J. DIFFERENTIAL GEOMETRY
21 (1985) 269–281

# CONFORMAL DEFORMATIONS OF COMPLETE MANIFOLDS WITH NEGATIVE CURVATURE

PATRICIO AVILES & ROBERT McOWEN

## Introduction

A basic problem in Riemannian geometry is that of studying the set of curvature functions that a manifold possesses. In this generality there has been such a great deal of work that we cannot here record the different contributions. (For a fairly complete account, see [23].) However, in this paper we shall be concerned with the special case of ("pointwise") conformal deformations of metrics which we shall call problem ($\kappa$):

> Let $M$ be an $n$-dimensional Riemannian manifold with metric $g$. If we are given a real-valued function on $M$, does there exist a metric $\tilde{g}$ on $M$, conformal to $g$, with the given function as its curvature (i.e., Gaussian curvature if $n = 2$, and scalar curvature if $n \geqslant 3$)?

This problem has been extensively studied for compact manifolds with or without boundary (see [6], [7], [9], [13], [14], [15], and [18]). However there are still some unsettled questions, even for $M = S^2$ with the standard metric (see [18]), or on more general manifolds. The special case of deforming to constant scalar curvature is known as the Yamabe Problem and has recently been completely resolved for compact manifolds by R. Schoen [21] (see also [6]).

On the other hand, if $M$ is a complete but noncompact Riemannian manifold, very little is known. In the special case $M = \mathbf{R}^n$ with Euclidean metric $g$, problem ($\kappa$) has been studied in [4], [16], [17], [19], and [20]. The purpose of this paper is to study ($\kappa$) for simply-connected manifolds with negative curvature. The model case is $H^n(-1)$, the $n$-dimensional space form of curvature $-1$, and Kazdan has posed ($\kappa$) for $H^n(-1)$ and more general manifolds of negative curvature as an open problem in [12].

Received September 6, 1984 and, in revised form, April 26, 1985. The research of the first author was conducted while visiting Massachusetts Institute of Technology and was supported by NSF Grant 8403666.

This type of manifold has recently been the object of study by many authors: see [2], [3] and [22]. The basic topics they considered are the existence of bounded harmonic functions and some aspects of function theory on these spaces. In particular, they proved that if $(M, g)$ has sectional curvatures bounded between two finite negative constants, then there are bounded harmonic functions.

The problem $(\kappa)$ however is studied by means of a partial differential equation. For two-dimensional manifolds $(M, g)$ with Gaussian curvature $K$, the Gaussian curvature $\tilde{K}$ of the conformal metric $\tilde{g} = e^{2u}g$ satisfies

$$\Delta_g u + \tilde{K}e^{2u} = K. \tag{1}$$

If $n \geqslant 3$, let $S$ denote the scalar curvature of $(M, g)$; then $\tilde{g} = u^{4/(n-2)}g$ (where $u(x) > 0$) has scalar curvature $\tilde{S}$ satisfying

$$\frac{4(n-1)}{(n-2)}\Delta_g u + \tilde{S}u^{(n+2)/(n-2)} = Su. \tag{2}$$

Hence $(\kappa)$ is equivalent to solving (1) or (2) for $u$. We shall consider the existence of $C^2$ solutions of (1) and (2) which are *bounded* on $M$ (and bounded away from zero in (2)). This guarantees that the conformal metrics $g$ and $\tilde{g}$ are uniformly equivalent ($C_1 g_{ij} \leqslant \tilde{g}_{ij} \leqslant C_2 g_{ij}$); in particular $\tilde{g}$ is also complete.

In dimension two, because of the uniformization theorem as well as the Ahlfors-Schwarz Lemma [1], it is enough to consider the model space $H^2(-1)$ which we take to be the unit disk $D$ with the Poincaré metric. We analyze this case in §1, and obtain the following result.

**Theorem A.** *Let $\tilde{K} \in C^\infty(D)$ satisfy $\tilde{K}(x) \leqslant 0$ for $x \in D$ and $-a^2 \leqslant \tilde{K}(x) \leqslant -b^2 < 0$ for $x \in D \setminus D_0$, where $D_0$ is compact and $a \geqslant b$ are positive constants. Then there is a unique complete metric $\tilde{g}$ which is conformal and uniformly equivalent to the Poincaré metric, having $\tilde{K}$ as its Gaussian curvature.*

In higher dimensions we must consider the general case which includes $H^n(-1)$; this is done in §2 and yields the following.

**Theorem B.** *Let $(M, g)$ be a simply-connected, complete Riemannian manifold with sectional curvatures $K$ satisfying $-A^2 \leqslant K(x) \leqslant -B^2$ for $x \in M$, where the positive constants $A$ & $B$ satisfy $1 \leqslant A^2B^{-2} \leqslant (n-1)^2/n(n-2)$. If $\tilde{S} \in C^\infty(M)$ satisfies $\tilde{S}(x) \leqslant 0$ for $x \in M$ and $-a^2 \leqslant \tilde{S}(x) \leqslant -b^2 < 0$ for $x \in M \setminus M_0$, where $M_0$ is a compact set and $a \geqslant b$ are positive constants, then there is a unique complete metric $\tilde{g}$ which is conformal and uniformly equivalent to $g$, having $\tilde{S}$ as its scalar curvature.*

In these theorems we have assumed that $g$, $K$, and $S$ are all $C^\infty$. By local regularity, the function $u$ and hence the metric $\tilde{g}$ are also $C^\infty$. It is clear from the analysis in §§1 and 2, however, how to proceed with less regularity.

It is of interest to inquire when additional regularity at infinity can be concluded for the solution. To formulate this, let $\rho(x)$ denote the distance of $x$ to a fixed point 0 and write

$$\lim_{\rho(x)\to\infty} u(x) = u_\infty, \tag{3}$$

where $u_\infty$ is a constant, if for every $\varepsilon > 0$ there is a compact set $M_\varepsilon$ such that $|u(x) - u_\infty| < \varepsilon$ whenever $x \in M \setminus M_\varepsilon$. In §§1 and 2 we find conditions on $\tilde{K}$ and $\tilde{S}$ under which the solutions $u$ of (1) and (2) satisfy (3) for some constant $u_\infty$ (cf. Theorems 3 and 5).

We prove all these theorems using the method of upper and lower solutions. One technical difficulty comes from the fact that (1) and (2) are not uniformly elliptic equations, in fact degenerate along the "boundary of $M$" (see [3]); this difficulty of course does not occur when $M$ is a compact manifold. Another difficulty arises when $\tilde{S}$ vanishes inside $M$ (if $\tilde{S} \leqslant -b^2 < 0$ throughout $M$ then a constant could serve as an upper solution and no restriction on the sectional curvature ratio $A/B$ need be made). In fact, the results of §§1 and 2 show that we can still solve (1) and (2) when a certain amount of positivity of $\tilde{K}$ or $\tilde{S}$ is allowed inside of $M$. This phenomenon is explored in greater detail for $H^2(-1)$ in §1 where a necessary (but not sufficient) solvability condition is found to be $\int \tilde{K}(x)\,dx < 0$. Thus the prescribed curvature functions cannot be "too positive in $M$". A condition on $\tilde{K}$ or $\tilde{S}$ which is both necessary and sufffficient for solvability is probably very difficult to find; note that this is not even well-understood when $M$ is compact (cf. [15]).

Finally, we should mention that we have also studied the special case of deforming to constant *negative* scalar curvature (as in the Yamabe Problem) on a complete noncompact Riemannian manifold. This will appear in a separate publication [5].

## 1. Dimension 2: Poincaré disk

If $M$ is complete with Gaussian curvature $K(x) \leqslant -b^2 < 0$ then $M$ is conformally equivalent to the Poincaré disk by the Ahlfors-Schwarz Lemma [1]. Thus we may restrict our attention to $M = D = \{x \in \mathbf{R}^2\colon\ r = |x| < 1\}$ with the metric $g_{ij} = 4(1 - r^2)^{-2}\delta_{ij}$ which has constant Gaussian curvature $K \equiv -1$. If $\tilde{K}(x)$ is a function on $D$ then the prescribed curvature equation (1) is

$$\frac{(1 - r^2)}{4}\Delta u + \tilde{K}(x)e^{2u} = -1, \tag{1.1}$$

where $\Delta$ is the ordinary Euclidean Laplacian. Notice that (1.1) is a degenerate elliptic equation in $D$ which we shall show admits bounded solutions provided

$\tilde{K}$ is negative at infinity and "not too positive inside $D$". Let $D_0$ be a compact subset of $D$.

**Theorem 1.** *Suppose $\tilde{K}$ is a Hölder continuous function on $D$ satisfying $-a^2 \leqslant \tilde{K}(x) \leqslant -b^2$ for $x \in D \setminus D_0$.*

(a) *If $\tilde{K}(x) \leqslant 0$ for $x \in D$, then there exists a unique bounded $C^2$-solution of* (1.1).

(b) *If $\tilde{K}(x_0) > 0$ for some $x_0 \in D$, let $D^{\pm} = \{x \in D_0\colon \pm\tilde{K}(x) > 0\}$ and suppose there exists $\alpha > 0$ such that:*

(i) $\alpha \geqslant \sup\{(\tilde{K}(x) + 1)(1 - r^2)^{-2}\colon x \in D^-\}$;

(ii) $\tilde{K}(x)b^{-2}e^{2\alpha(1-r^2)} \leqslant -1 + \alpha(1 - r^2)^2$ *for* $x \in D^+$.

*Then there exists a bounded $C^2$-solution of* (1.1).

*Proof.* We shall construct an upper solution $u_+$ and a lower solution $u_-$ (i.e., functions satisfying the differential inequalities obtained by replacing = in (1.1) by $\leqslant$ and $\geqslant$ respectively) which are bounded on $D$ and satisfy $u_+ \geqslant u_-$ in $D$. If (1.1) were uniformly elliptic we could apply the Monotone Iteration Scheme to conclude the existence of a $C^2$-solution $u$ satisfying $u_- \leqslant u \leqslant u_+$. Instead, however, we must take a sequence of open sets $D_j$ with $\overline{D}_j \subset D_{j+1}$ whose infinite union is $D$; then obtain a $C^2$-solution $u_j$ on each $D_j$ with $u_- \leqslant u_j \leqslant u_+$. The uniform bounds on $D_1$ enable us to select a subsequence $u_{j1}$ converging to a solution on $D_1$, and a further refinement yields a subsequence converging to a solution on $D_2$, etc. Hence the function $u(x) = \lim_{k\to\infty} u_{kk}(x)$ gives the desired solution on all of $D$ (cf. [19], Proof of Theorem 2.10).

Let $u_- = -\log a$. Then $\tilde{K}e^{2u_-} \geqslant -a^2e^{-2\log a} = -1$ so $u_-$ is a lower solution for both (a) and (b).

For (a) let $u_+ = \alpha(1 - r^2) + c_1$ where $c_1 = -\log b \geqslant 0$ since we may assume $0 < b < 1$, and $\alpha > 0$ is chosen so that $\alpha \geqslant \{(\tilde{K}(x) + 1)(1 - r^2)^{-2}\colon x \in D_0\}$. On $D_0$ use $\tilde{K}e^{2u_+} \leqslant \tilde{K}$ and $\Delta u_+ = -4\alpha$ to check upper solution:

$$\Delta_g u_+ + \tilde{K}e^{2u_+} \leqslant \Delta_g u_+ + \tilde{K} = -(1 - r^2)^2\alpha + \tilde{K}(x) \leqslant -1.$$

On $D \setminus D_0$ use $\tilde{K}e^{2c_1} = \tilde{K}b^{-2} \leqslant -1$ to check upper solution:

$$\Delta_g u_+ + \tilde{K}e^{2u_+} \leqslant -\alpha(1 - r^2)^2 - e^{2\alpha(1-r^2)} \leqslant -1.$$

For (b) again let $u_+ = \alpha(1 - r^2) + c_1$ where $c_1 - \log b \geqslant 0$ and $\alpha$ satisfies (i) and (ii). It is easy to verify $u_+$ is an upper solution by using (i) in $D^-$ and (ii) in $D^+$.

Finally, we can show that the solution in (a) is unique by invoking the generalized maximum principle (see [6]). Namely, suppose $u$ and $v$ are two bounded solutions so that $w = u - v$ is bounded and satisfies

$$\Delta_g w = -\tilde{K}[e^{2u} - e^{2v}].$$

Notice that $w$ cannot attain a negative minimum at $x_0 \in D$. (If $w(x_0) < 0$ is a minimum then $w < 0$ in a neighborhood $U$ of $x_0$. But then $\Delta_g w \leqslant 0$ in $U$; since $w(x_0)$ is a minimum we must have $w$ constant in $U$. Enlarging $U$ we find $w$ must be a negative constant in $D$. But this would require $\tilde{K} \equiv 0$, a contradiction.) However the generalized maximum principle implies the existence of points $x_i \in D$ with $w(x_i) \to \inf\{w(x)\colon x \in D\}$ and $\overline{\lim}\, \Delta w(x_i) \geqslant 0$. Taking a subsequence, we may assume either $x_i \to x_0 \in D$ or $x_i$ approach $\partial D$. If $x_i \to x_0$ then $w(x_0) \geqslant 0$ by the above argument, so $w \geqslant 0$ in $D$. If $x_i$ approach $\partial D$ we may assume $K(x_i) \leqslant -b^2$. If $\inf\{w(x)\colon x \in D\} < 0$ then there exists a subsequence $x_i'$ with $w(x_i') \leqslant -\varepsilon < 0$ so $e^{2u(x_i')} - e^{2v(x_i')} \leqslant -\eta < 0$. But then $\Delta_g w(x_i') \leqslant -\eta b^2 < 0$, contradicting $\overline{\lim}\, \Delta w(x_i) \geqslant 0$. Thus $w \geqslant 0$ in $D$. Similarly, we can show $w \leqslant 0$ in $D$, so $u \equiv v$ in $D$.

**Remark.** The consideration of bounded solutions in Theorem 1 is not at all restrictive since the hypothesis $-a^2 \leqslant \tilde{K}(x) \leqslant -b^2 < 0$ for $x \in D \setminus D_0$ implies by the local Schwarz lemma that every solution of (1.1) is bounded (cf. [8]).

Thus $\tilde{K} \leqslant 0$ is a sufficient condition for the existence of bounded solutions of (1.1), but not a necessary condition: it is easy to construct functions $\tilde{K}$ with $\tilde{K}(0) \geqslant 0$ which satisfy (ii). Nevertheless, it seems that $\tilde{K}$ cannot be too positive in $D$: for example, if $\tilde{K}(0) > 0$ then (ii) can only be satisfied if $\tilde{K}(0) \leqslant b^2/(2e^3)$ since this is the maximum value of $h(\alpha) = b^2(\alpha - 1)e^{-2\alpha}$. We now prove an integral condition which expresses the restriction on the positivity of $\tilde{K}$.

**Theorem 2.** *A necessary condition for the existence of a bounded solution of* (1.1) *is*

$$\int_D \tilde{K}(x)\, dx < 0. \tag{1.2}$$

*Proof.* (1.1) implies $4\tilde{K}(x) = -4e^{-2u} - (1 - r^2)^2(\Delta u)e^{-2u}$. We integrate this equation and use integration by parts to get

$$\begin{aligned} \int_D \tilde{K}(x)\, dx &= -4\int_D e^{-2u}\, dx - \int_D (1 - r^2)^2(\Delta u)e^{-2u}\, dx \\ &= -4\int_D e^{-2u}\, dx - 4\int_D (1 - r^2)e^{-2u}(x \cdot \nabla u)\, dx \\ &\quad - 2\int_D (1 - r^2)^2 e^{-2u}|\nabla u|^2\, dx \\ &\quad + \lim_{r\to 1}\int_{|x|=r} (1 - r^2)^2 e^{-2u}\nabla u \cdot ds. \end{aligned} \tag{1.3}$$

The last term is zero by

$$|\nabla u(x)| = O\left((1 - r^2)^{-1}\right) \quad \text{as } r = |x| \to 1 \tag{1.4}$$

which follows from formula (3.16) in [11] (use $\Omega$ = the disk of radius $(1 - |x|)/2$ centered at $x \in D$). Using

$$-(1 - r^2)x \cdot \nabla u \leqslant \frac{|x|^2}{2} + \frac{|\nabla u|^2}{2}(1 - r^2)^2,$$

we find (1.3) becomes

$$4\int_D \tilde{K}(x)\,dx \leqslant -4\int_D e^{-2u}\,dx + 2\int_D e^{-2u}|x|^2\,dx.$$

Using $|x|^2 \leqslant 1$ we obtain (1.2).

We shall use the following proposition to construct an example showing that the necessary condition in Theorem 2 is not a sufficient condition.

**Proposition 1.5.** *If there exists a bounded solution $u$ of* (1.1) *and if there exists a solution $\phi$ of the inequalities*

$$\frac{(1 - r^2)^2}{4}\Delta\phi - 2\phi \leqslant 2\tilde{K} \quad \text{in } D, \tag{1.6}$$

$$\phi > e^{-2u} \quad \text{near } \partial D, \tag{1.7}$$

*then $\phi > 0$ in $D$.*

*Proof*. Let $v = e^{-2u}$ which satisfies

$$\frac{(1 - r^2)^2}{4}\Delta v - 2v = e^{-2u}(1 - r^2)^2|\nabla u|^2 + 2\tilde{K} \geqslant 2\tilde{K}.$$

Combining with (1.6) we have

$$\frac{(1 - r^2)^2}{4}\Delta(\phi - v) - 2(\phi - v) \leqslant 0 \quad \text{in } D.$$

For $0 < R < 1$, let $D_R = \{x \in D\colon |x| < R\}$. For $R$ close to 1 we have $\phi \geqslant e^{-2u} = v$ on $\partial D_R$ by (1.7), so the maximum principle implies $\phi \geqslant v > 0$ in $D_R$. Letting $R \to 1$ we have $\phi > 0$ in $D$.

**Example 1.8.** Let $\phi_b(r) = -\ln(b^2 - r^2)$ where $b > 1$. A calculation shows

$$\frac{(1 - r^2)^2}{4}\Delta\phi_b - 2\phi_b = 2\tilde{K}_b,$$

where

$$\tilde{K}_b(x) = \frac{(1 - r^2)^2}{2(b^2 - r^2)^2}b^2 + \ln(b^2 - r^2).$$

Notice that $I_b = \int \tilde{K}_b(x)\,dx \to -\infty$ as $b \to 1$, so choose $b_0$ such that $I_{b_0} < 0$ and let $\tilde{K} = \tilde{K}_{b_0}$. For fixed $x$ we claim that $\tilde{K}_b(x)$ is an increasing function of $b$:

$$\frac{d}{db}\tilde{K}_b(x) = \frac{b}{(b^2 - r^2)^3}\left[2(b^2 - r^2)^2 - (1 - r^2)^2(b^2 + r^2)\right] \geqslant 0,$$

since $(1 - r^2)^2 \geqslant 0 \Rightarrow 1 + r^2 + r^4 \geqslant 3r^2 \Rightarrow b^2(1 + r^2) + r^4 \geqslant 3r^2 \Rightarrow 2(b^2 - r^2) \geqslant (1 - r^2)(b^2 + r^2) \Rightarrow 2(b^2 - r^2)^2 \geqslant (1 - r^2)^2(b^2 + r^2)$. Thus

$$\frac{(1 - r^2)^2}{4}\Delta\phi_b - 2\phi_b \leqslant 2\tilde{K}$$

for all $b_0 \geqslant b > 1$. Now suppose there is a bounded solution $u$ of (1.1). Since $\phi_b(1) \to \infty$ as $b \to 1$ we can choose $b_1$ such that $\phi = \phi_{b_1} > e^{-2u}$ on $\partial D$. But $\phi(0) = -\ln(b_1^2) < 0$ contradicting Proposition 1.5. Thus (1.1) admits no bounded solutions even though $\int_D \tilde{K}(x)\,dx < 0$.

**Remark.** In fact the example shows that any condition of the form

$$\int_D \tilde{K}(x)\,dx < -c^2 \qquad (c \geqslant 0)$$

cannot be a sufficient condition for solvability of (1.1).

Theorem 1 only asserts the existence of bounded solutions but says little about their behavior as $|x| \to 1$: For example, does the solution $u$ admit a continuous extension to the boundary? The following example shows that additional regularity assumptions on $\tilde{K}$ as $|x| \to 1$ are required in order to conclude regularity of the solution as $|x| \to 1$.

**Example 1.9.** The function $u(x) = \frac{1}{2}\sin(\ln(1 - |x|^2))$ is bounded but oscillatory as $|x| \to 1$ so does not extend continuously to $\partial D$. A calculation shows that $u$ solves (1.1) with

$$\tilde{K}(x) = \left[\frac{|x|^2}{2}\sin\left(\ln(1 - |x|^2)\right) + \frac{1}{2}\cos\left(\ln\left(1 - |x|^2\right)\right) - 1\right]e^{-\sin(\ln(1 - |x|^2))},$$

which is bounded between negative constants, but of course also oscillatory as $|x| \to 1$.

On the other hand, if $\tilde{K}(x) \leqslant 0$ and approaches a negative constant as $|x| \to 1$, then we can solve (1.1) with $u(x)$ approaching a certain constant as $|x| \to 1$.

**Theorem 3.** *Suppose $\tilde{K}(x) \leqslant 0$ for $x \in D$ and $\tilde{K}(x) = -c^2 + H(x)$ where $c > 0$ and*

$$\sup\left\{(1 - r)^{-\beta}H(x) \colon x \in D\right\} < \infty \tag{1.10}$$

*for some $0 < \beta < 1$, then there is a unique bounded solution of* (1.1) *satisfying*

$$\lim_{|x| \to 1} u(x) = -\ln c. \tag{1.11}$$

*In fact, $u(x) = -\ln c + v(x)$ where $\sup\{(1 - r)^{-\beta}v(x); x \in D\} < \infty$ for the same $\beta$ as in* (1.10).

*Proof.* Let $u_+ = \alpha(1 - r^2)^\beta - \ln c$ where $\alpha, \beta > 0$. A calculation shows

$$\frac{(1 - r^2)^2}{4} \Delta u_+ = \alpha\beta(\beta r^2 - 1)(1 - r^2)^\beta,$$

$$\tilde{K} e^{2u_+} = K c^{-2} e^{2\alpha(1-r^2)^\beta} \leqslant -1 + Hc^{-2},$$

so $u_+$ is an upper solution provided

$$\alpha\beta(\beta r^2 - 1) + H(x)c^{-2}(1 - r^2)^{-\beta} \leqslant 0. \tag{1.12}$$

Since $\beta < 1$, we may take $\alpha$ sufficiently large to achieve (1.12).

Similarly, $u_- = -\alpha(1 - r^2)^\beta - \ln c$ is a lower solution if $\alpha$ is large enough that

$$-\alpha\beta(\beta r^2 - 1) + H(x)c^{-2}(1 - r^2)^{-\beta} \geqslant 0.$$

As in the proof of Theorem 1 we conclude there is a solution $u$ of (1.1) with $u_- \leqslant u \leqslant u_+$ in $D$. From the definition of $u_\pm$ we find $u = -\ln c + v(x)$ with $(1 - r)^{-\beta} v(x)$ bounded on $D$, so in particular (1.11) holds.

## 2. Higher dimensions

In this section we shall prove the results stated in the introduction in a slightly stronger version.

The following facts will be used in the proofs. Let $(M, g)$ be a simply connected complete Riemannian manifold of dimension $n \geqslant 3$ with sectional curvatures $K$ satisfying $-A^2 \leqslant K \leqslant -B^2$. Let $\rho$ denote the distance function in $M$ to a fixed point 0. Then

$$B(\coth B\rho)H \leqslant D^2\rho \leqslant A(\coth A\rho)H, \tag{2.1}$$

where $H = g - d\rho \otimes d\rho$ and $D^2\rho$ denotes the covariant Hessian of $\rho$. This estimate is well known (see [3]). Also we have

$$|\nabla\rho|^2 = 1; \tag{2.2}$$

from (2.1) we get

$$\Delta_g \rho \geqslant B(n - 1), \tag{2.3}$$

where $\Delta_g$ is the Laplace-Beltrami operator in $M$; and if $S(x)$ denotes the scalar curvature associated to $(M, g)$ then

$$-A^2 n(n - 1) \leqslant S(x) \leqslant -B^2 n(n - 1). \tag{2.4}$$

**Theorem 4.** *Let $(M, g)$ be as above with positive constants $A$ & $B$ satisfying $1 \leqslant A^2B^{-2} < (n - 1)^2/n(n - 2)$ and let $\tilde{S}(x)$ be a Hölder continuous function defined on $M$ such that $-a^2 \leqslant \tilde{S}(x) \leqslant -b^2 < 0$ for $x \in M \setminus M_0$, where $M_0$ is a* *compact set and $a \geqslant b$ are positive constants. Then there exists $\varepsilon > 0$ such that there is a $C^2$-solution of (2) which is bounded between two positive constants provided $\tilde{S}(x) \leqslant \varepsilon$ for all $x \in M$. If $\tilde{S}(x) \leqslant 0$ for all $x \in M$, then the solution is unique.*

*Proof.* As in §1 we use the method of lower and upper solutions. Without loss of generality we may assume that $M$ is connected. Let

$$u_+(x) = \alpha(1 + e^{-\delta\rho(x)})^{\beta} \tag{2.5}$$

with $\alpha$, $\beta$, $\delta > 0$ to be chosen. A calculation shows (using (2.2))

$$\Delta_g u_+ = \alpha\beta(1 + e^{-\delta\rho})^{\beta-2}\left[(1 + e^{-\delta\rho})e^{-\delta\rho}\left(\delta^2 - \delta\Delta_g\rho\right) + (\beta - 1)e^{-2\delta\rho}\delta^2\right].$$

If we fix $\beta\delta = B(n-1)$ and then choose $\delta > 0$ small enough we have $\Delta_g u_+ \leqslant 0$. Choosing $\rho_0$ so that $\tilde{S}(x) \leqslant -b^2$ for $\rho_0 \leqslant \rho(x)$, we can choose $\alpha$ large so that

$$c_n\Delta_g u_+ + \tilde{S}(x)u_+^N \leqslant Su_+ \quad \text{for } \rho \geqslant \rho_0, \tag{2.6}$$

where $c_n = 4(n-1)/(n-2)$ and $N = (n+2)/(n-2)$. We shall now show that $\delta$, $\varepsilon > 0$ can be chosen small so that

$$c_n\Delta_g u_+ + \varepsilon u_+^N \leqslant Su_+ \quad \text{for } 0 < \rho < \rho_0. \tag{2.7}$$

Using the fact that $\tilde{S}(x) \leqslant \varepsilon$ we find $u_+$ is an upper solution.

To prove (2.7) we need to show

$$c_n\beta\delta\frac{e^{-\delta\rho}}{1 + e^{-\delta\rho}}\left[\frac{(\beta - 1)e^{-\delta\rho}}{1 + e^{-\delta\rho}}\delta + \delta - \Delta_g\rho\right] + \varepsilon\alpha^{4/(n-2)}(1 + e^{-\delta\rho})^{4\beta/(n-2)} \leqslant S(x).$$

Using (2.3) and (2.4) it suffices to prove

$$c_n\beta\delta\frac{e^{-\delta\rho}}{1 + e^{-\delta\rho}}\left[\frac{(\beta - 1)e^{-\delta\rho}}{1 + e^{-\delta\rho}}\delta + \delta - B(n-1)\right] + \varepsilon\alpha^{4/(n-2)}(1 + e^{-\delta\rho})^{4\beta/(n-2)} \leqslant -A^2n(n-1). \tag{2.8}$$

Since $\beta\delta = B(n-1)$, letting $\delta \to 0$ we find

$$c_n\beta\delta\frac{e^{-\delta\rho}}{1 + e^{-\delta\rho}}\left[\frac{(\beta - 1)e^{-\delta\rho}}{1 + e^{-\delta\rho}}\delta + \delta - B(n-1)\right] \to -\frac{(n-1)^3}{n-2}B^2.$$

Thus to prove (2.8) it is enough to show

$$-\frac{(n-1)^3}{n-2}B^2 + \varepsilon\alpha^{4/(n-2)}C \leqslant -A^2n(n-1), \tag{2.9}$$

where $C = 2^{4\beta/(n-2)}$. This follows by selecting $\varepsilon$ sufficiently small, since $A^2B^{-2} < (n-1)^2/n(n-2)$.

Since $\tilde{S}$ is bounded below on $M$, it is enough to choose a small *positive* constant as a lower solution. This establishes the existence of the desired solution. (Since $u_+$ is not $C^1$ at 0 we technically should consider our solution $u$ on $M \setminus \{0\}$, or weakly on $M$. But local regularity then shows $u$ is $C^2$ at 0.)

To prove uniqueness we invoke the generalized maximum principle as in the proof of Theorem 1. (The function $\phi^x(y) = \psi(\rho_x^2(y))$, where $\rho_x(y)$ denotes the distance from $x$ to $y$, and $\psi(t)$ is a $C^\infty$-function with $\psi(t) = 1$ for $t \leqslant 1/2$ and $\psi(t) = 0$ for $t \geqslant 1$, satisfies the hypotheses of Theorem 3.76 in [6] since $\phi^x$ and $|\nabla\phi^x|$ are bounded independently of $x$ and (2.1) and (2.2) may be used to verify $\phi_{ii}^x \geqslant -Cg_{ii}$ with constant $C$ independent of $x$.) Indeed, letting $v = \ln u$, the equation (2) becomes

$$c_n\left(\Delta_g v + |\nabla v|^2\right) + \tilde{S}e^{4v/(n-2)} = S.$$

If this equation has two bounded solutions $v_1$ and $v_2$, let $w = v_1 - v_2$. Then

$$\Delta_g w = |\nabla v_2|^2 - |\nabla v_1|^2 - c_n^{-1}\tilde{S}\left(e^{4v_1/(n-2)} - e^{4v_2/(n-2)}\right).$$

Applying the generalized maximum principle we have

$$\Delta_g w(x_i) = c_n^{-1}\tilde{S}(x_i)\left(e^{4v_1(x_i)/(n-2)} - e^{4v_2(x_i)/(n-2)}\right),$$

since

$$|\nabla v_2(x_i)|^2 - |\nabla v_1(x_i)|^2 = -\nabla\left(v_1(x_i) + v_2(x_i)\right) \cdot \nabla w(x_i) \to 0,$$

and we can proceed as in the proof of Theorem 1.

**Remark.** The restriction $A^2B^{-2} < (n-1)^2/n(n-2)$ is only necessary to allow $\tilde{S}$ to be positive inside $M_0$; if $\tilde{S} \leqslant 0$ on $M$, then we can allow $A^2B^{-2} = (n-1)^2/n(n-2)$. As mentioned in the introduction, if $-a^2 \leqslant \tilde{S}(x) \leqslant -b^2$ for all $x \in M$, constants may be used as upper and lower solutions and no restriction on the ratio $A/B$ is needed.

Next, we shall show the equivalent of Theorem 3.

**Theorem 5.** *Let $M$ be as in Theorem 4 and let $\tilde{S}(x) = S(x) + H(x)$ where $\sup\{e^{\delta_0\rho(x)}H(x): x \in M\} < \infty$ for some $\delta_0 > 0$. If $\tilde{S}(x) \leqslant 0$, then there exists a unique solution $u$ of (2) such that $\lim_{\rho(x)\to\infty} u(x) = 1$.*

*Proof.* As in Theorem 4 we shall assume that $M$ is connected. Also as in the proof of Theorem 4 it suffices to construct lower and upper solutions. Let

$$u_+(x) = \left(1 + e^{-\delta\rho(x)}\right)^\beta.$$

Since $u_+(x) \geqslant 1$ and $\tilde{S}(x) \leqslant 0$, we have $u_+^N \geqslant u_+$ and

$$c_n\Delta_g u_+ + \tilde{S}(x)u_+^N \leqslant c_n\Delta_g u_+ + S(x)u_+ + H(x)u_+.$$

Therefore, to show that $u_+$ is an upper solution it is enough to have

$$c_n\Delta_g u_+ + H(x)u_+ \leqslant 0. \tag{2.10}$$

We first observe that (2.10) holds if

$$\text{(2.11)}\qquad \frac{c_n\beta\delta}{1+e^{-\delta\rho}}\left[\frac{(\beta-1)\delta e^{-\delta\rho}}{1+e^{-\delta\rho}}+\delta-B(n-1)\right]+e^{\delta\rho}H(x)\leqslant 0.$$

Letting $\beta\delta = B(n-1)$ with $\beta - 1 \geqslant 0$, and using $e^{-\delta\rho}(1+e^{-\delta\rho})^{-1} \leqslant 1/2$ and $(1+e^{-\delta\rho}) \leqslant 2$, we can choose $\delta_1 < \min(\delta_0, B(n-1)/2)$ and observe that to obtain (2.11) it is enough to have

$$\text{(2.12)}\qquad -c_n\frac{B^2(n-1)^2}{8}+e^{\delta\rho}H(x)\leqslant 0$$

for some $\delta$ satisfying $0 < \delta \leqslant \delta_1$. But $e^{\delta_0\rho(x)}H(x)$ being bounded implies there exists $\rho_0$ such that (2.12) holds for all $\rho \geqslant \rho_0$ and all $\delta$ with $0 < \delta \leqslant \delta_1$. To prove (2.10) for $\rho < \rho_0$ we proceed as in the proof of Theorem 4. Indeed we use the fact that $\tilde{S}(x) \leqslant 0$ to obtain $-H(x) \geqslant S(x) \geqslant -A^2n(n-1)$, then we write (2.10) as (2.8) (with $\varepsilon = 0$) and let $\delta \to 0$.

As lower solution we consider

$$u_-(x) = (1-\alpha e^{-\delta\rho(x)})^{\beta}, \qquad \tfrac{1}{2}\leqslant \alpha < 1.$$

A calculation shows

$$\begin{aligned}\Delta_g u_- = {}&\alpha\beta\delta(1-\alpha e^{-\delta\rho})^{\beta-2}e^{-\delta\rho}\\ &\times\left[(1-\alpha e^{-\delta\rho})\Delta_g\rho-(1-\alpha e^{-\delta\rho})\delta+(\beta-1)e^{-\delta\rho}\alpha\delta\right].\end{aligned}$$

Since $0 < u_- \leqslant 1$ and $\tilde{S}(x) \leqslant 0$ it suffices to prove

$$\text{(2.13)}\qquad c_n\Delta_g u_- + Hu_- \geqslant 0.$$

Calculations show that to have (2.13) it suffices to show

$$\text{(2.14)}\qquad \frac{c_n\alpha\beta\delta}{(1-\alpha e^{-\delta\rho})}\left[B(n-1)-\delta+(\beta-1)\frac{e^{-\delta\rho}}{(1-\beta e^{-\delta\rho})}\alpha\delta\right] + H(x)e^{\delta\rho}\geqslant 0.$$

But

$$(1-\alpha e^{-\delta\rho})^{-1}\geqslant 1.$$

Thus choosing $\beta\delta = B(n-1)$, $\delta_1 < \min(\delta_0, \frac{1}{2}B(n-1))$, and $(\beta - 1) \geqslant 0$, we observe that to have (2.14) it is enough to obtain

$$c_n\alpha\frac{B^2(n-1)^2}{2}+H(x)e^{\delta\rho}\geqslant 0$$

for some $\delta$ satisfying $0 < \delta \leqslant \delta_1$. Since $\alpha \geqslant \frac{1}{2}$ there exists $\rho_0$ such that for $\rho \geqslant \rho_0$ and $0 < \delta \leqslant \delta_1$, (2.13) is satisfied, independent of the choice of $\alpha$. To prove (2.13) for $\rho < \rho_0$, let $\delta \to 0$: the left-hand side of (2.14) converges to $c_n\alpha(1-\alpha)^{-1}B^2(n-1)^2$, so it suffices to decrease $\delta$ and to take $\alpha$ close to 1.

**Remark.** It is interesting to observe that we may weaken the hypothesis on the ratio $A/B$ if $\tilde{S}$ is strictly negative. In fact, if $\tilde{S}(x) \leqslant -l^2$ for $x \in M$ then Theorem 5 holds provided:

$$\frac{B^2(n-1)^3}{(n-2)} + l^2 \geqslant A^2 n(n-1).$$

For $l = 0$ this reduces to the case of Theorem 5.

**Corollary.** *If $M = H^n(-1)$ realized as the unit ball in $\mathbf{R}^n$ with Poincaré metric, and $\tilde{S}(x) = -n(n-1) + H(x)$ where*

$$\sup\left\{(1-|x|)^{-\delta_0} H(x) \colon x \in H^n(-1)\right\} < \infty$$

*for some $\delta_0 > 0$, then the conclusions of the theorem hold.*

The proof of this corollary follows from the facts that if we choose 0 to be the origin, then $\rho(x)$ in $H^n(-1)$ is given by

$$\rho(x) = \log\frac{(1+r)}{(1-r)}, \qquad r = |x|;$$

and also because the scalar curvature is given by $-n(n-1)$.

**Acknowledgements.** Wei-Ming Ni and Carlos Kenig obtained independently the existence statement of part (a) in Theorem 1. We would also like to thank Ni for bringing the preprint [8] to our attention during the final preparation of this manuscript.

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