PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 9, September 1995

CONFORMAL DIFFEOMORPHISMS PRESERVING THE RICCI TENSOR

W. KÜHNEL AND H.-B. RADEMACHER

(Communicated by Christopher B. Croke)

ABSTRACT. We characterize semi-Riemannian manifolds admitting a global conformal transformation such that the difference of the two Ricci tensors is a constant multiple of the metric. Unless the conformal transformation is homothetic, the only possibilities are standard Riemannian spaces of constant sectional curvature and a particular warped product with a Ricci flat Riemannian manifold.

We consider semi-Riemannian manifolds (M, g) and *conformal diffeomorphisms* $f: (M, g) \to (\overline{M}, \overline{g})$ between them meaning that $f^*\overline{g}$ is pointwise a positive scalar multiple of g. If this factor is constant f is called a *homothety.* A *conformal structure* on M is a class of conformally equivalent metrics. In this paper we study the behavior of the Ricci tensor Ric_g within one conformal class A classical theorem of Liouville determines all possible conformal diffeomorphisms between euclidean metrics. As a generalization we call a conformal transformation $g \to \overline{g}$ a *Liouville transformation* if $\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = 0$. In Theorems 1 and 2 we classify complete semi-Riemannian manifolds (M, g) admitting a non-homothetic conformal transformation $\overline{g} = \varphi^{-2}g$ such that the difference $\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g$ of the Ricci tensors is a constant multiple of the metric g or $\overline{g}$. We show that M is Riemannian and that M is either a standard space of constant sectional curvature or is a warped product $\mathbb{R} \times_{\exp} M_*$ of the real line and a Ricci flat manifold M_*. As a special case we obtain in Corollary 1 that a globally defined Liouville transformation is a homothety. This result is due to Liouville [Liv] in the case of E^3, generalized by Lie [Lie] to the case of E^n (see also [S, p. 173]) and by Haantjes [H] to the case of pseudo-euclidean space E^n_k. For Riemannian manifolds Corollary 1 has been obtained by Ferrand [Fe], where a Liouville transformation is called a quasi-similarity. For the compact Riemannian case, compare the recent paper [X] but be careful with the signs in formula (2) there. In Theorem 3 we show that for a complete semi-Riemannian manifold admitting a global non-homothetic concircular transformation between two metrics of constant scalar curvature the same conclusions as in Theorem 1 hold.

Received by the editors August 25, 1993 and, in revised form, February 3, 1994.
1991 *Mathematics Subject Classification.* Primary 53C20, 53C50; Secondary 53A30, 58G30.
Key words and phrases. Semi-Riemannian manifolds, Ricci tensor, conformal mapping, Hessian.

General assumption. (M^n, g) is a semi-Riemannian manifold of dimension $n \geq 3$ and of class C^3 such that the number of negative eigenvalues of g is not greater than $\frac{n}{2}$.

Notation. $\operatorname{grad}\varphi$ denotes the gradient of φ, $\nabla^2\varphi = \nabla \operatorname{grad}\varphi$ is the Hessian of φ, and $\Delta\varphi = \operatorname{trace}\nabla^2\varphi$ is the Laplacian. Let us introduce the notation $[h]$ for the class of all tensors which are pointwise scalar multiples of a given $(0, 2)$-tensor h. This includes the zero tensor—as well as negative multiples—which is not in the conformal structure itself. (M, g) is called an *Einstein space* if $[\operatorname{Ric}_g] = [g]$. Two metrics $g, \overline{g}$ are *concircular* to one another if $[g] = [\overline{g}] = [\operatorname{Ric}_{\overline{g}} - \operatorname{Ric}_g]$.

Lemma 1. *Two conformally equivalent metrics g and $\overline{g} = \frac{1}{\varphi^2} g$ satisfy the relation*

$$[\operatorname{Ric}_{\overline{g}} - \operatorname{Ric}_g] = [g] = [\overline{g}]$$

if and only if the function φ satisfies the equation

$$\nabla^2\varphi = \frac{\Delta\varphi}{n} \cdot g.$$

Proof. This follows from the well-known formula [S, p. 168], [Be, Sect. 1J], [Kü, Sect. A]

$$\operatorname{Ric}_{\overline{g}} - \operatorname{Ric}_g = \frac{1}{\varphi^2}[(n-2)\varphi\nabla^2\varphi + (\varphi\Delta\varphi - (n-1)g(\operatorname{grad}\varphi, \operatorname{grad}\varphi)) \cdot g].$$

Lemma 1 holds only under the assumption $n \geq 3$.

Remark. A transformation $g \to \overline{g}$ as in Lemma 1 is called *concircular* because it preserves the curves of constant geodesic curvature and vanishing geodesic torsion (so-called *geodesic circles*); see [Y]. In this case the conformal geodesics are geodesic circles; see [Fi, p. 454]. Any conformal transformation between two Einstein spaces is automatically concircular. A concircular transformation $g \to \overline{g}$ satisfies $\operatorname{Ric}_{\overline{g}} - \operatorname{Ric}_g = \frac{1}{n}(\frac{\overline{S}}{\varphi^2} - S) \cdot g$ where $S, \overline{S}$ denote the scalar curvatures of $g, \overline{g}$.

Lemma 2. *A function $\varphi: M \to \mathbb{R}$ satisfies $\nabla^2\varphi = \lambda \cdot g$ for some $\lambda: M \to \mathbb{R}$ in a neighborhood of a point with $g(\operatorname{grad}\varphi, \operatorname{grad}\varphi) \neq 0$ if and only if g is locally a warped product metric $ds^2 = \eta dt^2 + \varphi^2(t)ds_*^2$, where $\eta \in \{+1, -1\}$ denotes the sign of $g(\operatorname{grad}\varphi, \operatorname{grad}\varphi)$, φ, λ are functions depending only on t satisfying $\varphi'' = \eta \cdot \lambda$, and ds_*^2 is independent of t.*

This lemma can be found in [Fi, Sect. 12] and also in [T] for the Riemannian case. For the special case of Einstein metrics it is due to Brinkmann [Br].

Particular consequences of Lemma 2 are the following:

1. If $\nabla^2\varphi = \lambda g$ and if $\operatorname{grad}\varphi$ is a non-null vector, then the trajectories of $\operatorname{grad}\varphi$ are geodesics (up to parametrization).

2. If $\nabla^2\varphi = \lambda g$, then along every such non-null geodesic $\gamma(t)$ in direction $\operatorname{grad}\varphi$ with $g(\dot\gamma, \dot\gamma) = \eta$ the function $\varphi(t) := \varphi(\gamma(t))$ satisfies $\varphi'' = \eta \cdot \lambda$. Along a null geodesic this function $\varphi(t)$ satisfies $\varphi'' = 0$ according to the proof of Lemma 3 below.

Definition. A semi-Riemannian manifold (M, g) is called (geodesically) *complete* if every geodesic can be defined over $\mathbb{R}$ [O, p. 68]. It is called *null complete* if this holds for every null geodesic.

Theorem 1. *Let (M, g) be complete and admitting a global conformal transformation $\overline{g} = \frac{1}{\varphi^2} \cdot g$ satisfying*

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = c \cdot (n-1) \cdot g$$

for some constant c. Then one of the following three cases occurs:

1. *φ is constant.*
2. *(M, g) and $(M, \overline{g})$ are simply connected Riemannian spaces of constant sectional curvature.*
3. *(M, g) is a warped product $\mathbb{R} \times_{e^t} M_*$ where $\varphi(t) = e^t$ and (M_*, g_*) is a complete Ricci-flat $(n-1)$-dimensional Riemannian manifold.*

Remark. In Theorem 1 case 1 corresponds to $c = 0$, and cases 2 and 3 correspond to $c > 0$. In case 2 $\overline{g}$ must be flat and g must be hyperbolic, and in case 3 $\mathrm{Ric}_{\overline{g}} = 0$. A non-constant φ occurs only for Einstein space. Therefore we compare this to another theorem.

Theorem 1*. *Let (M, g) be complete and assume that both g and $\overline{g} = \frac{1}{\varphi^2} \cdot g$ are Einstein metrics. Then the same conclusion as in Theorem* 1 *holds, i.e. one of the cases* 1, 2, 3 *occurs.*

Remark. In Theorem 1* case 2 occurs for various combinations of the signs of the constant curvatures of g and $\overline{g}$.

Theorem 2. *Let (M, g) be complete and admitting a global conformal transformation $\overline{g} = \frac{1}{\varphi^2} \cdot g$ satisfying*

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = c \cdot (n-1) \cdot \overline{g} = \frac{c \cdot (n-1)}{\varphi^2} \cdot g$$

for a constant c. Then either φ is constant or (M, g) is isometric with the euclidean space.

Remark. To pass from Theorem 1 to Theorem 2 one just has to interchange the roles of g and $\overline{g}$. However, this is not quite symmetric because at most one of them can be complete (unless φ is constant). In Theorem 2 a constant φ corresponds to $c = 0$; the other case corresponds to $c > 0$. In particular, $c < 0$ is impossible in Theorem 1 and in Theorem 2.

Corollary 1. *A globally defined Liouville transformation of a complete semi-Riemannian manifold is a homothety.*

This result is just the case $c = 0$ in Theorems 1 and 2.

Corollary 2. *Assume that two semi-Riemannian metrics in the same conformal class have pointwise the same Ricci tensor. If one of them is complete, then they are homothetic to one another.*

This is a uniqueness result for the problem of prescribing a Ricci tensor in a conformal class.

Lemma 3. *Let g be a null complete indefinite metric admitting a globally defined nonconstant solution φ of $\nabla^2\varphi = \frac{\Delta\varphi}{n}\cdot g$. Then φ has a zero.*

Proof. Along any null geodesic $\gamma(s)$ one calculates

$$\begin{aligned}\frac{d^2}{ds^2}(\varphi(\gamma(s))) &= \frac{d}{ds}g(\operatorname{grad}\varphi, \dot\gamma)\\ &= g(\nabla_{\dot\gamma}\operatorname{grad}\varphi, \dot\gamma) + g(\operatorname{grad}\varphi, \nabla_{\dot\gamma}\dot\gamma)\\ &= g\left(\frac{\Delta\varphi}{n}\cdot\dot\gamma, \dot\gamma\right)\\ &= 0.\end{aligned}$$

Therefore $\varphi(\gamma(s))$ is linear in s with $\frac{d}{ds}\varphi(\gamma(s)) = g(\operatorname{gradf}\varphi, \dot\gamma)$. If we choose γ such that $g(\operatorname{grad}\varphi, \dot\gamma) \neq 0$ at a point p, then it follows that φ has a zero along γ.

Corollary 3. *The only globally defined concircular transformations of a null complete indefinite metric are the homotheties.*

Theorem 3. *Let (M, g) be complete and admitting a globally defined concircular transformation $\overline{g} = \frac{1}{\varphi^2}\cdot g$. Assume that $S, \overline{S}$ are constant. Then one of the three cases* 1, 2, 3 *as in Theorem* 1 *occurs.*

Proof of Theorem 1. By assumption $\varphi: M \to \mathbb{R}$ is a function which is positive everywhere. By Lemma 3 we may assume that φ is a non-constant function on a manifold with positive definite metric g. Let $p \in M$ be a point with $\operatorname{grad}\varphi(p) \neq 0$. By Lemma 1 the equation

$$\begin{aligned}c(n-1)\cdot g &= \mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g\\ &= \frac{1}{\varphi^2}[(n-2)\varphi\cdot\nabla^2\varphi + (\varphi\Delta\varphi - (n-1)\|\operatorname{grad}\varphi\|^2)\cdot g]\end{aligned}$$

implies

$$\frac{2(n-1)}{n}\varphi\Delta\varphi - (n-1)\|\operatorname{grad}\varphi\|^2 - c(n-1)\varphi^2 = 0. \tag{1}$$

By Lemma 2 along the unit speed geodesic $\gamma(t)$ in the direction of $\operatorname{grad}\varphi = \varphi'\cdot\frac{\partial}{\partial t}$ we have

$$2\varphi\varphi'' - \varphi'^2 - c\varphi^2 = 0. \tag{2}$$

Differentiating once more we get

$$2\varphi\varphi''' - 2c\varphi\varphi' = 0 \tag{3}$$

or, since $\varphi \neq 0$,

$$\varphi''' = c\cdot\varphi'. \tag{4}$$

Therefore there is a constant a satisfying

$$\varphi'' = c\cdot\varphi + a. \tag{5}$$

$\varphi'(p) \neq 0$ implies equivalently

$$(\varphi'^2)' = c\cdot(\varphi^2)' + 2a\varphi'. \tag{6}$$

This means that there is a constant b satisfying

(7) $$\varphi'^2 = c \cdot \varphi^2 + 2a\varphi + b.$$

Now (5), (7), and (2) together imply $b = 0$; thus

(8) $$\varphi'^2 = \varphi(c \cdot \varphi + 2a).$$

Case I: $c = 0$. By (5) φ is a polynomial of degree at most 2. Then (8) reduces to $\varphi'^2 = 2a\varphi$, which implies that φ is quadratic and that φ and φ' have a common zero along γ. By the completeness of g the metric $\overline{g} = \frac{1}{\varphi^2} \cdot g$ has a singularity there, a contradiction. Alternatively, if $\varphi(t) = At^2 + Bt + C$, then (8) implies that the discriminant $4AC - B^2$ is zero. Therefore $\varphi(t)$ is the square of a linear function.

Case II: $c < 0$. In this case every solution φ of (5) is periodic and therefore attains its minimum and maximum. At each of these points the equation $0 = \varphi(c\varphi + 2a)$ is satisfied by (8). Hence $\varphi = 0$ and $c\varphi + 2a = 0$ must be satisfied at the minimum and maximum, respectively. This leads to a contradiction as in Case I.

Alternatively, the general solution of (5)

$$\varphi(t) = \alpha \cos(\sqrt{-c}t) + \beta \sin(\sqrt{-c}t) - \frac{a}{c}$$

satisfies

$$\alpha^2 + \beta^2 = \frac{a^2}{c^2}$$

by (8). A typical solution looks like $\varphi(t) = \cos t + 1$.

Case III: $c > 0$. In this case the general solution of (5) is

$$\varphi(t) = \alpha\cosh(\sqrt{c}t) + \beta\sinh(\sqrt{c}t) - \frac{a}{c}.$$

Then (8) implies

$$\alpha^2 - \beta^2 = \frac{a^2}{c^2},$$

in particular $\alpha^2 \geq \beta^2$.

If $\alpha^2 > \beta^2$, then φ has a critical point along γ. This implies that γ has a point q with $\operatorname{grad}\varphi(q) = 0$ (note that $\operatorname{grad}\varphi = \varphi' \cdot \frac{\partial}{\partial t}$). Furthermore φ satisfies globally $\nabla^2\varphi = (c\varphi + a) \cdot g$ with $c > 0$. A result of Tashiro [T] implies that (M, g) is isometric with the hyperbolic space of constant sectional curvature $-c$. Roughly the argument is the following: In geodesic polar coordinates around q the metric g looks like $ds^2 = dt^2 + \sinh^2(\sqrt{c}t) \cdot ds_1^2$ where ds_1^2 is the metric of a round sphere of appropriate radius; compare [Kü, Lemma 18].

If $\alpha^2 = \beta^2$, then $a = 0$ and $\varphi(t) = \alpha \cdot e^{\mp\sqrt{c}\cdot t}$ is a solution without a critical point along γ. This implies that

(9) $$ds^2 = dt^2 + e^{2\sqrt{c}\cdot t} ds_*^2$$

is a complete metric on $M = \mathbb{R} \times M_*$. It follows that $\overline{g} = e^{-2\sqrt{c}t} g$ is the product metric $dt^2 + ds_*^2$ on $(0, \infty) \times M_*$. For an arbitrary tangent vector

X in the direction of M_* the standard formulas for the curvature of warped products imply

$$\begin{aligned} c(n-1)g(X, X) &= (\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g)(X, X) \\ &= (1 - c\cdot e^{2\sqrt{c}t})\mathrm{Ric}_{g_*}(X, X) + c(n-1)g(X, X), \end{aligned}$$

which is impossible unless $\mathrm{Ric}_{g_*} = 0$. This completes the proof of Theorem 1.

Note that in the case of an indefinite metric (9) does not define a complete warped product metric; compare [O, p. 209]. Compare also [Kb] for global solutions of $\nabla^2\varphi = c\cdot\varphi\cdot g$, $c > 0$, in the indefinite case if φ has at least one critical point.

Proof of Theorem 1*. In the case of indefinite metrics φ is constant by Lemma 3, using the same argument as in Theorem 1. The Riemannian case has been treated in [Kü, Theorem 27]. The local considerations in this case are due to Brinkmann [Br]. Compare also [Be, 9.110].

Remark. Geodesic mappings of the same kind as in Theorem 1 have been studied in [V]. For the case of conformal vector fields on Einstein spaces compare [YN] and [Kan] in the Riemannian case and [Ke1], [Ke2] in the non-Riemannian case. Brinkmann describes in [Br, §4] indefinite Einstein metrics (M, g) carrying a non-constant positive function φ such that the conformally equivalent metric $\overline{g} := \varphi^{-2}g$ is also Einstein and the gradient $\mathrm{grad}\,\varphi$ is everywhere null. Then it follows that $\mathrm{Ric}_{\overline{g}} = \mathrm{Ric}_g = 0$ and $\nabla^2\varphi = 0$, i.e. $\mathrm{grad}\,\varphi$ is parallel. By Lemma 3 (M, g) cannot be null complete.

In general relativity these metrics were studied in several papers. They are called *pp-waves* or *gravitational plane waves*; see e.g. [Hf].

Locally all the considerations in the proofs of Theorems 1–3 remain valid also in the case of an indefinite metric. This includes a local classification and the existence of various examples which, however, cannot be null complete.

Proof of Theorem 2. This follows the pattern of the proof of Theorem 1. In particular g must be positive definite if φ is not constant. We start with the equation

$$\frac{c\cdot(n-1)}{\varphi^2}\cdot g = \mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \frac{1}{\varphi^2}[(n-2)\varphi\cdot\nabla^2\varphi + (\varphi\Delta\varphi - (n-1)\|\mathrm{grad}\,\varphi\|^2)\cdot g]$$

which implies

$$\frac{2(n-1)}{n}\varphi\Delta\varphi - (n-1)\|\mathrm{grad}\,\varphi\|^2 - c(n-1) = 0. \tag{10}$$

If $\mathrm{grad}\,\varphi \neq 0$ at p, then along the geodesic γ in direction $\mathrm{grad}\,\varphi$ we have

$$2\varphi\varphi'' - \varphi'^2 - c = 0, \tag{11}$$

which implies

$$2\varphi\varphi''' = 0 \tag{12}$$

or

$$\varphi(t) = At^2 + Bt + C. \tag{13}$$

If we put this into (11) we get

$$4AC - B^2 = c. \tag{14}$$

The case $c = 0$ leads to a zero of φ as in the proof of Theorem 1; the case $c < 0$ leads to two zeros of φ, a contradiction. If $c > 0$, then φ has no zero but it has a critical point along γ. This is a critical point for φ on M. φ satifies the equation $\nabla^2\varphi = 2A \cdot g$. By a theorem of Tashiro [T] this implies that (M, g) is isometric with the euclidean space. Around the critical point the geodesic polar coordinates coincide with the euclidean polar coordinates.

In particular, if φ is non-constant, then c must be positive and $\overline{g}$ is a space of constant sectional curvature c.

Proof of Theorem 3. The case of an indefinite metric can be ruled out by Lemma 3. In the Riemannian case the equation

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \frac{1}{n}\left(\frac{\overline{S}}{\varphi^2} - S\right) \cdot g \tag{15}$$

implies

$$2\varphi\varphi'' - \varphi'^2 + \frac{S}{n(n-1)} \cdot \varphi^2 - \frac{\overline{S}}{n(n-1)} = 0 \tag{16}$$

along a unit speed geodesic in direction $\operatorname{grad}\varphi$. Differentiating once more leads to

$$2\varphi\varphi''' + \frac{2S}{n(n-1)}\varphi\varphi' = 0$$

or

$$\varphi''' + \rho \cdot \varphi' = 0 \tag{17}$$

where $\rho := \frac{S}{n(n-1)}$ denotes the normalized scalar curvature.

As in the proof of Theorem 1 we conclude

$$\varphi'^2 = -\rho\varphi^2 + 2a \cdot \varphi - \overline{\rho} \tag{18}$$

for a certain constant a.

In any case the solution φ of (17) and (18) either has a zero (which is impossible because φ is a conformal factor) or a critical point, except for solutions of the type

$$\varphi'(t) = \alpha \cdot e^{\sqrt{-\rho}t} \tag{19}$$

leading to the same warped product metric as in (9). If there is a critical point, then the levels around it are round spheres and thus (M, g) is a standard space of constant sectional curvature [T], [Kü, Lemmas 13 and 18]. This completes the proof of Theorem 3.

The local part of this calculation is due to Tachibana [Tb, Theorem 8.1]. In the compact case the following holds: a compact Riemannian manifold with constant scalar curvature admitting a non-constant solution of

$$\nabla^2\varphi = \frac{\Delta\varphi}{n} \cdot g$$

is isometric with a round sphere [Kü, Theorem 24].

References

[Be] A. L. Besse, *Einstein manifolds*, Ergebnisse Math. Grenzgeb., 3. Folge, Band 10, Springer, Berlin, Heidelberg, and New York, 1987.

[Br] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Math. Ann. **94** (1925), 119–145.

[Fe] J. Ferrand, *Sur une classe de morphismes conformes*, C. R. Acad. Sci. Paris **296** (1983), 681–684.

[Fi] A. Failkow, *Conformal geodesics*, Trans. Amer. Math. Soc. **45** (1939), 443–473.

[H] J. Haantjes, *Conformal representations of an n-dimensional euclidean space with a non-dimensional euclidean space with a non-definite fundamental form on itself*, Proc. Kon. Nederl. Akad. Amsterdam **40** (1937), 700–705.

[Hf] W. D. Halford, *Brinkmann's theorem in general relativity*, Gen. Relativity Gravitation **14** (1982), 1193–1195.

[Kan] M. Kanai, *On a differential equation characterizing a Riemannian structure of a manifold*, Tokyo J. Math. **6** (1983), 143–151.

[Kb] Y. Kerbrat, *Transformations conformes des variétés pseudo-Riemanniennes*, J. Differential Geom. **11** (1976), 547–571.

[Ke1] M. G. Kerckhove, *Conformal transformations of pseudo-Riemannian Einstein manifolds*, thesis, Brown Univ., 1988.

[Ke2] ——, *The structure of Einstein spaces admitting conformal motions*, Classical Quantum Gravity **8** (1991), 819–825.

[Kü] W. Kühnel, *Conformal transformations between Einstein spaces*, Conformal Geometry (R. S. Kulkarni and U. Pinkall, eds.), Aspects of Math., vol. E12, Braunschweig, 105–146, Vieweg-Verlag, 1988, pp. 105–146.

[Lie] S. Lie, *Komplexe, insbesondere Linien und Kugelkomplexe mit Anwendung auf die Theorie partieller Differentialgleichungen*, Math. Ann. **5** (1872), 145–246.

[Liv] J. Liouville, *Extension au cas des trois dimensions de la question du tracé géographique*, Note VI, Applications de l'Analyse à la Géométrie (G. Monge, ed.), Paris, 1850, pp. 609–617.

[O] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

[S] J. A. Schouten, *Der Ricci-Kalkül*, Springer-Verlag, Berlin, 1924.

[Tb] S. Tachibana, *On concircular geometry and Riemann spaces with constant scalar curvatures*, Tôhoku Math. J. **3** (1951), 149–158.

[T] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275.

[V] P. Venzi, *Klassifikation der geodätischen Abbildungen mit* $\overline{\mathrm{Ric}} - \mathrm{Ric} = \Delta \cdot g$, Tensor (N.S.) **37** (1982), 137–147.

[X] X. Xu, *Prescribing a Ricci tensor in a conformal class of Riemannian metrics*, Proc. Amer. Math. Soc. **115** (1992), 455–459; corrigenda, ibid. **118** (1993), 333.

[Y] K. Yano, *Concircular geometry* I–V, Proc. Imperial Acad. Japan **16** (1940), 195–200, 354–360, 442–448, 505–511; ibid. **18** (1942), 446–451.

[YN] K. Yano and T. Nagano, *Einstein spaces admitting a one-parameter group of conformal transformations*, Ann. of Math. (2) **69** (1959), 451–460.

Fachbereich Mathematik, Universität Duisburg, 47048-Duisburg, Germany
Current address: Mathematisches Institut B, Universität Stuttgart, 70550 Stuttgart, Germany
E-mail address: `kuehnel@morse.physik.uni-stuttgart.de`

Institut für Mathematik, Universität Augsburg, 86135-Augsburg, Germany
Current address: Mathematisches Institut, Universität Leipzig, 04009 Leipzig, Germany
E-mail address: `rademacher@uni-augsburg.de`