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CONFORMAL DEFORMATION OF METRICS ON S^2

SUN-YUNG A. CHANG & PAUL C. YANG

1. Introduction

On the sphere $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ with its canonical metric $g_0 = \sum_{i=1}^3 dx_i^2$ the problem of conformal deformation of metric is to find conditions on the function $K(x)$ so that $K(x)$ is the Gauss curvature of a conformally related metric $g = e^{2u}g_0$. In terms of the Laplacian in the canonical metric this is expressed by the differential equation

$$\Delta u + Ke^{2u} = 1. \tag{1.1}$$

This equation has a solution when K is an even function; in this case Moser [13] showed that there is in fact an even function u solving the differential equation. Moser's approach was to maximize a functional within the class of even functions in H^1. The crucial ingredient for proof of convergence is a sharp version of Trudinger's inequality [12]: for an even $\mathscr{C}^1$ function u with $\int_{S^2} u = 0$, and $\int_{S^2} |\nabla u|^2 = 1$, we have $\int_{S^2} \exp(8\pi u^2) \leqslant c_0$ where c_0 is a universal constant. Without the evenness condition the inequality is weakened to $\int_{S^2} \exp(4\pi u^2) \leqslant c_0'$. Kazdan and Warner gave in [9] the necessary condition

$$\int_{S^2} (\nabla K, \nabla x_i) e^{2u} = 0, \qquad i = 1,2,3,$$

where x_i is any of the ambient coordinate functions on S^2.

In our previous article [4] we have obtained the following

Theorem. *On S^2, let $K > 0$ be a smooth function with nondegenerate critical points, and in addition $\Delta K(Q) \neq 0$ where Q is any critical point. Suppose there are at least two local maxima and that at all saddle points of K, $\Delta K(Q) > 0$. Then K admits a solution to equation* (1.1).

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Our approach was to employ the variational functional used by Moser: $F[u] = \log \fint Ke^{2u} - (\fint |\nabla u|^2 + 2\fint u)$. It turns out that unless K is identically constant, $F[u]$ does not have any local maximum in the Sobolev space $H^1(S^2)$. We therefore used a max-min scheme to locate saddle points of F. As usual, a maximizing sequence of a max-min scheme need not converge, and there is the possibility of a sequence of masses e^{2u} concentrating at a point. The main analysis deals with the description of such a concentrating sequence and the basic tool is a sharpened form of an inequality of Onofri which can be derived from Moser's inequality.

In this paper we give two results. The first solves a boundary value problem associated to equation (1.1), the second completes the analysis of [4] and yields an index counting criteria for solvability of (1.1).

Theorem I. *Suppose D is a smooth domain of S^2 and K a smooth function defined on D. Then for the Neumann boundary problem*

$$\begin{cases} \Delta u + Ke^{2u} = 1 & \text{on } D, \\ \partial u/\partial n = 0 & \text{on } \partial D \end{cases} \tag{1.2}$$

to have a solution u, we have

(a) *When* Area$(D) < 2\pi$, *it is sufficient that K be positive somewhere.*

(b) *When D = hemisphere, it is sufficient that*

$$\frac{1}{\text{Area}(D)} \int_D K\,dA > \max_{x \in \partial D} (K(x), 0).$$

Remarks. (a) In case D = hemisphere H, say $H = \{(x_1, x_2, x_3) \in S^2,\ x_3 > 0\}$, the boundary condition in (1.2) means geometrically that the equator $\{x_3 = 0\}$ remains a geodesic in the new conformal metric $g = e^{2u}g_0$. We point out some obvious necessary conditions for the equation to be solvable in this case, namely (1) K should be positive somewhere and (2) Kazdan-Warner's implicit condition $\int_H \langle \nabla K, \nabla x_i \rangle e^{2u} = 0$ for $i = 1, 2$ still holds for K satisfying $\partial K/\partial n = 0$; this can be seen either by reflecting K and u over the lower hemisphere or by direct computation using the condition $\partial u/\partial n = 0$ and $\partial x_i/\partial n = 0$ $(i = 1, 2)$ on ∂H. Of course the function $K(x) = 1 + \varepsilon x_1$ does not satisfy the hypothesis of our theorem.

(b) Naturally if K is a function on S^2 which is reflection symmetric w.r.t the equator $\partial H = \{x_3 = 0\}$, i.e., $K(x_1, x_2, x_3) = K(x_1, x_2, -x_3)$, and K satisfies the condition in case (b) of Theorem I, then equation (1.1) allows a solution on S^2 which is also reflection symmetric.

Theorem II. *Let K be a positive smooth function with only nondegenerate critical points, and in addition $\Delta K(Q) \neq 0$ where Q is any critical point. Suppose there are $p + 1$ local maximum points of K, and q saddle points of K with $\Delta K(Q) < 0$. If $q \neq p$, then K admits a solution to the equation* (1.1).

We briefly describe the analysis required for these two results. In §2, we relate (in Proposition 2.1) the best exponent in the Moser-Trudinger inequality to a constant which appears in an isoperimetric inequality. This relation will make it clear that for even functions defined on S^2, the best exponent in the inequality is exactly twice the usual exponent for arbitrary functions. We then apply the proposition to study Moser's inequality for more general domains and obtain:

Proposition 2.3. *Suppose D is a piecewise $\mathscr{C}^2$, bounded, finitely connected domain in the plane with finite number of vertices. Let θ_D be the minimum interior angle at the vertices of D. There exists a constant c_D such that for all $u \in \mathscr{C}^1(\bar{D})$ with*

$$\int_D |\nabla u|^2 \, dx \leqslant 1, \int_D u \, dx = 0, \tag{1.3}$$

we have

$$\int_D e^{\beta u^2} dx \leqslant c_D, \tag{1.4}$$

where $\beta \leqslant 2\theta_D$. As in the case of Moser's result, the integral is finite for all positive β, but if $\beta > 2\theta_D$ it can be made arbitrarily large by appropriate choice of u.

As mentioned above, the proof of Proposition 2.3 depends on an isoperimetric inequality (Proposition 2.1). We apply the inequality directly to the evaluation of the exponential integral using the distribution of u. This replaces the symmetrization procedure used by Moser. We make a change of variable which reduces the inequality to a calculus inequality (see Theorem A in §2) given in Moser's original paper.

In cases where there are no corners on ∂D, then $\theta_D = \pi$, and we find $\beta = 2\pi$ which is half the index in case of Moser's Theorem for Dirichlet boundary condition. Since it will be apparent that the arguments used to derive Proposition 2.3 for plane domains also apply for domains of S^2, as an immediate corollary of Proposition 2.3 we have:

Corollary. *Suppose D is a smooth finitely connected subdomain of S^2, there is a constant C_D depending only on D such that for all $u \in \mathscr{C}^1(D)$ we have*

$$\frac{1}{m(D)} \int_D \exp(2u) \leqslant C_D \exp\left(\frac{1}{2\pi} \int_D |\nabla u|^2 + \frac{2}{m(D)} \int_D u \right), \tag{1.5}$$

where $m(D)$ denotes the measure of D.

In §3 we give elementary properties of the functionals $F[u]$ and recall the basic inequality of Onofri as well as the sharpened versions which are used to describe the concentration phenomenon. We introduce two sets of parameters which measure the extent of concentration of masses and prove a concentration lemma. We apply the results in §2 and the concentration lemma to prove Theorem I, then recall the asymptotic formula for the integral of a $\mathscr{C}^2$ function with respect to concentrated masses. We outline here the argument to these technical results but refer the reader to [4] for complete details. In §4 we give an improved version of the lifting lemma in [4]. This will be crucial for analysis of concentration at saddle points of K. In §5 we define the variational schemes used to prove Theorem II, and analyze the concentration of a maximizing sequence if there is no convergence. In §6 we give the proof of Theorem II. For Theorem II, in case $p > q$, we use the precise information about the nature of failure of the one-dimensional max-min scheme to show that, in a rough sense, there are more one-dimensional schemes available than those that can fail. Thus we get an index 1 solution of (1.1). In case $q > p$, we first prove in §6 that certain 2-dimensional schemes associated with some simple closed curves Γ on S^2, which satisfy some conditions to be specified in §5, cannot have concentrating maximizing sequence. Thus the proof consists in a counting argument which shows that when $q > p$, there is always a simple closed curve Γ meeting the required conditions and thus we get an index 2 solution of (1.1). In §7, we discuss examples of K with $q = p$ where our schemes fail as well as examples where our schemes give more than one solution.

While Theorems I and II and the previously cited work give sufficient conditions for existence of solutions to equation (1.1), there is another result of Kazdan-Warner [10] which states that for any K positive somewhere on S^2, there always exists some diffeomorphism φ so that the equation

$$\Delta u + K \circ \varphi e^{2u} = 1$$

is solvable. It is therefore of interest to find some analytic conditions on the class of functions K which is topologically simple (e.g., K has only a global maximum and a global minimum) that ensures existence of a solution of (1.1).

In related developments for the analogous equation of prescribing scalar curvature on a compact manifold M of dimension n, $n \geqslant 3$, the corresponding equation becomes

$$4(n-1)(n-2)^{-1}\Delta u + Ru^{(n+2)/(n-2)} = R_0 u,$$

where R_0 is the scalar curvature of the underlying metric ds_0^2 and R is the prescribed scalar curvature of the conformally related metric $ds^2 = u^{4/(n-2)} ds_0^2$. When $R =$ constant, this was Yamabe's problem and is recently solved by Aubin [1] and Schoen [18]. While for the analogous problem of prescribing R on S^n ($n \geqslant 3$) with ds_0^2 the standard metric on S^n, Escobar and Schoen gave in [6] the analogue of Moser's theorem, Bahri and Coron [3] have announced an analogue of our Theorem II on S^3.

2. An isoperimetric inequality and Moser's inequality with Neumann condition

Let D be a bounded, piecewise $\mathscr{C}^2$ finitely connected domain in the plane with a finite number of vertices, and let θ_D be the minimum interior angle at the vertices of D. In this section we will first establish an isoperimetric inequality and then apply it to prove the version of Moser's inequality stated in the introduction.

Let D be as before and consider a curve γ separating D into two regions, say D_1 and D_2, with Area$(D_1) \leqslant$ Area(D_2). Define the isoperimetric constant $I(D, A)$ as

$$I(D, A) = \inf_{\gamma} (L(\gamma))^2/A,$$

where $A = A(D_1) =$ area of D_1 and $L(\gamma) =$ length of γ. Then we have

Proposition 2.1. *The function $I(D, A)$ satisfies*:

(2.1) *$I(D, A)$ is bounded from below by a positive constant c_1 for all*

$$A \in \left[0, \tfrac{1}{2}\text{ Area}(D)\right],$$

(2.2) *as $A \to 0$ we have $I(D, A) \geqslant 2\theta_D(I + \varepsilon(A))$,*

where $\varepsilon(A) \leqslant 0$, $|\varepsilon(A)| = O(A^{1/2})$, and $\underline{\lim}_{A \to 0} I(D, A) = 2\theta_D$.

Proof. For the first assertion, in view of the classical isoperimetric inequality $L((\partial D_1))^2/\text{Area}(D_1) \geqslant 4\pi$ for any domain in D_1, it suffices to show that there exists a constant C such that

$$\text{length}(\gamma) \geqslant C\,\text{length}(\partial D_1 \setminus \gamma).$$

It suffices to check this inequality for a component γ' of the curve γ, length$(\gamma') \geqslant C$ length$(\partial D_1 \setminus \gamma)$. There are two cases:

(1) If γ' joins two different components of the complement of D, say Ω_1 and Ω_2, then length$(\gamma') \geqslant$ distance(Ω_1, Ω_2) and the inequality follows.

(2) If γ' ends on the same component of ∂D, say at P_1, P_2, then length $(\gamma') \geqslant$ distance(P_1, P_2), the latter is bounded below by a fixed multiple of length of ∂D between P_1 and P_2, by a well-known property of chord-arc relation on a piecewise $\mathscr{C}^1$ curve.

To further simplify our consideration, we observe that we may assume D_1 consists of one component, for if $D_1 = D_1' \cup D_2''$ then $L = L' + L''$ and $L'^2/A' \geqslant \theta$, $L''^2/A'' \geqslant \theta$ implies that $(L' + L'')^2 \geqslant L'^2 + L''^2 + 2L'L'' \geqslant \theta(A' + A'')$.

For the second assertion, we need only consider those D_1 with small area, hence only those γ with short lengths. In this case we notice that components of γ must begin and end on the same component of ∂D, say Γ. Γ being a piecewise $\mathscr{C}^1$ curve, there is a global constant c such that $\operatorname{dist}(q_1, q_2) \geqslant c$ length (arc on Γ between $q_1 q_2$). Thus we may assume further that each component of γ begins and ends on points q_1, q_2 which are close along Γ. From the previous paragraph we may without loss of generality assume D_1 consists of one component. Replacing, if necessary, γ by γ_1 the shortest component of ∂D_1 with end points on $\partial D \cap \partial D_1$, we may assume without loss of generality that γ consists of only one component. We assert that we may assume γ is a convex with respect to D_1. For if not, we may replace D_1 by (convex hull of D_1) $\cap\, D$ (recall diam D_1 is small, hence the convex hull of D_1 still has small area). This has the effect of increasing the area and decreasing the boundary length, hence decreasing the ratio L^2/A.

Assume γ is a convex curve (with respect to D_1) with the ratio $L^2(\gamma)/A$ close to $I(D, A)$. We may assume also without loss of generality that γ meets the boundary of ∂D at two points, say P_1, P_2. Choose a boundary point $P_0 \in \partial D_1 \setminus \gamma$ which is equidistant from P_1, P_2. Then we can represent γ as a graph γ: $r = r(\theta)$, $0 \leqslant \theta \leqslant \alpha$, in the polar coordinates based at P_0. Let θ_{P_0} represents the corner angle of D at P_0. Thus

$$L(\gamma) = \int_0^\alpha \left(r^2 + \left(\frac{dr}{d\theta} \right)^2 \right)^{1/2} d\theta,$$

$$A = \text{Area of } D_1 = \frac{1}{2} \int_0^\alpha r^2 d\theta + E,$$

where E represents the area of the excess region of $D \setminus$ cone over P_0. To estimate $L^2(\gamma)/A$ we apply the following lemma.

Lemma 2.2. (a)

$$\frac{\int_0^\alpha r^2 d\theta}{\left(\int_0^\alpha \left(r^2 + (dr/d\theta)^2 \right)^{1/2} d\theta \right)^2} \leqslant \frac{1}{\alpha}.$$

(b) $E \leqslant c_2 (L(\gamma))^3$ *for some constant* $c_2 > 0$.

Assuming the lemma for a moment, we may easily finish the proof of Proposition 2.1:

$$\frac{L^2(\gamma)}{A} \geqslant \frac{1}{1/(2\alpha) + c_2 L(\gamma)} = 2\alpha \left(1 + 2\alpha c_2 L(\gamma)\right)^{-1}$$

$$\geqslant 2\alpha\left(1 - 2\alpha c_2 L(\gamma)\right) \geqslant 2\alpha\left(1 - 2\alpha c_2 c_1^{1/2} A^{1/2}\right).$$

In the last inequality we have applied statement (1) in Proposition 2.1. We now claim $\alpha = \theta_{P_0} - O(A^{1/2})$: Let Δ be the triangle spanned by P_0, P_1, and P_2. Obviously

$$A \geqslant \text{Area}(\Delta) \geqslant |P_1 - P_2|^2 \sin\alpha,$$

hence

$$|P_1 - P_2| = O(A^{1/2}).$$

The piecewise $\mathscr{C}^2$ condition implies that

$$\cot\left(\frac{\alpha}{2}\right) = \cot\left(\frac{\theta_{P_0}}{2}\right)(1 + O|P_1 - P_2|)$$
$$= \cot\frac{\theta_{P_0}}{2}(1 + O(A^{1/2})).$$

Hence $\alpha = \theta_{P_0} - O(A^{1/2})$ as claimed. Thus we find

$$\frac{L^2(\gamma)}{A} \geqslant 2\theta_{P_0} - O(A^{1/2}).$$

To see that $\underline{\lim}_{A\to 0} I(D, A) = 2\theta_D$, let P be some vertex in ∂D with $\theta_p = \theta_D$. Choose a circle C_ρ centered at P with radius ρ, and let $\gamma = C_\rho \cap D$. Then $(L(\gamma))^2/A \cong 2\theta_D$ when ρ is sufficiently small. Thus we have finished the proof of Proposition 2.1 assuming Lemma 2.2.

Proof of Lemma 2.2. For (a), because of homogeneity of the ratio, we may consider the variational problem of minimizing $(L(\gamma))^2$ subject to the constraint $\frac{1}{2}\int_0^\alpha r^2\, d\theta = 1$.

The variational equation for the extremal is

$$\text{(1)} \qquad \frac{d}{d\theta}\left[\frac{dr}{d\theta}\left(\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{-1/2}\right)\right] - r\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{-1/2} + \lambda r = 0,$$

$$\text{(2)} \qquad \left.\frac{dr}{d\theta}\right|_{\theta=0} = \left.\frac{dr}{d\theta}\right|_{\theta=\alpha} = 0,$$

where λ is the constant of the Lagrange multiplier.

Equation (1) is the equation for a curve γ of constant curvature; that is γ is a circular arc while the boundary condition (2) makes γ orthogonal to the two sides of the cone with angle α at $P_0 \in \partial D$. Hence the optimal ratio of (a) is achieved by a circular arc centered at P_0, with the value $\alpha/2$.

For (b), write $E = E_1 + E_2$ where E_i, $i = 1, 2$, is the excess area to each side of the angle at P_0. Each E_i is estimated by length l_i of the segment P_0P_i (where P_i, $i = 1, 2$, are the end points of the arc γ) multiplied by the width w_i

of the area. Because of the piecewise $\mathscr{C}^2$ assumption we have on ∂D, $w_i = O(l_i^2)$. Since $l_i \leqslant \text{length}(\partial D_1 \setminus \gamma)$ we find

$$E_i \leqslant \left(\text{length}(\partial D_1 \setminus \gamma)\right)^3 < \tfrac{1}{2}c(\text{length}\,\gamma)^3,$$

where c is a constant depending on the $\mathscr{C}^1$ property of ∂D. We have thus proved (b).

Remarks. (1) The proof of Proposition 2.1 is very likely known. since we need Proposition 2.1 in its precise form, we give the proof here for the sake of completeness.

(2) When the domain D is contained in S^2, similar arguments as in Proposition 2.1 also hold. In this case, the isoperimetric inequality becomes $L^2 \geqslant A(4\pi - A)$. While for two small regions on S^2 symmetric under the antipodal map, we have $L = L_1 + L_2$, $A = A_1 + A_2$ with $L_1 = L_2$, $A_1 = A_2$, and $L_i^2 \geqslant A_i(4\pi - A_i)$ for $i = 1, 2$. Thus $L^2 = 4L_1^2 \geqslant 4A_i(4\pi - A_i) = A(8\pi - A)$. As we will see in the proof of Proposition 2.3 below, this explains that for even functions defined on S^2, the best exponent in the Moser-Trudinger inequality is 8π instead of 4π.

As we have stated in the introduction, our proof of Proposition 2.3 depends on a calculus inequality which also appeared in Moser's paper [12].

Theorem A (*Moser*). *Suppose $w(t)$ is a monotonically increasing function defined on the real line $(-\infty, \infty)$ satisfying*

$$\int_{-\infty}^{\infty} \dot{w}^2(t)\,dt \leqslant 1, \qquad \int_{-\infty}^{\infty} w(t)\rho(t)\,dt = 0 \tag{2.3}$$

with $\rho(t)$ a positive continuous function satisfying

$$\rho(t) \leqslant c_0 e^{-|t|}, \qquad \int_{-\infty}^{\infty} \rho(t)\,dt = 1 \tag{2.4}$$

for some constant c_0. Then

$$\int_{-\infty}^{\infty} e^{w^2(t)}\rho(t)\,dt \tag{2.5}$$

is uniformly bounded.

Proof of Proposition 2.3. (Proposition 2.3 was stated in §1.) We will break the proof of the theorem into two steps. First we will prove it for $\mathscr{C}^2$ functions u defined on $\overline{D}$ which are Morse functions (i.e., u has only isolated nondegenerate critical points on $\overline{D}$). This will be established by a change of variable based on the distribution of u and by applying Theorem A above. In the second step we will use an approximation argument and prove the theorem for all $\mathscr{C}^1$ functions u defined on D.

Step I. Assume u is a $\mathscr{C}^2$ Morse function defined on $\overline{D}$. For each real number M, let

$$L_M = \text{length of the level curve } \{u = M\},$$
$$A_M = \text{area of the region } \{u \leqslant M\}.$$

Now assume u satisfies (1.3), that is

$$\int_D |u_x|^2\,dx \leqslant 1 \quad \text{and} \quad \int_D u\,dx = 0. \tag{2.6}$$

We begin the proof by rewriting (1.3) in terms of L_M, A_M:

$$\int_D |u_x|^2\,dx = \int_{-\infty}^{\infty}\left(\int_{u=M}|du|\,ds\right)dM$$
$$\geqslant \int_{-\infty}^{\infty}\frac{\left(\int_{u=M}ds\right)^2}{\int_{u=M}(1/|du|)\,ds}\,dM$$

(by Holder's inequality, and the fact that u is a Morse function)

$$= \int_{-\infty}^{\infty}\frac{L_M^2}{(dA_M/dM)}\,dM,$$

$$\int_D u\,dx = \int_{-\infty}^{\infty} M\,dA_M.$$

Thus (2.6) is equivalent to

$$\int_{-\infty}^{\infty}\frac{L_M^2}{(dA_M/dM)}\,dM \leqslant 1, \tag{2.6$'$}$$

$$\int_{-\infty}^{\infty} M\,dA_M = 0. \tag{2.6$''$}$$

To estimate $\int_D e^{\beta u^2}\,dx$ for $0 < \beta \leqslant \beta_D = 2\theta_D$, again we rewrite the integral in terms of M, A_M:

$$\int_D e^{\beta u^2}\,dx = \int_{-\infty}^{\infty} e^{\beta M^2}\,dA_M. \tag{2.7}$$

Now make a change of variable in (2.6), (2.6)$''$, and (2.7) by choosing $t \in (-\infty, \infty)$ as a function of M which satisfies

$$\beta\frac{dA_M}{dt} = \phi(D, A_M)A_M(A(D) - A_M), \tag{2.8}$$

where $\phi(D, A_M) \leqslant I(D, A_M)$ is a modification of the function $I(D, A_M)$ defined in Proposition 2.1 for $A_M \in [0, A(D)]$ ($A(D)$ denotes the area of D) which satisfies

$$\phi(D, A_M) \geqslant c_1 \quad \text{for } A_M \in [0, A(D)], \tag{2.9}$$

$$\phi \text{ is a } \mathscr{C}^1 \text{ function for } A_M \in (D, A(D)), \tag{2.10}$$

$$\phi(D, A_M) = \frac{2\theta_D}{A(D)}(1 + \varepsilon(A_M)), \tag{2.11}$$

where $\varepsilon(A_M) = O(A_M^{1/2})$ as $A_M \to 0$ or $A_M \to A(D)$).

To show the variable t is well defined in (2.8), notice that by (2.9), we may define t as

$$t = \beta \int_{A(D)/2}^{A_M} \left(\frac{1}{\phi(D, A_M) A_M (A(D) - A_M)} \right) dA_M. \tag{2.12}$$

We will postpone the proof that t is well defined (i.e., as A_M increases from 0 to $A(D)$, t increases from $-\infty$ to ∞) until the end of the section. Since u is a $\mathscr{C}^2$ Morse function, it then follows that A_M is a piecewise $\mathscr{C}^1$ function of M and also t is a piecewise $\mathscr{C}^1$ function of M, hence of A_M.

We now define the function w as an increasing function of t by $w(t) = \beta^{1/2} M$. Then

$$\begin{aligned} \left(\frac{dw}{dt}\right)^2 dt &= \beta \left(\frac{dM}{dt}\right)^2 dt = \beta \left(\frac{dM}{dt}\right) dM = \beta \left(\frac{dA_M}{dt} \Big/ \frac{dA_M}{dM}\right) dM \\ &= \left((\phi(D, A_M) A_M (A(D) - A_M)) \Big/ \frac{dA_M}{dM} \right) dM \quad \text{(by (2.8))}. \end{aligned} \tag{2.13}$$

Since $L_M^2 \geqslant (\phi(D, A_M) A_M (A(D) - A_M))$ by Proposition 2.1, we have from (2.6)′ and (2.13) that

$$\int_{-\infty}^{\infty} \left(\frac{dw}{dt}\right)^2 dt \leqslant 1. \tag{2.14}$$

Finally define $\rho(t) = dA_M/dt$, then ρ is a piecewise $\mathscr{C}^1$ function of t and we can rewrite (2.6)″ and (2.7) as

$$\int_{-\infty}^{\infty} w(t)\rho(t)\, dt = 0, \tag{2.15}$$

$$\int_D e^{\beta u^2} dt = \int_{-\infty}^{\infty} e^{w^2(t)} \rho(t)\, dt. \tag{2.16}$$

From (2.14), (2.15), (2.16), it is clear that we are ready to apply Theorem A, provided we can verify

$$\rho(t) = \frac{dA_M}{dt} \leqslant c_0 e^{-|t|} \quad \text{for some constant } c_0 \text{ for all } \beta \leqslant 2\theta_D. \tag{2.17}$$

Thus to finish the proof of the theorem, it remains to verify (2.17) and to check that the variable t in (2.12) is well defined.

Checking of (2.12). Since A_M is an increasing function of M, without ambiguity we will write A for A_M, $|D|$ for $A(D)$, and $\phi(D, A_M)$ as $\phi(A)$.

Define $y = A/(|D| - A)$ and integrate (2.12) by part to get $t = \mathrm{I} + \mathrm{II}$ where

$$\mathrm{I} = \frac{\beta}{|D|}(\log y)/\phi(A), \tag{2.18}$$

$$\mathrm{II} = \beta \int_1^y \frac{(\log y)\phi'(A)}{(\phi(A))^2(1+y)^2}\,dy. \tag{2.19}$$

Since $\phi(A) \geqslant c_1$ by (2.9), it is clear that as A changes from 0 to $|D|$, the value of I ranges from $-\infty$ to ∞. Thus to check that t is well defined it suffices to verify that the integral in II is uniformly bounded for all $y \geqslant 0$ and for all $\beta \geqslant 0$.

When $y \geqslant 1$, i.e., when $A \geqslant |D|/2$, applying (2.10) and (2.11) we get

$$\frac{|\phi'(A)|}{(\phi(A))^2} = 0\left(\frac{1}{|D| - A}\right)^{1/2} = O\big((1+y)^{1/2}\big),$$

as $A \to |D|$ or $y \to \infty$.

Thus $|\mathrm{II}| \leqslant$ constant $\int_1^y (\log y)(1+y)^{-3/2}\,dy$ which is uniformly bounded for all $y \geqslant 1$.

Since a similar argument clearly applies to the estimate in II when $y \leqslant 1$, we have verified that $|\mathrm{II}|$ is uniformly bounded for all y, hence all A.

Proof of (2.17). We will first assume $t > 0$, i.e., $A > |D|/2$ or $y \geqslant 1$. From the definition of $\rho(t)$ and the identity (2.8), we have, by (2.18) and (2.19),

$$\begin{aligned}\rho(t)e^t = \frac{dA_M}{dt}e^t &= \frac{1}{\beta}\phi(A)A(|D| - A)e^t \\ &\leqslant \frac{1}{\beta}\phi(A)A^2\frac{1}{y}e^{(\beta/|D|)(\log y/\phi(A))}\cdot e^{\mathrm{II}} \\ &\leqslant \text{constant } \frac{1}{\beta}\phi(A)A^2\frac{1}{y}e^{(\beta/|D|)(\log y/\phi(A))}.\end{aligned}$$

Thus to verify (2.17) for t large it suffices to check that the term $(1/y)e^{(\beta/|D|)(\log y/\phi(A))}$ is bounded for all $0 < \beta \leqslant 2\theta_D$ and for y large. To see this, we apply (2.9), (2.11) to get

$$\begin{aligned}\frac{1}{y}e^{(\beta/|D|)(\log y/\phi(A))} &= \left(\frac{\beta}{|D|\phi(A)} - 1\right)\log y \leqslant \left(\frac{2\theta_D}{|D|\phi(A)} - 1\right)\log y \\ &\leqslant \frac{1}{c_1}O(|D| - A)^{1/2}\log\frac{A}{|D| - A},\end{aligned}$$

which is uniformly bounded for all $A \geqslant |D|/2$. Hence (2.17) holds for all $t > 0$, t large. Since a similar argument applies for all $t < 0$, (2.17) is established, and we have finished the proof of the theorem for $\mathscr{C}^2$ Morse functions.

Step II. For a general $\mathscr{C}^1$ function u defined on $\overline{D}$ with $\int_D |u|^2\,dx = 1$, we may approximate u by a sequence u_n of $\mathscr{C}^2$ Morse functions defined on $\overline{D}$ such that $\int_D |(u_n - u)|^2\,dx \to 0$ and $\int_D u_n\,dx \to \int_D u\,dx$ as $h \to \infty$ (cf., for example, [10, Corollary 6.8]). Denote for all $\beta \leqslant 2\theta_D$,

$$I(u) = \int_D e^{\beta(u-\bar{u})^2/\int_D|u|^2\,dx}\,dx,$$

and $I(u_n) = I_n$. Then $I_n \leqslant C_D$ by the proof of Step I above. And if we denote $E = \beta(u-\bar{u})^2/\int_D |u|^2\,dx$, $E_n = \beta(u_n - \bar{u}_n)^2/\int_D |u_n|^2\,dx$ we may estimate $I(u)$ as:

$$\begin{aligned} I(u) &= \int_{E\leqslant E_n} e^E\,dx + \int_{E\geqslant E_n} e^E\,dx \leqslant C_D + \int_{E\geqslant E_n}\left[(e^E - e^{En}) + e^{En}\right]dx \\ &\leqslant 2C_D + \int_{E\geqslant E_n}(e^E - e^{En})\,dx \leqslant 2C_D + \int_D |E - E_n| e^E e^{E-En}\,dx \\ &\leqslant 2C_D + \left(\int_D e^{4E}\,dx\right)^{1/2}\left(\int_D |E - En|^2\,dx\right)^{1/2}. \end{aligned}$$

Since

$$\begin{aligned} &\int_D |E - En|^2\,dx \\ &\quad\leqslant C\beta^2\left[\int_D |u - \bar{u} - (u_n - \bar{u}_n)|^2\,dx + \left(\int_D |u_n|^2\,dx - \int_D |u|^2\,dx\right)^2\right] \\ &\quad\leqslant C\beta^2\int_D |(u_n - u)|^2\,dx \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

we have $I(u) \leqslant 2C_D$.

We have thus finished the proof of Proposition 2.3.

For geometric application, we remark that with suitable modification of Proposition 2.1, the same proof of Proposition 2.3 gives, in the case of smooth subdomain D contained in S^2, the following corollary:

Corollary 2.4. *Suppose u is a $\mathscr{C}^1$ function defined on a subdomain D of S^2 satisfying*

$$\int_D |\nabla u|^2\,dA \leqslant 1 \quad \text{and} \quad \int_D u\,dA = 0.$$

Then there exists an absolute constant C_D such that

$$\int_D e^{2\pi u^2}\,dA \leqslant C_D m(D),$$

where dA is the volume form on D corresponding to the metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$, and $|\nabla u|$ denotes the gradient of u with respect to this metric.

For a $\mathscr{C}^1$ function u defined on D let $\bar{u}$ denote $\int_D u\, dA/m(D)$. As has been pointed out in [12], an immediate consequence of Corollary 2.4 based on the inequality

$$2(u-\bar{u}) \leqslant \left((u-\bar{u})^2\Big/\left(\frac{1}{2\pi}\int_D |\nabla u|^2 dA\right)\right) + \frac{1}{2\pi}\int_D |\nabla u|^2 dA$$

is the following.

Corollary 2.5. *Suppose $u \in \mathscr{C}^1(D)$. Then*

$$(2.20)\quad \frac{1}{m(D)}\int_D e^{2u} dA \leqslant C_D \exp\left(\frac{1}{2\pi}\int_D |\nabla u|^2 dA + \frac{2}{m(D)}\int_D u\, dA\right).$$

In §3 below, we will find that the constant C_D which appears in (2.20) has the optimal value 1 when D is a hemisphere.

3. Properties of $F[u], J[u]$

The standard 2-sphere S^2 is usually represented as $\{x \in \mathbb{R}^3 \mid |x|^2 = 1\}$. Relative to any orthonormal frame e_1, e_2, e_3 of $\mathbb{R}^3$ we have the Euclidean coordinates $x_i = x \cdot e_i$ and we call $(0,0,1)$ (respectively $(0,0,-1)$) the north pole (respectively south pole). Through the stereographic projection to the x_1, x_2 plane we have the complex stereographic coordinates

$$z = \frac{x_1 + ix_2}{1 - x_3},$$

which has the inverse transformation

$$x_1 = \frac{2}{1+|z|^2}\operatorname{Re} z, \quad x_2 = \frac{2}{1+|z|^2}\operatorname{Im} z, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

The conformal transformations of S^2 are thus identified with fractional linear transformations

$$w = \frac{az+b}{cz+d}, \quad ad - bc = 1, \qquad a, b, c, d, \text{ complex numbers},$$

which form a six-dimensional Lie group. For our purpose we need the following set of conformal transformation: Given $P \in S^2$, $t \in (0, \infty)$ we choose a frame $e_1, e_2, e_3 = P$; then using the stereographic coordinates with P at infinity we denote the transformation

$$(3.1)\qquad \phi_{P,t}(z) = tz.$$

Observe that $\phi_{P,1} \equiv \text{id}$ and $\phi_{P,t^{-1}} = \phi_{-P,t}$, hence the set of conformal transformation $\{\phi_{P,t} \mid P \in S^2, t \geqslant 1\}$ is parametrized by $B^3 \cong S^2 \times [1,\infty)/S^2 \times \{1\}$, where B^3 is the unit ball in $\mathbb{R}^3$ with each point $(Q,t) \in S^2 \times [1,\infty)$ identified with $(t-1)t^{-1}Q \in B^3$.

$H^1 = H^1(S^2)$ is the Sobolev space of L^2 functions on S^2 whose gradient lie in L^2 with

$$\|u\|_1 = \left(\frac{1}{4\pi}\int |\nabla u|^2 + u^2\right)^{1/2};$$

we also denote

$$\|u\| = \left(\frac{1}{4\pi}\int |\nabla u|^2\right)^{1/2}.$$

We adopt the notation $⨍ f$ to mean average integral $(4\pi)^{-1}\int f\,d\mu$, also written as $⨍ f = \bar{f}$.

Definition. For $u \in H^1(S^2)$ let

$$S[u] = ⨍ |\nabla u|^2 + 2 ⨍ u, \tag{3.2}$$

$$J[u] = \log ⨍ e^{2u} - S[u], \tag{3.3}$$

$$F[u] = F_K[u] = \log ⨍ Ke^{2u} - S[u]. \tag{3.4}$$

For $H =$ the hemisphere $= \{x \in S^2, x_3 \geqslant 0\}$ we have similarly

$$\hat{S}[u] = ⨍_H |\nabla u|^2 + ⨍_H 2u,$$

$$\hat{J}[u] = \log ⨍_H e^{2u} - \hat{S}[u],$$

$$F[u] = \log ⨍_H Ke^{2u} - \hat{S}[u], \quad \text{where } ⨍_H = (1/2\pi)\int_H.$$

For functions $u \in H^1(S^2)$ which are symmetric with respect to the x_1x_2 plane we have $S[u] = \hat{S}[u\,|_H]$, $J[u] = \hat{J}[u\,|_H]$, and $F[u] = \hat{F}[u\,|_H]$.

The critical points of $J[u]$ satisfy the Euler equation

$$\Delta u + e^{2u} = 1, \tag{3.5}$$

where Δ denotes the Laplacian with respect to the standard metric. All solution of (3.5) are of the form $u = (1/2)\log\det|d\phi|$, ϕ a conformal map of S^2. Similarly the critical points of $F_K[u]$ satisfy the Euler equation

$$\Delta u + Ke^{2u} = 1. \tag{3.6}$$

The functional $S[u]$ enjoys the following invariance property:

Definition. Given $u \in H^1$ and ϕ a conformal transformation let

$$(3.7) \qquad u_\phi = u \circ \phi + (1/2) \log \det |d\phi|,$$

also written as $T^t(Q)(u)$ when $\phi = \phi_{Q,t}$.

Proposition 3.1. $S[u] = S[u_\phi]$, *also if* ϕ *leaves* ∂H *invariant and* u *is symmetric with respect to the* $x_1 x_2$ *plane, then so is* u_ϕ *and* $\hat{S}[u|_H] = \hat{S}[u_\phi|_H]$.

The proof is left as an exercise in integration by parts, using the equation (3.5) for the term $(1/2)\log\det|d\phi|$.

The implicit condition found by Kazdan-Warner [9] is a consequence of $S[u] = S[u_\phi]$:

Corollary 3.2. *If u satisfies* (3.6) *then*

$$⨍ \langle \nabla K, \nabla x_j \rangle e^{2u} = 0, \qquad j = 1,2,3.$$

More generally, Kazdan and Warner [9, p. 33] found the following implicit consequence by a tricky partial integration. If $\Delta v + he^v = c$, then

$$(3.8) \qquad ⨍ e^v \nabla h \cdot \nabla x_i = (2 - c) ⨍ e^v h x_i, \qquad i = 1,2,3.$$

Given $u \in H^1(S^2)$, e^{2u} may be thought of as a mass distribution. So we define the center of mass of e^{2u}: $\text{C.M.}(e^{2u}) = ⨍ \vec{x} e^{2u} / ⨍ e^{2u}$.

Definition. $\mathscr{S} = \{u \in H^{1,2} \mid \text{C.M.}(e^{2u}) = \vec{0}\}$, $\mathscr{S}_0 = \{u \in \mathscr{S} \mid ⨍ e^{2u} = 1\}$. For each $Q \in S^2$, $0 < t < \infty$,

$$\mathscr{S}_{Q,t} = \{u \in H^1 \mid T^t(Q)(u) \in \mathscr{S}_0\}.$$

For each $P \in S^2$, $0 < \delta < 1$,

$$C_{P,\delta} = \left\{ u \in H^1 \mid \frac{\text{C.M.}(e^{2u})}{|\text{C.M.}(e^{2u})|} = P;\ 1 - \delta = ⨍ P \cdot \vec{x} e^{2u} \right\}.$$

The center of mass is a parameter that measures the extent to which the mass e^{2u} is concentrated. The (Q,t) parameter serves the same purpose, and it is more natural for our setting because of the invariance property (Proposition 3.1).

Proposition 3.2 [4, *Proposition* 2.2]. *Given a continuous map u*: $\mathbb{R}$ (*or* Δ: *the unit disc in the complex plane*) $\to H^1(S^2)$, *there is a continuous map* (Q,t): $\mathbb{R}$ (*or* Δ) $\to S^2 \times [1,\infty)/S^2 \times \{1\} \cong B^3$, *so that* $u(s) \in \mathscr{S}_{Q(s),t(s)}$ *for all* $s \in \mathbb{R}$ (*or* $s \in \Delta$). *If u is symmetric with respect to the* $x_1 x_2$ *plane, then Q lies on the equator*: $x_3 = 0$.

The importance of the class $\mathscr{S}$ lies in the following:

Proposition 3.3 [1]. *Suppose $u \in H^1$ with $⨍ e^{2u}x_j = 0$ for $j = 1, 2, 3$. Then for every $\varepsilon > 0$, there exists a constant C_ε with*

$$(3.9) \qquad ⨍ e^{2u} \leqslant C_\varepsilon \exp\left[\left(\tfrac{1}{2} + \varepsilon\right) ⨍ |\nabla u|^2 + 2 ⨍ u\right].$$

Corollary 3.4. *Suppose $u \in \mathscr{S}_0$. Then $⨍ |\nabla u|^2 \leqslant 4(S[u] + \log C_{1/4})$ where $C_{1/4}$ is the same constant as in* (3.9) *with $\varepsilon = 1/4$.*

Corollary 3.4 indicates that $\mathscr{S}_0$ forms a compact family in H^1 in the subset where $S[u]$ stays bounded. This is a key fact which was used in the following inequality of Onofri:

Proposition 3.5 [14]. *Given $u \in H^1(S^2)$ we then have $J[u] \leqslant 0$ with the equality holding only for $u = (1/2)\log\det|d\phi|$ where ϕ is a conformal map of S^2.*

By a simple change of variable, we often refer to Onofri's inequality in the following form:

$$(3.10) \qquad ⨍ e^{Cu} \leqslant \exp\left(\tfrac{1}{4}C^2 ⨍ |\nabla u|^2 + C ⨍ u\right) \quad \text{for any real } C.$$

We state the first important consequence of Corollary (3.4):

Proposition 3.6 (*Concentration lemma*). *Given a sequence of functions $u_j \in H^1(S^2)$ with $⨍ e^{2u_j} = 1$ and $S[u_j] \leqslant C$ then either*

(i) *there exist a constant C' such that $⨍ |\nabla u_j|^2 \leqslant C'$ or*

(ii) *a subsequence concentrates at a point $P \in S^2$, i.e., given $\varepsilon > 0$ $\exists N$ large such that*

$$(1/4\pi)\int_{B(P,\varepsilon)} \exp 2u_j \geqslant (1 - \varepsilon) \quad \text{for } j \geqslant N,$$

where $B(P, \varepsilon)$ is the ball in S^2 of radius ε, centered at P.

In case u is symmetric with respect to the x_1x_2 plane in case (ii), *we have the point of concentration P lying on the equator $x_3 = 0$.*

Proof. Since $u_j \in \mathscr{S}_{Q_j, t_j}$, then $v_j = (u_j)_{\phi_j} \in \mathscr{S}_0$, where $\phi_j = \phi_{Q_j, t_j}$ and $S[u_j] = S[v_j] \leqslant C$. It follows from Corollary 3.4 above that $⨍ |\nabla v_j|^2 \leqslant C'$. We have two possibilities. Either all ϕ_j lie in a compact set, i.e., $t_j \leqslant C''$, in which case it follows easily that $⨍ |\nabla u_j|^2 \leqslant C(C', C'')$ or the t_j do not remain bounded, in which case a subsequence still denoted u_j has $t_j \to \infty$ and $Q_j \to P$. Further since $⨍ |\nabla v_j|^2 \leqslant C$, a subsequence converges weakly to $v_\infty \in \mathscr{S}_0$. Since

$$\int_{B(P,\varepsilon)} \exp 2u_j = \int_{\phi_j^{-1}(G(P,\varepsilon))} \exp 2v_j,$$

the right-hand side converges to $\int_{\phi_j^{-1}(B(P,\varepsilon))} \exp 2v_\infty$, which for j large is greater than $1 - \varepsilon$. This proves the concentration lemma in the general case.

In case u_j is symmetric in the $x_1 x_2$ plane, Q_j lies on the equator, hence the point of concentration $P = \lim Q_j$ must also lie on the equator $x_3 = 0$.

Proof of Theorem I. In case (a), Area$(D) < 2\pi$. Consider the functional $F_D(u) = \log \fint_D Ke^{2u} - (\fint_D |\nabla u|^2 + 2\fint_D u)$. We maximize the functional F_D subject to the constraint $\fint_D u = 0$, then its maxima satisfy the Euler equation

$$(1.2)' \qquad \Delta u + \frac{Ke^{2u}}{\fint_D Ke^{2u}} = \lambda, \quad \frac{\partial u}{\partial n} = 0.$$

Integrating the equation yields $\lambda = 1$, and shifting u by a constant $v = u + c$ gives the solution of (1.2). Thus it suffices to show $\sup_{\fint u = 0} F_D[u]$ is achieved. It follows immediately from Corollary (2.5) that $F_D[u]$ is bounded from above:

$$\begin{aligned} F_D[u] &= \log \fint_D Ke^{2u} - \left(\fint_D |\nabla u|^2 + 2\fint u \right) \\ &\leqslant \log(\max K) + \log C_D + [1/2\pi - 1/\text{meas}(D)] \int_D |\nabla u|^2 \\ &\leqslant \log(\max K) + \log C_D. \end{aligned}$$

And for a maximizing sequence, $F_D[u_j] \geqslant C$; hence

$$C + [1/\text{meas}(D) - 1/2\pi] \int_D |\nabla u|^2 \leqslant \log(\max K) + \log C_D$$

gives immediately the compactness, hence the existence of maximum of F_D.

In case (b), D is the hemisphere H, we maximize the functional F_H subject to the constraint $\fint_H u = 0$. We define the symmetric extension $\tilde{u}$, $\tilde{K}$ of u, K to the entire sphere and we find that $F[\tilde{u}] = F_H[u]$. According to the concentration lemma, a maximizing sequence $\{u_j\}$ for F_H either stays bounded in the H^1 norm, hence achieves a maximum for F_H or concentrates at a boundary point P. In the latter case we find $\fint_H Ke^{2u_j}$ converges to $K(P)$, hence $\lim F[\tilde{u}_j] \leqslant \log K(P)$. However, according to the assumption $F_H[0] = \log \fint K > \log \max\{K(P), 0\}$, this is a contradiction. This finishes the proof of Theorem I.

In preparation for the proof of Theorem II we recall a few facts from [4]. First we recall a sharpened form of Onofri's inequality.

Proposition 3.7 [4, *Proposition B*]. *These exists some* $a < 1$ *such that for all* $u \in \mathscr{S}$,

$$(3.11) \qquad \fint e^{2u} \leqslant \exp\left(a\fint |\nabla u|^2 + 2\fint u \right).$$

Since $(1 - a)⨍|\nabla u|^2 = S[u] - (a⨍|\nabla u|^2 + 2⨍ u)$, we have as a direct consequence:

Corollary 3.8. *If $u \in \mathscr{S}_0$, then $⨍|\nabla u|^2 \leqslant (1 - a)^{-1}S[u]$.*

The corollary says that $S[u]$ serves as a "norm" for the class of function in $\mathscr{S}_0$, and more generally, for $u \in H^1$ with $⨍ e^{2u} = 1$, it measures the deviation of u from a solution of (3.5) with the same (Q, t) parameters. The following asymptotic formula is based on this idea:

Proposition 3.9 [4, *Proposition D*]. *Suppose $u \in \mathscr{S}_{Q,t}$ with $S[u] = O(t^{-\alpha})$ for $\alpha > 0$ and t sufficiently large. Then $u \in C_{P,\delta}$ where $\delta = 4t^{-2}\ln t + O(t^{-2})$ and $|P - Q| = O(t^{-1})$, and for every $\mathscr{C}^2$ function f defined on S^2 we have*

$$(3.12) \qquad ⨍ fe^{2u} = f(P) + 2\Delta f(P)t^{-2}\log t + O(t^{-2})$$

$$= f(P) + (1/2)\Delta f(P)\delta + O\left(\delta\left(|\log\delta|\right)^{-1}\right).$$

We also want to remark that we can run the above parameter changes from (Q, t) to (P, δ) backwards, and obtain:

Corollary 3.10. *Suppose $u \in C_{P,\delta}$ with $S[u] = O(\delta^{\beta})$ for some $\beta > 0$ and δ sufficiently small. Then $u \in \mathscr{S}_{Q,t}$ where $\delta = 4t^{-2}\log t + O(t^{-2})$ and $|P - Q| = O(t^{-1})$, and (3.12) holds for any $\mathscr{C}^2$ function f.*

Corollary 3.11. *Suppose $u \in \mathscr{S}_{Q,t}$ with $S[u] = O(t^{-\alpha})$ for some $\alpha > 0$ and t large. Then for any $f \in \mathscr{C}^2(S^2)$ we have*

$$(3.12)' \qquad ⨍ fe^{2u} = f(Q) + 2\Delta f(Q)t^{-2}\log t + O(t^{-2}) + O\left(|\nabla f(Q)|(t^{-2}\log t)^{1/2}(S[u])^{1/2}\right).$$

4. A Lifting Lemma

When a function $u \in H^1$ or a parametrized family of functions $u_s \in H^1$ with $⨍ e^{2u_s} = 1$ is sufficiently concentrated, namely $u \in \mathscr{S}_{Q,t}$ with $t \geqslant t_0$, we can compare the functional $F[u]$ with $J[u]$. In fact we can compare $F'[u]$ with $J'[u]$ to construct a continuous lifting process which increases the value of $F[u]$ and $J[u]$ simultaneously until $S[u]$ becomes suitably small while leaving fixed the class $\mathscr{S}_{Q,t}$ to which u or u_s belongs. We formulate this process as the following Lifting Lemma:

Proposition 4.1 [4, *Proposition C*]. *Given u_s a continuous family in H^1, where $u_s \in \mathscr{S}_{Q_s t_s}$ with t_s large and $S[u_s] \leqslant C_1$, there exists a continuous path $u_{s,\gamma}$, $\gamma \in [0, \gamma_0]$, with $u_{s,0} = u_s$, $u_{s,\gamma} \in \mathscr{S}_{Q_s,t_s}$ for all $\gamma \in [0, \gamma_0]$ such that $J[u_{s,\gamma}]$, $F[u_{s,\gamma}]$ both increase in γ and $S[u_{s,\gamma_0}] = O(t_s^{-1}(\log t_s)^2)$.*

It turns out that in order to handle the concentration behavior of a sequence $\{u_s\}$ in H^1 near a point $P \in S^2$, which is a saddle point of K with $\Delta K(P) < 0$, we need to improve the above lifting process till the order of $S[u_{s,\gamma_0}]$ reaches $O(|\nabla K(Q_s)|^2 t_s^{-2} \log t_s)$. We state this as:

Proposition 4.2. *Given $u_0 \in \mathcal{S}_{Q,t}$ with t large and $S[u_0] \leqslant \eta$, there exists a continuous path u: $[0, \gamma_0] \to H^1$ with $u(0) = u_0$, $u_\gamma \in \mathcal{S}_{Q,t}$ for all $\gamma \in [0, \gamma_0]$, and $J[u_\gamma]$, $F[u_\gamma]$ both increasing functions of γ and with $S[u_{\gamma_0}] = O(|\nabla K(Q)|^2 t^{-2} \log t + O(t^{-2}))$.*

To prove Proposition 4.2, we first observe that applying Proposition 4.1 we may assume that we have lifted u_γ up to $S[u_\gamma] = O(t^{-1}(\log t)^2)$. Thus it suffices to continue the lifting process assuming $S[u]$ is sufficiently small. As in the case in the proof of Proposition 4.1 the key step in establishing the proposition is to verify the following lemma:

Lemma 4.3. *There exists some $\eta > 0$, given $u \in \mathcal{S}_{Q,t}$ with t large and with $S[u] \leqslant \eta$. Then there exists some $v_u \in H^1$ with $\|v_u\| \leqslant 1$ and such that*:

$$J'[u](v_u) \geqslant C(\eta)(S[u])^{1/2}, \tag{4.1}$$

$$F'[u](v_u) \geqslant C(\eta)\left(S[u]^{1/2}\right) - O\left(|\nabla K(Q)|t^{-1}(\log t)^{1/2} + O(t^{-1})\right), \tag{4.2}$$

$$\frac{d}{ds}\left(\fint \exp[2T'(Q)(u + sv_u)]x_j\right)\Big|_{s=0} = 0 \quad \text{for } j = 1,2,3, \tag{4.3}$$

for some constant $C(\eta)$, depending only on η.

Assuming Lemma 4.3, the lifted path u_γ required in Proposition 4.2 could be constructed by solving the ordinary differential equation $(d/d\gamma)(u_\gamma) = v_{u_\gamma}$ with u_0 given and normalizing the solution by setting $\fint \exp(2u_\gamma) = 1$ for all γ.

To prove Lemma 4.3, we first recall some facts from [4] (Lemma 4.3 in [4]). For the sake of completeness of this paper, we will also outline the proofs of these facts in the following lemma:

Lemma 4.4. *There exists a constant $\eta > 0$ (sufficiently small). Whenever $w \in \mathcal{S}_0$ with $S[w] \leqslant \eta$, then for the function v_w defined as*

$$v_w = -\left(w - \overline{w} - \sum_{j=1}^{3} \beta_j x_j\right), \tag{4.4}$$

where $\beta = (\vec{\beta}_j)$ satisfies $\Lambda(w)\beta = \gamma$ with $\Lambda(w) = (\Lambda_{ij}(w))$, $\Lambda_{ij}(w) = \fint e^{2w} x_i x_j$, $\gamma = (\gamma_i)$, $\gamma_i = \fint e^{2w} x_i$, we have the following:

$$\fint |\nabla v_w|^2 = \fint |\nabla w|^2 + O(S[w])^2, \tag{4.5}$$

$$\fint \nabla w \cdot \nabla v_w = \fint |\nabla w|^2 + O(S[w])^2, \tag{4.6}$$

(4.7) $$\int e^{2w} v_w \leqslant C_1(\eta) \int |\nabla w|^2 \quad \textit{for some } C_1(\eta) < 1,$$

(4.8) $$J'[w](v_w)/\|v_w\| \geqslant C(\eta)(S[w])^{1/2},$$

for some constant $C(\eta)$ *depending only on* η.

Proof. Fix $w \in \mathcal{S}_0$, and let v_w be as in (4.4). Denote $S = S[w]$, and $\alpha_j = \int w x_j$, $j = 1, 2, 3$. Then

(4.5)′ $$\int |\nabla v_w|^2 = \int |\nabla w|^2 - 4 \sum_{j=1}^{3} \beta_j \alpha_j + \frac{2}{3} \sum_{j=1}^{3} \beta_j^2,$$

(4.6)′ $$\int \nabla w \cdot \nabla v_w = \int |\nabla w|^2 - 2 \sum_{j=1}^{3} \beta_j \alpha_j.$$

Thus to verify (4.5) and (4.6) it suffices to verify that $\alpha_j = O(S)$ and $\beta_j = O(S)$ for all $j = 1, 2, 3$. To do this, we apply the sharpened form of the Onofri inequality (3.11) to the function $w \in \mathcal{S}_0$ and conclude

$$|\alpha_j| \leqslant -\int w \leqslant \tfrac{1}{2} \int |\nabla w|^2 \leqslant (2(1-a))^{-1} S[w],$$

hence $\alpha_j = O(S)$. To estimate β_j, we notice that

$$\Lambda_{ij}(w) = \int e^{2w} x_i x_j = \int (e^{2w} - 1) x_i x_j \quad \text{if } i \neq j,$$

$$\Lambda_{ii}(w) = \int e^{2w} x_i x_i = \int (e^{2w} - 1) x_i^2 + 1/3 \quad \text{for } i = 1, 2, 3,$$

and applying Onofri's inequality and (3.11) we have

(4.9) $$\begin{aligned} \int |e^{2w} - 1| &\leqslant \left(\int (e^{2w} - 1)^2 \right)^{1/2} = \left(\int (e^{4w} - 1) \right)^{1/2} \\ &\leqslant \left[\exp\left(2 \int |\nabla w|^2 + 2S[w] \right) - 1 \right]^{1/2} \\ &\leqslant 2 \left(\int |\nabla w|^2 \right)^{1/2} \exp\left(\frac{2}{1-a} S[w] \right) = O(S^{1/2}). \end{aligned}$$

Hence for $w \in \mathcal{S}_0$ we conclude $\Lambda_{ij}(w) = O(S^{1/2})$ for $i \neq j$, $\Lambda_{ii}(w) = 1/3 + O(S^{1/2})$ for $i = 1, 2, 3$, and $\beta = \Lambda^{-1}\gamma = (3 + O(S^{1/2}))\gamma$. Thus to estimate β_j it suffices to estimate γ_j. To do so, we may rewrite

$$\begin{aligned} \gamma_j = \alpha_j + \gamma_j - \alpha_j &= \alpha_j + \int (e^{2w} - 1)(w - \bar{w}) x_j \\ &\leqslant \alpha_j + \left(\int (e^{2w} - 1)^2 \right)^{1/2} \left(\int (w - \bar{w})^2 \right)^{1/2}. \end{aligned}$$

Apply the estimate in (4.9) to get $|\gamma_j| \leqslant |\alpha_j| + O(S) = O(S)$ and $\beta_j = O(S)$ as well. We have thus obtained the desired estimates as in (4.5) and (4.6).

To see (4.7), notice that, by definition, $\bar{v}_w = \fint v_w = 0$ and $w \in \mathscr{S}_0$; thus

$$(4.7)' \quad \fint e^{2w} v_w = \fint e^{2w}(v_w - \bar{v}_w) = \fint e^{2w}(w - \bar{w}) = \fint e^{2w}(w' - \bar{w}'),$$

where $w' = w - 3\sum_{j=1}^{3} \alpha_j x_j$ with $\fint w' x_j = 0$. Since the next eigenvalue of $-\Delta$ on S^2 greater than 2 is 6, we have

$$(4.10) \qquad \fint (w' - \bar{w}')^2 \leqslant (1/6) \fint |\nabla w'|^2 \leqslant (1/6) \fint |\nabla w|^2.$$

Thus from (4.9) and (4.10) we get

$$\fint e^{2w} v_w \leqslant \left[\fint (e^{2w} - 1)^2\right]^{1/2} \left[\fint (w' - \bar{w}')^2\right]^{1/2}$$
$$\leqslant 2(6)^{-1/2} \left(\fint |\nabla w|^2\right) \exp((2/(1 - a))S[w]).$$

It is then obvious that if $S[w] \leqslant \eta$ is small, we may choose $C_1(\eta) = 2(6)^{-1/2} \exp[(2/(1 - a))\eta] < 1$. We have thus established inequality (4.7).

Finally since $2J'[w](v_w) = \fint \nabla w \cdot \nabla v_w - \fint e^{2w} v_w$ and for $w \in \mathscr{S}_0$, it follows from Corollary 3.8 that $\fint |\nabla w|^2 \sim S[w]$. Inequality (4.8) is a direct consequence of the estimates in (4.5), (4.6), and (4.7) with $C(\eta)$ any positive constant less than $1 - C_1(\eta)$. We have thus finished the proof of Lemma 4.3.

We are now ready to verify the main Lemma 4.2.

Proof of Lemma 4.2. Given $u \in \mathscr{S}_{Q,t}$ with t large and with $S[u] \leqslant \eta$ (with η as in Lemma 4.3), we may choose the desired function v_u as follows: Let $w = T'(Q)(u)$; then $w \in \mathscr{S}_0$ and $S[w] = S[u]$. Choose v_w as in Lemma 4.3 and let $\tilde{v}_u = v_w \circ \phi_{Q,t}^{-1}$ and $v_u = \tilde{v}_u / \|\tilde{v}_u\|$. Then by a direct computation we have:

$$(4.11) \qquad \begin{aligned} J'[u](\tilde{v}_u) &= J'[w](v_w) = 2\left((A_1/B_1) - \fint \nabla w \cdot \nabla v_w\right), \\ F'[u](\tilde{v}_u) &= F'_{K'}[w](v_w) = 2\left((A/B) - \fint \nabla w \cdot \nabla v_w\right), \end{aligned}$$

where

$$K' = K \circ \phi_{Q,t}, \quad A = \fint K' e^{2w} v_w, \quad B = \fint K' e^{2w},$$

$$A_1 = K(Q) \fint e^{2w} v_w, \qquad B_1 = K(Q) \fint e^{2w}.$$

It is then clear that since $\fint |\nabla v_w|^2 = \fint |\nabla \tilde{v}_u|^2$, (4.1) in Lemma 4.2 follows directly from inequality (4.8) in Lemma 4.3 and the equality in (4.11).

To verify (4.2) in view of (4.1), it suffices to estimate the difference between $F'[u](\tilde{v}_u)$ and $J'[u](\tilde{v}_u)$, i.e., the term $((A/B) - (A_1/B_1))$. For this purpose we write

$$(4.12)\quad |(A/B) - (A_1/B_1)| \leqslant |A_1/B_1|\,|(B_1 - B)/B| + |B|^{-1}|A - A_1|$$

and observe $|B| \geqslant \inf K > 0$, and reduce the estimates to that of $|A - A_1|$, $|B - B_1|$, and $|A_1/B_1|$. To do so, we first compute $⨍ |K \circ \phi_{Q,t} - K(Q)|^2$ using the Taylor series expansion of K around Q, we obtain (this can be done either by a direct computation or by applying Lemma 5.1 in [4])

$$(4.13)\quad ⨍ |K \circ \phi_{Q,t} - K(Q)|^2 = O\left(|\nabla K(Q)|^2 t^{-2}\log t\right) + O(t^{-2}).$$

Thus

$$(4.14)\quad \begin{aligned} |B - B_1| &\leqslant ⨍ |K \circ \phi_{Q,t} - K(Q)| e^{2w} \\ &\leqslant \left(⨍ |K \circ \phi_{Q,t} - K(Q)|^2\right)^{1/2}\left(⨍ e^{4w}\right)^{1/2} \\ &= O\left(|\nabla K(Q)|(t^{-2}\log t)^{1/2}\right) + O(t^{-1}) \quad \text{(by (4.13))}, \end{aligned}$$

$$(4.15)\quad \begin{aligned} |A - A_1| &= \left|⨍ (K \circ \phi_{Q,t} - K(Q)) e^{2w} v_w\right| \\ &\leqslant \left(⨍ |K \circ \phi_{Q,t} - K(Q)|^2\right)^{1/2}\left(⨍ e^{8w}\right)^{1/2}\left(⨍ v_w^4\right)^{1/4} \\ &= \left[O\left(|\nabla K(Q)|(t^{-2}\log t)^{1/2}\right) + O(t^{-1})\right] \cdot \left(⨍ |\nabla v_w|^2\right)^{1/2}. \end{aligned}$$

It also follows directly from (4.7) and (4.5) that

$$(4.16)\quad \begin{aligned} |A_1/B_1| &\leqslant C_1(\eta) ⨍ |\nabla w|^2 \leqslant C_1(\eta)(1 - a)^{-1/2}(S[w])^{1/2}\left(⨍ |\nabla w|^2\right)^{1/2} \\ &\leqslant C_2(\eta)\|v_w\| \leqslant C_2(\eta)\|\tilde{v}_u\|. \end{aligned}$$

Combining (4.14), (4.15), and (4.16) into (4.11) we get

$$\begin{aligned} F'[u](v_u) &\geqslant J'[u](v_u) - |A/B - A_1/B_1|\,\|\tilde{v}_u\|^{-1} \\ &\geqslant C(\eta)(S[u])^{1/2} - O\left(|\nabla K(Q)|t^{-1}(\log t)^{1/2} + O(t^{-1})\right), \end{aligned}$$

which establishes inequality (4.2) in the lemma.

Finally to verify (4.3), we observe that for $w \in \mathscr{S}_0$, we can choose v_w satisfy $\int e^{2w} v_w x_j = 0$ for all $j = 1, 2, 3$. Thus for $\phi = \phi_{Q,t}$,

$$\left[\frac{d}{ds}\int \exp(2(u + sv_u)_\phi) x_j\right]\Big|_{s=0} = \left[\frac{d}{ds}\int \exp\left(2\left(u_\phi + sv_u \circ \phi\right)\right) x_j\right]\Big|_{s=0}$$
$$= 2\int \exp(2u_\phi)(v_u \circ \phi) x_j$$
$$= 2\int \exp(2w) v_w x_j = 0 \quad \text{for all } j = 1, 2, 3.$$

We have thus finished the proof of Lemma 4.3, and hence Proposition 4.2.

5. Analysis of concentration near critical points

In this section we apply §3 and §4 to analyze the phenomenon of concentration near critical points of K. We will consider the concentration which occurs in two types of variational schemes, $\operatorname{Var}(P_\alpha, P_\beta)$ and $\operatorname{Var}(\Gamma)$, which we now define.

Given two points P_α, P_β on S^2, we formulate the one-dimensional scheme $\operatorname{Var}(P_\alpha, P_\beta)$ as follows. Let $\mathscr{P}(P_\alpha, P_\beta) = \{u\colon (-\infty, \infty) \to H^1(S^2),\ u_p\colon -\infty < p < \infty$ is a continuous one-parameter family of functions in $H^1(S^2)$ with $\int \exp(2u_p) = 1$ and satisfy:

(1) $S[u_p] \to 0$ as $|p| \to \infty$,

(2) $\lim_{p\to\infty} \text{C.M.}(\exp 2u_p) = P_\alpha$, $\lim_{p\to-\infty} \text{C.M.}(\exp 2u_p) = P_\beta\}$.

Let $c = \sup\{\min_p F[u_p] \mid u \in \mathscr{P}(P_\alpha, P_\beta)\}$. Clearly if a maximizing path $u^{(k)}$ which assumes its minimum at p_k is denoted by $u_{p_k}^{(k)}$, then if $u_{p_k}^{(k)}$ converges weakly in H^1, the limit u will satisfy weakly the Euler equation (1.1). Consequently by the regularity theory for elliptic equations u will be a smooth solution of (1.1).

For Theorem II we require in addition to $\operatorname{Var}(P_\alpha, P_\beta)$ a two-dimensional scheme $\operatorname{Var}(\Gamma)$, which we now define. Let Γ be a simple closed curve on S^2 satisfying the following condition:

(5.1) $\min\{K(Q) \mid Q \in \Gamma\}$ is achieved only at saddle points Q_γ of K with $\Delta K(Q_\gamma) < 0$, so that near Q_γ, Γ is a $\mathscr{C}^1$ curve of the following form: rotate coordinates to arrange to have $Q_\gamma = (0, 0, 1)$ and K has the following Taylor expansion in (x_1, x_2) near Q_γ.

$$K(x_1, x_2, x_3) = K(Q_\gamma) + Ax_1^2 - Cx_2^2 + O\left(\left(x_1^2 + x_2^2\right)^{3/2}\right), \qquad 0 < A < C.$$

We require that Γ be tangent to x_1 axis at Q_γ. We parametrize Γ by $\gamma\colon \partial\Delta \to \Gamma$ where $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

Definition. $\mathscr{P}(\Gamma) = \{u\colon \Delta \to H^1(S^2)$ a continuous map with $\int e^{2u_p} = 1$, $p \in \Delta$; and satisfying the asymptotic conditions: for all $p_0 \in \partial\Delta$ (1) $\lim_{p \to p_0 \in \partial\Delta} S[u_p] = 0$, (2) $\lim_{p \to p_0 \in \partial\Delta}$C.M. $(\exp 2u_p) = \gamma(p_0)$.$\}$ Let $c = \sup_{u \in \mathscr{P}(\Gamma)} \min_p F[u_p]$. As previously, if a maximizing sequence of minima $u_{p_k}^{(k)}$ converges weakly, the limit will be an index 2 saddle point solution of the equation (1.1).

We first remark that in either problem Var(P_α, P_β) or Var(Γ), we may restrict the class of competing paths in such a way that if concentration occurs along the paths then the functional S must be small at such points, so that we may apply the lifting results of §4. More precisely we define the following class of paths:

$$\mathscr{P}'_{t_0}(P_\alpha, P_\beta) = \Big\{u \in \mathscr{P}(P_\alpha, P_\beta) \,|\, u_p \in \mathscr{S}_{Q,t},\ t \geqslant 2t_0$$

$$\Rightarrow S[u_p] \leqslant c|\nabla K(Q)|^2 t^{-2}\log t + O(t^{-2})\Big\},$$

$$\mathscr{P}'_{t_0}(\Gamma) = \Big\{u \in \mathscr{P}(\Gamma) \,|\, u_p \in \mathscr{S}_{Q,t},\ t \geqslant 2t_0$$

$$\Rightarrow S[u_p] \leqslant c|\nabla K(Q)|^2 t^{-2}\log t + O(t^{-2})\Big\}.$$

Choose t_0 and a constant C large so that the lifting Proposition C″ holds for all $u \in \mathscr{S}_{Q,t}$, $t \geqslant t_0$. For each $u_s \in \mathscr{S}_{Q,t}$, there exists $u_{s,\tau}$, $0 \leqslant \tau \leqslant \tau(u_s)$, continuous in τ with $u_{s,0} = u_s$, $u_{s,\tau} \in \mathscr{S}_{Q,t}$, $F[u_{s,\tau}]$ and $J[u_{s,\tau}]$ both monotone increasing in τ, and such that at $\tau = \tau(u_s)$ we have $S[u_{s,\tau}] \leqslant c|\nabla K(Q)|^2 t^{-2}\log t + O(t^{-2})$.

Let $\rho(t) = \min\{1, (t - t_0)/t_0\}$ for $t \in [t_0, \infty)$. For $u_s \in \mathscr{S}_{Q,t}$ define

$$u'_s = \begin{cases} u_{s,\rho(t)\tau(u_s)} & \text{if } t \geqslant t_0, \\ u_s & \text{if } t < t_0. \end{cases}$$

Then $u'_s \in \mathscr{P}'_{t_0}(P_\alpha, P_\beta)$ or $u'_s \in \mathscr{P}'_{t_0}(\Gamma)$ while $F[u'_s] \geqslant F[u_s]$. Hence it follows that

$$\sup_{u \in \mathscr{P}'_{t_0}} \min F[u_s] = \sup_{u \in \mathscr{P}} \min F[u_s].$$

In view of the equality above, we will assume all paths u in both schemes belong to the lifted path class $\mathscr{P}'_{t_0}(P_\alpha, P_\beta)$ or $\mathscr{P}'_{t_0}(\Gamma)$.

Assuming u_k is an unbounded sequence in H^1 and is a max-mini sequence for either the scheme Var(P_α, P_β) or Var(Γ), it then follows from Proposition 3.2 (the concentration lemma) that the masses $\exp(2u_k)$ ($u_k = u_{p_k}^{(k)}$) converge (perhaps on a subsequence) to a delta function concentrated at $P_\infty \in S^2$. By constructing suitable variations of the max-mini sequence we have managed to show:

Proposition 5.1 [4, *Propositions G, F*]. *In either problem* Var(P_α, P_β), *where* P_α, P_β *are local maxima of* K, *or* Var(Γ), *where* Γ *is a simple closed curve satisfying condition* (5.1), *if a maximizing sequence of paths* $u^{(k)} \in \mathscr{P}'_{t_0}(P_\alpha, P_\beta)$ *or* $u^{(k)} \in \mathscr{P}'_{t_0}(\Gamma)$ *has minima* $\exp(2u^{(k)}_{p_k})$ *concentrating at a point* P_∞, *then without loss of generality we may assume that*

(a) P_∞ *must be a critical point of* K;

(b) P_∞ *cannot be a local maximum of* K;

(c) P_∞ *cannot be a local minimum or a saddle point of* K *where* $\Delta K(P_\infty) > 0$.

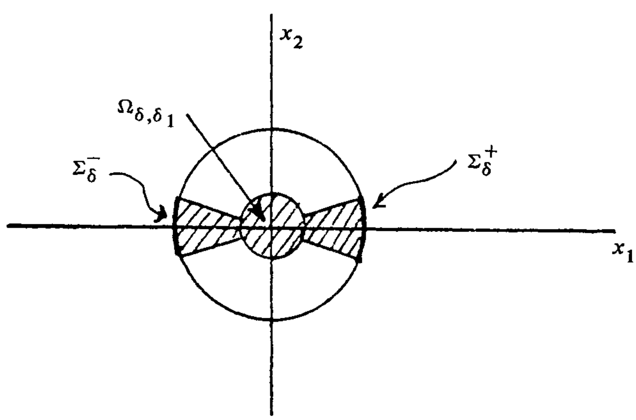
We consider the remaining possibility that concentration occurs at a saddle point P_∞ where $\Delta K(P_\infty) < 0$. Assume $P_\infty = (0,0,1)$, and that Taylor expansion for K near P_∞ is of the form

$$K(x_1, x_2, x_3) = K(P_\infty) + Ax_1^2 - Cx_2^2 + O\left(|x'|^3\right),$$
$$A, C > 0,\ A - C = (1/2)\Delta K(P_\infty) < 0. \tag{5.2}$$

For $\delta_0 = 4t_0^{-2}\log t_0$, let $B((\delta_0)^{1/2}) = \{x \in S^2 \,|\, x_1^2 + x_2^2 \leqslant \delta_0, x_3 > 0\}$.

Definition.

$$\Omega_\delta = \left\{x \in B\left((\delta)^{1/2}\right), |x_1/x_2| \geqslant 2(C/A)^{1/2}\right\},$$
$$\Omega_{\delta,\delta_1} = \Omega_\delta \cup B\left((\delta_1)^{1/2}\right), \qquad 0 < \delta_1 < \delta,$$
$$\Sigma_\delta^+ = \left\{x \in \Omega_\delta, x_1^2 + x_2^2 = \delta,\ x_1 > 0\right\},$$
$$\Sigma_\delta^- = \left\{x \in \Omega_\delta,\ x_1^2 + x_2^2 = \delta, x_1 < 0\right\}. \tag{5.3}$$



In the next proposition we show that for the variational problem Var(P_α, P_β), if concentration occurs at a saddle point P_∞ where $\Delta K(P_\infty) < 0$, we can arrange to have good control of the location of the concentration parameter Q for functions u_p when u_p is close to the minima $u_{p_k}^{(k)}$.

Proposition 5.2. *For the variational scheme* $\operatorname{Var}(P_\alpha, P_\beta)$, *if a maximizing sequence of minima* $u_{p_k}^{(k)}$ *concentrates at a saddle point* P_∞ *with* $\Delta K(P_\infty) < 0$, *then there exists some* δ_0 *small and constant* c *large but fixed, such that we can find a competing sequence* $\tilde{u}^{(k)}$ *achieving their minima at* $p = 0$, *and that over the interval* $[-1, 1]$, *we have*

(i) *the associated* (Q, t) *parameter of* $\tilde{u}_p^{(k)}$ *denoted by* $(Q_p^{(k)}, t_p^{(k)})$ *belongs to* $\Omega_{c\delta_0, c^{-1}\delta_0} \times [t_0, \infty)$ $(\delta_0 = 4t_0^{-2}\log t_0)$,

(ii) *at the endpoints* $p = 1, -1$, *we have* $Q_1^{(k)}, Q_{-1}^{(k)} \in \Sigma_{c\delta_0}^+ \cup \Sigma_{c\delta_0}^-$, *and*

(iii) $S[\tilde{u}_1^{(k)}]$, $S[\tilde{u}_{-1}^{(k)}]$ *can be made as small as wanted so that* $F[\tilde{u}_1^{(k)}]$, $F[\tilde{u}_{-1}^{(k)}] \geqslant \log K(P_\infty) + c_{\delta_0}$ *for* k *large and* $c_{\delta_0} > 0$ *depends only on* δ_0.

Proof. We will construct such a sequence in two steps. The first step uses the flow associated to the gradient flow ∇K. Let $\rho \in \mathscr{C}^1((B(2(c\delta_0)^{1/2}) \times [t_0, \infty))$ be a function with $0 \leqslant \rho \leqslant 1$, $\rho = 1$ on the region $(B((c\delta_0)^{1/2}) - B((c^{-1}\delta_0)^{1/2})) \times [2t_0, \infty)$, decreasing linearly in $(|x_1|^2 + |x_2|^2)^{1/2}$ to zero at $\partial B(2(c\delta_0)^{1/2})$ and $\partial B(2^{-1}(c^{-1}\delta_0)^{1/2})$, as well as decreasing in t to zero at $t = t_0$. For each $u \in \mathscr{S}_{Q,t}$ with $Q \in B(2(c\delta_0)^{1/2})$, $t \geqslant t_0$, let $R_{Q,\theta}$ denote the rotation in the plane determined by the span of Q and $\nabla K(Q)$ in the direction $\overline{Q, \nabla K(Q)}$ with angle of rotation θ. Consider the 1-parameter flow $\tilde{\Psi}_s(u) = u_s$ given by solutions of the ordinary differential equation:

$$\frac{d}{ds} u_s = \frac{d}{d\tau}\bigg|_{\tau=0} (u_s) R_{Q_s, -\tau\theta_s}$$

where $u_s \in \mathscr{S}_{Q_s t_s}$ and $\theta_s = \rho(Q_s, t_s)|\nabla K(Q_s)|$.

The motion of the Q parameter Q_s under this flow corresponds to exactly the motion of the flow Ψ_s associated to the gradient field $\rho \cdot \nabla K$ on the (x_1, x_2) plane, and the t parameter being preserved under the flow. We require the following qualitative result on the gradient flow $\rho \cdot \nabla K$ in the disk $B((c\delta_0)^{1/2})$.

Lemma 5.3. *For* δ_0 *sufficiently small and* c *large, given a curve* X: $[-1, 1] \to B((c\delta_0)^{1/2})$, *there exists* $r_0 > 0$ *so that for* $r \geqslant r_0$, *denoting by* $\Psi_r \circ X = X_r$ *the segment of the curve lying in* $B((c\delta_0)^{1/2})$, *say* $X_r^{-1}(B((c\delta_0)^{1/2}) = [p_{-1}, p_1]$, *then one of the following holds*:

(i) $X_r([p_{-1}, p_1]) \subset \Omega_{c\delta_0, c^{-1}\delta_0}$ *with* $X_r(p_{-1})$, $X_r(p_1) \in \Sigma_{c\delta_0}^+ \cup \Sigma_{c\delta_0}^-$.

(ii) $X_r([-1, 1]) \subset B(2(c\delta_0)^{1/2}) - B((c\delta_0)^{1/2})$.

(iii) *There exists a point* $p_0 \in [-1, 1]$ *such that*

$$X(p_0) \in \left(B\left((c\delta_0)^{1/2}\right) - B\left((c^{-1}\delta_0)^{1/2}\right)\right) \cap \left\{|x_1/x_2| \leqslant 2^{-1}(C/A)^{1/2}\right\}.$$

(iv) $X_r([-1, 1]) \subset B((c^{-1}\delta_0)^{1/2})$.

Proof. Suppose in a neighborhood of $P_\infty = (0,0,1)$, K is of the form

$$K(x_1, x_2, x_3) = K(P_\infty) + Ax_1^2 - Cx_2^2, \qquad 0 < A < C.$$

Then it is easy to check the result of the lemma. In general when K has the power series expansion in a neighborhood of P_∞,

$$K(x_1, x_2, x_3) = K(P_\infty) + Ax_1^2 - Cx_2^2 + O(|x'|^3) \qquad (x' = (x_1, x_2)), \tag{5.4}$$

we can apply a perturbation argument based on the linearization theorem for vector fields (cf., [7, Chapter 9]).

We now apply Lemma 5.3 to prove claims (i) and (ii) in the statement of Proposition 5.2. To do so, we apply the lemma to each segment $u_p^{(k)}$, $p \in [a_k, b_k]$, of a maximizing sequence $u^{(k)}$ which concentrates with its (Q, t) parameter in $B((c\delta_0)^{1/2}) \times [t_0, \infty)$, we will first verify that along the flow $\tilde{\Psi}_s$, F is nondecreasing: ($u_s = \tilde{\Psi}_s(u)$, $S[u_s] = S[u]$)

$$\begin{aligned}\frac{dF}{ds}[u_s] &= \frac{d}{ds}\log \int K\exp(2u_s)\\ &= \left(\int K\exp(2u_s)\right)^{-1} \int \left[\frac{d}{d\tau}\left(K\circ R_{Q_s,-\tau\theta_s}\right)\right]_{\tau=0} \exp(2u_s)\\ &= \rho(\theta_s, t_s)|\nabla K(Q_s)|\left[f(P_s) + \tfrac{1}{2}\Delta f(P_s)d_s + O(d_s)\right], \int K\exp(2u_s)\end{aligned}$$

where $u \in \mathscr{S}_{Q_s,t_s} \cap C_{P_s,\delta_s}$, $\delta_s = 4t_s^{-2}\log t_s + O(t_s^{-2})$, and

$$f = \langle \nabla K, (d/d\tau)|_{\tau=0}(R_{Q_s,-\tau\theta_s})\rangle.$$

Since $|P_s - Q_s| = O(t_s^{-1}) = O(\delta_s|\log\delta_s|^{1/2})$, we have for $\rho(\theta_s, t_s) \neq 0$,

$$f(P_s) = f(Q_s) + O(t_s^{-1}) = |\nabla K(Q_s)| + O(t_s^{-1}) \approx (c^{-1}\delta_0)^{1/2} + o((\delta_0)^{1/2}).$$

Thus

$$\begin{aligned}\frac{dF[u_s]}{ds} &\geqslant \rho(\theta_s, t_s)|\nabla K(Q_s)|\left((c^{-1}\delta_0)^{1/2} + O((\delta_0)^{1/2})\right)\\ &\geqslant 0, \quad \int K\exp 2u_s\end{aligned}$$

and $\tilde{\Psi}_s(u^{(k)})$ yields a competing maximizing sequence of paths. We will now apply Lemma 5.3 to the sequence $u^{(k)}$ and indicate that only case (i) of the lemma could happen.

If for a maximizing sequence $u^{(k)}$ we are in case (ii) of Lemma 5.3, then we have a competing sequence of paths which do not concentrate at P_∞, contradicting our hypothesis.

If for some maximizing sequence $u^{(k)}$ we have $\tilde{u}_s^{(k)} = \tilde{\Psi}_s(u^{(k)})$ for some subsequence in case (iii) for Lemma 5.3, denote by p_k the point such that $\tilde{u}^{(k)}_{p_k} \in \mathscr{S}_{Q_k, t_k}$ with $Q_k \in (B((c\delta_0)^{1/2}) - B((c^{-1}\delta_0)^{1/2})) \cap \{|x_1/x_2| \leqslant 2^{-1}(C/A)^{1/2}\}$, and $t_k \geqslant t_0$. Thus

$$\begin{aligned}
F\left[u^{(k)}_{p_k}\right] &= \log \int K\exp\left(2\tilde{u}^{(k)}_{p_k}\right) - S\left[\tilde{u}^{(k)}_{p_k}\right] \\
&\leqslant \log \int K\exp\left(2\tilde{u}^{(k)}_{p_k}\right) \quad \left(\text{by } (3.12)'\right) \\
&\leqslant \log\Big[K(Q_k) + 2^{-1}\Delta K(Q_k)\delta_k + o(\delta_k) \\
&\qquad\qquad + \left(S\left[u^{(k)}_{p_k}\right]\right)^{1/2}\delta_k^{1/2}|\nabla K(Q_k)|\Big] \\
&\leqslant \log\left[K(Q_k) + (A - C)\delta_k + o(\delta_k)\right] \\
&\leqslant \log\left[K(P_\infty) + Ax_1^2(Q_k) - Cx_2^2(Q_k) + o(\delta_k)\right] \\
&\leqslant \log\left[K(P_\infty) - (3AC\delta_0)/c(4A + C)\right] \quad \text{for } k \text{ sufficiently large.}
\end{aligned}$$

Hence $\min_p F[u^{(k)}_p] \leqslant \log[K(P_\infty) - O(\delta_0)] < \log K(P_\infty)$, contradicting our hypothesis that $\sup_u \min F[u] = \log K(P_\infty)$. Thus case (iii) of Lemma 5.2 cannot occur.

If for a maximizing sequence $u^{(k)}$ we are (after applying $\tilde{\Psi}_s$) in case (iv), we may estimate $F[u^{(k)}_{p_k}]$, the first point of entry of the (Q, t) parameter into the region $B((c^{-1}\delta_0)^{1/2}) \times [t_0, \infty)$, as follows:

$$\begin{aligned}
F\left[u^{(k)}_{p_k}\right] &= \log \int K\exp\left(2u^{(k)}_{p_k}\right) - S\left[u^{(k)}_{p_k}\right] \\
&\leqslant \log\left[K(P_k) + (1/2)\Delta K(P_k)\delta_0 + o(\delta_0)\right] \\
&\leqslant \log\Big[K(P_\infty) + |K(P_\infty) - K(Q_k)| + |K(Q_k) - K(P_k)| \\
&\qquad\qquad + (1/4)\Delta K(P_\infty)\delta_0 + o(\delta_0)\Big] \\
&\leqslant \log\left[K(P_\infty) + (1/2)(A - C)\delta_0 + c^{-1}\delta_0 + o(\delta_0)\right] \\
&< \log K(P_\infty)
\end{aligned}$$

where $u^{(k)}_{p_k} \in \mathscr{S}_{Q_k, t_0} \cap C_{P_k, \delta_0}$; and c is chosen so that $c^{-1} \ll (A - C)/2$, then by the same reasoning as in case (iii), we see that case (iv) cannot occur.

Thus we must be in the remaining case (i) of Lemma 5.3 for all large k. By reparametrizing $\tilde{u}^{(k)} = \tilde{\Phi}_s(u^{(k)})$, we may arrange to have their Q parameter curve $Q^{(k)}_p$ entering the region $\Omega_{c\delta_0, c^{-1}\delta_0}$ in the segment $[-1, 1]$ so that $Q^{(k)}_1$, $Q^{(k)}_{-1} \in \Sigma^+_{c\delta_0} \cup \Sigma^-_{c\delta_0}$ and achieving their minima at $p = 0$, this establishes (i) and (ii) of Proposition 5.2.

Our next step is to perform a second deformation on the part of the curve $u^{(k)}_p$ whose Q parameter falls into the region $\Omega_{c\delta_0, c^{-1}\delta_0}$ to ensure that $u^{(k)}_1$ and $u^{(k)}_{-1}$ have their t parameter as large as we wish, thus $S[u^{(k)}_1]$ and $S[u^{(k)}_{-1}]$ could be made as small as needed. To do so, let ρ be a function $0 \leqslant \rho \leqslant 1$ supported in $\Omega_{c\delta_0, c^{-1}\delta_0} \times [t_0, \infty)$ with $\rho = 1$ on $(\Sigma^+_{c\delta_0} \cup \Sigma^-_{c\delta_0}) \times [2t_0, \infty)$, and let $w_u = (d/ds)T^s(Q)(u)|_{s=1}$ and v_u be the function associated with u as in Lemma 4.3. Define the flow $\tilde{\Phi}$ by solving the ordinary differential equation for $\tilde{\Phi}_s[u] = u_s$ with $(\int \exp 2u_s = 1)$ $(d/ds)u_s = w_{u_s} + v_{u_s}$. Observe that if $u \in \mathscr{S}_{Q,t}$ then $u_s \in \mathscr{S}_{Q,t}$ and

$$-(d/ds)S[u_s] = (d/ds)J[u_s] = J'[u_s](w_{u_s}) + J'[u_s](v_{u_s})$$
$$= J'[u_s](v_{u_s}) \geqslant c(\eta)(S[u_s])^{1/2} \quad (\text{by } (4.1)).$$

Hence $S[u_s] \leqslant e^{-cs}S[u_0]$ for $s \geqslant 0$. Also we have

$$\frac{d}{ds}F[u_s] = F'[u_s](w_{u_s}) + F'[u_s](v_{u_s}) \tag{5.5}$$

where

$$F'[u_s](w_{u_s}) = \left(\int Ke^{2u_s}\right)^{-1} \int \langle \nabla K, \nabla(x \cdot Q)\rangle e^{2u_s}$$
$$= \left(\int Ke^{2u_s}\right)^{-1} \Big(f(Q) + 2^{-1}\Delta f(Q)\delta_s + o(\delta_s)$$
$$+ O\big((S[u_s])^{1/2}\big)\delta_s^{1/2}|\nabla f(Q)|\Big)$$

where $f = \langle \nabla K, \nabla(x \cdot Q)\rangle$, $\delta_s = 4t_s^{-2}\log t_s$.

Taking Taylor series expansion of K as in formula (5.4) and compute ∇K, $\nabla(x \cdot Q)$ around $P_\infty = (0, 0, 1)$ we find

$$\nabla K(x_1, x_2) = \big(2Ax_1 + O(|x'|^2), -2Cx_2 + O(|x'|^2)\big), \qquad x' = (x_1, x_2),$$
$$x \cdot Q = x_1(Q)x_1 + x_2(Q)x_2 + x_3(Q)(1 - x_1^2 - x_2^2)^{1/2},$$
$$\nabla(x \cdot Q) = \big(x_1(Q) - x_3(Q)x_1 + O(|x'|^2), x_2(Q) - x_3(Q)x_2 + O(|x'|^2)\big),$$
$$\langle \nabla K, \nabla(x \cdot Q)\rangle = 2Ax_1(Q)x_1 - 2Cx_2(Q)x_2 - 2Ax_3(Q)x_1^2$$
$$+ 2Cx_3(Q)x_2^2 + O(|x'|^2).$$

We have $f(Q) = 0$ (by definition of f)

$$|\nabla f(Q)| = O(|x'(Q)|),$$
$$\Delta f(Q) = (-4A + 4C)x_3(Q) + O(|x'|).$$

Thus

$$F'[u_s](w_{u_s}) \geqslant \left(\int Ke^{2u_s}\right)^{-1}\Big(2(C - A)\delta_s + o(\delta_s) + O\big(\delta_s^{1/2}(S[u_s])^{1/2}\big)\Big). \tag{5.6}$$

Also we have by Proposition 4.2,

$$F'[u_s](v_{u_s}) \geqslant (1 - c_1(\eta))(S[u_s])^{1/2} - O(|\nabla K(Q)|\delta_s^{1/2}) \tag{5.7}$$

for some $c_1(\eta) < 1$, η the same constant as in Lemma 4.3. From (5.5), (5.6), and (5.7) we conclude that we may apply Proposition 4.2 to fixed function u_s until $S[u_s] = O(|\nabla K(Q)|^2 \delta_s)$. Then (5.6) implies that $(d/ds)F[u_s] > 0$.

Evaluating $F[u_s]$ for $s \gg 1$ (so that $\delta_s \ll \delta_0$), $u \in \mathscr{S}_{Q,t}$, $Q \in \Sigma^+_{c\delta_0} \cup \Sigma^-_{c\delta_0}$ we find by using the asymptotic formula (3.12)′ that for $u_s \in \mathscr{S}_{Q_s,t_s}$

$$\begin{aligned} F[u_s] &= \log \fint Ke^{2u_s} - S[u_s] \\ &= \log\left[K(Q) + 2^{-1}\Delta K(Q)\delta_s + o(\delta_s)\right. \\ &\qquad \left. - O\left(|\nabla K(Q)|\delta_s\right)\right] - O\left(|\nabla K(Q)|^2 \delta_s\right) \\ &\geqslant \log\left[K(P_\infty) + (3/4)AC(1+(A/4C))^{-1}c\delta_0 + O(\delta_s)\right] - O(\delta_0\delta_s) \\ &\geqslant \log\left[K(P_\infty) + 2^{-1}AC(1+(A/4C))^{-1}c\delta_0\right] \quad \text{as } s \to \infty. \end{aligned}$$

This establishes (iii) and finishes the proof of Proposition 5.2.

Corollary 5.4. *Given u_1, $u_2 \in H^1(S^2)$ with $u_i \in \mathscr{S}_{Q_i,t_i}$, $Q_i \in \Sigma^+_{c\delta_0}$, $t_i \geqslant t_0$, $S[u_i] \leqslant ct_i^{-2}\log t_i$, and $F[u_i] \geqslant \log K(P_\infty) + c_{\delta_0}$ for $i = 1,2$, there exists a 1-parameter family of functions u_s, $1 \leqslant s \leqslant 2$, so that $u_s \in \mathscr{S}_{Q_s,t_s}$ where $Q_s \in \Sigma^+_{c\delta_0}$ and $F[u_s] \geqslant \log K(P_\infty) + c_{\delta_0}$.*

Proof. By rotation along the curve if necessary, we may assume without loss of generality that $u_i \in \mathscr{S}_{Q,t_i}$ with a common $Q \in \Sigma^+_{c\delta_0}$. Applying the flow $\tilde{\Phi}_s$ of Proposition 5.2, we have constructed a continuous family $\tilde{\Phi}_s(u_i)$ along which (1) the Q parameter is fixed, (2) $S[\tilde{\Phi}_s(u_i)]$ becomes arbitrarily small, and (3) $F[\tilde{\Phi}_s(u_i)] \geqslant \log K(P_\infty) + c\delta_0$. Thus by the conformal invariance of the functional S and the asymptotic formula (3.12)′ applied to evaluate $\fint Ke^{2u}$, it suffices to prove the following elementary lemma to establish the corollary.

Lemma 5.5. *There exists some $\tilde{\eta} > 0$ so that for given $v_0, v_1 \in \mathscr{S}_0$ with $S[v_i] \leqslant \tilde{\eta}$ there is a continuous curve $v_s \in \mathscr{S}_0$, $0 \leqslant s \leqslant 1$, such that $S[v_s] \leqslant c_a\tilde{\eta}$, c_a is a constant depending on the constant of Proposition* 3.7.

Proof. Let $e^{2v_s} = se^{2v_1} + (1-s)e^{2v_0}$. Then $v_0, v_1 \in \mathscr{S}_0$ implies that $v_s \in \mathscr{S}_0$ for all $0 \leqslant s \leqslant 1$. Furthermore

$$|\nabla v_s| \leqslant e^{-2v_s}\left(s|\nabla v_1|e^{2v_1} + (1-s)|\nabla v_0|e^{2v_0}\right) \leqslant |\nabla v_0| + |\nabla v_1|.$$

Thus

$$\begin{aligned} \fint |\nabla v_s|^2 &\leqslant 2\left(\fint |\nabla v_0|^2 + \fint |\nabla v_1|^2\right) \\ &\leqslant 2(1-a)^{-1}(S[v_0] + S[v_1]) \quad \text{(Corollary 3.8)} \\ &\leqslant 4(1-a)^{-1}\tilde{\eta}, \\ S[v_s] &\leqslant \fint |\nabla v_s|^2 \leqslant 4(1-a)^{-1}\tilde{\eta}. \end{aligned}$$

Corollary 5.6. *Suppose a maximizing sequence $u^{(k)}$ of minima for* $\mathrm{Var}(P_\alpha, P_\beta)$ *concentrates at a saddle point P_∞ with $\Delta K(P_\infty) < 0$. Then there is a competing sequence of paths $\tilde{u}^{(k)}$ satisfying the conclusion of Proposition* 5.2 *and in addition we have for $\tilde{u}^{(k)}_{\pm 1} \in \mathscr{S}_{Q_{\pm 1}, t_{\pm 1}}$ that either*

(1) $Q^{(k)}_{+1} \in \Sigma^+_{c\delta_0}$, $Q^{(k)}_{-1} \in \Sigma^-_{c\delta_0}$ *or*

(2) $Q^{(k)}_{+1} \in \Sigma^-_{c\delta_0}$, $Q^{(k)}_{-1} \in \Sigma^+_{c\delta_0}$.

Proof. Suppose $Q^{(k)}_{\pm 1} \in \Sigma^+_{c\delta_0}$. Then Corollary 5.4 allows us to construct a new path $\tilde{\tilde{u}}^{(k)}$ by joining $\tilde{u}^{(k)}_{-1}$ to $\tilde{u}^{(k)}_1$ so that the new path either achieves its minimum outside the interval $[-1,1]$ or in $[-1,1]$, but the value of F on the path is greater than $\log K(P_\infty)$; in either case we do not have concentration at P_∞. (Actually along each path $\tilde{u}^{(k)}$ there may be several intervals, but finite in number, in which concentration occurs at perhaps several distinct saddle points P_∞ with $\Delta K(P_\infty) < 0$ and $K(P_\infty)$ have the same value, but the same analysis would eliminate all such intervals.)

The next proposition eliminates the possibility that a maximizing sequence for $\mathrm{Var}(\Gamma)$ can concentrate anywhere.

Proposition 5.7. *If Γ is a simple closed curve on S^2 satisfying condition* (5.1), *then the variational problem* $\mathrm{Var}(\Gamma)$ *does not have maximizing sequence $u^{(k)}_{p_k}$ of minima which concentrates at a saddle point P_∞ with $\Delta K(P_\infty) < 0$.*

Proof. It suffices to prove the following two statements:

(i) A maximizing sequence of minima for $\mathrm{Var}(\Gamma)$ cannot concentrate at a saddle point $P_\infty \in \Gamma$ with $\Delta K(P_\infty) < 0$,

(ii) The same max-min sequence cannot occur in the interior $\gamma(\Delta)$ of Γ.

To prove (i), assume the contrary, then we observe that (by the max-min procedure) such a saddle point P_∞ can only occur where $\min\{K(P) \mid P \in \Gamma\}$ is attained. Thus it follows from condition (5.1) that we may assume (by taking δ_0 sufficiently small) that $\Gamma \cap B((c\delta_0)^{1/2})$ lies in $\Omega_{c\delta_0}$ and that $\Gamma \cap \Sigma^+_{c\delta_0}$, $\Gamma \cap \Sigma^-_{c\delta_0}$ are single points. We also observe the following topological fact:

Lemma 5.8. *For each $u \in \mathscr{P}(\Gamma)$, there exists $p_0 \in \Delta$ where $C.M.(e^{2u_{p_0}}) \in K_{c\delta_0}$;*

$$K_\delta = \text{the part of the circle in the } x_2x_3 \text{ plane centered at } (0,0,1) \text{ of radius } \delta \text{ that is contained in the unit ball } B = \{x_1^2 + x_2^2 + x_3^2 < 1\}. \tag{5.8}$$

Proof of Lemma 5.8. The center of mass surface C.M.: $\Delta \to B$ given by $\mathrm{C.M.}(p) = ⨍ x e^{2u_p}$ is a continuous map which extends continuously to $\bar{\Delta}$, mapping $\partial\Delta$ to Γ, agreeing with the parametrization γ. Let P_1, P_2 denote the end points of the circular arc $K_{c\delta_0}$. Making a conformal transformation ϕ of $\mathbb{R}^3$ sending P_1 to $(0,0,1)$ and P_2 to $(0,0,-1)$ and $\phi\gamma$ to the diameter $\{(0,0,x_3), -1 < x_3 < 1\}$, then $\phi(B((c\delta_0)^{1/2})$ covers the hemisphere $x_1 \geqslant 0$. Taking projection of $\phi(\Gamma)$ to the x_1x_2 plane it follows that index $(\phi(\Gamma), 0) \neq 0$.

Therefore every continuous extension of the map $\gamma\colon \partial\Delta \to \Gamma$ to $\bar{\Delta} \to B$ must meet Γ as claimed.

Take $p_0 \in \Delta$ given by the above lemma; we compute $F[u_{p_0}]$ using the asymptotic formula

$$F[u_{p_0}] = \log \fint K \exp 2u_{p_0} - S[u_{p_0}] \leqslant \log \fint K \exp 2u_{p_0}$$
$$\leqslant \log\left[K(P) + 2^{-1}\Delta K(P)\delta + O(\delta)\right], \quad \text{where } u_{p_0} \in C_{P,\delta}.$$

Since $x_1(P) = 0$ and $P \in K_{c\delta_0}$, it follows that

$$F[u_{p_0}] \leqslant \log K(p_\infty) - c_{\delta_0},$$

which contradicts our assumption that $\sup_{u \in \mathscr{P}'_{t_0}} \min_{p \in \Delta} F[u_p] = \log K(P_\infty)$. This finishes the proof of claim (i) in Proposition 5.7.

To prove (ii), we assume that the maximizing sequence $u^{(k)}$ with minima $u^{(k)}_{p_k}$ occur in the interior of Δ; hence by reparametrizing, we may assume $p_k = 0$ for all k. Restricting $u^{(k)}$ to each diameter $u_{\theta,p}$: $p = re^{i\theta}$, $-1 < r < 1$, we may apply Proposition 5.2 to each path $u^{(k)}_\theta$ and obtain a competing path $\tilde{u}^{(k)}_\theta$ which may be reparametrized so that $Q(u^{(k)}_{\theta,\pm 1/2}) \in \Sigma^{\pm}_{c\delta_0}$. Since the path $\tilde{u}_\theta$ was obtained from u_θ by a flow which is a continuous process depending continuously on initial data, we see that we must have either both Q parameters of $u^{(k)}_{\theta,\pm 1/2}$ lying in $\Sigma^{+}_{c\delta_0}$ or both lying in $\Sigma^{-}_{c\delta_0}$. In either case, we apply the procedure of Corollary 6.1 to obtain a map $\hat{u}\colon B(1/2) \to H^1(S^2)$, such that $\hat{u}^{(k)} | \partial B(1/2) = u^{(k)} | \partial B(1/2)$ and $Q(\hat{u}^{(k)}_p) \in \Sigma^{+}_{c\delta_0}$ for all $p \in B(1/2)$ so that $F[\hat{u}^{(k)}_p] \geqslant \log K(P_\infty) + c_{\delta_0}$. This contradicts our assumption that $u^{(k)}_p$ is a maximizing sequence for $\mathrm{Var}(\Gamma)$. We have thus proved statement (ii) and hence finished the proof of Proposition 5.7.

6. Proof of Theorem II

First we label the local maxima of K by $P_0, P_1, \cdots, P_p$ according to the increasing order of the value of K. Group the saddle points of K according to whether ΔK is positive or negative. For those saddle points with $\Delta K < 0$, we label them as $Q_1, \cdots, Q_q$ (the order of Q_j in terms of the value of $K(Q_j)$ will be fixed later.)

We divide the proof into two cases according to whether $p > q$ or $q > p$. In the first case, we will show that some one-dimensional scheme $\mathrm{Var}(P_\alpha, P_\beta)$, $0 \leqslant \alpha, \beta \leqslant p$, succeeds. In the second case we will show that it is possible to select a simple closed curve Γ which satisfies condition (5.1) so that the 2-dimensional scheme $\mathrm{Var}(\Gamma)$ succeeds.

Proof of Theorem II *for the case* $p > q$. For the $p > q$ case, we begin by analyzing when a scheme Var(P_α, P_β) fails. According to Proposition 5.1, a max-min sequence for such a scheme must concentrate at a saddle point with $\Delta K(Q_\gamma) < 0$. Let us label the Q's according to increasing values of K:

$$(6.1)\qquad \begin{aligned} K_1 &= K(Q_1) = \cdots = K(Q_{q_1}) < K_2 = K(Q_{q_1+1}) \\ &= \cdots < K_m = K(Q_{q_1+\cdots+q_m}), \qquad q_1 + \cdots + q_m = q. \end{aligned}$$

We define the following equivalence relations on the set of local maxima $\{P_0, P_1, \cdots, P_p\}$ indexed by the K_j's.

Definition. We say $P_\alpha \sim_j P_\beta$ if there exists a path $u_p \in \mathscr{P}(P_\alpha, P_\beta)$ so that $\inf F[u_p] > \log K_j$. Denote by $[P_\alpha]_j$ the $\sim_j$ equivalence class containing P_α. Declare $P_\alpha \sim_0 P_\beta$ and $P_\alpha \sim_j P_\alpha$.

Assume that there are no solutions to all schemes Var(P_α, P_β), and set n_j = number of $\sim_j$ equivalence classes. Then we claim:

$$(6.2)\qquad n_j \leqslant q_1 + q_2 + \cdots + q_j + 1, \qquad 0 \leqslant j \leqslant q.$$

We proceed to prove (6.2) inductively. The statement for $j = 0$ is trivially true. For the inductive step, assume $S_1, \cdots, S_n$ are the $\sim_j$ equivalence classes, then the $\sim_{j+1}$ equivalence classes are partitions of the $\sim_j$ equivalence classes. We need to count the number of new $\sim_{j+1}$ equivalence classes each $\sim_j$ equivalence class can contain. To do this it will be convenient to introduce ideal points Q^+, Q^- into our consideration and to extend the notion of $\sim_j$ equivalence. For each saddle point Q with $\Delta K(Q) < 0$, fix a coordinate system (x_1, x_2, x_3) with $Q = (0, 0, 1)$. In a neighborhood of Q, K has the Taylor expansion

$$K(x_1, x_2, x_3) = K(Q) + Ax_1^2 - Cx_2^2 + O(|x'|^2), \qquad (x' = (x_1, x_2))$$

with

$$Q^+ = \left((c\delta_0)^{1/2}, 0, (1 - c\delta_0)^{1/2}\right), \qquad Q^- = \left(-(c\delta_0)^{1/2}, 0, (1 - c\delta_0)^{1/2}\right),$$

so that $Q^+ \in \Sigma^+_{c\delta_0}(Q)$, $Q^- \in \Sigma^-_{c\delta_0}(Q)$ as in Proposition (5.2) and Corollary (5.6). We extend the notion of $\sim_{j+1}$ equivalence to that of $\approx_{j+1}$ equivalence on the set

$$\{P_1, \cdots, P_p\} \cup \left\{Q^\pm_{q_1+\cdots+q_j+1}, Q^\pm_{q_1+\cdots+q_j+2}, \cdots, Q^\pm_{q_1+\cdots+q_{j+1}}\right\},$$

where we say $P_\alpha \approx_{j+1} P_\beta$ if $P_\alpha \sim_{j+1} P_\beta$ as before, and $Q^+_\gamma \approx_{j+1} P_\alpha$ if there exists a path $u_p \in \mathscr{P}((Q^+_\gamma, P_\alpha)$ such that $\inf F[u_p] \geqslant \log K_{j+1}$; and similarly define $Q^-_\gamma \approx_{j+1} P_\alpha$ as well as $Q^+_\gamma \approx_{j+1} Q^-_\gamma$.

Lemma 6.1. *Assume that there are no solutions to the equation* (1.1). *Then for any* $P_\alpha \sim_j P_\beta$, $P_\alpha \sim_{j+1} P_\beta$ *with* $K(P_\alpha)$, $K(P_\beta) > K_{j+1}$, *we can find a test function* $u_p \in \mathscr{P}(P_\alpha, P_\beta)$ *such that there is at most a finite number of disjoint intervals* I_γ, $1 \leqslant \gamma \leqslant N$, *a corresponding set of* Q_γ *with* $K(Q_\gamma) = K_{j+1}$ *so that* u_p *has its* Q, t *parameter lying in the region* $\Omega_{c\delta, c^{-1}\delta}(Q_\gamma) \times [t_0, \infty)$ *over the intervals* I_γ, *and that over the complement* $\bigcup I' = (\bigcup I_\gamma)^c$, $F[u_p] \geqslant K_{j+1} + \varepsilon(\delta_0)$, *where* $\varepsilon(\delta_0) > 0$.

Proof. Take a max-min sequence of test functions $u^{(k)}$ for the scheme $\mathrm{Var}(P_\alpha, P_\beta)$ which must concentrate at (perhaps several) Q_γ with $K(Q_\gamma) = K_{j+1}$. The Q,t parameter of the test functions $u^{(k)}$ will fall into the regions $\Omega_{c\delta_0, c^{-1}\delta_0}(Q_\gamma) \times [t_0, \infty)$ over a corresponding interval I_γ. In the complementary intervals $\bigcup I' = (I_\gamma)^c$ either $t \leqslant t_0$ or if $t > t_0$, the Q parameter lies in the region $\{K(Q) \geqslant K_{j+1} - \varepsilon\} - \bigcup_\gamma \Omega_{c\delta_0, c^{-1}\delta_0}(Q_\gamma)$. Consider the gradient flow associated to the functional $F'[u]$. We claim that $|F'[u]| \geqslant C_{t_0} > 0$ for $u \in \mathscr{S}_{Q,t}$, $t \leqslant t_0$, and $F[u] \geqslant$ Constant. For if not, then there exists a sequence $u_j \in \mathscr{S}_{Q_j, t_j}$, $t_j \leqslant t_0$, with $F'[u_j] \to 0$. Set $u_j' = u_{\phi_j}$ with $\phi_j = \phi_{Q_j, t_j}$. Then we have $S[u_j'] = S[u_j] \leqslant$ Constant, hence it follows from the Onofri's inequality that a subsequence $u_j' \to u_\infty'$. From the compactness of the family of conformal transformations we have a subsequence u_j converging weakly in H^1 to u_∞. This means that $F'[u_\infty] = \lim F'[u_j] = 0$, contradicting the hypothesis.

Hence we may combine the flow Φ_s, associated to $F'[u]$ for those u with their (Q, t) parameter $t \leqslant t_0$, with the flow Ψ_s associated to ∇K as in Proposition (5.2) to those u with $t \geqslant t_0$, Q lying in the set $\{K(Q) \geqslant K_{j+1} - \varepsilon\} - \bigcup_\gamma \Omega_{c\delta_0, c^{-1}\delta_0}(Q_\gamma)$, arranged with a suitable cutoff function in t to yield a test path with the required properties. This finishes the proof of the lemma.

Each $\sim_j$ equivalence class S_k is thus partitioned into sets

$$S_k = \bigcup_\gamma \left(\left(S_k \cap [Q_\gamma^+]_{j+1} \right) \cup \left(S_k \cap [Q_\gamma^-]_{j+1} \right) \right),$$

and we say Q_γ separates S_k if $[Q_\gamma^+]_{j+1} \neq [Q_\gamma^-]_{j+1}$ and $[Q_\gamma^+]_{j+1} \cap S_K \neq \varnothing$ (hence $[Q_\gamma^-]_{j+1} \cap S_k \neq \varnothing$). It is clear that Q_γ cannot separate two distinct S_k's. Let $Q(S_k) = \{Q \mid K(Q) = K_{j+1}$ and Q_γ separates $S_k\}$. To prove (6.2) it suffices to show each S_k is decomposed into at most $|Q(S_k)| + 1 \sim_{j+1}$ equivalence classes.

To see this, associate to S_k the connected graph G: the vertices of G are $\approx_{j+1}$ equivalence classes and the edges are the saddle points that separate S_k. Two $\approx_{j+1}$ equivalence classes, say vertices S_i' and S_j', are joined by an edge Q if there is a variational scheme $\mathrm{Var}(P_i', P_j')$, $P_i' \in S_i'$ and $P_j' \in S_j'$ with a maximizing sequence concentrating only at Q, thus $S_i' = [Q^+]_{j+1}$, $S_j' = [Q^-]_{j+1}$ or vice versa.

Since we assume no variational scheme produces a convergent solution, it follows that G is connected. Thus $\#\{\text{vertices}\} \leqslant \#\{\text{edges}\} + 1$, and this is the inequality (6.2) claimed, for it follows that $n_{j+1} \leqslant q_{j+1} + n_j \leqslant q_1 + q_2 + \cdots + q_{j+1} + 1$.

Applying this inequality for $j = m$, we conclude that for $p > q$, there is a pair of maxima P_α, P_β which are $\sim_m$ equivalent. For such a pair, $\operatorname{Var}(P_\alpha, P_\beta)$ will clearly yield a solution of (1.1).

Proof of Theorem II *for the case $p > p$.* We will use the two-dimensional scheme $\operatorname{Var}(\Gamma)$ with Γ satisfying condition (5.1). We will prove that under the hypothesis $q > p$, such a curve Γ exists. Recall that Proposition 5.7 indicates that for such Γ, $\operatorname{Var}(\Gamma)$ has a convergent max-min sequence.

Since K is nondegenerate, the level sets are points which are local maxima or minima or smooth curves whose intersections occur only at saddle points where exactly two level curves meet transversally. The superlevel sets $U_c = \{P \in S^2 \mid K(P) \geqslant c\}$ consist of several components each of which is bounded by level sets $L_c = \{P \in S^2 \mid K(P) = c\}$. Listing the critical points $\{Q_j\}$ with $\Delta K(Q_j) < 0$ according to decreasing values of K:

$$(6.3)\quad \begin{aligned} K_1 = K(Q_1) = \cdots = K(Q_{q_1}) > K_2 = K(Q_{q+1}) = \cdots = K(Q_{q_1+q_2}) \\ > \cdots > K_m = \cdots = K(Q_{q_1+\cdots+q_m}), \qquad q_1 + \cdots + q_m = q. \end{aligned}$$

We proceed to count the number N_j of components in the super-level sets U_{K_j}. We claim:

(6.4) If there is no curve Γ satisfying condition (5.1) with $\min\{K(P) \mid P \in \Gamma\} \geqslant K_j$ then

$$N_j \leqslant \#\{P_\alpha \mid K(P_\alpha) \geqslant K_j\} - \#\{Q_\gamma \mid K(Q_\gamma) > K_j\}.$$

Proof of claim (6.4). We do this inductively. For $j = 1$, the superlevel set $U_{k_1+\varepsilon}$ has at most $\#\{P_\alpha \mid K(P_\alpha) > K_1\}$ components. Since there are no Γ satisfying (5.1) with $\min K(P) \geqslant K_1$ at each Q_i with $K(Q_i) = K_1$, when we reach the superlevel set U_{K_1} disjoint components are attached at the saddle points $Q_1, \cdots, Q_q$, hence the assertion (6.4) for $j = 1$, where we have taken into account the possibility that some $K(P_\alpha) = K_1$ as well.

Assuming (6.4) for the jth level, there are at the next level $K_{j+1} + \varepsilon$ at most $N_j + \#\{P_\alpha \mid K_{j+1} < K(P_\alpha) < K_j\}$ components which are then joined at the level K_{j+1} to q_{j+1} saddle points (under the assumption that there is no curve satisfying (5.1) with $\min\{K(P) \mid P \in \Gamma\} \geqslant K_{j+1}$), verifying (6.4) for the $j + 1$ stage.

It follows from claim (6.4) that since $q > p$ there will be some level K_j where $N_j = 0$; this is a contradiction ($N_j \geqslant 1$ for each j). Thus at some saddle point Q with $\Delta K(Q) < 0$ a path satisfying (5.1) can be constructed, which as we have stated before, gives a solution of equation (1.1) by Proposition (5.7). We have thus finished the proof of Theorem II.

7. Examples and remarks

As we have mentioned in §1, although the hypothesis of Theorem II is satisfied by an open set of functions K in the $\mathscr{C}^2$ topology, it will only be sufficient but not necessary for the existence of solution of equation (1.1). It is apparent, as the Kazdan-Warner theorem [10] indicates, that some condition of analytic nature would be necessary conditions.

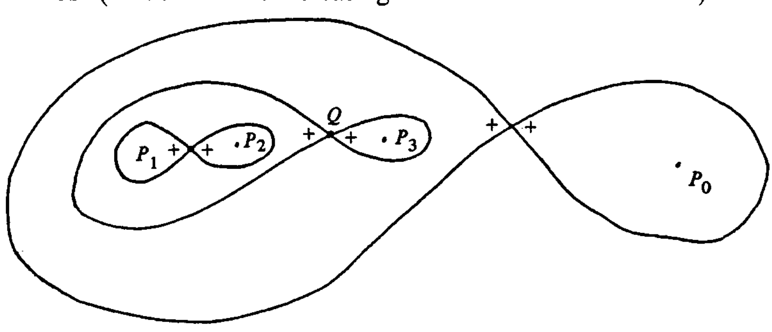
In this section, we will describe several examples, the first two examples indicate that when $|p - q|$ is large, our method can give more than one distinct solution of equation (1.1). The last example gives a plausible argument, which indicates that, for our variational method to work, $p \neq q$ may be a necessary condition in Theorem II.

Examples. (1) When $q = 1$, $p = 3$, and the relevant critical points of K (in Theorem II) satisfy

(1.a) $K(P_0) < K(Q) < K(P_1) < K(P_2) < K(P_3)$ and

(1.b) for any pair (P_i, P_j), $1 \leqslant i, j \leqslant 3$, there exist a path on S^2 joining P_i to P_j along which $K \geqslant K(Q)$,

Picture for the level curve of such a
K: (+ denotes the increasing direction of the value of K).

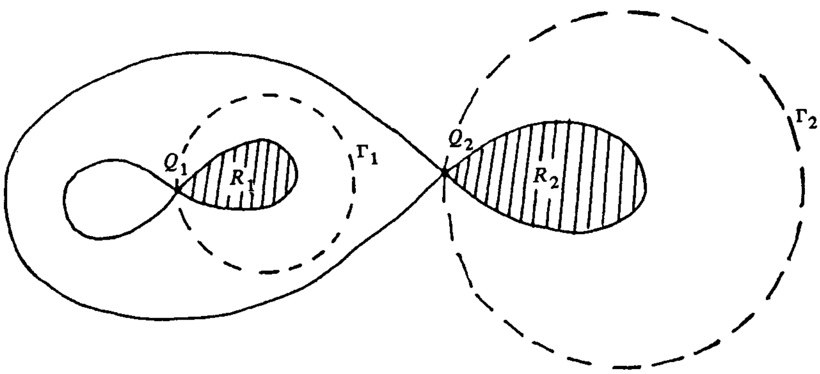


Then obviously, the scheme $\mathrm{Var}(P_0, P_1)$ yields a solution u_0 with $F[u_0] < \log K(P_0)$. Now under the assumption (1.b), the scheme $\mathrm{Var}(P_i, P_j)$ has optimal value $\geqslant \log K(Q)$. Thus we have $[Q^+] \cup [Q^-] = \{P_1, P_2, P_3\}$, hence at least one of the schemes $\mathrm{Var}(P_i, P_j)$, $1 \leqslant i, j \leqslant 3$, produces a solution u_1 with $F[u_1] > \log K(Q)$, which is therefore distinct from u_0.

(2) When $p = 0$, $q = 2$ with

(2.a) the level set structure given in the figure with the unique maximum P_0 at infinity and

(2.b) $K(Q_1) < \inf\{K(P) \mid P \in R_2\}$.



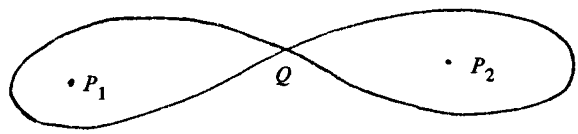
Then the variational schemes $\mathrm{Var}(\Gamma_1)$ and $\mathrm{Var}(\Gamma_2)$, where Γ_1, Γ_2 are the pictured dotted curves, both produce convergent solutions, say u_1, u_2. Under assumption (2.b) we have $F[u_1] < \log K(Q_1) < F[u_2]$. Hence u_1, u_2 are distinct solutions.

Naturally the above examples can be modified to give functions K with any desired number of index 1 or index 2 solutions.

(3) When $p = q = 1$, we will present a plausible argument that both variational schemes in our paper would concentrate at some saddle point Q with $\Delta K(Q) < 0$.

We remark that when $p = q = 0$, the Kazdan-Warner type example $K(x) = 1 + \varepsilon x_1$ indicates that in this situation we cannot anticipate a solution.

When $p = q = 1$, say, the function K has exactly two local maxima P_1, P_2, a saddle point Q with $\Delta K(Q) < 0$, and a global minimum M. We picture its level set through Q as in the figure (as a narrow figure 8) in the plane with the point M identified with the point at infinity. After a conformal mapping sending P_1 to the south pole S, P_2 to the north pole N, Q to a point on the equator, the figure 8 becomes a narrow curve approximating a longitudinal line. Analytically $1 - \varepsilon < K \leqslant 1$ inside the figure 8 with $K(P_1) = K(P_2) = 1$, $K(Q) = 1 - \varepsilon$ and outside a δ-neighbourhood ($\delta \ll \varepsilon$) of the figure 8, we have $0 < K < \varepsilon$. For such a function K and any $u \in H^1(S^2)$ with area of $\exp(2u)$ normalized to 1 we have $S[u] \geqslant 0$, and if $\int K \exp(2u) \geqslant 1 - \varepsilon$, $\exp(2u)$ will



have its mass concentrated inside the figure 8, hence any path in $\mathscr{P}(P_1, P_2)$ with $F[u] \geqslant \log(1-\varepsilon)$ should be a concentrated path moving within the figure 8. Thus if $\Delta K(Q) < 0$, we would expect the max-min sequence to concentrate at Q.

Added in proof. 1. Since the submission of the manuscript, we were informed that Chen and Ding [5] have obtained results which partially overlapped with our Theorem 1 in [4].

2. Recently, Osgood, Phillips and Sarnak [16] have given another interesting proof of the Onofri's inequality in their work [16], [17] on the isospectral problem.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES
UNIVERSITY OF SOUTHERN CALIFORNIA