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CONFORMAL η-RICCI SOLITON IN LORENTZIAN-PARA KENMOTSU MANIFOLD

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ABSTRACT. The objective of the present paper is to study conformal η-Ricci soliton on Lorentzian-Para Kenmotsu manifolds with some curvature conditions. We obtained some results of conformal η-Ricci soliton on Lorentzian Para-Kenmotsu manifolds satisfying $R(\xi, X).S = 0$, $C(\xi, X).S = 0$ and the condition of quasi conformaly flatness. Finally, we give examples of Lorentzian-Para Kenmotsu manifold which admits conformal η- Ricci soliton.

1. INTRODUCTION

The concept of Ricci flow was introduced by R.S. Hamilton [7] in 1982 to find a canonical metric on a smooth manifold. It was formed to giving the answer of thrustons geometric conjucture, according to this each closed three manifold admits a geometric decomposition. After which the Ricci flow become one of the powerful tool to study Riemannian manifolds. The Ricci flow equation for metrics on a Riemannian manifold is given as follows

$$\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}. \tag{1.1}$$

The solution for the Ricci flow equation is known as Ricci soliton in case it driven only by a one parameter family of diffeomorphism and scaling. M. M. Tripathi,[20] C.L. Bejan, M. Crasmareanu, [3] analysed Ricci soliton in contact metric manifolds. A Ricci soliton is a natural generalization of Einstein metric. A Ricci soliton $(g.V, \lambda)$ on a Riemannian manifold (M, g) is defined as

$$\mathcal{L}_V g + 2S + 2\lambda g = 0 \tag{1.2}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectivily. In 1972 K. Kenmotsu [9] studied a class of contact Riemannian manifolds satisfying some special conditions, we call it Kenmotsu manifold.
A.E Fisher [6] in 2004 introduced a new concept known as a conformal Ricci flow a variation of the classical Ricci flow equation which has revised the unit volume

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constraint of that equation to a scalar curvature constraint. In the classical Ricci flow equation, the unit volume constraint plays an important role but in the conformal Ricci flow equation, the scalar curvature r is considered as a constraint. The conformal Ricci flow on M is defined by the following equation

$$\frac{\partial g}{\partial t} + 2\{Ric(g) + \frac{g}{n}\} = -pg, \tag{1.3}$$
$$r(g) = -1$$

where r is the scalar curvature of the manifold, p is a scalar non dynamical field (time-dependent scalar field), p is also known as conformal pressure. For the study of curvature conditions on Lorentzian para-Kenmotsu and Lorentzian para-Sasakian manifolds, we request to the reader please refer ([1],[8], [12],[15], [17][18]).

N. Basu and A. Bhattacharya, [2, 4, 5] in 2015 establish the notion of conformal Ricci soliton and conformal η-Ricci soliton and both equations are defined in the following manner

$$\mathcal{L}_V g + 2S = [2\lambda - (p + \frac{2}{n})]g,$$
$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0. \tag{1.4}$$

Recently, the Ricci soliton on almost contact metric manifolds has been studied by various authors such as [10, 11, 13, 14, 19].

In this paper, we studied conformal $\eta-$ Ricci soliton in a Lorentzian-Para Kenmotsu manifolds. The paper is organized in the following ways. In section 2, we discribe a brief introduction about Lorentzian-Para Kenmotsu manifold. Section 3, deals with the study of conformal η- Ricci soliton in Lorentzian-Para Kenmotsu manifolds. In section 4 and 5, we study curvature conditions $R.S = 0$, $C.S = 0$. In section 6, we discuss quasi conformally flat condition for Lorentzian-Para Kenmotsu manifold. We also study conformal $\eta-$ Ricci soliton in Lorentzian-Para Kenmotsu manifold admitting Codazzi type of Ricci tensor and cyclic parallel Ricci tensor. At last, we construct two examples of Lorentzian-Para Kenmotsu manifolds which admits conformal η- Ricci soliton.

2. PRELIMINARIES

Let $\mathcal{M}$ be an n-dimensional Lorentzian metric manifold. If it is endowed with a structure (ϕ, ξ, η, g), where ϕ ia a (1,1) tensor field, ξ ia a vector field, η is a 1-form on $\mathcal{M}$ and g is a Lorentzian metric, satisfying

$$\phi^2 X = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X) \tag{2.2}$$

for any $X, Y \in \mathfrak{X}(\mathcal{M})$, thus it is called Lorentzian almost paracontact manifold. In the Lorentzian almost paracontact manifold, the following relations holds:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.3}$$

$$\Phi(X, Y) = \Phi(Y, X), \tag{2.4}$$

where $\Phi(X,Y) = g(X,\phi Y)$.
If ξ is a killing vector field, the (para) contact structure is called K-(para) contact. In such a case, we have

$$\nabla_X \xi = \phi X. \tag{2.5}$$

A Lorentzian almost paracontact manifold $\mathcal{M}$ is called Lorentzian-Para Sasakian manifold if

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.6}$$

for any vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.
Now, we define a new manifold called Lorentzian-Para Kenmotsu manifold:

Definition 2.1. A Lorentzian almost paracontact manifold $\mathcal{M}$ is called Lorentzian-Para Kenmotsu manifold if

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.7}$$

for any vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.
In Lorentzian-Para Kenmotsu manifold, we have

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)Y = -g(X,Y) - \eta(X)\eta(Y), \tag{2.9}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.
Further, on a Lorentzian-Para Kenmotsu manifold $\mathcal{M}$, the following relations holds:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.10}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{2.13}$$

$$S(X,\xi) = (n-1)\eta(X), \qquad S(\xi,\xi) = -(n-1), \tag{2.14}$$

$$Q\xi = (n-1)\xi, \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y). \tag{2.16}$$

Now we define:

Definition 2.2. A Lorentzian-Para Kenmotsu manifold $\mathcal{M}$ is said to be an η-Einstein manifold if its Ricci tensor S is of the form-

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.17}$$

where a and b are scalar functions on $\mathcal{M}$.

Definition 2.3. A Lorentzian-Para Kenmotsu manifold $\mathcal{M}$ is said to be a generalized η- Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\Phi(X,Y), \tag{2.18}$$

where a, b and c are scalar functions on $\mathcal{M}$ and $\Phi(X,Y) = g(\phi X, Y)$. If $c = 0$, then the manifold reduces to an η- Einstein manifold.

Definition 2.4. The Concircular curvature tensor C in an n-dimensional manifold $\mathcal{M}$ is defined by [16]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.19}$$

where R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

Definition 2.5. The quasi-conformal curvature tensor $\mathcal{C}$ in an n-dimensional Lorentzian-Para Kenmotsu manifold $\mathcal{M}$ is defined by[15]

$$\begin{aligned}\mathcal{C}(X,Y)Z = aR(X,Y)Z + b\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ - \frac{r}{n}\{(\frac{a}{n-1} + 2b)(g(Y,Z)X - g(X,Z)Y)\},\end{aligned} \tag{2.20}$$

where a and b are constants such that $ab \neq 0$ and R, S, Q and r are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator defined by $g(QX,Y) = S(X,Y)$, and the scalar curvature of the manifold respectivily.

3. Conformal η-Ricci soliton in Lorentzian-Para Kenmotsu manifold

Let an n-dimensional Lorentzian-Para Kenmotsu manifold admits conformal η- Ricci soliton, then (1.4) holds, and thus we have

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + (2\lambda - (p + \frac{2}{n}))g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.1}$$

We know that

$$(\mathcal{L}_\xi g)(X,Y) = -2g(X,Y) - 2\eta(X)\eta(Y). \tag{3.2}$$

Using (3.2) in (3.1), we get

$$-2g(X,Y) - 2\eta(X)\eta(Y) + 2S(X,Y) + (2\lambda - (p + \frac{2}{n}))g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{3.3}$$

In vertue of (3.3), we have

$$S(X,Y) = \{1 - \lambda + \frac{p}{2} + \frac{1}{n}\}g(X,Y) + (1-\mu)\eta(X)\eta(Y), \tag{3.4}$$

which is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where $a = (1 - \lambda + \frac{p}{2} + \frac{1}{n})$ and $b = (1-\mu)$.
Putting $Y = \xi$ in (3.4), we have

$$S(X,\xi) = (\mu - \lambda + \frac{p}{2} + \frac{1}{n})\eta(X). \tag{3.5}$$

From equation (2.14) and (3.5), we obtain

$$(n-1) = (\mu - \lambda + \frac{p}{2} + \frac{1}{n}) \tag{3.6}$$

which implies,

$$(\lambda - \mu) = (1 + \frac{p}{2} + \frac{1}{n} - n). \tag{3.7}$$

Thus we have the following:

Theorem 3.1. *If an n-dimensional Lorentzian-Para Kenmotsu manifold admits conformal η- Ricci soliton, then the manifold is an η- Einstein manifold of the form (3.4) and the scalars λ, p and μ are related by (3.7).*

4. Conformal η-Ricci soliton in Lorentzian-Para Kenmotsu manifold satsfying $R(\xi, X).S = 0$

Let an n-dimensional Lorentzian-Para Kenmotsu manifold admitting conformal η-Ricci soliton and satisfying $R(\xi, X).S = 0$. Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0, \tag{4.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. By using (2.11) in (4.1), we have

$$S(g(X, Y)\xi - \eta(Y)X, Z) + S(Y, g(X, Z)\xi - \eta(Z)X) = 0, \tag{4.2}$$

taking $Z = \xi$ and using (3.5) in (4.2), we get

$$S(X, Y) = (\mu - \lambda + \frac{p}{2} + \frac{1}{n})g(X, Y). \tag{4.3}$$

Now from (3.4) and (4.3), we get

$S(X, Y) = (1-\mu)[g(X, Y)+\eta(X)\eta(Y)] = 0. \implies S(X, Y) = (1-\mu)[g(\phi X, \phi Y)] = 0$

From which it follows that $\mu = 1$ and $g(\phi X, \phi Y) \neq 0$.
From (4.3), we get

$$S(X, Y) = (1 - \lambda + \frac{p}{2} + \frac{1}{n})g(X, Y). \tag{4.4}$$

Thus we have the following important result:

Theorem 4.1. *If an n-dimensional Lorentzian-Para-Kenmotsu manifold admitting conformal η -Ricci soliton satisfies $R(\xi, X).S = 0$, then the manifold is an Einstein manifold of the form (4.4).*

5. Conformal η-Ricci soliton in Lorentzian-Para Kenmotsu manifold satsfying $C(\xi, X).S = 0$

Let an n-dimensional Lorentzian-Para Kenmotsu manifold admitting conformal η-Ricci soliton and satisfying $C(\xi, X).S = 0$. Let's suppose $C(X, Y)f = 0$, where f is C^∞ function. Then $C(\xi, X).S = 0$, is defined. Consider $C(\xi, X).S(Y, Z) = 0$, which implies

$$S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0. \tag{5.1}$$

From (2.19), we find that

$$C(\xi, X)Y = (1 - \frac{r}{n(n-1)})(g(X,Y)\xi - \eta(Y)X). \tag{5.2}$$

Now using (5.2) in (5.1), we have

$$(1 - \frac{r}{n(n-1)})[g(X,Y)S(\xi,\xi) - \eta(Y)S(X,\xi) + g(X,\xi)S(Y,\xi) - S(X,Y)], \tag{5.3}$$

which by putting $y = \xi$ and using (3.5) and (2.2), reduces to

$$(1 - \frac{r}{n(n-1)})[S(X,Y) - (\mu - \lambda + \frac{p}{2} + \frac{1}{n})g(X,Y)] = 0. \tag{5.4}$$

Therefore we have, either $r = n(n-1)$, or

$$S(X,Y) = (\mu - \lambda + \frac{p}{2} + \frac{1}{n})g(X,Y). \tag{5.5}$$

Now from (3.4) and (5.5), we get

$$S(X,Y) = (1-\mu)[g(X,Y)+\eta(X)\eta(Y)] = 0. \implies S(X,Y) = (1-\mu)[g(\phi X, \phi Y)] = 0.$$

From which it follows that $\mu = 1$ and $g(\phi X, \phi Y) \neq 0$. Using $\mu = 1$ in (5.5), we get

$$S(X,Y) = (1 - \lambda + \frac{p}{2} + \frac{1}{n})g(X,Y). \tag{5.6}$$

Thus we have the following important result:

Theorem 5.1. *If an n-dimensional Lorentzian-Para-Kenmotsu manifold admitting conformal η- Ricci soliton satisfies $C(\xi, X).S = 0$, then either the scalar curvature r is constant or the manifold becomes an Einstein manifold of the form (5.6).*

Definition 5.2. A Lorentzian-Para-Kenmotsu manifolld is said to have Codazzi type of Ricci tensor if its Ricci tensor S of type (0,2) is non zero and satisfies the following condition

$$(\nabla_Z S)(X,Y) = (\nabla_X S)(Y,Z), \tag{5.7}$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.
Taking covariant derivative of (3.4) along any vector field Z and using (2.9), we get

$$(\nabla_Z S)(X,Y) = (1-\mu)\{-g(Z,X)\eta(Y) - g(Z,Y)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}. \tag{5.8}$$

If the Ricci tensor S is of Codazzi type, then we have from (5.7) and (5.8)

$$(1-\mu)\{g(X,Y)\eta(Z) - g(Z,Y)\eta(X)\} = 0, \tag{5.9}$$

putting $Z = \xi$, in (5.9), we obtain

$$(1-\mu)\{g(X,Y) + \eta(X)\eta(Y)\} = 0. \tag{5.10}$$

From which, it follows that $\mu = 1$. (since $g(X, Y) + \eta(X)\eta(Y) \neq 0$).
Putting $\mu = 1$ in (3.4), we get

$$S(X, Y) = (1 - \lambda + \frac{p}{2} + \frac{1}{n})g(X, Y). \tag{5.11}$$

Therefore the manifold becomes an Einstein manifold. Thus we have the following:

Theorem 5.3. *An n-dimensional Lorentzian-Para Kenmotsu manifold whose Ricci tensor is of Codazzi type admitting conformal η- Ricci soliton is an Einstein manifold of the form (5.11).*

Corollary 5.4. *An n-dimensional Lorentzian-Para Kenmotsu manifold whose Ricci tensor is of Codazzi type admitting conformal η- Ricci soliton is a manifold of constatnt curvature.*

Definition 5.5. A Lorentzian-Para Kenmotsu manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type (0,2) is non zero and satisfies the following condition

$$(\nabla_Z S)(X, Y) + (\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) = 0, \tag{5.12}$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$.

Let an n-dimensional Lorentzian-Para Kenmotsu manifold admiting conformal η- Ricci soliton and the manifold has cyclic parallel Ricci tensor, So (5.11) holds. By taking covariant derivative of (3.4) along vector field Z and using (2.9), we get

$$(\nabla_Z S)(X, Y) = (1-\mu)\{-g(Z, X)\eta(Y)-g(Z, Y)\eta(X)-2\eta(X)\eta(Y)\eta(Z)\}, \tag{5.13}$$

simillarly, we have

$$(\nabla_X S)(Y, Z) = (1-\mu)\{-g(X, Y)\eta(Z)-g(X, Z)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)\}, \tag{5.14}$$

and

$$(\nabla_Y S)(Z, X) = (1-\mu)\{-g(Y, Z)\eta(X)-g(Y, X)\eta(Z)-2\eta(X)\eta(Y)\eta(Z)\}. \tag{5.15}$$

Now using (5.13), (5.14) and (5.15) in (5.12), we get

$$(1-\mu)\{g(X, Y)\eta(Z)+g(Y, Z)\eta(X)+g(Z, X)\eta(Y)+3\eta(X)\eta(Y)\eta(Z)\} = 0. \tag{5.16}$$

Putting $Z = \xi$ and using (2.2) in (5.16), we get

$$(1 - \mu)\{g(X, Y) + \eta(X)\eta(Y)\} = 0, \implies (1 - \mu)g(\phi X, \phi Y) = 0.$$

From which it follows that $\mu = 1$. Since ($g(\phi X, \phi Y) \neq 0$). Now using $\mu = 1$, in (3.4), we get

$$S(X, Y) = (1 - \lambda + \frac{p}{2} + \frac{1}{n})g(X, Y). \tag{5.17}$$

Thus, we have the following:

Theorem 5.6. *If an n-dimensional Lorentzian-Para Kenmotsu manifold admits conformal η-Ricci soliton and the manifold has a cyclic parallel Ricci tensor, then the manifold is an Einstein manifold of the form (5.17).*

6. Conformal η-Ricci solitons on quasi conformally flat Lorentzian-Para-Kenmotsu manifolds

Lets assume that the manifold $\mathcal{M}$ admitting the conformal η- Ricci solitons is quasi conformally flat, that is, $\mathcal{C} = 0$, then from (2.20), it follows that,

$$R(X,Y)Z = -\frac{b}{a}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{n}(\frac{a}{n-1} + 2b)(g(Y,Z)X - g(X,Z)Y). \quad (6.1)$$

Taking the inner product of (6.1) with ξ and using equation (2.2), (3.4) and (3.5), we get

$$\eta(R(X,Y)Z) = -\frac{b}{a}[(1-\lambda+\frac{p}{2}+\frac{1}{n})(g(Y,Z)+(1-\mu)\eta(X)\eta(Z))\eta(X) - (1-\lambda+\frac{p}{2}+\frac{1}{n})(g(X,Z)+(1-\mu)\eta(X)\eta(Z))\eta(Y) + (\mu-\lambda+\frac{1}{n}+\frac{p}{2})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \frac{r}{n}(\frac{a}{n-1}+2b)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)), \quad (6.2)$$

which takes form,

$$\eta(R(X,Y)Z) = -\frac{b}{a}\{(g(Y,Z)\eta(X) - (g(X,Z)\eta(Y))(1-\lambda+\frac{p}{2}+\frac{1}{n}) + (\mu-\lambda+\frac{p}{2}+\frac{1}{n})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\} + \frac{r}{n}(\frac{a}{n-1}+2b)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)), \quad (6.3)$$

after some straight forward calculation, we get

$$\eta(R(X,Y)Z) = \{\frac{(1-\mu)b}{a} + \frac{r}{n}(\frac{a}{n-1}+2b)\}(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)), \quad (6.4)$$

using equation (2.10), (6.4) become

$$\{\frac{(1-\mu)b}{a} + \frac{r}{n}(\frac{a}{n-1}+2b) - 1\}(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) = 0. \quad (6.5)$$

Putting $X = \xi$, we get

$$\{\frac{(1-\mu)b}{a} + \frac{r}{n}(\frac{a}{n-1}+2b) - 1\}(-g(Y,Z) - \eta(Z)\eta(Y)) = 0,$$
$$\implies \{\frac{(1-\mu)b}{a} + \frac{r}{n}(\frac{a}{n-1}+2b) - 1\}g(\phi Y, \phi Z) = 0, \quad (6.6)$$

from which it follows that

$$\{\frac{(1-\mu)b}{a}+\frac{r}{n}(\frac{a}{n-1}+2b)-1\}=0 \quad \textit{becouse} \quad [g(\phi Y,\phi Z)\neq 0].$$

Thus, we can state the following theorem:

Theorem 6.1. *A quasi-conformally flat Lorentzian-Para-Kenmotsu manifold admits a conformal η- Ricci soliton then* $\lambda=\{2-\frac{a}{b}+\frac{ar}{bn}(\frac{a}{n-1+2b})+\frac{p}{2}+\frac{1}{n}-n\}$ *and* $\mu=\{1-\frac{a}{b}+\frac{ar}{bn}(\frac{a}{n-1+2b})\}$.

Corollary 6.2. *If we take scalar curvature* $r=\frac{n(a-b)(2b+n-1)}{a^2}$, *then* $\mu=0$ *and* $\lambda=(1+\frac{p}{2}+\frac{1}{n}-n)$.

7. Examples

Example 7.1. We consider the 5-dimensional manifold

$$\mathcal{M}=\{(x_1,x_2,x_3,x_4,z)\in R^5, z>0\}, \tag{7.1}$$

where (x_1,x_2,x_3,x_4,z) are the standard coordinates in R^5. Let e_1,e_2,e_3,e_4 and e_5 be the vector fields on $\mathcal{M}$ given by

$e_1=z\frac{\partial}{\partial x_1}$, $e_2=z\frac{\partial}{\partial x_2}$, $e_3=z\frac{\partial}{\partial x_3}$

$e_4=z\frac{\partial}{\partial x_4}$, $e_5=z\frac{\partial}{\partial z}=\xi$,

which are linearly independent at each point p of $\mathcal{M}$, and hence form a basis of T_pM. Define a Lorentzian metric g on $\mathcal{M}$ defined by $g(e_i,e_j)=0, i\neq j$ where $i,j=1,2,3,4,5$ and

$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=g(e_4,e_4)=1$ and $g(e_5,e_5)=-1$.

Let η be the 1-form on $\mathcal{M}$ defined by $\eta(X)=g(X,e_5)=g(X,\xi)$ for all $X\in\mathfrak{X}(\mathcal{M})$ and let ϕ be the (1,1)-tensor field on $\mathcal{M}$.

By applying the linearty of ϕ and g, we have

$\eta(\xi)=g(\xi,\xi)=-1$, $\phi^2X=X+\eta(X)\xi$, $\eta(\phi X)=0$,

$g(X,\xi)=\eta(X)$, $g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y)$,

for all X, $Y\in\mathfrak{X}(\mathcal{M})$.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. So, we have

$[e_1,e_2]$=$[e_1,e_3]$=$[e_1,e_4]$=$[e_2,e_3]$=$[e_2,e_4]$=$[e_3,e_4]$=0,

$[e_1,e_5]=-e_1$, $[e_2,e_5]=-e_2$, $[e_3,e_4]=-e_3$, $[e_4,e_5]=-e_4$.

The Riemannian connection ∇ of the metric g is given by

$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y])$,

which is known as Koszul's formula. Using Koszul's formula we easily calculate

$$\nabla_{e_1} e_1 = e_5, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0, \nabla_{e_1} e_4 = 0, \nabla_{e_1} e_5 = -e_1, \tag{7.2}$$

$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = e_5, \nabla_{e_2} e_3 = 0, \nabla_{e_2} e_4 = 0, \nabla_{e_2} e_5 = -e_2,$$

$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = e_5, \nabla_{e_3} e_4 = 0, \nabla_{e_3} e_5 = -e_3,$$

$$\nabla_{e_4} e_1 = 0, \nabla_{e_4} e_2 = 0, \nabla_{e_4} e_3 = 0, \nabla_{e_4} e_4 = e_5, \nabla_{e_4} e_5 = -e_4,$$

$$\nabla_{e_5} e_1 = 0, \nabla_{e_5} e_2 = 0, \nabla_{e_5} e_3 = 0, \nabla_{e_5} e_4 = 0, \nabla_{e_5} e_5 = 0.$$

Also one can easily verify that

$\nabla_X \xi = -X - \eta(X)\xi$, and $(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X$.

Hence the manifold is Lorentzian para-Kenmotsu manifold of dimension 5. Now let

$$X = \sum_{i=1}^{5} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5,$$
$$Y = \sum_{i=1}^{5} Y^i e_i = Y^1 e_1 + Y^2 e_2 + Y^3 e_3 + Y^4 e_4 + Y^5 e_5,$$
$$Z = \sum_{i=1}^{5} Z^i e_i = Z^1 e_1 + Z^2 e_2 + Z^3 e_3 + Z^4 e_4 + Z^5 e_5.$$

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.3}$$

From (7.2) and (7.3), we can be easily calculate that

$$R(e_1, e_2)e_1 = e_2, R(e_1, e_3)e_1 = e_3, R(e_1, e_4)e_1 = e_4, R(e_1, e_5)e_1 = e_5, \tag{7.4}$$
$$R(e_2, e_3)e_1 = R(e_2, e_4)e_1 = R(e_2, e_5)e_1 = 0,$$
$$R(e_3, e_4)e_1 = R(e_3, e_5)e_1 = R(e_4, e_5)e_1 = 0,$$

$$R(e_1, e_2)e_2 = -e_1, R(e_2, e_3)e_2 = e_3, R(e_2, e_4)e_2 = e_4, R(e_2, e_5)e_2 = e_5, \tag{7.5}$$
$$R(e_1, e_3)e_2 = R(e_1, e_4)e_2 = R(e_1, e_5)e_2 = 0,$$
$$R(e_3, e_4)e_2 = R(e_3, e_5)e_2 = R(e_4, e_5)e_2 = 0,$$
$$R(e_1, e_3)e_3 = -e_1, R(e_2, e_3)e_3 = -e_2, R(e_3, e_4)e_3 = e_4, R(e_3, e_5)e_3 = e_5, \tag{7.6}$$

$$R(e_1,e_3)e_2 = R(e_1,e_4)e_2 = R(e_1,e_5)e_2 = 0,$$
$$R(e_3,e_4)e_2 = R(e_3,e_5)e_2 = R(e_4,e_5)e_2 = 0,$$

$$R(e_1,e_4)e_4 = -e_1, R(e_2,e_4)e_4 = -e_2, R(e_3,e_4)e_4 = -e_3, R(e_4,e_5)e_4 = e_5, \quad (7.7)$$
$$R(e_1,e_2)e_4 = R(e_1,e_3)e_4 = R(e_1,e_5)e_5 = 0,$$
$$R(e_2,e_3)e_4 = R(e_2,e_5)e_5 = R(e_3,e_5)e_4 = 0,$$

$$R(e_1,e_5)e_5 = -e_1, R(e_2,e_5)e_5 = -e_2, R(e_3,e_5)e_5 = -e_3, R(e_4,e_5)e_5 = -e_5, \quad (7.8)$$
$$R(e_1,e_2)e_5 = R(e_1,e_3)e_5 = R(e_1,e_4)e_5 = 0,$$
$$R(e_2,e_3)e_5 = R(e_2,e_4)e_5 = R(e_3,e_4)e_4 = 0.$$

With the help of above expressions of the curvature tensors, it follows that

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \quad (7.9)$$

From which, we get $S(Y,Z) = 4g(Y,Z) \implies r = 20$.
Now, from equation (3.4), we get

$$\sum_{i=1}^{5} \epsilon_i S(e_i,e_i) = \{1 - \lambda + \frac{p}{2} + \frac{1}{n}\} \sum_{i=1}^{5} \epsilon_i g(e_i,e_i) - \sum_{i=1}^{5} \epsilon_i \eta(e_i)\eta(e_i), \quad (7.10)$$

after some calculation, we get

$$\lambda - \frac{\mu}{5} = \frac{p}{2} - 3. \quad (7.11)$$

Now, from equation (3.7) and (7.11), we get $\mu = 1$ and $\lambda = (\frac{p}{2} - \frac{14}{5})$.
Hence, the data (g, ξ, λ, μ) for $\mu = 1$ and $\lambda = (\frac{p}{2} - \frac{14}{5})$, defines a conformal $\eta-$ Ricci soiton on a Lorentzian-Para-Kenmotsu manifold $\mathcal{M}$.

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Example 7.2. We consider the 3-dimensional manifold

$\mathcal{M} = \{(x,y,z) \in R^3, z > 0\}$,

where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2, and e_3 be the vector fields on $\mathcal{M}$ given by

$e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$, $e_3 = z\frac{\partial}{\partial z} = \xi$,

which are linearly independent at each point p of $\mathcal{M}$ and hence form a basis of $T_p\mathcal{M}$. Define a Lorentzian metric g on $\mathcal{M}$ such that

$g(e_1,e_1) = g(e_2,e_2) = 1$ and $g(e_3,e_3) = -1$.

Let η be the 1-form on $\mathcal{M}$ defined by $\eta(X) = g(X,e_3) = g(X,\xi)$, for all $X \in \mathfrak{X}(\mathcal{M})$ and let ϕ be the (1,1)-tensor field on $\mathcal{M}$ defined as

$\phi e_1 = -e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$.

By applying linearty of ϕ and g, we have

$\eta(\xi) = g(\xi, \xi) = -1$, $\phi^2 X = X + \eta(X)\xi$, $\eta(\phi X) = 0$,

$g(X, \xi) = \eta(X)$, $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$,

for all X, $Y \in \mathfrak{X}(\mathcal{M})$.

Let ∇ be the Levi Civita connection with respect to the Lorentzian metric g. So, we have

$[e_1, e_2]$=$[e_2, e_1]$=0, $[e_1, e_3]$= $-e_1$, $[e_3, e_1]$= e_1, $[e_2, e_3]$= $-e_2$, $[e_3, e_2]$= e_2.

The Riemannian connection ∇ of the metric g is given by

$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y])$,

which is known as Koszul's formula. Using Koszul's formula, we easily calculate

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = -e_1, \nabla_{e_2} e_1 = 0, \tag{7.12}$$

$$\nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = -e_2, \nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0.$$

Also one can easily verify that

$\nabla_X \xi = -X - \eta(X)\xi$, and $(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X$.

Hence the manifold is Lorentzian para-Kenmotsu manifold of dimension 3. Now, let

$$X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3,$$

$$Y = \sum_{i=1}^{3} Y^i e_i = Y^1 e_1 + Y^2 e_2 + Y^3 e_3 +,$$

$$Z = \sum_{i=1}^{3} Z^i e_i = Z^1 e_1 + Z^2 e_2 + Z^3 e_3$$

it is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.13}$$

From equation (7.12) and (7.13), we easily verified that

$$R(e_1, e_2)e_1 = -e_2, R(e_1, e_3)e_1 = -e_3, R(e_2, e_3)e_1 = 0, \tag{7.14}$$

$$R(e_1, e_2)e_2 = -e_1, R(e_1, e_3)e_2 = -0, R(e_2, e_3)e_2 = e_3,$$
$$R(e_1, e_2)e_3 = 0, R(e_1, e_3)e_3 = -e_1, R(e_2, e_3)e_3 = e_2.$$

With the help of above exression of the curvature tensors, it follows that

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y,$$

from which, we get

$$S(Y, Z) = 2g(Y, Z). \tag{7.15}$$

The Ricci tensor S is given by

$$S(e_1, e_1) = S(e_2, e_2) = 2 \quad and \quad S(e_3, e_3) = -2. \tag{7.16}$$

From (3.4) and (7.16), we get
$\lambda = (\frac{p}{2} - \frac{2}{3})$ and $\mu = 1$. Thus the data (g, ξ, λ, μ) for $\lambda = (\frac{p}{2} - \frac{2}{3})$ and $\mu = 1$, defines conformal η-Ricci soliton on the Lorentzian-para Kenmotsu manifold $\mathcal{M}$.

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