, \tag{2.51}$$

where

$$\begin{aligned}\bar{F}(P,\mathbf{\Theta})\equiv{}&\Theta^{(1)}_{A_1}\dots\Theta^{(1)}_{A_{h_1}}\Theta^{(2)}_{A_{h_1+1}}\dots\Theta^{(2)}_{A_{h_1+h_2}}\\&\dots\Theta^{(n_\Theta)}_{A_{h_1+\dots+h_{n_\Theta-1}+1}}\dots\Theta^{(n_\Theta)}_{A_{h_1+\dots+h_{n_\Theta}}}Z^{(1)}_{A_{|\lambda|-\lambda_1+1}}\dots Z^{(1)}_{A_{|\lambda|}}F^{A_1\dots A_{|\lambda|}}(P)\,,\end{aligned} \tag{2.52}$$

with

$$\mathbf{\Theta}\equiv\left(\Theta^{(1)},\Theta^{(2)}\dots,\Theta^{(n_\Theta)},Z^{(1)}\right). \tag{2.53}$$

For traceless transverse tensors one can, without loss of information, drop scalar products of any two polarizations or of one polarization and the corresponding embedding space coordinate, i.e.

$$\begin{aligned}F^{A_1\dots A_{|\lambda|}}(P)\text{ traceless \& transverse }&\leftrightarrow\bar{F}(P,\mathbf{\Theta})\big|_{\substack{\Theta^{(p)}\cdot\Theta^{(q)}=\Theta^{(p)}\cdot Z^{(1)}=Z^{(1)2}=0\\ \Theta^{(p)}\cdot P=Z^{(1)}\cdot P=0}}\,,\\&\leftrightarrow F(P,\mathbf{Z})\big|_{\substack{Z^{(p)}\cdot Z^{(q)}=Z^{(p)}\cdot\Theta^{(1)}=0\\ Z^{(p)}\cdot P=\Theta^{(1)}\cdot P=0}}\,.\end{aligned} \tag{2.54}$$

This means that transverse polynomials satisfy the transversality condition

$$F(P,\mathbf{Z}+\mathbf{c}P)=F(P,\mathbf{Z})\,, \tag{2.55}$$

JHEP02(2015)151

for any set $\mathbf{c} = (c_1, \ldots, c_{n_Z}, \gamma)$ of n_Z commuting numbers c_i and one anti-commuting number γ.

It is also possible to relate the polynomial $f(x, \mathbf{z})$ to the embedding polynomial $F(P, \mathbf{Z})$, as well as $\bar{f}(x, \boldsymbol{\theta})$ to $\bar{F}(P, \boldsymbol{\Theta})$. The procedure is entirely analogous to that described in [10]: in the case of the Poincaré patch where $P_x = (1, x^2, x)$, each embedding polarization can be written as

$$Z^{(p)}_{z,x} = \left(0, 2x \cdot z^{(p)}, z^{(p)}\right) \quad \text{and} \quad \Theta^{(p)}_{\theta,x} = \left(0, 2x \cdot \theta^{(p)}, \theta^{(p)}\right), \tag{2.56}$$

so that the relation between the polynomials is simply

$$f(x, \mathbf{z}) = F\left(P_x, \mathbf{Z}_{z,x}\right) \quad \text{and} \quad \bar{f}(x, \boldsymbol{\theta}) = \bar{F}\left(P_x, \boldsymbol{\Theta}_{\theta,x}\right). \tag{2.57}$$

The projector $\pi_\lambda^{a_1 \ldots a_{|\lambda|}, b_1 \ldots b_{|\lambda|}}$ to the irrep λ lifts to the projector $\Pi_\lambda^{A_1 \ldots A_{|\lambda|}, B_1 \ldots B_{|\lambda|}}$ in embedding space. The only case we need here is when it is inserted between two transverse tensors, since we will always work with polynomials that are transverse.[3] In this case $\Pi_\lambda^{A_1 \ldots A_{|\lambda|}, B_1 \ldots B_{|\lambda|}}$ is obtained from $\pi_\lambda^{a_1 \ldots a_{|\lambda|}, b_1 \ldots b_{|\lambda|}}$ by replacing all Kronecker deltas $\delta^{a_i b_j}, \delta^{a_i a_j}$ and $\delta^{b_i b_j}$ by embedding space metrics $\eta^{A_i B_j}, \eta^{A_i A_j}$ and $\eta^{B_i B_j}$. This implies that the operators $\mathbf{D_z}$ can also be carried over to embedding space by replacing $\mathbf{z}$ by $\mathbf{Z}$, when they are used between two transverse polynomials. The contraction of two traceless transverse tensors $F^{A_1 A_2 A_3}$ and $G^{B_1 B_2 B_3}$ in the irrep $\yng(2,1)$ is, as in the example (2.41), given by

$$F^{A_1 A_2 A_3} G_{A_1 A_2 A_3} = F(D_Z, \partial_\Theta) G(Z, \Theta), \tag{2.58}$$

with

$$D_Z^A = \frac{1}{\sqrt{6}} \left(\frac{\partial}{\partial Z_A} - \frac{3}{2(d-1)} Z^A \frac{\partial^2}{\partial Z \cdot \partial Z} \right). \tag{2.59}$$

In general, contractions will be written as

$$F^{A_1 \ldots A_{|\lambda|}} G_{A_1 \ldots A_{|\lambda|}} = F(\mathbf{D_Z})\, G(\mathbf{Z}). \tag{2.60}$$

3 Correlation functions

In this section we address the main kinematic problem that is to be solved when thinking about correlation functions of arbitrary tensor irreps: to count, and to construct, all independent tensor structures.

3.1 Tensor-product coefficients

One part of the problem is finding all the possible ways a given set of mixed-symmetry tensors can be contracted. A more mathematical way to pose this question is to ask for the multiplicity of the scalar representation in the tensor product of the tensors in question. Fortunately, this problem is already solved. Here we shall review the relevant results for our purposes; for a comprehensive introduction to the general properties of tensor-product coefficients see [18].

[3] The general form of Π_λ can be obtained analogously as it was done for symmetric tensors in [10].

JHEP02(2015)151

Let G be $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$ and λ, μ, ν irreducible G-modules which are enumerated by Young diagrams. These are the vector spaces of tensors with the index symmetries described in section 2.4. They will often be called representations instead of modules in the following. λ^* denotes the vector space dual to λ, i.e. if λ contains tensors with lower indices, λ^* contains tensors with upper indices. Upper and lower indices can be contracted and the result will then transform under G as indicated by the remaining indices.

Let ${\mathcal{N}_{\lambda\mu}}^{\nu}$ be the tensor-product coefficients of G. They count the multiplicity with which the irrep ν appears in the tensor product of λ and μ

$$\lambda \otimes \mu = \bigoplus_{\nu} {\mathcal{N}_{\lambda\mu}}^{\nu} \, \nu \,, \tag{3.1}$$

and satisfy

$$ {\mathcal{N}_{\lambda\bullet}}^{\nu} = \delta^{\nu}_{\lambda} \,, \qquad {\mathcal{N}_{\lambda\lambda^*}}^{\bullet} = 1 \,, \qquad {\mathcal{N}_{\lambda\mu}}^{\nu} = {\mathcal{N}_{\lambda\nu^*}}^{\mu^*} \,, \tag{3.2}$$

where $\bullet$ denotes the scalar representation. Let us also denote by $\mathcal{N}_{\lambda\mu\nu}$ the multiplicity of the scalar representation in the triple product

$$\lambda \otimes \mu \otimes \nu = \mathcal{N}_{\lambda\mu\nu} \bullet \oplus \text{ other irreps} \,. \tag{3.3}$$

This notation has the advantage of being symmetric in its three labels and contains the same information due to

$${\mathcal{N}_{\lambda\mu}}^{\nu} = \mathcal{N}_{\lambda\mu\nu^*} \,. \tag{3.4}$$

The multiplicity of a given representation μ in products of more than two tensors will be denoted by ${\mathcal{N}_{\lambda_1 \ldots \lambda_n}}^{\mu}$

$$\lambda_1 \otimes \ldots \otimes \lambda_n = \bigoplus_{\mu} {\mathcal{N}_{\lambda_1 \ldots \lambda_n}}^{\mu} \, \mu \,, \tag{3.5}$$

and can be calculated by recursively using (3.1)

$${\mathcal{N}_{\lambda_1 \ldots \lambda_n}}^{\mu} = \sum_{\nu_3, \ldots, \nu_n} {\mathcal{N}_{\lambda_1\lambda_2}}^{\nu_3} \prod_{i=3}^{n-1} \left({\mathcal{N}_{\nu_i \lambda_i}}^{\nu_{i+1}} \right) {\mathcal{N}_{\nu_n \lambda_n}}^{\mu} \,. \tag{3.6}$$

This also computes the multiplicity of the scalar representation in the product $\lambda_1 \otimes \ldots \otimes \lambda_n \otimes \mu^*$.

3.1.1 Unitary groups

When specializing to $G = \mathrm{SU}(n)$ the tensor-product coefficients are the famous Littlewood-Richardson coefficients ${c_{\lambda\mu}}^{\nu}$,

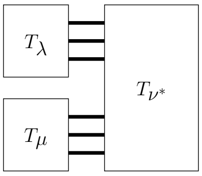
$${\mathcal{N}_{\lambda\mu}}^{\nu} = {c_{\lambda\mu}}^{\nu} \quad \text{for } G = \mathrm{SU}(n) \,. \tag{3.7}$$

The only allowed contraction in this group is between upper and lower indices, so the number of indices adds up when the tensor product between two tensors with lower indices is formed

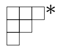
$${c_{\lambda\mu}}^{\nu} = 0 \quad \text{for } |\lambda| + |\mu| \neq |\nu| \,. \tag{3.8}$$

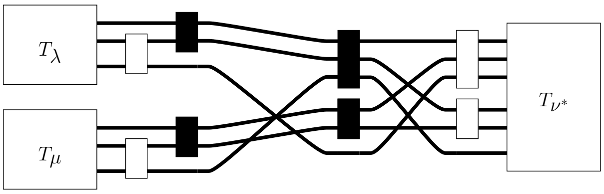
JHEP02(2015)151

This implies that the product of three tensors can only contain the scalar representation if one of them is in a dual representation relative to the other two. This can be illustrated by the following schematic contraction of tensor indices

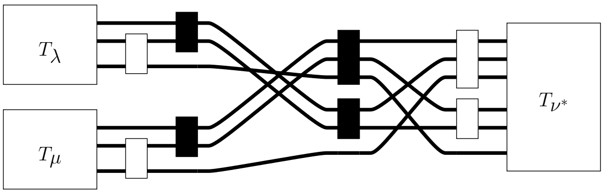
 $\propto\ c_{\lambda\mu}{}^{\nu}\,\bullet\ .$ (3.9)

The coefficients $c_{\lambda\mu}{}^{\nu}$ can be calculated using the Littlewood-Richardson rule [19].[4]

For simple examples one can often find the possible contractions for a given tensor product quickly using birdtracks. For example, one can easily convince oneself that the only two inequivalent ways to contract $\lambda = \mu =$  and $\nu^* =$  are

 , (3.10)

and

 . (3.11)

The Littlewood-Richardson coefficient is thus $c_{\lambda\mu}{}^{\nu} = 2$.

3.1.2 Orthogonal and symplectic groups

Following the reasoning of [13], the orthogonal and symplectic groups can be obtained from the unitary groups by taking into account the fact that these groups have by definition additional group invariants. For $\mathrm{SO}(d)$ this is a symmetric quadratic form g_{ab} and its inverse g^{ab}, while for $\mathrm{Sp}(d)$ the invariant is skew symmetric $f_{ab} = -f_{ba}$. In both cases these invariants can be used to raise and lower indices, which implies that the distinction between the two becomes unnecessary, the representations are self-dual $\lambda^* = \lambda$. Any two indices can be contracted and this leads to different tensor-product coefficients

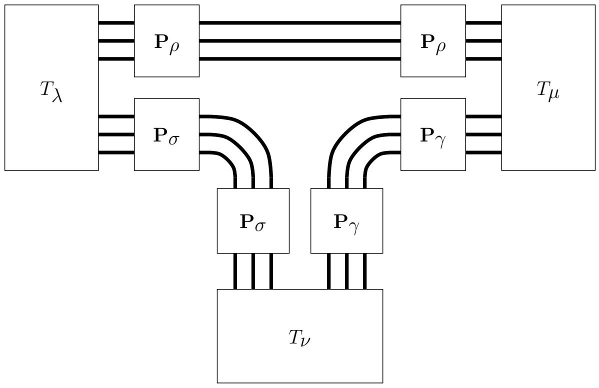
$$\mathcal{N}_{\lambda\mu}{}^{\nu} = \mathcal{N}_{\lambda\mu\nu} = b_{\lambda\mu\nu} \quad \text{for } G \in \left\{\mathrm{SO}(2n), \mathrm{SO}(2n+1), \mathrm{Sp}(2n)\right\}. \tag{3.12}$$

Because of the self-duality of the representations the position of the indices of the tensor-product coefficients becomes meaningless, so these coefficients are always written with only lower indices. It is not hard to convince oneself that the counting of tensor structures here can be broken down to the counting that was relevant in the SU(n) case where the

[4]The algorithm has been implemented for instance in Anders Skovsted Buch's lrcalc program, which is available at http://www.math.rutgers.edu/ asbuch/lrcalc/.

restriction $|\lambda| + |\mu| = |\nu|$ applied. The following figure shows how three sets of indices can be contracted with each other, by first dividing each set of indices into two,

$$\sum_{\rho,\sigma,\gamma}$$



$$\propto \; b_{\lambda\mu\nu} \bullet\,, \tag{3.13}$$

where $\mathbf{P}_\rho$ is a projector to the irrep ρ, and so on. The number of tensor structures obtained in such a way is

$$b_{\lambda\mu\nu} = \sum_{\rho,\sigma,\gamma} c_{\sigma\rho}{}^{\lambda}\, c_{\rho\gamma}{}^{\mu}\, c_{\gamma\sigma}{}^{\nu}. \tag{3.14}$$

This formula is known as the Newell-Littlewood formula [20, 21] and holds if the sum of the heights of two of the three irreps λ, μ and ν does not exceed n, i.e. for

$$h_1^\lambda + h_1^\mu + h_1^\nu - \max\left(h_1^\lambda, h_1^\mu, h_1^\nu\right) \leq n = \left\lfloor \frac{d}{2} \right\rfloor. \tag{3.15}$$

Otherwise even the tensor product of the two irreps with the smallest h_1 contains Young diagrams that violate (2.1) and hence do not correspond to irreps of SO(d) or Sp(d). In this case (3.14) can be used anyway by transforming these Young diagrams into diagrams that correspond to irreps using modification rules [12] and taking the additional contributions that arise in this way into account. Then also the statement (3.12) that the tensor-product coefficients are the same for SO$(2n)$, SO$(2n+1)$ and Sp$(2n)$ does not hold true anymore. For simplicity, we will assume (3.15) to be satisfied throughout the paper. Note that this implies that explicit examples in this paper hold only for d sufficiently large.

The coefficients descibing the decomposition of the tensor product of more than two irreps are given by (3.6)

$$b_{\lambda_1\ldots\lambda_n} = \sum_{\nu_3,\ldots,\nu_n} b_{\lambda_1\lambda_2\nu_3} \prod_{i=3}^{n-2} \left(b_{\nu_i\lambda_i}{}^{\nu_{i+1}}\right) b_{\nu_{n-1}\lambda_{n-1}\lambda_n}\,. \tag{3.16}$$

For SO(d) or Sp(d) the same coefficients also count the multiplicity of the scalar representation in the tensor product $\lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_n$.

A notation that will be used below is the restriction of a tensor product to irreps that have the same number of indices as both irreps in the product. This operation will be denoted with square brackets and amounts to using the SU(n) Littlewood-Richardson coefficients as tensor-product coefficients,

$$[\lambda \otimes \mu] \equiv \bigoplus_\nu b_{\lambda\mu\nu}\, \nu\Big|_{|\nu|=|\lambda|+|\mu|} = \bigoplus_\nu c_{\lambda\mu}{}^{\nu}\, \nu\,. \tag{3.17}$$

The second equality can be found for instance in [12]. To wrap up this section consider the following example

$$[\lambda\otimes\mu\otimes\nu]\otimes\rho\otimes\sigma=\left(\sum_{\gamma,\kappa}c_{\lambda\mu}{}^{\gamma}\,c_{\gamma\nu}{}^{\kappa}\,b_{\kappa\rho\sigma}\,\bullet\right)\oplus\text{ other irreps}\,. \tag{3.18}$$

3.2 Two-point functions

Unitary irreducible representations of the conformal group $\mathrm{SO}(d+1,1)$ will be labeled by $\chi\equiv[\lambda,\Delta]$, where Δ is the conformal dimension and λ an irreducible representation of $\mathrm{SO}(d)$. The two-point function of the primary corresponding to χ is, up to a normalization constant, a tensor depending on two points in the embedding space with components

$$G^{A_1\dots A_{|\lambda|}B_1\dots B_{|\lambda|}}\big(P_1,P_2\big)\,. \tag{3.19}$$

It is encoded, as described above, by a polynomial

$$G_\chi\big(P_1,P_2;\mathbf{Z}_1,\mathbf{Z}_2\big)=\prod_{p=1}^{n_Z}\prod_{q=1}^{\min(l_p,n_\Theta)}\left(Z_1^{(p)}\cdot\partial_{\Theta_1^{(q)}}\right)\left(Z_2^{(p)}\cdot\partial_{\Theta_2^{(q)}}\right)\bar{G}_\chi(P_1,P_2;\mathbf{\Theta}_1,\mathbf{\Theta}_2)\,, \tag{3.20}$$

where

$$\begin{aligned}&\bar{G}_\chi\big(P_1,P_2;\mathbf{\Theta}_1,\mathbf{\Theta}_2\big)=\\&=\Theta^{(1)}_{1A_1}\dots\Theta^{(n_\Theta)}_{1A_{|\lambda|-\lambda_1}}\dots Z^{(1)}_{1A_{|\lambda|}}\Theta^{(1)}_{2B_1}\dots\Theta^{(n_\Theta)}_{2B_{|\lambda|-\lambda_1}}\dots Z^{(1)}_{2B_{|\lambda|}}G^{A_1\dots A_{|\lambda|}B_1\dots B_{|\lambda|}}\big(P_1,P_2\big)\,.\end{aligned} \tag{3.21}$$

To construct the two-point function one has to find $\bar{G}_\chi$, which is subject to the following conditions. Firstly, it is homogeneous of degree $-\Delta$ in the embedding space coordinates

$$\bar{G}_\chi\big(\{\alpha_iP_i;\mathbf{\Theta}_i\}\big)=(\alpha_1\alpha_2)^{-\Delta}\bar{G}_\chi\big(\{P_i;\mathbf{\Theta}_i\}\big)\,, \tag{3.22}$$

for α_i arbitrary positive constants. Secondly, it is a polynomial in the polarizations with degrees given by the shape of the Young diagram λ,

$$\bar{G}_\chi\big(\{P_i;\boldsymbol{\beta}_i\mathbf{\Theta}_i)\}\big)=\left(\beta_1^{(1)}\beta_2^{(1)}\right)^{h_1}\dots\left(\beta_1^{(n_\Theta)}\beta_2^{(n_\Theta)}\right)^{h_{n_\Theta}}\left(\beta_1^{(Z)}\beta_2^{(Z)}\right)^{\lambda_1}\bar{G}_\chi\big(\{P_i;\mathbf{\Theta}_i\}\big)\,, \tag{3.23}$$

where we defined

$$\boldsymbol{\beta}_i\mathbf{\Theta}_i=\left(\beta_i^{(1)}\Theta_i^{(1)},\dots,\beta_i^{(n_\Theta)}\Theta_i^{(n_\Theta)},\beta_i^{(Z)}Z_i^{(1)}\right)\,, \tag{3.24}$$

for arbitrary (commuting) constants $\beta_i^{(p)}$.

Finally, $\bar{G}_\chi$ has to be transverse

$$\bar{G}_\chi\big(\{P_i;\mathbf{\Theta}_i+\boldsymbol{\gamma}_iP_i\}\big)=\bar{G}_\chi\big(\{P_i;\mathbf{\Theta}_i\}\big)\,. \tag{3.25}$$

where

$$\boldsymbol{\gamma}_i=\left(\gamma_i^{(1)},\dots,\gamma_i^{(n_Z)},c_i\right)\,, \tag{3.26}$$

is a set of n_Z anticommuting numbers and one commuting number. This last condition has to be satisfied modulo $O\big(P^2\big)$ terms. An identically transverse function $\bar{G}_\chi$ can be obtained by dropping terms proportional to $\Theta^{(p)} \cdot \Theta^{(q)}$ and $\Theta^{(p)} \cdot P$, where $p = 1, \ldots, n_\Theta, Z$. Notice that we are using the notation $\Theta^{(Z)} = Z^{(1)}$ to make equations more compact. We are left to constructing $\bar{G}_\chi$ from the tensors

$$C^{(p)}_{iAB} = \Theta^{(p)}_{iA} P_{iB} - \Theta^{(p)}_{iB} P_{iA} = \begin{cases} \Theta^{(p)}_{iA} P_{iB} - \Theta^{(p)}_{iB} P_{iA}\,, & p = 1, \ldots, n_\Theta\,, \\ Z^{(1)}_{iA} P_{iB} - Z^{(1)}_{iB} P_{iA}\,, & p = Z\,, \end{cases} \tag{3.27}$$

with $i = 1, 2$. Contracting two such tensors with the same index i leads to terms of the type that do not appear in transverse functions, so the only possible terms are traces of a string of C's with alternating i's, i.e. of the form

$$\mathrm{Tr}\left(C^{(p)}_1 \cdot C^{(q)}_2 \ldots C^{(r)}_1 \cdot C^{(s)}_2\right), \tag{3.28}$$

the shortest one being

$$H^{(p,q)}_{ij} \equiv \mathrm{Tr}\left(C^{(p)}_i \cdot C^{(q)}_j\right) = 2\left(\left(P_j \cdot \Theta^{(p)}_i\right)\left(P_i \cdot \Theta^{(q)}_j\right) - \left(\Theta^{(p)}_i \cdot \Theta^{(q)}_j\right)\left(P_i \cdot P_j\right)\right). \tag{3.29}$$

Recall that both p and/or q can also take the value Z, for which case they describe the commuting polarization $Z^{(1)}$.

Traces of more than two alternating C_1's and C_2's can always be expressed in terms of $H^{(p,q)}_{12}$. This can be seen by considering

$$\left(C^{(p)}_1 \cdot C^{(q)}_2 \cdot C^{(r)}_1 \cdot C^{(s)}_2\right)_{AB} = \frac{1}{2}\left(C^{(p)}_{1AC} H^{(q,r)}_{21} C^{(s)}_{2CB} + P_{1A} P_{2B} R\right), \tag{3.30}$$

where

$$\begin{aligned} R = {} & \Theta^{(p)}_{1A} H^{(q,r)}_{21} \Theta^{(s)}_{2A} + C^{(p)}_{1AB}\left(\Theta^{(q)}_2 \cdot \Theta^{(r)}_1\right) C^{(s)}_{2BA} - \left(\Theta^{(p)}_1 \cdot \Theta^{(q)}_2\right) H^{(r,s)}_{12} - H^{(p,q)}_{12}\left(\Theta^{(r)}_1 \cdot \Theta^{(s)}_2\right) \\ & + 2(P_1 \cdot P_2)\left[\Theta^{(p)}_{1A}\left(\Theta^{(q)}_2 \cdot \Theta^{(r)}_1\right)\Theta^{(s)}_{2A} - \left(\Theta^{(p)}_1 \cdot \Theta^{(q)}_2\right)\left(\Theta^{(r)}_1 \cdot \Theta^{(s)}_2\right)\right], \end{aligned} \tag{3.31}$$

which satisfies

$$(P_1 \cdot P_2) R = \frac{1}{2}\left(H^{(p,q)}_{12} H^{(r,s)}_{12} - C^{(p)}_{1AB} H^{(q,r)}_{21} C^{(s)}_{2BA}\right). \tag{3.32}$$

Using also that

$$\left(P_2 \cdot C^{(p)}_1 \cdot C^{(q)}_2\right)_A = \frac{1}{2} H^{(p,q)}_{12} P_{2A}\,, \tag{3.33}$$

one sees that multiplying (3.30) by any number of factors $C_1 \cdot C_2$ produces only more terms of the same structure that turn into products of H_{12}'s when the trace is closed.

Naively one could imagine that the different ways to distribute polarizations among H_{12}'s lead to different tensor structures, e.g. for the diagram ⊞ one could consider

$$\left(H^{(1,1)}_{12} H^{(2,2)}_{12}\right)^2, \quad H^{(1,1)}_{12} H^{(2,2)}_{12} H^{(1,2)}_{12} H^{(2,1)}_{12} \quad \text{and} \quad \left(H^{(1,2)}_{12} H^{(2,1)}_{12}\right)^2. \tag{3.34}$$

However, the tensor product of two copies of an irrep contains the scalar representation with multiplicity one, as written in (3.2), so there can be only one tensor structure for each two-point function. Indeed, all possible ways to distribute the polarizations among the

H_{12}'s lead to the same result after Young symmetrization (this can be checked explicitly by considering (3.20)). With the weights of coordinates and polarizations being fixed by (3.22) and (3.23), we choose a convenient set of H_{12}'s and find that the unique tensor structure for the two-point function is given by (3.20) with

$$\bar{G}_\chi(P_1, P_2; \mathbf{\Theta}_1, \mathbf{\Theta}_2) = \frac{1}{(P_{12})^{\Delta+|\lambda|}} \prod_{r=1}^{n_\Theta} \left(H_{12}^{(r,r)}\right)^{h_r} \left(H_{12}^{(Z,Z)}\right)^{\lambda_1} . \tag{3.35}$$

3.2.1 Example: p-form field

As an example, let us write explicitly the two-point function of a p-form field. The Young diagram of a p-form field consists of one column of p boxes, therefore $|\lambda| = p$, $n_Z = 0$ and $n_\Theta = 1$. Since there are no rows with more than one box and hence there are no indices to symmetrize, there is no need to introduce commuting polarizations. There is a single anti-commuting polarization vector, which we denote by Θ. The correlation function can be read off from (3.35) to be

$$\bar{G}_\chi(P_1, P_2; \Theta_1, \Theta_2) = \frac{\left(H_{12}^{(\Theta,\Theta)}\right)^p}{(P_{12})^{\Delta+p}} = \frac{1}{(P_{12})^{\Delta}} \left((\Theta_1 \cdot \Theta_2) - \frac{(P_2 \cdot \Theta_1)(P_1 \cdot \Theta_2)}{P_1 \cdot P_2} \right)^p . \tag{3.36}$$

Then, using the maps (2.56) and (2.57), it is simple to find the polynomial $\bar{g}_\chi(x_1, x_2; \theta_1, \theta_2)$ that describes this tensor structure in physical space.

Note also that, acting with the Θ derivatives $\partial_{\Theta_1^{A_1}} \dots \partial_{\Theta_1^{A_p}} \partial_{\Theta_2^{B_1}} \dots \partial_{\Theta_2^{B_p}}$, one can write explicitly the components of the tensor in the embedding space as

$$G_\chi^{A_1 \dots A_p B_1 \dots B_p} = \frac{1}{(P_{12})^{\Delta}} \delta^{A_1}_{[C_1} \dots \delta^{A_p}_{C_p]} \delta^{B_1}_{[D_1} \dots \delta^{B_p}_{D_p]} \prod_{k=1}^{p} \left(\eta^{C_k D_k} - \frac{P_2^{C_k} P_1^{D_k}}{P_1 \cdot P_2} \right) , \tag{3.37}$$

whose projection to physical space gives the components

$$g_\chi^{a_1 \dots a_p b_1 \dots b_p} = \frac{1}{(x_{12}^2)^{\Delta}} \delta^{a_1}_{[c_1} \dots \delta^{a_p}_{c_p]} \delta^{b_1}_{[d_1} \dots \delta^{b_p}_{d_p]} \prod_{k=1}^{p} \left(\delta^{c_k d_k} - 2 \frac{(x_{12})^{c_k} (x_{12})^{d_k}}{x_{12}^2} \right) , \tag{3.38}$$

where $x_{12} = x_1 - x_2$.

3.2.2 Example: smallest hook diagram

As another example let us consider the irrep corresponding to the diagram . This is the simplest example where the Young symmetrization operator appears. Here we have $n_Z = 1$ and $n_\Theta = 1$, with polarization vectors $\mathbf{Z} = (Z, \Theta)$ and $\mathbf{\Theta} = (\Theta, Z)$. Thus, the polynomials encoding the tensor structure for the two-point function of these operators are

$$\bar{G}_\chi(P_1, P_2; \mathbf{\Theta}_1, \mathbf{\Theta}_2) = \frac{1}{(P_{12})^{\Delta+3}} \left(H_{12}^{(\Theta,\Theta)}\right)^2 H_{12}^{(Z,Z)} , \tag{3.39}$$

and

$$\begin{aligned} G_\chi(P_1, P_2; \mathbf{Z}_1, \mathbf{Z}_2) &= (Z_1 \cdot \partial_{\Theta_1}) (Z_2 \cdot \partial_{\Theta_2}) \bar{G}_\chi(P_1, P_2; \mathbf{\Theta}_1, \mathbf{\Theta}_2) \\ &= \frac{2}{(P_{12})^{\Delta+3}} \left(H_{12}^{(\Theta,\Theta)} H_{12}^{(Z,Z)} - H_{12}^{(\Theta,Z)} H_{12}^{(Z,\Theta)} \right) H_{12}^{(Z,Z)} . \end{aligned} \tag{3.40}$$

JHEP02(2015)151

Using the differential operator (2.59), it is a simple exercise to derive the components of the physical tensor associated to this polynomial, which were already derived in [22, 23] for all hook shaped Young diagrams. We shall not pursue this here, and work instead with embedding polynomials.

3.3 Three-point functions

Next we consider the tensor structures allowed in a three-point functions with each operator in the SO$(d+1,1)$ irrep labelled by $\chi_j \equiv [\lambda_j, \Delta_j]$, for $j=1,2,3$. Such three-point function is conveniently written as

$$G_{\chi_1\chi_2\chi_3}(\{P_i;\mathbf{Z}_i\}) = \prod_{j=1}^{3}\prod_{p=1}^{n_Z^j}\prod_{q=1}^{\min(l_p^j,n_\Theta^j)}\left(Z_j^{(p)}\cdot\partial_{\Theta_j^{(q)}}\right)\frac{\bar{Q}_{\lambda_1\lambda_2\lambda_3}(\{P_i;\mathbf{\Theta}_i\})}{(P_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}}(P_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}}(P_{31})^{\frac{\tau_3+\tau_1-\tau_2}{2}}}, \tag{3.41}$$

where $\tau_i = \Delta_i + |\lambda_i|$. The factor in the denominator was included to give $\bar{Q}_{\lambda_1\lambda_2\lambda_3}$ the same overall weight in embedding space coordinates as in polarizations, therefore simplifying its construction out of building blocks that have the same property. The conditions on $\bar{Q}_{\lambda_1\lambda_2\lambda_3}$ are otherwise analogous to (3.22)–(3.25), i.e.

$$\begin{aligned}&\bar{Q}_{\lambda_1\lambda_2\lambda_3}(\{\alpha_i P_i;\boldsymbol{\beta}_i(\mathbf{\Theta}_i+\boldsymbol{\gamma}_i P_i)\}) = \\ &= \bar{Q}_{\lambda_1\lambda_2\lambda_3}(\{P_i;\mathbf{\Theta}_i\})\prod_i \alpha_i^{|\lambda_i|}\left(\beta_i^{(1)}\right)^{h_1^i}\cdots\left(\beta_i^{(n_\Theta^i)}\right)^{h_{n_\Theta^i}^i}\left(\beta_i^{(Z)}\right)^{(\lambda_i)_1}.\end{aligned} \tag{3.42}$$

In addition to $H_{ij}^{(p,q)}$ given in (3.29), there are now other building blocks that can appear in the polynomial $\bar{Q}_{\lambda_1\lambda_2\lambda_3}(\{P_i;\mathbf{\Theta}_i\})$, which are

$$V_{i,jk}^{(p)} \equiv \frac{P_j\cdot C_i^{(p)}\cdot P_k}{P_j\cdot P_k} = \frac{\left(P_j\cdot\Theta_i^{(p)}\right)\left(P_i\cdot P_k\right)-\left(P_j\cdot P_i\right)\left(\Theta_i^{(p)}\cdot P_k\right)}{P_j\cdot P_k}. \tag{3.43}$$

Because of the property $V_{i,jk}^{(p)} = -V_{i,kj}^{(p)}$ there is only one independent $V^{(p)}$ for each operator i. They will be denoted

$$V_1^{(p)} = V_{1,23}^{(p)}, \qquad V_2^{(p)} = V_{2,31}^{(p)}, \qquad V_3^{(p)} = V_{3,12}^{(p)}. \tag{3.44}$$

Other terms of the form $P\cdot C\cdot\ldots\cdot C\cdot P$ can always be expressed in terms of $V_i^{(p)}$ and $H_{ij}^{(p,q)}$ due to (3.33). One could imagine that traces of more than two C's result in independent terms, but it was proven in [10] that this is not the case. This means that parity invariant three-point functions can be completely constructed out of $V_i^{(p)}$ and $H_{ij}^{(p,q)}$.[5]

Let us first consider the terms in the polynomial $\bar{Q}_{\lambda_1\lambda_2\lambda_3}(\{P_i;\mathbf{\Theta}_i\})$ that are constructed only out of $H_{ij}^{(p,q)}$'s. The number of independent structures that can arise from such terms is given by the tensor product coefficient $b_{\lambda_1\lambda_2\lambda_3}$ of SO(d), which was introduced in section 3.1.2. We shall denote by $\bar{W}_{\lambda_1\lambda_2\lambda_3}$ the linear combination (with arbitrary coefficients)

[5] Additional parity odd tensor structures can be constructed using the fully antisymmetric ϵ-tensor [10]. The number of such tensor structures depends on the dimension and is not considered here.

of these $b_{\lambda_1\lambda_2\lambda_3}$ combinations of $H_{ij}^{(p,q)}$'s that lead to independent tensor structures and scale as in (3.42). Such a function can easily be constructed for any example by constructing terms and checking if they give rise to independent tensor structures after the full Young symmetrization.

As an example consider one of the first combinations of irreps where the Littlewood-Richardson coefficient is larger than one, $\lambda_1 = \lambda_2 = \yng(2,1)$, $\lambda_3 = \yng(2)$. The corresponding Littlewood-Richardson coefficient is $b_{\yng(2,1)\,\yng(2,1)\,\yng(2)} = 2$. Indeed, there are two combinations of the $H_{ij}^{(p,q)}$ that lead to different tensor structures

$$\begin{aligned}
&(Z_1\cdot\partial_{\Theta_1})\,(Z_2\cdot\partial_{\Theta_2})\left(H_{12}^{(\Theta,\Theta)}\right)^2 H_{13}^{(Z,Z)}H_{23}^{(Z,Z)} = \\
&\qquad = 2\left(H_{12}^{(Z,Z)}H_{12}^{(\Theta,\Theta)} - H_{12}^{(\Theta,Z)}H_{12}^{(Z,\Theta)}\right)H_{13}^{(Z,Z)}H_{23}^{(Z,Z)}, \\
&(Z_1\cdot\partial_{\Theta_1})\,(Z_2\cdot\partial_{\Theta_2})\,H_{12}^{(Z,Z)}H_{12}^{(\Theta,\Theta)}H_{13}^{(\Theta,Z)}H_{23}^{(\Theta,Z)} = \\
&\qquad = H_{12}^{(Z,Z)}\left[\left(H_{12}^{(\Theta,\Theta)}H_{13}^{(Z,Z)} - H_{12}^{(Z,\Theta)}H_{13}^{(\Theta,Z)}\right)H_{23}^{(Z,Z)}\right. \\
&\qquad\quad \left. + \left(H_{12}^{(Z,Z)}H_{13}^{(\Theta,Z)} - H_{12}^{(\Theta,Z)}H_{13}^{(Z,Z)}\right)H_{23}^{(\Theta,Z)}\right].
\end{aligned} \tag{3.45}$$

Thus, we conclude that for this example

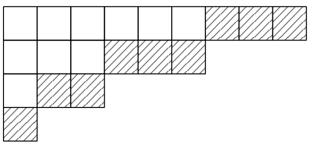
$$\bar{W}_{\yng(2,1)\,\yng(2,1)\,\yng(2)} = c_1\left(H_{12}^{(\Theta,\Theta)}\right)^2 H_{13}^{(Z,Z)}H_{23}^{(Z,Z)} + c_2\,H_{12}^{(Z,Z)}H_{12}^{(\Theta,\Theta)}H_{13}^{(\Theta,Z)}H_{23}^{(\Theta,Z)}\,, \tag{3.46}$$

with c_1 and c_2 constants.

Next we describe how to construct the general terms containing both $H_{ij}^{(p,q)}$'s and $V_i^{(p)}$'s. A given term may have an arbitrary number of $V_i^{(p)}$'s. However, since for $p \in \{1,\dots,n_\Theta\}$ the $V_i^{(p)}$ are linear in the Grassmann variables and inherit their property,

$$V_i^{(p)}V_j^{(q)} = (-1)^{\delta^{pq}\delta^{ij}}V_j^{(q)}V_i^{(p)}\,, \qquad p,q\in\{1,\dots,n_\Theta\}\,, \tag{3.47}$$

each $V_i^{(p)}$ can appear only once in a given term.[6] Thus, to the p-th column of the irrep λ_i there may be only one $V_i^{(p)}$, and therefore there can be at most l_1^i of the $V_i^{(p)}$'s in a given tensor structure. To illustrate this, the boxes of the following Young diagram that may be assigned to $V_i^{(p)}$'s are shaded


. (3.48)

Now consider the tensor structures that contain q of the $V_i^{(p)}$ building blocks. To the remaining boxes in the Young diagrams we assign a linear combination of the $H_{ij}^{(p,q)}$'s, therefore q is even (odd) if the total number of boxes in all diagrams $|\lambda_1|+|\lambda_2|+|\lambda_3|$ is even (odd). It is also clear that q can take values in the range

$$q \in \left\{0,1,\dots,l_1^1+l_1^2+l_1^3\right\}\,. \tag{3.49}$$

[6]A simple corollary is the well-known fact that two scalar operators couple only to fully symmetric representations.

The number of such independent tensor structures, containing q of the $V_i^{(p)}$ building blocks, is given by the multiplicity of the scalar representation in the tensor product of the three irreps under consideration and one Young diagram consisting of one row of length q, i.e. in the product

$$\lambda_1 \otimes \lambda_2 \otimes \lambda_3 \otimes [q] \,. \tag{3.50}$$

Hence the total number of tensor structures in a three-point function of operators in irreps λ_1, λ_2 and λ_3 is

$$\sum_{q=0}^{l_1^1+l_1^2+l_1^3} b_{\lambda_1\lambda_2\lambda_3[q]} \,, \tag{3.51}$$

with $b_{\lambda_1\lambda_2\lambda_3[q]}$ given in (3.16). Notice that the term in the sum with $q=0$ counts structures made only out of the $H_{ij}^{(p,q)}$'s considered first above.

To prove the result (3.51), we resort to the correspondence between three-point functions and leading OPE coefficients established in [24, 25] and discussed in the context of the embedding formalism in [10]. We start with the leading terms in the OPE of operators $\mathcal{O}_i$ in arbitrary irreps labeled by $[\lambda_i, \Delta_i]$ using physical space coordinates x_i^a and polarizations $\mathbf{z}_i^a$, following the discussion in [10]

$$\mathcal{O}_1(x_1,\mathbf{z}_1)\,\mathcal{O}_2(x_2,\mathbf{z}_2) \sim \sum_k \mathcal{O}_k(x_1,\mathbf{D}_{\mathbf{z_k}})\, t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_k)\, \left(x_{12}^2\right)^{-\frac{\Delta_1+\Delta_2-\Delta_k+|\lambda_1|+|\lambda_2|+|\lambda_k|}{2}} \,. \tag{3.52}$$

When this is inserted into a three-point function $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$, only $\mathcal{O}_k = \mathcal{O}_3$ contributes. $t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3)$ is a rotationally invariant polynomial which scales as

$$t(\alpha x_{12},\boldsymbol{\beta}_1\mathbf{z}_1,\boldsymbol{\beta}_2\mathbf{z}_2,\boldsymbol{\beta}_3\mathbf{z}_3) = t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3)\prod_{i=1}^{3}\alpha^{|\lambda_i|}\left(\beta_i^{(1)}\right)^{l_1^i}\cdots\left(\beta_i^{(n_Z)}\right)^{l_{n_Z}^i}\left(\beta_i^{(\theta)}\right)^{(\lambda_i^t)_1} , \tag{3.53}$$

where we defined

$$\boldsymbol{\beta}_i\mathbf{z}_i = \left(\beta_i^{(1)}z_i^{(1)},\ldots,\beta_i^{(n_Z)}z_i^{(n_Z)},\beta_i^{(\theta)}\theta_i^{(1)}\right) , \tag{3.54}$$

for arbitrary constants $\beta_i^{(p)}$. The number of independent tensor structures in $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$ is now equal to the number of strucutures in $t(x_{12},\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3)$, which is clearly given by (3.51). Note that the sum in (3.51) arises because the combination of vectors x_{12}^a is a symmetric power of the fundamental representation, which decomposes into traceless symmetric tensors of the same or smaller rank (with the ranks being all even or all odd).

$$\mathrm{Sym}^n(\,\square\,) = [n]\oplus[n-2]\oplus[n-4]\oplus\ldots \,. \tag{3.55}$$

Next we analyze some examples.

3.3.1 Example: (two-form)-vector-scalar

We start with a simple example of a two-form, a vector and a scalar, $\lambda_1 = \begin{smallmatrix}\square\\\square\end{smallmatrix}, \lambda_2 = \square$, $\lambda_3 = \bullet$. As already explained for the two-point function of a p-form, there is no need to introduce commuting polarizations for the two-form. Also, for the vector, there is obviously

JHEP02(2015)151

no need to introduce any symmetrization or antisymmetrization. It has $n_{Z_2} = n_{\Theta_2} = 0$, therefore one can freely choose whether to use Z_2 or Θ_2 as polarization. In this case the only possible tensor structure has $q = 1$, hence there is one V_i building block. This is simple to see, since

$$\ydiagram{1,1} \otimes \ydiagram{1} \otimes \bullet = \ydiagram{1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1}\,, \tag{3.56}$$

whose product with $[q]$ has a scalar representation only for $q = 1$. The corresponding tensor structure gives a three-point function of the form

$$\begin{aligned} G_{\chi_1\chi_2\chi_3}\big(\{P_i\};\Theta_1, Z_2\big) &= \frac{V_1^{(\Theta)} H_{12}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+3}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1-1}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+1}{2}}} \qquad (3.57)\\ &= \frac{-4\Big((P_2\cdot\Theta_1)(P_1\cdot P_3)-(P_2\cdot P_1)(\Theta_1\cdot P_3)\Big)\Big((P_2\cdot\Theta_1)(P_1\cdot Z_2)-(\Theta_1\cdot Z_2)(P_1\cdot P_2)\Big)}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+3}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1+1}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+1}{2}}}\,. \end{aligned}$$

It is a mechanical computation to act on this polynomial with the derivatives $\partial_{\Theta_1^A}\partial_{\Theta_1^B}\partial_{Z_2^C}$ to obtain the components of the corresponding tensor in the embedding space.

3.3.2 Example: two-form-vector-vector

Next we consider the three-point function of a two-form and two vectors, $\lambda_1 = \ydiagram{1,1}$, $\lambda_2 = \lambda_3 = \ydiagram{1}$. In this case there are three possible tensor structures,

$$\begin{aligned} q=0 &\qquad \to \qquad H_{12}^{(\Theta,Z)}H_{13}^{(\Theta,Z)}\,,\\ q=2 &\qquad \to \qquad V_1^{(\Theta)}V_2^{(Z)}H_{13}^{(\Theta,Z)} \text{ and } V_1^{(\Theta)}V_3^{(Z)}H_{12}^{(\Theta,Z)}\,. \end{aligned} \tag{3.58}$$

This can be seen from the product

$$\ydiagram{1,1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \bullet \oplus 2\,\ydiagram{2} \oplus 3\,\ydiagram{1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus 2\,\ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1}\,, \tag{3.59}$$

which contains the scalar and $\ydiagram{2}$ representations with multiplicities one and two, respectively. The corresponding three-point function has the form

$$G_{\chi_1\chi_2\chi_3}\big(\{P_i\};\Theta_1, Z_2, Z_3\big) = \frac{c_1 H_{12}^{(\Theta,Z)}H_{13}^{(\Theta,Z)} + c_2 V_1^{(\Theta)}V_2^{(Z)}H_{13}^{(\Theta,Z)} + c_3 V_1^{(\Theta)}V_3^{(Z)}H_{12}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+2}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+2}{2}}}\,, \tag{3.60}$$

with c_1, c_2 and c_3 constants.

3.3.3 Example: hook-scalar-vector

The polynomial that encodes the correlator of a small hook diagram $\lambda_1 = \ydiagram{2,1}$, a scalar $\lambda_2 = \bullet$ and a vector $\lambda_3 = \ydiagram{1}$ consists of a single tensor structure, as can easily seen by considering the product

$$\ydiagram{2,1} \otimes \bullet \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}\,. \tag{3.61}$$

q	$b_{\lambda_1\lambda_2\lambda_3[q]}$	tensor structures
0	1	$H_{12}^{(Z,Z)}H_{12}^{(\Theta,Z)}H_{13}^{(\Theta,Z)}$
2	4	$V_1^{(\Theta)}V_1^{(Z)}H_{12}^{(\Theta,Z)}H_{23}^{(Z,Z)},\ V_1^{(\Theta)}V_3^{(Z)}H_{12}^{(Z,Z)}H_{12}^{(\Theta,Z)},$ $V_1^{(\Theta)}V_2^{(Z)}H_{12}^{(Z,Z)}H_{13}^{(\Theta,Z)},\ V_1^{(\Theta)}V_2^{(Z)}H_{12}^{(\Theta,Z)}H_{13}^{(Z,Z)}$
4	2	$V_1^{(\Theta)}V_1^{(Z)}V_2^{(Z)}V_3^{(Z)}H_{12}^{(\Theta,Z)},\ V_1^{(\Theta)}V_1^{(Z)}(V_2^{(Z)})^2H_{13}^{(\Theta,Z)}$

Table 1. All seven tensor structures appearing in a three-point function of irreps $\yng(2,1)$, $\yng(2)$ and $\yng(1)$.

Recall that for the small hook diagram we have $n_{Z_1} = 1$ and $n_{\Theta_1} = 1$, with polarization vectors $\mathbf{Z}_1 = (Z_1, \Theta_1)$ and $\mathbf{\Theta}_1 = (\Theta_1, Z_1)$, so the tensor structure is obtained by acting with a derivative $Z_1 \cdot \partial_{\Theta_1}$ on a polynomial of the $V_i^{(p)}$'s and $H_{ij}^{(p,q)}$'s. In this case the single tensor structure has the form

$$\begin{aligned} G_{\chi_1\chi_2\chi_3}\big(\{P_i\}; \mathbf{Z}_1, Z_3\big) &= \frac{(Z_1\cdot\partial_{\Theta_1})\, V_1^{(\Theta)}V_1^{(Z)}H_{13}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+2}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1-2}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+4}{2}}} \\ &= \frac{V_1^{(\Theta)}V_1^{(Z)}H_{13}^{(Z,Z)} - \left(V_1^{(Z)}\right)^2 H_{13}^{(\Theta,Z)}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3+2}{2}}(P_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1-2}{2}}(P_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2+4}{2}}}\,. \end{aligned} \tag{3.62}$$

3.3.4 Example: hook-spin 2-vector

Let us finally consider the example $\lambda_1 = \yng(2,1)$, $\lambda_2 = \yng(2)$, $\lambda_3 = \yng(1)$. Table 1 contains all independent tensor structures for this case. Notice that for $q = 2$ there is another tensor structure constructed from $V_1^{(Z)}V_2^{(Z)}H_{12}^{(\Theta,Z)}H_{13}^{(\Theta,Z)}$, but this is not linear independent since

$$\begin{aligned} &(Z_1\cdot\partial_{\Theta_1})\, V_1^{(Z)}V_2^{(Z)}H_{12}^{(\Theta,Z)}H_{13}^{(\Theta,Z)} \\ &\quad = (Z_1\cdot\partial_{\Theta_1})\left(V_1^{(\Theta)}V_2^{(Z)}H_{12}^{(Z,Z)}H_{13}^{(\Theta,Z)} - V_1^{(\Theta)}V_2^{(Z)}H_{12}^{(\Theta,Z)}H_{13}^{(Z,Z)}\right). \end{aligned} \tag{3.63}$$

In this case the product of the three representations λ_1, λ_2 and λ_3 contains the following representations consisting of a single row

$$\yng(2,1) \otimes \yng(2) \otimes \yng(1) = \bullet \oplus 4\,\yng(2) \oplus 2\,\yng(4) \oplus \dots\,, \tag{3.64}$$

in agreement with table 1.

3.4 Four-point functions

Starting from four-point functions, correlation functions can depend on functions of the conformally invariant cross-rations. For four points there are two cross-ratios that can be defined to be

$$u = \frac{P_{12}P_{34}}{P_{13}P_{24}}\,, \qquad v = \frac{P_{14}P_{23}}{P_{13}P_{24}}\,. \tag{3.65}$$

JHEP02(2015)151

Then a generic four-point function can be written as

$$G_{\chi_1\chi_2\chi_3\chi_4}(\{P_i;\mathbf{Z}_i\}) = \frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\tau_1-\tau_2}{2}}\left(\frac{P_{14}}{P_{13}}\right)^{\frac{\tau_3-\tau_4}{2}}}{(P_{12})^{\frac{\tau_1+\tau_2}{2}}(P_{34})^{\frac{\tau_3+\tau_4}{2}}}\times \prod_{j=1}^{4}\prod_{p=1}^{n_Z^j}\prod_{q=1}^{\min(l_p^j,n_\Theta^j)}\left(Z_j^{(p)}\cdot\partial_{\Theta_j^{(q)}}\right)\sum_k f_k(u,v)\,\bar{Q}^{(k)}_{\chi_1\chi_2\chi_3\chi_4}(\{P_i;\mathbf{\Theta}_i\})\,, \tag{3.66}$$

where $\tau_i = \Delta_i + |\lambda_i|$ and the sum over k runs over all independent tensor structures. Each tensor structure is multiplied by a function of the cross-ratios $f_k(u,v)$ and the pre-factor is chosen in such a way that each $\bar{Q}^{(k)}_{\chi_1\chi_2\chi_3\chi_4}$ scales analogously to (3.42),

$$\begin{aligned}&\bar{Q}^{(k)}_{\chi_1\chi_2\chi_3\chi_4}(\{\alpha_i P_i;\boldsymbol{\beta}_i(\mathbf{\Theta}_i+\boldsymbol{\gamma}_i P_i)\}) =\\ &= \bar{Q}^{(k)}_{\chi_1\chi_2\chi_3\chi_4}(\{P_i;\mathbf{\Theta}_i\})\prod_i \alpha_i^{|\lambda_i|}\left(\beta_i^{(1)}\right)^{h_1^i}\dots\left(\beta_i^{(n_\Theta^i)}\right)^{h^i_{n^i_\Theta}}\left(\beta_i^{(Z)}\right)^{(\lambda_i)_1}\,.\end{aligned} \tag{3.67}$$

For an n-point function, $n-2$ of the building blocks $V^{(p)}_{i,jk}$ are linearly independent [10]. In the case of $n=4$, this is due to the relations $V^{(p)}_{i,jk} = -V^{(p)}_{i,kj}$ and

$$(P_2\cdot P_3)(P_1\cdot P_4)V^{(p)}_{1,23} + (P_3\cdot P_4)(P_1\cdot P_2)V^{(p)}_{1,34} + (P_4\cdot P_2)(P_1\cdot P_3)V^{(p)}_{1,42} = 0\,. \tag{3.68}$$

In general one can choose for instance the basis

$$\mathcal{V}^{(p)}_{ij} \equiv V^{(p)}_{i,(i+1)(i+1+j)}\,,\qquad j\in\{1,2,\dots,n-2\}\,, \tag{3.69}$$

where the external point labels i etc. are meant to be interpreted modulo n.

The tensor structures can be counted by inserting the OPE (3.52) twice into the four-point function

$$\begin{aligned}&\mathcal{O}_1(x_1,\mathbf{z}_1)\,\mathcal{O}_2(x_2,\mathbf{z}_2)\,\mathcal{O}_3(x_3,\mathbf{z}_3)\\ &\sim \mathcal{O}_1(x_1,\mathbf{z}_1)\sum_k \mathcal{O}_k(x_2,\mathbf{D}_{\mathbf{z_k}})\,t(x_{23},\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_k)\,(x_{23}^2)^{-\frac{\Delta_2+\Delta_3-\Delta_k+|\lambda_2|+|\lambda_3|+|\lambda_k|}{2}}\\ &\sim \sum_j \mathcal{O}_j(x_1,\mathbf{D}_{\mathbf{z_j}})\sum_k \frac{t(x_{12},\mathbf{z}_1,\mathbf{D}_{\mathbf{z_k}},\mathbf{z}_j)\,t(x_{23},\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_k)}{(x_{12}^2)^{\frac{\Delta_1+\Delta_k-\Delta_j+|\lambda_1|+|\lambda_k|+|\lambda_j|}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_k+|\lambda_2|+|\lambda_3|+|\lambda_k|}{2}}}\,.\end{aligned} \tag{3.70}$$

When this is inserted into $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$, only $\mathcal{O}_j = \mathcal{O}_4$ contributes. When summed over all possible irreps k, the terms $t(x_{12},\mathbf{z}_1,\mathbf{D}_{\mathbf{z_k}},\mathbf{z}_4)\,t(x_{23},\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_k)$ clearly contain all possible contractions of x_{12}, x_{23} and the four polarizations $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ and $\mathbf{z}_4$. To exclude contractions between x_{12} and x_{23}, which do not lead to new tensor structures, counting can be performed using the restricted tensor product defined in (3.17), that keeps only irreps that have the same number of indices as both irreps in the product. The number of tensor structures in a four-point function are then given by the multiplicity of the scalar representation in the tensor product

$$\lambda_1\otimes\lambda_2\otimes\lambda_3\otimes\lambda_4\otimes\Big[[q_1]\otimes[q_2]\Big]\,, \tag{3.71}$$

JHEP02(2015)151

for non-negative integers q_1 and q_2 satisfying

$$q_1, q_2 \in \left\{0, 1, \ldots, \sum_{i=1}^{4} l_1^i\right\}. \tag{3.72}$$

Hence the total number of tensor structures is

$$\sum_{q_1=0}^{\sum_i l_1^i} \sum_{q_2=0}^{\sum_i l_1^i} \sum_{\mu} c_{[q_1][q_2]}{}^{\mu}\, b_{\mu\lambda_1\lambda_2\lambda_3\lambda_4} \equiv \sum_{q_1=0}^{\sum_i l_1^i} \sum_{q_2=0}^{\sum_i l_1^i} d(q_1, q_2)\,, \tag{3.73}$$

where in the last equality we defined the number of structures $d(q_1, q_2)$ for a given pair (q_1, q_2). In fact, when constructing the tensor structures, it is helpful to treat the contributions for each combination (q_1, q_2) separately. We make the assignment that the tensor structures corresponding to (q_1, q_2) contain q_1 building blocks $\mathcal{V}_{i1}^{(p)}$ and q_2 building blocks $\mathcal{V}_{i2}^{(p)}$. To make sense of this assignment one can consider two boxes in the same column p of one of the Young diagrams λ_i. The only way to assign these two boxes to $\mathcal{V}_{ij}^{(p)}$ building blocks is to assign one to $\mathcal{V}_{i1}^{(p)}$ and one to $\mathcal{V}_{i2}^{(p)}$, because of (3.47). Correspondingly, in the decomposition of the tensor product of $[q_1] \otimes [q_2]$ vertically aligned boxes always consist of one box belonging to $[q_1]$ and one to $[q_2]$. Finally, notice that for a given pair (q_1, q_2) there may be more than $d(q_1, q_2)$ possible combinations of building blocks $H_{ij}^{(p,q)}$ and $\mathcal{V}_{ij}^{(p)}$ that contain q_1 building blocks $\mathcal{V}_{i1}^{(p)}$ and q_2 building blocks $\mathcal{V}_{i2}^{(p)}$, however only $d(q_1, q_2)$ of these combinations will give raise to linearly independent structures after antisymmetrization (just as in the example of section 3.3.4).

3.4.1 Example: scalar-vector-scalar-vector

As an example, table 2 lists the five tensor structures in a four-point function of irreps $\lambda_1 = \lambda_3 = \bullet$ and $\lambda_2 = \lambda_4 = \square$, which were already given in [10, 17]. Let us see explicitly how these structures arise from the scalar degeneracy in the tensor product (3.71). The tensor product of two scalars and two vectors decomposes as

$$\bullet \otimes \square \otimes \bullet \otimes \square = \square \otimes \square = \bullet \oplus \square\!\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}. \tag{3.74}$$

That the scalar representation appears with multiplicity one here means that there is one tensor structure for $(q_1, q_2) = (0, 0)$, i.e. $d(0, 0) = 1$. For $(q_1, q_2) = (2, 0)$ or $(q_1, q_2) = (0, 2)$ we have to consider the tensor product

$$\bullet \otimes \square \otimes \bullet \otimes \square \otimes \square\!\square = \left(\bullet \oplus \square\!\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) \otimes \square\!\square = \bullet \oplus \text{other irreps}\,. \tag{3.75}$$

Thus $d(2, 0) = d(0, 2) = 1$. Finally, for $(q_1, q_2) = (1, 1)$ one needs to consider

$$\bullet \otimes \square \otimes \bullet \otimes \square \otimes \left[\square \otimes \square\right] = \left(\bullet \oplus \square\!\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) \otimes \left(\square\!\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = 2\,\bullet \oplus \text{other irreps}\,, \tag{3.76}$$

so that $d(1, 1) = 2$.

JHEP02(2015)151

q_1	q_2	$d(q_1,q_2)$	tensor structures
0	0	1	$H_{24}^{(Z,Z)}$
2	0	1	$\mathcal{V}_{21}^{(Z)}\mathcal{V}_{41}^{(Z)}$
1	1	2	$\mathcal{V}_{21}^{(Z)}\mathcal{V}_{42}^{(Z)}$, $\mathcal{V}_{22}^{(Z)}\mathcal{V}_{41}^{(Z)}$
0	2	1	$\mathcal{V}_{22}^{(Z)}\mathcal{V}_{42}^{(Z)}$

Table 2. All five tensor structures in a four-point function of irreps $\bullet$, $\square$, $\bullet$, $\square$.

q_1	q_2	tensor structures
2	0	$\mathcal{V}_{11}^{(\Theta)}\mathcal{V}_{11}^{(Z)}H_{12}^{(\Theta,Z)}$
1	1	$\mathcal{V}_{11}^{(Z)}\mathcal{V}_{12}^{(\Theta)}H_{12}^{(\Theta,Z)}$, $\mathcal{V}_{11}^{(\Theta)}\mathcal{V}_{12}^{(Z)}H_{12}^{(\Theta,Z)}$
0	2	$\mathcal{V}_{12}^{(\Theta)}\mathcal{V}_{12}^{(Z)}H_{12}^{(\Theta,Z)}$
3	1	$\mathcal{V}_{11}^{(\Theta)}\mathcal{V}_{11}^{(Z)}\mathcal{V}_{21}^{(Z)}\mathcal{V}_{12}^{(\Theta)}$
2	2	$\mathcal{V}_{11}^{(\Theta)}\mathcal{V}_{11}^{(Z)}\mathcal{V}_{12}^{(\Theta)}\mathcal{V}_{22}^{(Z)}$, $\mathcal{V}_{11}^{(\Theta)}\mathcal{V}_{21}^{(Z)}\mathcal{V}_{12}^{(\Theta)}\mathcal{V}_{12}^{(Z)}$
1	3	$\mathcal{V}_{11}^{(\Theta)}\mathcal{V}_{12}^{(\Theta)}\mathcal{V}_{12}^{(Z)}\mathcal{V}_{22}^{(Z)}$

Table 3. All eight tensor structures in a four-point function of irreps $\ydiagram{2,1}$, $\square$, $\bullet$, $\bullet$.

3.4.2 Example: hook-vector-scalar-scalar

For this example we consider the irreps $\lambda_1 = \ydiagram{2,1}$, $\lambda_2 = \square$, $\lambda_3 = \lambda_4 = \bullet$. Table 3 shows all the eight tensor structures for this correlator. This counting can be confirmed by looking at the OPE in the channel $\mathcal{O}_3 \times \mathcal{O}_4 \sim \mathcal{O}_l$. Then the number of possible structures in a three-point function of $\lambda_1 = \ydiagram{2,1}$, $\lambda_2 = \square$ and a spin l symmetric traceless tensor is also seen to be eight.[7]

Let us see again explicitly how these structures arise from the scalar degeneracy in the tensor product (3.71). First we consider the product

$$\ydiagram{2,1} \otimes \ydiagram{1} \otimes \bullet \otimes \bullet = \ydiagram{2} \oplus \ydiagram{1,1} \oplus \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1}\,. \tag{3.77}$$

Since there is no scalar irrep in this sum we have $d(0,0) = 0$. For the other values of (q_1, q_2), (3.77) must be multiplied by $\big[[q_1] \otimes [q_2]\big]$, and then $d(q_1,q_2)$ is just the multiplicity of the scalar irrep in the overall product. Table 4 shows the different possibilities.

3.4.3 Example: vector-vector-vector-vector

Finally, the correlation function of four vectors illustrates how the tensor product also generates the number of possible contractions between H's, i.e. those corresponding to

[7]This is a simple generalization of the example in subsection 3.3.4 which has $l = 2$. For $l \geq 3$, the three-point function of $\lambda_1 = \ydiagram{2,1}$, $\lambda_2 = \ydiagram{4}\cdots$ with $|\lambda_2| = l$ and $\lambda_3 = \square$ has the structures listed in table 1 each multiplied by $\left(V_2^{(Z)}\right)^{l-2}$ and the additional structure $H_{12}^{(\Theta,Z)}H_{12}^{(Z,Z)}H_{23}^{(Z,Z)}V_1^{(\Theta)}\left(V_2^{(Z)}\right)^{l-3}$.

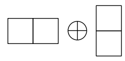
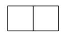
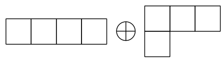
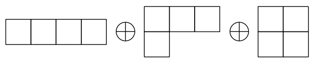
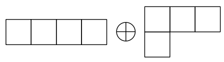
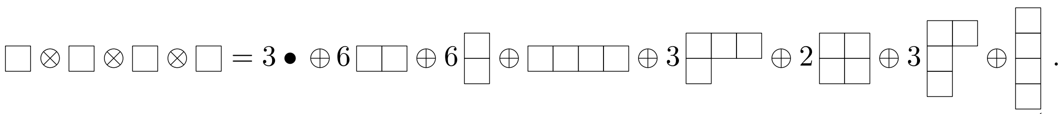
q_1	q_2	$\big[[q_1] \otimes [q_2]\big]$	$d(q_1, q_2)$
2	0		1
1	1		2
0	2		1
3	1		1
2	2		2
1	3		1

Table 4. From the product of $\big[[q_1] \otimes [q_2]\big]$ with (3.77) it is straightforward to extract the scalar multiplicity $d(q_1, q_2)$, which counts the independent tensor structures given in table 3.

$q_1 = q_2 = 0$. The number of such tensor structures is calculated using the SO(d) tensor product



(3.78)

Correspondingly, there are three tensor structures that can be built out of H's, namely

$$H_{12}^{(Z,Z)} H_{34}^{(Z,Z)}, \quad H_{13}^{(Z,Z)} H_{24}^{(Z,Z)} \quad \text{and} \quad H_{14}^{(Z,Z)} H_{23}^{(Z,Z)} . \tag{3.79}$$

There are $3!2^2$ other structures with two $\mathcal{V}$'s and one H and 2^4 other structures with four $\mathcal{V}$'s. Thus, in total for this case there are 43 independent tensor structures. As in the previous example, this counting is done by considering the scalar multiplicity in the product of $\big[[q_1] \otimes [q_2]\big]$ with (3.78). Table 5 shows the different possibilities to which it is trivial to assign the independent tensor structures.

3.5 n-point functions

Let us comment briefly on the general construction of n-point functions. It is analogous to the construction of four-point functions. Generically one can write,

$$G_{\chi_1 \cdots \chi_n}\big(\{P_i; \mathbf{Z}_i\}\big) = \prod_{g<h}^{n} P_{gh}^{-\alpha_{gh}} \prod_{j=1}^{n} \prod_{p=1}^{n_Z^j} \prod_{q=1}^{\min(l_p^j, n_\Theta^j)} \Big(Z_j^{(p)} \cdot \partial_{\Theta_j^{(q)}}\Big) \sum_k f_k(u_a)\, \bar{Q}^{(k)}_{\chi_1 \cdots \chi_n}\big(\{P_i; \mathbf{\Theta}_i\}\big) , \tag{3.80}$$

where u_a are the $n(n-3)/2$ independent conformally invariant cross-ratios,

$$\alpha_{gh} = \frac{\tau_g + \tau_h}{n-2} - \frac{1}{(n-1)(n-2)} \sum_{i=1}^{n} \tau_i , \tag{3.81}$$

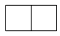
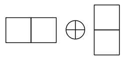
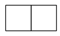
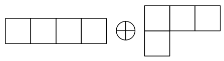
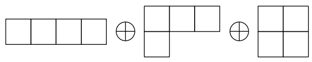
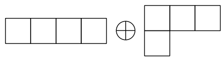
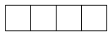
q_1	q_2	$[[q_1] \otimes [q_2]]$	$d(q_1,q_2)$
0	0	$\bullet$	3
2	0		6
1	1		12
0	2		6
4	0		1
3	1		4
2	2		6
1	3		4
0	4		1

Table 5. Multiplicity $d(q_1, q_2)$ counting tensor structures for the correlation function of four vectors.

and the pre-factor is chosen to let the functions $\bar{Q}^{(k)}_{\chi_1\ldots\chi_n}$ scale in the already familiar way

$$\begin{aligned}\bar{Q}^{(k)}_{\chi_1\ldots\chi_n}\left(\{\alpha_i P_i; \boldsymbol{\beta}_i(\boldsymbol{\Theta}_i + \boldsymbol{\gamma}_i P_i)\}\right) = \\ = \bar{Q}^{(k)}_{\chi_1\ldots\chi_n}\left(\{P_i;\boldsymbol{\Theta}_i\}\right) \prod_i \alpha_i^{|\lambda_i|} \left(\beta_i^{(1)}\right)^{h_1^i} \ldots \left(\beta_i^{(n_\Theta^i)}\right)^{h^i_{n^i_\Theta}} \left(\beta_i^{(Z)}\right)^{(\lambda_i)_1} .\end{aligned} \tag{3.82}$$

These functions are again constructed from the building blocks $H_{ij}^{(p)}$ and $\mathcal{V}_{ij}^{(p)}$ defined in (3.29) and (3.69). The counting of tensor structures is done as described for four-point functions in the previous section, but now the tensor product contains $n-2$ additional representations for counting all the combinations of $\mathcal{V}_{ij}^{(p)}$ building blocks, since there are that many independent $\mathcal{V}_{ij}^{(p)}$ for each i. The resulting number of tensor structures in a correlator of n operators in the irreps $\chi_i = [\lambda_i, \Delta_i]$ is the multiplicity of the scalar representation in the product

$$\lambda_1 \otimes \ldots \otimes \lambda_n \otimes \Big[[q_1] \otimes \ldots \otimes [q_{n-2}]\Big] , \tag{3.83}$$

which is given by

$$\sum_{q_1,\ldots,q_{n-2}} \sum_{\mu} c_{[q_1]\ldots[q_{n-2}]}{}^{\mu}\, b_{\mu\lambda_1\ldots\lambda_n} , \tag{3.84}$$

where the sums run over non-negative q_j's with

$$q_j \in \left\{0, 1, \ldots, \sum_{i=1}^{n} l_1^i\right\} . \tag{3.85}$$

As for four-point functions, to construct the tensor structures it is helpful to assign to each $(q_1, \ldots, q_{n-2})$ the tensor structures with q_1 copies of $\mathcal{V}_{i1}^{(p)}$, q_2 copies of $\mathcal{V}_{i2}^{(p)}$, and so on.

4 Conserved tensors

Let us now consider conserved tensors in arbitrary irreducible SO(d) representations. Recall that the unitarity bound for mixed-symmetry tensors [26, 27], that must be satisfied in unitary CFTs, restricts the conformal dimension of primaries in the irrep λ to satisfy the condition

$$\Delta \geq l_1^\lambda - h_{l_1}^\lambda + d - 1\,, \tag{4.1}$$

where $h_{l_1}^\lambda$ is the height of the rightmost column (the number of upper rows with the same number of boxes). The dimension for which (4.1) is saturated is called the critical dimension.

Let us first recall that, at the critical dimension, the conservation condition on fully symmetric or fully antisymmetric tensors $f_{a_1\ldots a_l}(x)$,

$$\frac{\partial}{\partial x_{a_1}} f_{a_1\ldots a_l}(x) = 0\,, \tag{4.2}$$

is conformally invariant. The question which equations are conformally invariant for more general representations of the conformal group was discussed in [28], and specifically for mixed-symmetry tensors of hook diagram type in [22]. We will show below that, for general mixed-symmetry tensors in irrep λ, the analogue of the conservation condition (4.2) can only be imposed with respect to indices that correspond to boxes in one of the lowest columns in the Young tableau, i.e. they can be written as

$$\frac{\partial}{\partial x_{g_1}} f_{[a_1\ldots a_{h_1}][b_1\ldots b_{h_2}]\ldots[g_1\ldots g_{h_{l_1}}]}(x) = 0\,. \tag{4.3}$$

We will see that this equation can be imposed directly in embedding space. At the same time this will allow us to see that it is conformally invariant only when the unitarity bound (4.1) is saturated, and that similar equations with the derivative contracted with a different index are not conformally invariant.

The computation was done in [10] for symmetric tensors and the only part that changes is when the index symmetries are used. Let us first write

$$\begin{aligned}\frac{\partial}{\partial x_{a_{|\lambda|}}} f_{a_1\ldots a_{|\lambda|}}(x) &= \frac{\partial}{\partial x_{a_{|\lambda|}}}\left(\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|}}}{\partial x^{a_{|\lambda|}}} F_{A_1\ldots A_{|\lambda|}}(P_x)\right)\\ &= \frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}} S_{A_1\ldots A_{|\lambda|-1}}(P_x) + T_{a_1\ldots a_{|\lambda|-1}}(x)\,,\end{aligned} \tag{4.4}$$

where the projection from $F_{A_1\ldots A_{|\lambda|}}$ to $f_{a_1\ldots a_{|\lambda|}}$ given in (2.49) was inserted and

$$S_{A_1\ldots A_{|\lambda|-1}}(P) = \left[\frac{\partial}{\partial P_{A_{|\lambda|}}} - \frac{1}{P\cdot\bar{P}}\left(\bar{P}\cdot\frac{\partial}{\partial P}\right)P^{A_{|\lambda|}} - (d-1-\Delta)\frac{\bar{P}^{A_{|\lambda|}}}{P\cdot\bar{P}}\right] F_{A_1\ldots A_{|\lambda|}}(P)\,, \tag{4.5}$$

is obtained in the same way as in [10], with $\bar{P} = (0, 2, 0)$ in the light-cone coordinates introduced in (2.45). The part $T_{a_1\ldots a_{|\lambda|-1}}(x)$ comprises terms where $\frac{\partial}{\partial x_a}$ acts on the $\frac{\partial P^A}{\partial x^b}$ and can be simplified using

$$\frac{\partial}{\partial x_a}\frac{\partial P^A}{\partial x^b} = \delta_{ab}\bar{P}^A. \tag{4.6}$$

This is the part where the index symmetries are important

$$T_{a_1\ldots a_{|\lambda|-1}}(x) = -\frac{1}{P\cdot\bar{P}}\frac{\partial P^{A_{|\lambda|}}}{\partial x^{a_{|\lambda|}}}\Bigg[\delta_{a_{|\lambda|}a_1}\bar{P}^{A_1}\frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}}+\ldots$$
$$+\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-2}}}{\partial x^{a_{|\lambda|-2}}}\delta_{a_{|\lambda|}a_{|\lambda|-1}}\bar{P}^{A_{|\lambda|-1}}\Bigg]F_{A_1\ldots A_{|\lambda|}}(P)$$
$$= -\frac{1}{P\cdot\bar{P}}\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}}\bar{P}^{A_{|\lambda|}}$$
$$\Big[F_{A_{|\lambda|}A_2\ldots A_{|\lambda|-1}A_1}+F_{A_1A_{|\lambda|}A_3\ldots A_{|\lambda|-1}A_2}+\ldots+F_{A_1\ldots A_{|\lambda|-2}A_{|\lambda|}A_{|\lambda|-1}}\Big]. \tag{4.7}$$

The second identity here is just a relabelling of indices. The sum in the last brackets simplifies due to the index symmetries (2.18) and becomes

$$\big((l_1-1)-(h_{l_1}-1)\big)F_{A_1\ldots A_{|\lambda|}}\,. \tag{4.8}$$

Note that this step is only possible since the derivative in (4.3) is contracted with an index in the rightmost column of the Young tableau. The lift of the conservation condition with respect to the last index is then

$$0=\frac{\partial}{\partial x_{a_{|\lambda|}}}f_{a_1\ldots a_{|\lambda|}}(x)=\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_{|\lambda|-1}}}{\partial x^{a_{|\lambda|-1}}}R_{A_1\ldots A_{|\lambda|-1}}(P_x)\,, \tag{4.9}$$

where

$$R_{A_1..A_{|\lambda|-1}}(P)=\left[\frac{\partial}{\partial P_{A_{|\lambda|}}}-\frac{1}{P\cdot\bar{P}}\left(\bar{P}\cdot\frac{\partial}{\partial P}\right)P^{A_{|\lambda|}}-(l_1-h_{l_1}+d-1-\Delta)\frac{\bar{P}^{A_{|\lambda|}}}{P\cdot\bar{P}}\right]F_{A_1..A_{|\lambda|}}(P). \tag{4.10}$$

This generalises the result derived in [10] for symmetric tensors. As discussed in [10], the first two terms in (4.10) are $SO(d+1,1)$ invariant. The last term is not, but it vanishes for conserved tensors which saturate the unitarity bound (4.1). Because of the index symmetries (2.19) the derivative in the conservation condition (4.9) can be contracted with any index that belongs to a column in the Young diagram of the same height as the rightmost one. In particular it may be contracted with any index in the case of rectangular Young diagrams.

There is actually a second conformally invariant condition that can be imposed on mixed-symmetry tensors. This was found for hook diagrams in [22] and requires a value for Δ different from the critical dimension. It is now very easy to find the dimension where this condition can be imposed for general mixed-symmetry tensors simply by lifting the conservation condition to the embedding space. This is most easily seen in the symmetric basis, so now take f to be in the symmetric basis as in (2.20) and consider the conservation condition

$$\frac{\partial}{\partial x_{g_1}}f_{(a_1\ldots a_{l_1})(b_1\ldots b_{l_2})\ldots(g_1\ldots g_{l_{h_1}})}(x)=0\,. \tag{4.11}$$

The lift to embedding space (4.4)–(4.7) works exactly as before. Now (2.21) is used to bring the last bracket in (4.7) into a form analogous to (4.10),

$$-\big((h_1-1)-(l_{h_1}-1)\big)F_{A_1\ldots A_{|\lambda|}}\,. \tag{4.12}$$

The conservation condition (4.11) becomes

$$0=\left[\frac{\partial}{\partial P_{A_{|\lambda|}}}-\frac{1}{P\cdot\bar{P}}\left(\bar{P}\cdot\frac{\partial}{\partial P}\right)P^{A_{|\lambda|}}-\left(l_{h_1}-h_1+d-1-\Delta\right)\frac{\bar{P}^{A_{|\lambda|}}}{P\cdot\bar{P}}\right]F_{A_1\ldots A_{|\lambda|}}(P)\,. \tag{4.13}$$

This is conformally invariant for

$$\Delta=l_{h_1}-h_1+d-1\,. \tag{4.14}$$

For rectangular Young diagrams, where $h_{l_1}=h_1$ and $l_{h_1}=l_1$ this is again the critical dimension. However, in general, we have $h_{l_1}\leq h_1$ and $l_{h_1}\leq l_1$, hence the unitarity bound (4.1) is violated for non-rectangular diagrams and the operators for which (4.13) is conformally invariant are non-unitary.

5 Conformal blocks

In this section we shall show how the above methods can be used to compute the conformal blocks for arbitrary irreducible tensor representations of the conformal group. The basic idea is that a conformal block in the channel $\mathcal{O}_1\,\mathcal{O}_2\to\mathcal{O}_3\,\mathcal{O}_4$ can be written as a conformal integral of the product of the 3-point function of the operators $\mathcal{O}_1$, $\mathcal{O}_2$ and the exchanged operator $\mathcal{O}$ of dimension Δ, times the 3-point function of the operators $\mathcal{O}_3$, $\mathcal{O}_4$ and the shadow of the exchanged operator $\tilde{\mathcal{O}}$ of dimension $d-\Delta$ [15, 29, 30]. This method makes use of the shadow formalism of [31–34]. In practice, however, one needs to remove from the final expression the contribution of the shadow operator exchange to the conformal block, which has the wrong OPE limit. This can be done rather efficiently by doing a monodromy projection of the above conformal integral, as proposed in [11].[8]

Conformal blocks are known for many cases involving external scalar operators and the exchange of spin l symmetric tensors. These results can be reused for correlators of external spin l operators by acting with differential operators on the conformal blocks for external scalars [36], but new exchanged tensor representations can not be taken care of in this way. Here we will follow closely the approach detailed in [11] to compute the conformal blocks, and show with a non-trivial example that the embedding methods here presented can be used to compute conformal blocks with external and exchanged operators in arbitrary tensor representations of the conformal group.

The idea is to define a projector to the conformal multiplet of a given operator which, when inserted into a four-point function, produces the conformal partial wave for the exchange of that operator (and its descendants). For an operator $\mathcal{O}$ with conformal dimension Δ this projector has the form

$$|\mathcal{O}|=\frac{1}{\mathcal{N}_{\mathcal{O}}}\int D^dP_0D^dP_5\,|\mathcal{O}(P_0;\mathbf{D}_{\mathbf{Z}_0})\rangle\,\langle\mathcal{O}(P_0;\mathbf{Z}_0)\mathcal{O}(P_5;\mathbf{D}_{\mathbf{Z}_5})\rangle\big|_{\Delta\to\tilde{\Delta}}\,\langle\mathcal{O}(P_5;\mathbf{Z}_5)|\,. \tag{5.1}$$

Note that we are schematically representing the index contraction of $\mathcal{O}$ with a differential operator acting on the polarization vectors, as explained in (2.60). The integrals appearing

[8]Such split of the operator and its shadow exchanges can also be done using the Mellin space representation of the conformal partial wave [35].

JHEP02(2015)151

here are called conformal integrals and are defined as

$$\int D^d P = \frac{1}{\text{Vol } GL(1,\mathbb{R})^+} \int_{P^+ + P^- \geq 0} \mathrm{d}^{d+2} P \; \delta(P^2) \,. \tag{5.2}$$

Explicit expressions for these integrals are known for all functions that appear in the computation of conformal blocks (see for instance appendix A.5 in [37]).

The projector (5.1) can be more compactly expressed in terms of the shadow operator $\tilde{\mathcal{O}}$, which is in the same SO(d) irrep as $\mathcal{O}$ and has conformal dimension $\tilde{\Delta} = d - \Delta$,

$$|\mathcal{O}| = \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d P_0 \, |\mathcal{O}(P_0; \mathbf{D}_{\mathbf{Z}_0})\rangle\langle\tilde{\mathcal{O}}(P_0; \mathbf{Z}_0)| \,, \tag{5.3}$$

where

$$\langle\tilde{\mathcal{O}}(P_0; \mathbf{Z}_0)| = \int D^d P_5 \, \langle\mathcal{O}(P_0; \mathbf{Z}_0)\mathcal{O}(P_5; \mathbf{D}_{\mathbf{Z}_5})\rangle\big|_{\Delta\to\tilde{\Delta}} \, \langle\mathcal{O}(P_5; \mathbf{Z}_5)| \,. \tag{5.4}$$

Consider for simplicity the case where the three-point functions have only one tensor structure. Inserting $|\mathcal{O}|$ into a four-point function one obtains the conformal partial wave

$$\begin{aligned} W_{\mathcal{O}} &= \langle\mathcal{O}_1(P_1; \mathbf{Z}_1)\mathcal{O}_2(P_2; \mathbf{Z}_2)|\mathcal{O}|\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle \\ &= \frac{1}{\mathcal{N}_{\mathcal{O}}} \int D^d P_0 \, \langle\mathcal{O}_1(P_1; \mathbf{Z}_1)\mathcal{O}_2(P_2; \mathbf{Z}_2)\mathcal{O}(P_0; \mathbf{D}_{\mathbf{Z}_0})\rangle\langle\tilde{\mathcal{O}}(P_0; \mathbf{Z}_0)\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle. \end{aligned} \tag{5.5}$$

Since $\tilde{\mathcal{O}}$ is in the same SO(d) irrep as $\mathcal{O}$, three-point functions containing either of them must be equal, up to an overall constant and to the conformal dimensions of the operators, i.e.

$$\langle\tilde{\mathcal{O}}(P_0; \mathbf{Z}_0)\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle = \mathcal{S}_\Delta \, \langle\mathcal{O}(P_0; \mathbf{Z}_0)\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle\big|_{\Delta\to\tilde{\Delta}} \,. \tag{5.6}$$

This constant $\mathcal{S}_\Delta$ is calculated by using the definition of the shadow operator (5.4) and by computing the corresponding conformal integral. The constant $\mathcal{N}_{\mathcal{O}}$ in (5.5) can then be calculated by demanding that $|\mathcal{O}|$ acts trivially when inserted into a three-point function, i.e. requiring

$$\langle\mathcal{O}(P_0; \mathbf{Z}_0)|\mathcal{O}|\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle = \langle\mathcal{O}(P_0; \mathbf{Z}_0)\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle \,. \tag{5.7}$$

Using (5.3) and (5.4) one sees that this insertion amounts to doing the shadow transformation twice, hence with (5.6) we have

$$\begin{aligned} \langle\mathcal{O}(P_0; \mathbf{Z}_0)|\mathcal{O}|\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle &= \frac{1}{\mathcal{N}_{\mathcal{O}}} \langle\tilde{\tilde{\mathcal{O}}}(P_0; \mathbf{Z}_0)\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle \\ &= \frac{\mathcal{S}_\Delta \mathcal{S}_{\tilde{\Delta}}}{\mathcal{N}_{\mathcal{O}}} \langle\mathcal{O}(P_0; \mathbf{Z}_0)\mathcal{O}_3(P_3; \mathbf{Z}_3)\mathcal{O}_4(P_4; \mathbf{Z}_4)\rangle \,, \end{aligned} \tag{5.8}$$

and thus $\mathcal{N}_{\mathcal{O}} = \mathcal{S}_\Delta \mathcal{S}_{\tilde{\Delta}}$.

5.1 Example: hook diagram exchange

As an example we will compute the conformal block $g_T^{\Delta_i}(u,v)$ for the exchange of the tensor T with irreducible representation $[\yng(2,1),\Delta]$ in the correlation function of two scalars and two vectors $\langle\phi_1 J_2^\mu \phi_3 J_4^\nu\rangle$. The conformal partial wave is

$$\begin{aligned} W_T &= \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_{34}}{2}}\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}}\frac{g_T^{\Delta_i}(u,v)}{P_{12}^{\frac{\Delta_1+\Delta_2}{2}}P_{34}^{\frac{\Delta_3+\Delta_4}{2}}} \\ &= \langle\phi_1(P_1)J_2(P_2;Z_2)|T|\phi_3(P_3)J_4(P_4;Z_4)\rangle \\ &= \frac{1}{\mathcal{S}_{\tilde\Delta}}\int D^dP_0\,\langle\phi_1(P_1)J_2(P_2;Z_2)T(P_0;D_{Z_0},\partial_{\Theta_0})\rangle\langle T(P_0;Z_0,\Theta_0)\phi_3(P_3)J_4(P_4;Z_4)\rangle|_{\Delta\to\tilde\Delta}\,, \end{aligned} \tag{5.9}$$

where we recall that u,v are the cross ratios defined in (3.65) and that the function $g_T^{\Delta_i}(u,v)$ also depends on the external polarization vectors Z_2 and Z_4. The ingredients for this calculation are the two- and three-point functions from (3.40) and (3.62), for which we choose the normalizations

$$\langle T(P_1;Z_1,\Theta_1)\,T(P_2;Z_2,\Theta_2)\rangle = \frac{2\left(H_{12}^{(\Theta,\Theta)}H_{12}^{(Z,Z)}-H_{12}^{(\Theta,Z)}H_{12}^{(Z,\Theta)}\right)H_{12}^{(Z,Z)}}{(P_{12})^{\Delta+3}}\,, \tag{5.10}$$

$$\langle T(P_0;Z_0,\Theta_0)\phi_3(P_3)J_4(P_4;Z_4)\rangle = \frac{V_{0,34}^{(\Theta)}V_{0,34}^{(Z)}H_{04}^{(Z,Z)}-\left(V_{0,34}^{(Z)}\right)^2H_{04}^{(\Theta,Z)}}{(P_{03})^{\frac{\Delta+\Delta_3-\Delta_4+2}{2}}(P_{34})^{\frac{\Delta_3+\Delta_4-\Delta-2}{2}}(P_{40})^{\frac{\Delta_4+\Delta-\Delta_3+4}{2}}}\,,$$

the differential operator D_Z from (2.59) which encodes the projection to the irrep $\yng(2,1)$, the constant $\mathcal{S}_\Delta$ and the solution of the conformal integrals.

The constant $\mathcal{S}_\Delta$ is computed using (5.6) and evaluating the conformal integral

$$\begin{aligned} &\langle\tilde T(P_0;Z_0,\Theta_0)\phi_3(P_3)J_4(P_4;Z_4)\rangle \\ &\quad= \int D^dP_5\,\langle T(P_0;Z_0,\Theta_0)\,T(P_5;D_{Z_5},\partial_{\Theta_5})\rangle|_{\Delta\to\tilde\Delta}\,\langle T(P_5;Z_5,\Theta_5)\phi_3(P_3)J_4(P_4;Z_4)\rangle\,. \end{aligned} \tag{5.11}$$

All the integrals here are of the type

$$\int D^dP_5\,\frac{P_5^{A_1}\dots P_5^{A_n}}{(P_{50})^a(P_{53})^b(P_{54})^c}\,, \tag{5.12}$$

and their explicit solution can be found for instance in [37, 38].[9] Comparing the integral in (5.11) with the three-point function, the resulting constant is

$$\mathcal{S}_\Delta^{\yng(2,1)} = \frac{\pi^h(\Delta-2)\Delta\,\Gamma(\Delta-h)}{\Gamma(\tilde\Delta+2)}\frac{\Gamma\left(\frac{\tilde\Delta+\Delta_{34}+2}{2}\right)\Gamma\left(\frac{\tilde\Delta-\Delta_{34}+2}{2}\right)}{\Gamma\left(\frac{\Delta+\Delta_{34}+2}{2}\right)\Gamma\left(\frac{\Delta-\Delta_{34}+2}{2}\right)}\,. \tag{5.14}$$

[9]To give an impression of how these integrals look like, here is the case with $n=1$

$$\begin{aligned} \int D^dP_5\frac{P_5^A}{(P_{50})^a(P_{53})^b(P_{54})^c} &= \frac{\Gamma\left(\frac{b+c-a+1}{2}\right)\Gamma\left(\frac{c+a-b+1}{2}\right)\Gamma\left(\frac{a+b-c+1}{2}\right)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}\frac{\pi^h}{(P_{34})^{\frac{b+c-a+1}{2}}(P_{40})^{\frac{c+a-b+1}{2}}(P_{03})^{\frac{a+b-c+1}{2}}} \\ &\quad\times\left(\frac{P_{34}P_0^A}{\frac12(b+c-a+1)}+\frac{P_{40}P_3^A}{\frac12(c+a-b+1)}+\frac{P_{03}P_4^A}{\frac12(a+b-c+1)}\right). \end{aligned} \tag{5.13}$$

JHEP02(2015)151

Note that this is very similar to the corresponding constant for the exchange of the anti-symmetric two-tensor $\boxminus$, given below in (5.21), which was calculated in [11]. As a small consistency check observe that the constant $\mathcal{N}_{\mathcal{O}} = \mathcal{S}_\Delta \mathcal{S}_{\tilde{\Delta}}$ appearing in (5.1) is independent of Δ_{34}.

To calculate the conformal partial wave (5.9) it is enough to know the conformal integrals

$$\begin{aligned} &\int D^d P_0 \frac{(P_0 \cdot Z_2)(P_0 \cdot Z_4)}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f}\,, \qquad \int D^d P_0 \frac{P_0 \cdot Z_2}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f}\,, \\ &\int D^d P_0 \frac{P_0 \cdot Z_4}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f}\,, \qquad \int D^d P_0 \frac{1}{(P_{01})^a (P_{02})^b (P_{03})^e (P_{04})^f}\,, \end{aligned} \tag{5.15}$$

which much like the example (5.13) can be brought into a form where the polarizations are contracted with P_1, P_2, P_3 and P_4, or with each other. Just as in [11], after doing the monodromy projection to eliminate the shadow block, the final expression depends on functions of the cross ratios u, v given by

$$J^{(i)}_{j,k,l} = \frac{\Gamma(h+i-f)\,\Gamma(f)\sin(\pi f)}{\sin\big(\pi(e+f-h-i)\big)} \int_0^\infty \frac{\mathrm{d}x}{x} \int_{x+1}^\infty \frac{\mathrm{d}y}{y} \frac{x^b y^e}{(y+vxy-ux)^{h+i-f}(y-x-1)^f}\,, \tag{5.16}$$

with

$$\begin{aligned} b &= \alpha + i + j - 1\,, \\ e &= \beta - \Delta + h + i + k - l\,, \\ f &= 1 - \beta + h - k\,, \end{aligned} \tag{5.17}$$

and

$$\alpha = \frac{\Delta - \Delta_{12} - 2}{2}\,, \qquad \beta = \frac{\Delta + \Delta_{34} - 2}{2}\,, \tag{5.18}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$ and $h = d/2$. In even dimensions, $h \in \mathbb{N}$, the functions $J^{(i)}_{j,k,l}$ can be expressed in terms of ${}_2F_1$ hypergeometric functions, see [11].

Doing the computation we arrived at the following expression for the conformal block defined in (5.9),

$$\begin{aligned} g_T^{\Delta_i}(u,v) = &\frac{u^{\Delta/2-1}\Gamma(\Delta+2)}{4P_{24}(\tilde{\Delta}-2)\tilde{\Delta}(2h-1)\,\Gamma(\alpha+2)\,\Gamma(\beta+2)\,\Gamma(\Delta-\alpha)\,\Gamma(\Delta-\beta)\,\Gamma(h-\Delta)} \\ &\times \left[V^{(Z)}_{2,14}V^{(Z)}_{4,12}\,uF_1 + V^{(Z)}_{2,14}V^{(Z)}_{4,23}\,vF_2 + V^{(Z)}_{2,34}V^{(Z)}_{4,12}\,uF_3 + V^{(Z)}_{2,34}V^{(Z)}_{4,23}\,vF_4 + \frac{1}{2}H^{(Z,Z)}_{24}F_H\right]. \end{aligned} \tag{5.19}$$

As expected, this conformal block is organized into tensor structures that are analogous to the ones discussed for this four-point correlator in section 3.4.1. The functions $F_i = F_i(u,v)$ depend on h, Δ, α and β, and are expressed in terms of a finite number of the integrals $J^{(i)}_{j,k,l}$ given in (5.16) above. For clarity of exposition we decided to present these functions in the appendix A.[10]

[10] A Mathematica notebook containing this result can be obtained from the authors upon request.

The example at hand shows that we have a well defined algorithm to compute any conformal block. However, before going on to compute even more complicated conformal blocks, it would be helpful to study the functions $J^{(i)}_{j,k,l}$ in detail. Once the relations among them are better understood, it may well be that much shorter expressions for the conformal blocks are possible. We hope to return to this question.

5.2 Example: two-form exchange

The conformal block for exchange of a two-form tensor F, which corresponds to the irrep $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$, was computed analogously in [11], however the result contained a few typos which we now correct.[11] The normalizations for the two- and three-point functions of [11] are in our notation

$$\begin{aligned}
\big\langle F(P_1,\Theta_1)F(P_2,\Theta_2)\big\rangle &= \frac{1}{4}\frac{\left(H_{12}^{(\Theta,\Theta)}\right)^2}{(P_{12})^{\Delta+2}}\,,\\
\big\langle F(P_0,\Theta_0)\phi_3(P_3)J_4(P_4,Z_4)\big\rangle &= \frac{V_{0,34}^{(\Theta)}H_{04}^{(\Theta,Z)}}{(P_{03})^{\frac{\Delta+\Delta_3-\Delta_4+1}{2}}(P_{34})^{\frac{\Delta_3+\Delta_4-\Delta-1}{2}}(P_{40})^{\frac{\Delta_4+\Delta-\Delta_3+3}{2}}}\,,
\end{aligned} \tag{5.20}$$

and the contraction of two-forms is now done using the normalized derivative $\partial_\Theta/\sqrt{2}$. The constant $\mathcal{S}_\Delta$ is given by

$$\mathcal{S}_\Delta^{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \frac{\pi^h(\Delta-2)\,\Gamma(\Delta-h)}{4\,\Gamma(\tilde\Delta+1)}\frac{\Gamma\left(\frac{\tilde\Delta+\Delta_{34}+1}{2}\right)\Gamma\left(\frac{\tilde\Delta-\Delta_{34}+1}{2}\right)}{\Gamma\left(\frac{\Delta+\Delta_{34}+1}{2}\right)\Gamma\left(\frac{\Delta-\Delta_{34}+1}{2}\right)}\,. \tag{5.21}$$

After doing carefully the conformal integrals we obtained a slightly shorter formula for this conformal block,

$$\begin{aligned}
g_F^{\Delta_i}(u,v) = {}& \frac{2u^{\Delta/2-1/2}\,\Gamma(\Delta+1)}{P_{24}(2-\tilde\Delta)\,\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\Delta-\alpha)\,\Gamma(\Delta-\beta)\,\Gamma(h-\Delta)}\\
&\times\Bigg[V^{(Z)}_{2,14}V^{(Z)}_{4,12}\,u\Big((v-1)J^{(2)}_{0,1,2}+v(\beta-h+1)J^{(1)}_{1,2,2}+(\beta-\Delta+h)J^{(1)}_{1,1,2}\Big)\\
&-V^{(Z)}_{2,14}V^{(Z)}_{4,23}\,v(v-1)J^{(2)}_{0,1,1}\\
&+V^{(Z)}_{2,34}V^{(Z)}_{4,12}\,u\Big((v-1)J^{(2)}_{0,1,1}-\alpha J^{(1)}_{0,1,1}+(\beta-h+1)\left(vJ^{(1)}_{1,2,1}-\alpha J^{(0)}_{1,2,1}\right)\\
&+(\beta-\alpha+h-1)J^{(1)}_{1,1,1}\Big)\\
&+V^{(Z)}_{2,34}V^{(Z)}_{4,23}\,v\Big(-(v-1)J^{(2)}_{0,1,0}+\alpha J^{(1)}_{0,1,0}+(\alpha-\Delta+1)J^{(1)}_{1,1,0}\Big)\\
&-\frac{1}{2}H^{(Z,Z)}_{24}\Bigg(-\frac{v-1}{h+1}\Big(J^{(2)}_{0,0,0}+J^{(2)}_{0,0,1}+J^{(2)}_{0,1,1}+J^{(2)}_{1,0,1}+vJ^{(2)}_{1,1,1}+uJ^{(2)}_{1,1,2}\Big)\\
&+(\beta-\Delta+h)\Big(\alpha J^{(0)}_{1,1,1}+(\alpha-\Delta+1)J^{(0)}_{2,1,1}\Big)\\
&+(\beta-h+1)\Big(\alpha J^{(0)}_{1,2,1}+(\alpha-\Delta+1)vJ^{(0)}_{2,2,1}\Big)\Bigg)\Bigg]\,,
\end{aligned} \tag{5.22}$$

[11] We thank David Simmons-Duffin for correspondence on this point.

JHEP02(2015)151

where $J^{(i)}_{j,k,l}$ is defined in (5.16), but now with

$$\alpha = \frac{\Delta - \Delta_{12} - 1}{2}\,, \qquad \beta = \frac{\Delta + \Delta_{34} - 1}{2}\,. \tag{5.23}$$

To compare this to the corrected result of [11], we took into account the different definitions for H_{ij} and $V_{i,jk}$, and used the three following identities which we checked numerically,

$$\begin{aligned}
&u\Big((v-1)J^{(2)}_{0,1,2} + v(\beta-h+1)J^{(1)}_{1,2,2} + (\beta-\Delta+h)J^{(1)}_{1,1,2}\Big)\\
&=(\beta-\Delta+h)\left(\alpha J^{(0)}_{1,1,1} - J^{(1)}_{1,0,1}\right) - v(\beta-h+1)\left(J^{(0)}_{2,2,1}(\alpha-\Delta+1) + J^{(1)}_{1,2,1}\right)\\
&\quad - \alpha J^{(1)}_{0,1,1} - 2v(\alpha+\beta-\Delta+1)J^{(1)}_{1,1,1} - v(\alpha-\Delta+1)\left(J^{(1)}_{1,1,0} + J^{(1)}_{2,1,1}\right)\\
&\quad - \alpha v J^{(1)}_{0,1,0} + (v-1)\left(vJ^{(2)}_{0,1,0} + vJ^{(2)}_{1,1,1} - J^{(2)}_{0,0,1}\right),
\end{aligned} \tag{5.24}$$

$$-uJ^{(2)}_{0,1,1} = J^{(2)}_{0,0,0} + vJ^{(2)}_{0,1,0} - \alpha J^{(1)}_{0,1,0}\,, \tag{5.25}$$

$$\begin{aligned}
&(v-1)\,J^{(2)}_{0,1,1} - \alpha J^{(1)}_{0,1,1} + (\beta-h+1)\left(vJ^{(1)}_{1,2,1} - \alpha J^{(0)}_{1,2,1}\right) + (\beta-\alpha+h-1)J^{(1)}_{1,1,1}\\
&= -(v-1)\left(J^{(2)}_{0,1,0} + J^{(2)}_{1,1,1}\right) + (\beta-h+1)\left(J^{(1)}_{1,2,1} + (\alpha-\Delta+1)J^{(0)}_{2,2,1}\right) + \alpha J^{(1)}_{0,1,0}\\
&\quad + (\alpha+\beta-\Delta+h)J^{(1)}_{1,1,1} + (\alpha-\Delta+1)\left(J^{(1)}_{1,1,0} + J^{(1)}_{2,1,1}\right).
\end{aligned} \tag{5.26}$$

6 S-matrix rule for counting structures

The matching of tensor structures in CFT correlators and scattering amplitudes that was found for symmetric tensors in [10] straightforwardly generalizes to general irreps when considering non-conserved operators. The general statement is: *The number of independent structures in a correlation function of n non-conserved operators of* SO(d) *irreps $\lambda_1, \ldots, \lambda_n$ is equal to to the number of independent structures in a n-point scattering amplitude of massive particles of the same irreps in $d+1$ dimensional flat Minkowski space.*

This is not surprising since particles have polarizations in irreps of the little group, which is SO(d) for massive particles in $d+1$ dimensions. The index-free notation introduced in section 2.4 can be employed by simply using the same Young-symmetrized polarizations. Thus, an n-point scattering amplitude of irreps $\lambda_1, \ldots, \lambda_n$ and momenta $k_1, \ldots, k_n$ can be written as

$$A_{\lambda_1\ldots\lambda_n}(\{k_i;\mathbf{z}_i\}) = \prod_{j=1}^{n}\prod_{p=1}^{n_Z^j}\prod_{q=1}^{\min(l_p^j,n_\Theta^j)}\left(z_j^{(p)}\cdot\partial_{\theta_j^{(q)}}\right)\sum_k f_k(v_a)\,\bar{R}^{(k)}_{\lambda_1\ldots\lambda_n}(\{k_i;\theta_i\})\,, \tag{6.1}$$

where $f_k(v_a)$ are functions of the $n(n-3)/2$ independent Mandelstams v_a. The momenta and polarizations are vectors in $(d+1)$-dimensional Minkowski space and the polarizations are transverse to the corresponding momenta

$$\theta_i^{(p)}\cdot k_i = z_i^{(p)}\cdot k_i = 0\,. \tag{6.2}$$

The scaling in the polarization vectors is fixed by the condition that the complete polarization tensor appears linearly in the amplitude. This translates to the following scaling of $\bar{R}^{(k)}_{\lambda_1\ldots\lambda_n}$ in the polarization vectors, which is equivalent to the one for CFT correlators in (3.82),

$$\bar{R}^{(k)}_{\lambda_1\ldots\lambda_n}\left(\{k_i;\boldsymbol{\beta}_i\theta_i\}\right) = \bar{R}^{(k)}_{\lambda_1\ldots\lambda_n}\left(\{k_i;\theta_i\}\right)\prod_i \left(\beta_i^{(1)}\right)^{h_1^i}\ldots\left(\beta_i^{(n_\Theta^i)}\right)^{h^i_{n^i_\Theta}}\left(\beta_i^{(Z)}\right)^{(\lambda_i)_1}. \tag{6.3}$$

The functions $\bar{R}^{(k)}_{\lambda_1\ldots\lambda_n}$ can be constructed from the two kinds of building blocks

$$\tilde{H}^{(p,q)}_{ij} \equiv \theta_i^{(p)}\cdot\theta_j^{(q)}\,, \qquad \tilde{\mathcal{V}}^{(p)}_{ij} \equiv \theta_i^{(p)}\cdot k_j\,, \tag{6.4}$$

where $\theta_i^{(p)}$ should be replaced by $z_i^{(1)}$ for $p = z$. There are $n-2$ independent $\tilde{\mathcal{V}}^{(p)}_{ij}$'s for each i, because one of the possible terms vanishes due to the transversality condition (6.2) and another one can be eliminated using momentum conservation

$$k_1 + k_2 + \ldots + k_n = 0\,. \tag{6.5}$$

Furthermore, the building blocks depend in the same way on Grassmann polarizations as their counterparts $H^{(p,q)}_{ij}$ and $\mathcal{V}^{(p)}_{ij}$ that appear in CFT correlators. Hence, there is a one-to-one correspondence between building blocks and the counting of tensor structures is the same as in CFT correlators.

A more thorough treatment of on-shell amplitudes of arbitrary SO(d) irreps (in the context of the open bosonic string) can be found in [39].

7 Concluding remarks

In this work we developed a formalism to elegantly describe irreducible tensor representations of SO(d) in terms of polynomials. With this formalism and the help of representation theory, tensor structures in CFT correlators and scattering amplitudes become tangible. We gave an algorithm for counting the number of independent tensor structures in any CFT correlator (or massive scattering amplitude) of bosonic operators (or particles), allowing for a systematic construction of the tensor structures for any given example.

The most obvious application for correlators of mixed-symmetry tensors is the construction of conformal blocks, which we reviewed using our new index-free notation. Once all conformal blocks appearing in a given correlator are known, it is possible to implement constrains that follow from conformal symmetry, using recent conformal bootstrap techniques, i.e. proving bounds on the CFT data (conformal dimensions Δ_i and OPE coefficients) by use of linear programming [3]. Since there are no further assumptions, such bounds are universal, they hold for any CFT. Until now, in lack of conformal blocks for mixed-symmetry tensor exchange, this has only be done for correlators of scalar operators.

While we only computed one conformal block of mixed-symmetry tensor exchange in a correlator of two scalars and two vectors, it would be much more interesting to consider correlators of stress-tensors. This is because the stress-tensor appears in any CFT and

JHEP02(2015)151

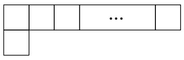
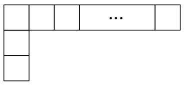
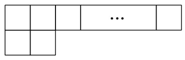
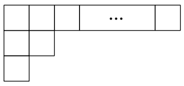
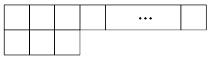
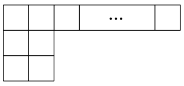
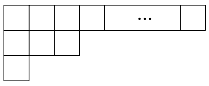
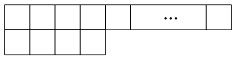
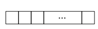
correlator	new exchanged SO(d) irreps
$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$	
$\langle \phi_1 J_2^\mu \phi_3 J_4^\nu \rangle$	
$\langle J_1^\mu J_2^\nu J_3^\rho J_4^\sigma \rangle$	 , 
$\langle J_1^\mu T_2^{\nu\rho} J_3^\sigma T_4^{\lambda\kappa} \rangle$	 , 
$\langle T_1^{\mu\nu} T_2^{\rho\sigma} T_3^{\lambda\kappa} T_4^{\tau\omega} \rangle$	 ,  , 

Table 6. Exchanged irreps in correlators of currents and stress-tensors, following the discussion of possible tensor structures for three-point functions in section 3.3 and the construction of conformal blocks in section 5.

thus could lead to truly universal bounds on CFT data. Another reason for interest in the stress-tensor is its connection to the graviton in AdS, via the AdS/CFT duality. As was pointed out already in [11], universal bounds on CFT data for external operators with spin may explain the weak gravity conjecture [40] or the bounds on a and c in [41].

With the insights about three-point correlators from this work it is easy to outline what needs to be done to compute all conformal blocks for the correlator of four stress-tensors. Table 6 contains all irreps that are exchanged in this correlator. Some conformal blocks can actually be written in terms of derivatives of conformal blocks for exchange of the same irrep in a simpler correlator, as it is the case for exchange of symmetric tensors [10]. For example, the conformal blocks for exchange of  in $\langle T_1^{\mu\nu} T_2^{\rho\sigma} T_3^{\lambda\kappa} T_4^{\tau\omega} \rangle$ are given by derivatives of the conformal blocks of $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$. For this reason, each line in the table displays for some correlator the irreps of exchanged operators that appear for the first time for that correlator. If one picks a correlator in one line and a single irrep from the same line, the computation of that conformal block is comparatively easy, since in those cases the three-point function between external operators and the exchanged operator has only one tensor structure. One can hope that the conformal blocks for all other cases are given by derivatives of those simpler cases.

An interesting generalisation of our work would be to extend the formalism to general spinor representations of SO(d). This would complete the counting and construction of tensor structures for all CFT correlators and facilitate the conformal bootstrap for combinations of operators that imply exchange of operators with half-integer spin.

Finally, note that most discussions of higher spin fields in AdS focus on the case of spin J symmetric tensors. However, it would be interesting to consider AdS fields dual to operators in arbitrary irreps of the conformal group. We expect that the techniques described in this paper can also be extended to the case of AdS fields, in the spirit of [42].

JHEP02(2015)151

Acknowledgments

We wish to thank Rutger Boels, Anders Skovsted Buch, Vasco Gonçalves, Christoph Horst, João Penedones and Emilio Trevisani for helpful discussions. T.H. thanks Universidade do Porto for hospitality. The research leading to these results has received funding from the German Science Foundation (DFG) within the Collaborative Research Center 676 “Particles, Strings and the Early Universe”, from the [European Union] Seventh Framework Programme [FP7-People-2010-IRSES] and [FP7/2007-2013] under grant agreements No 269217 and 317089, and from the grant CERN/FP/123599/2011. *Centro de Fisica do Porto* is partially funded by the Foundation for Science and Technology of Portugal (FCT).

JHEP02(2015)151

A Functions in the conformal block for hook diagram exchange

The following are the functions appearing in the conformal block (5.19) for exchange of the primary in the irreducible representation $[\yng(2,1),\Delta]$ in the correlator of two scalars and two vectors $\langle\phi_1 J_2^\mu \phi_3 J_4^\nu\rangle$.

$$
\begin{aligned}
F_1 = (\alpha-\Delta+1)\Big[&(\beta-\Delta+h+1)\Big(-(2h-1)J^{(1)}_{2,1,2}(\beta-\Delta+h)-(\alpha+1)J^{(1)}_{1,1,1}\\
&-J^{(2)}_{1,1,2}\big((2h-1)(v-1)+u\big)+2(2h-1)vJ^{(1)}_{2,2,2}(-\beta+h-1)\Big)\\
&+(\beta-h+1)\,v\Big((2h-1)\Big(vJ^{(1)}_{2,3,2}(-\beta+h-2)-(v-1)J^{(2)}_{1,2,2}\Big)+uJ^{(2)}_{1,2,2}\Big)\Big]\\
&+(\alpha+1)\Big[(\beta-\Delta+h+1)\Big(2\alpha J^{(0)}_{1,2,1}(-\beta+h-1)+(1-2h)J^{(1)}_{1,1,2}(\beta-\Delta+h)\\
&-J^{(1)}_{1,2,2}(-\beta+h-1)\big((1-2h)(v+1)+2u\big)+J^{(2)}_{0,1,2}\big((1-2h)(v-1)+u\big)-\alpha J^{(1)}_{0,1,1}\Big)\\
&+(\beta-h+1)\Big(J^{(2)}_{0,2,2}\big((1-2h)(v-1)-u\big)\\
&+(2h-1)\,vJ^{(1)}_{1,3,2}(-\beta+h-2)+\alpha J^{(1)}_{0,2,1}+v(\alpha-\Delta+1)J^{(1)}_{1,2,1}\Big)\Big]
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
F_2 = (\alpha-\Delta+1)\Big[&(\beta-\Delta+h+1)\Big(uJ^{(1)}_{2,2,2}(-\beta+h-1)+J^{(2)}_{1,1,1}\big((2h-1)(v-1)+u\big)\Big)\\
&+(\beta-h+1)\,v\Big((-\beta+h-2)\Big((\alpha+1)J^{(0)}_{2,3,1}+uJ^{(1)}_{2,3,2}\Big)\\
&-J^{(2)}_{1,2,1}\big((1-2h)(v-1)+u\big)-(\alpha+1)J^{(1)}_{1,2,0}\Big)\Big]\\
&+(\alpha+1)\Big[(\beta-\Delta+h+1)\Big((\beta-h+1)\Big(uJ^{(1)}_{1,2,2}-J^{(0)}_{2,2,1}(\alpha-\Delta+1)-\alpha J^{(0)}_{1,2,1}\Big)\\
&-J^{(2)}_{0,1,1}\big((1-2h)(v-1)+u\big)+J^{(1)}_{1,1,0}(\alpha-\Delta+1)+\alpha J^{(1)}_{0,1,0}\Big)\\
&+(\beta-h+1)\Big(-\alpha\Big(J^{(0)}_{1,3,1}(\beta-h+2)+J^{(1)}_{0,2,0}\Big)+uJ^{(1)}_{1,3,2}(\beta-h+2)\\
&+J^{(2)}_{0,2,1}\big((2h-1)(v-1)+u\big)\Big)\Big]
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
F_3 = (\alpha-\Delta+1)\Big[&(\beta-\Delta+h+1)\Big(J^{(2)}_{1,1,1}\big((1-2h)(v-1)-u\big)\\
&-(2h-1)\Big((\alpha+1)J^{(0)}_{2,2,1}(-\beta+h-1)+J^{(1)}_{2,1,1}(-\alpha+\beta+h-2)\Big)\Big)\\
&+(\beta-h+1)v\Big((2h-1)(\beta-h+2)\Big((\alpha+1)J^{(0)}_{2,3,1}-vJ^{(1)}_{2,3,1}\Big)\\
&-(2h-1)J^{(1)}_{2,2,1}(-\alpha+2\beta-\Delta+2h)+J^{(2)}_{1,2,1}\big((1-2h)(v-1)+u\big)\Big)\Big]\\
&+(\alpha+1)\Big[(\beta-\Delta+h+1)\Big(J^{(2)}_{0,1,1}\big((1-2h)(v-1)+u\big)\\
&-(2h-1)\Big(J^{(1)}_{1,1,1}(-2\alpha+\beta+\Delta+h-2)-\alpha J^{(0)}_{1,2,1}(\beta-h+1)-\alpha J^{(1)}_{0,1,1}\Big)\Big)\\
&+(\beta-h+1)\Big((2h-1)(\beta-h+2)\Big(\alpha J^{(0)}_{1,3,1}-vJ^{(1)}_{1,3,1}\Big)+\alpha(2h-1)J^{(1)}_{0,2,1}\\
&+J^{(1)}_{1,2,1}(\beta-\alpha+h)\big((1-2h)(v+1)+2u\big)-J^{(2)}_{0,2,1}\big((2h-1)(v-1)+u\big)\Big)\Big]
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
F_4 = (\alpha-\Delta+1)\Big[&(\beta-\Delta+h+1)\Big((2h-1)\Big(J^{(1)}_{2,1,0}(-\alpha+\Delta-2)-2(\alpha+1)J^{(1)}_{1,1,0}\Big)\\
&+uJ^{(1)}_{2,2,1}(-\beta+h-1)+J^{(2)}_{1,1,0}\big((2h-1)(v-1)+u\big)\Big)\\
&+(\beta-h+1)\Big((\alpha+1)J^{(1)}_{1,2,0}\big((1-2h)(v+1)+2u\big)+uvJ^{(1)}_{2,3,1}(-\beta+h-2)\\
&-vJ^{(2)}_{1,2,0}\big((1-2h)(v-1)+u\big)+(2h-1)vJ^{(1)}_{2,2,0}(-\alpha+\Delta-2)\Big)\Big]\\
&+(\alpha+1)\Big[(\beta-\Delta+h+1)\Big(\alpha(1-2h)J^{(1)}_{0,1,0}+J^{(2)}_{0,1,0}\big((2h-1)(v-1)-u\big)\Big)\\
&+(\beta-h+1)\Big(\alpha(1-2h)J^{(1)}_{0,2,0}+uJ^{(1)}_{1,3,1}(\beta-h+2)\\
&+J^{(2)}_{0,2,0}\big((2h-1)(v-1)+u\big)+uJ^{(1)}_{1,2,1}(\beta-\Delta+h+1)\Big)\Big]
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
F_H = \frac{1}{h+1}\Big\{&(\alpha-\Delta+1)\Big[(\beta-\Delta+h+1)\Big(\big((1-2h)(v-1)-u\big)\\
&\times\Big(J^{(2)}_{1,1,1}+J^{(2)}_{1,0,0}+J^{(2)}_{2,0,1}+uJ^{(2)}_{2,1,2}\Big)\Big)\\
&+(\beta-h+1)\Big(v\big((1-2h)(v-1)+u\big)\Big(J^{(2)}_{1,1,1}+uJ^{(2)}_{2,2,2}+vJ^{(2)}_{2,2,1}+J^{(2)}_{1,1,0}\Big)\Big)\\
&+vJ^{(2)}_{2,1,1}\Big((1-2h)\Big(u-(v-1)\big(\Delta-2(\beta+1)\big)\Big)+(\Delta-1)u\Big)\Big]\\
&+(\alpha+1)\Big[(\beta-\Delta+h+1)\big(u-(2h-1)(v-1)\big)\Big(J^{(2)}_{1,1,1}v+J^{(2)}_{0,0,0}+J^{(2)}_{0,0,1}+uJ^{(2)}_{1,1,2}\Big)\\
&+(\beta-h+1)\big((1-2h)(v-1)-u\big)\Big(uJ^{(2)}_{1,2,2}+J^{(2)}_{0,1,0}+J^{(2)}_{0,2,1}+J^{(2)}_{1,1,1}\Big)\\
&+J^{(2)}_{0,1,1}\Big((2h-1)\Big(u+(v-1)\big(\Delta-2(\beta+1)\big)\Big)-(\Delta-1)u\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+J^{(2)}_{1,0,1}(\beta-\Delta+h+1)\Big((2h-1)(v-1)\big(\Delta-2(\alpha+1)\big)+\Delta u\Big)\\
&-vJ^{(2)}_{1,2,1}(-\beta+h-1)\Big((2h-1)(v-1)\big(\Delta-2(\alpha+1)\big)-\Delta u\Big)\Big\}\\
&+2J^{(0)}_{2,2,1}(\alpha+1)(-\alpha+\Delta-1)(-\beta+h-1)(\beta-\Delta+h+1)\big((2h-1)(v+1)-2u\big)\\
&+(2h-1)\Big\{(\alpha-\Delta+1)\Big[(\beta-\Delta+h+1)\Big(2(\alpha+1)J^{(0)}_{2,1,1}(\beta-\Delta+h)\\
&-(\alpha-\Delta+2)\Big(J^{(0)}_{3,1,1}(\beta-\Delta+h)+2vJ^{(0)}_{3,2,1}(\beta-h+1)\Big)\Big)\\
&+(\beta-h+1)\Big(v(-\beta+h-2)\Big(vJ^{(0)}_{3,3,1}(-\alpha+\Delta-2)-2(\alpha+1)J^{(0)}_{2,3,1}\Big)\Big)\Big]\\
&+(\alpha+1)\Big[(\beta-\Delta+h+1)\Big(-2\alpha J^{(0)}_{1,2,1}(-\beta+h-1)+\alpha J^{(0)}_{1,1,1}(\beta-\Delta+h)\\
&+J^{(0)}_{1,3,1}(-\beta+h-2)(-\beta+h-1)\Big)\Big]\Big\}
\end{aligned}
\tag{A.5}
$$

References

[1] S. Ferrara, A.F. Grillo and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, *Annals Phys.* **76** (1973) 161 [INSPIRE].

[2] A.M. Polyakov, *Nonhamiltonian approach to conformal quantum field theory*, *Zh. Eksp. Teor. Fiz.* **66** (1974) 23 [INSPIRE].

[3] R. Rattazzi, V.S. Rychkov, E. Tonni and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031 [arXiv:0807.0004] [INSPIRE].

[4] S. El-Showk M.F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *Solving the 3D Ising Model with the Conformal Bootstrap*, *Phys. Rev.* **D 86** (2012) 025022 [arXiv:1203.6064] [INSPIRE].

[5] S. El-Showk, M.F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents*, *J. Stat. Phys.* **157** (2014) 869 [arXiv:1403.4545] [INSPIRE].

[6] F. Kos, D. Poland and D. Simmons-Duffin, *Bootstrapping Mixed Correlators in the 3D Ising Model*, *JHEP* **11** (2014) 109 [arXiv:1406.4858] [INSPIRE].

[7] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [hep-th/9711200] [INSPIRE].

[8] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208] [INSPIRE].

[9] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116] [INSPIRE].

[10] M.S. Costa, J. Penedones, D. Poland and S. Rychkov, *Spinning Conformal Correlators*, *JHEP* **11** (2011) 071 [arXiv:1107.3554] [INSPIRE].

[11] D. Simmons-Duffin, *Projectors, Shadows and Conformal Blocks*, *JHEP* **04** (2014) 146 [arXiv:1204.3894] [INSPIRE].

[12] K. Koike and I. Terada, *Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, *J. Algebra* **107** (1987) 466.

[13] P. Cvitanovic, *Group theory: Birdtracks, Lie's and exceptional groups*, Princeton University Press, Princeton U.S.A. (2008).

[14] W. Fulton, *London Mathematical Society Student Texts. Vol. 35: Young tableaux*, Cambridge University Press, Cambridge U.K. (1997).

[15] V.K. Dobrev, V.B. Petkova, S.G. Petrova and I.T. Todorov, *Dynamical Derivation of Vacuum Operator Product Expansion in Euclidean Conformal Quantum Field Theory*, *Phys. Rev.* **D 13** (1976) 887 [INSPIRE].

[16] P.A. Dirac, *Wave equations in conformal space*, *Annals Math.* **37** (1936) 429.

[17] L. Cornalba, M.S. Costa and J. Penedones, *Deep Inelastic Scattering in Conformal QCD*, *JHEP* **03** (2010) 133 [arXiv:0911.0043] [INSPIRE].

[18] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal field theory*, Springer, Berlin Germany (1997).

[19] D. Littlewood and A. Richardson, *Group characters and algebra*, *Phil. Trans. R. Soc.* **A 233** (1934) 99.

[20] D.E. Littlewood, *Products and plethysms of characters with orthogonal, symplectic and symmetric groups*, *Canad. J. Math.* **10** (1958) 17.

[21] K. Koike, *On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters*, *Adv. Math.* **74** (1989) 57.

[22] K. Alkalaev, *Mixed-symmetry tensor conserved currents and AdS/CFT correspondence*, *J. Phys.* **A 46** (2013) 214007 [arXiv:1207.1079] [INSPIRE].

[23] K. Alkalaev, *Massless hook field in AdS(d+1) from the holographic perspective*, *JHEP* **01** (2013) 018 [arXiv:1210.0217] [INSPIRE].

[24] G. Mack, *Convergence of Operator Product Expansions on the Vacuum in Conformal Invariant Quantum Field Theory*, *Commun. Math. Phys.* **53** (1977) 155.

[25] H. Osborn and A.C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311 [hep-th/9307010] [INSPIRE].

[26] R.R. Metsaev, *Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time*, *Phys. Lett.* **B 354** (1995) 78 [INSPIRE].

[27] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, *Adv. Theor. Math. Phys.* **2** (1998) 781 [hep-th/9712074] [INSPIRE].

[28] O.V. Shaynkman, I.Y. Tipunin and M.A. Vasiliev, *Unfolded form of conformal equations in M dimensions and $o(M+2)$ modules*, *Rev. Math. Phys.* **18** (2006) 823 [hep-th/0401086] [INSPIRE].

[29] F.A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, *Nucl. Phys.* **B 599** (2001) 459 [hep-th/0011040] [INSPIRE].

[30] F.A. Dolan and H. Osborn, *Conformal Partial Waves: Further Mathematical Results*, arXiv:1108.6194 [INSPIRE].

JHEP02(2015)151

[31] S. Ferrara and G. Parisi, *Conformal covariant correlation functions*, *Nucl. Phys.* **B 42** (1972) 281 [INSPIRE].

[32] S. Ferrara, A. Grillo and G. Parisi, *Nonequivalence between conformal covariant wilson expansion in euclidean and minkowski space*, *Lett. Nuovo Cim.* **5S2** (1972) 147.

[33] S. Ferrara, A. Grillo, G. Parisi and R. Gatto, *The shadow operator formalism for conformal algebra. vacuum expectation values and operator products*, *Lett. Nuovo Cim.* **4S2** (1972) 115.

[34] S. Ferrara, A.F. Grillo, G. Parisi and R. Gatto, *Covariant expansion of the conformal four-point function*, *Nucl. Phys.* **B 49** (1972) 77 [INSPIRE].

[35] M.S. Costa, V. Gonçalves and J. Penedones, *Conformal Regge theory*, *JHEP* **12** (2012) 091 [arXiv:1209.4355] [INSPIRE].

[36] M.S. Costa, J. Penedones, D. Poland and S. Rychkov, *Spinning Conformal Blocks*, *JHEP* **11** (2011) 154 [arXiv:1109.6321] [INSPIRE].

[37] V. Gonçalves, J. Penedones and E. Trevisani, *Factorization of Mellin amplitudes*, arXiv:1410.4185 [INSPIRE].

[38] L. Cornalba, M.S. Costa and J. Penedones, *Eikonal Methods in AdS/CFT: BFKL Pomeron at Weak Coupling*, *JHEP* **06** (2008) 048 [arXiv:0801.3002] [INSPIRE].

[39] R.H. Boels and T. Hansen, *String theory in target space*, *JHEP* **06** (2014) 054 [arXiv:1402.6356] [INSPIRE].

[40] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060 [hep-th/0601001] [INSPIRE].

[41] D.M. Hofman and J. Maldacena, *Conformal collider physics: Energy and charge correlations*, *JHEP* **05** (2008) 012 [arXiv:0803.1467] [INSPIRE].

[42] M.S. Costa, V. Gonçalves and J. Penedones, *Spinning AdS Propagators*, *JHEP* **09** (2014) 064 [arXiv:1404.5625] [INSPIRE].

JHEP02(2015)151