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Conformal evolution of phantom dominated final stages of the universe in higher dimensions

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Abstract

Friedmann solutions and in higher dimensions 5D Kaluza-Klein solutions using mathematical packages such as Sagemath and Cadabra are calculated. Modified Friedmann equation powered by Loop Quantum Gravity in higher dimensions is calculated in this work. Loop quantization in extra-dimensional space is predicted. Modified equation of state for non-interacting dark matter and dark energy is calculated. It has been predicted that the higher curvature due to phantom density would be a local kind of quantized curvature. The modified Friedman solutions with Kaluza Klein interpretation is found. To achieve the conformal exit the non-interacting solutions are discussed in this work. Obtained results are compared with LCDM and quintessence models. The work supports the conformal cyclic cosmology which predicts the conformal evolution of the universe without facing any singularity as the consequence of the topological effects.

Keywords Phantom energy , Late time universe, Baby universe, Scale factor quantization, Loop quantum cosmology , Kaluza Klein cosmology, Non interacting phantom cosmology , Conformal Cyclic Cosmology.

1 Introduction

In our previous work, we have investigated the final stages of the universe as conformal cyclic and phantom energy dominated evolution [1]. In that work, we have discovered that the late time universe will be dominated by phantom energy. And the universe will continue its evolution without completely ripping off. In such a scenario it has been explained that the future Aeon will continue its evolution in higher dimensions. To confirm this scenario, we investigate Kaluza-Klein cosmological solutions for higher dimensions. Friedman solutions for higher dimensions are reported in this work.

The mathematical structure of loop quantum cosmology is reported in [2]. In loop quantum cosmology the big bang is replaced by a big bounce. The theory confirms that when the density of the universe approaches the critical density, then the universe bounces back [3]. The loop quantum gravity predicts the discreet nature of the space-time at quantum level [4]. Modifications on Friedmann equations from loop quantum gravity for a universe with a free scalar field is reported in [5]. A modified version of Friedman equations provides big bounce solutions. To explain the evolution of the universe the Conformal Cyclic Cosmological model is predicted by Penrose [6]. In this formalism, it is predicted that evolution of the universe happens as conformal cycles, which are called Aeons. The formalism attempts to solve many problems of evolution of the universe [7]. Scalar phantom energy as a cosmological dynamical system is reported in [8]. The phantom energy is proposed as the consequence for a big rip in the late time universe. Dark energy is described with multiscalar fields such as quintessence and phantom $\omega < -1$ and $-1 \leq \omega \leq 0$ [9]. Kaluza-Klein cosmology with modified holographic dark energy is reported in [10]. The authors also discuss the equation of state parameter as well as the equation of evolution of the modified holographic dark energy.

The bouncing evolution in the view of loop quantum gravity is discussed in [11]. The bouncing universe in terms of quintom universe is discussed in [12]. Relationship between the bouncing universe and the vector field is discussed in [13]. The bouncing universe scenareo and its various aspects are analyzed in [14],[15], [16] and [17]. The bouncing cosmology with $f(R)$ cosmology is studied in [18]. The bouncing universe scenario and cyclic cosmology scenario is compared and analyzed in [19].

In the present work curvature components and metric component are reported using mathematical calculations. For these calculations, mathematical packages such as Cadabra [20] and Sagemath [20] is applied. Tensor calculus with open-source, SageManifolds software project is reported in [21]. Like Sagemath, SageManifolds is also free, opensource and it is based on the Python programming language. Riemann normal coordinate expansions of the metric and other geometrical quantities using Cadabra is presented in [22]. Here in section 2, the 5 dimensional Friedman solutions are reported. From the metric given, Riemann tensor and Ricci scalar are calculated. In which first and second Friedman solutions are found. In section 3 Einstein equations in terms of Kaluza Klein reduction is discussed. Kaluza Klein reduction using Cadabra software is discussed. The Einstein action is implemented in the Kaluza Klein solution. The Kaluza Klein equation is implemented by the substitution $1-kr^2$ in g_{44} and g^{44}. In section 4 evolution of the late time baby universe by the consequence of phantom energy is discussed. Friedman solutions which are obtained from the Kaluza Klein solution and late time phantom energy solutions are compared in section 4. Similarly, combined solutions of loop quantum gravity and classical evolution for scale factor are reported. Due to the non-interacting solutions of dark matter and phantom energy, the universe exit and evolve into higher dimensions by producing local curvature. Non-interacting solutions of dark matter and dark energy which lead to avoid future big bounce is discussed in section 5. The paper concludes with modified scale factor reduction using loop quantum cosmology and non-interacting phantom solutions.

2 5D Friedmann solutions

The Friedman solutions predict the equation of the motion of the universe. In higher dimensions, the Kaluza-Klein analysis offers the solutions for Einstein field equations. To calculate higher dimensional Friedmann solutions, (which are inspired by Kaluza Klein equations [23], [24], [25]) the metric is to be defined. The 5 dimensional Lorentz manifold consists the elements such as,

$$(M, (t, r, \theta, \phi, \psi)) \tag{1}$$

The metric representation of the Kaluza Klein universe [26] is given by

$$\begin{aligned} ds^2 = -\mathrm{d}t \otimes \mathrm{d}t + \left(-\frac{a\,(t)^2}{kr^2-1}\right) \mathrm{d}r \otimes \mathrm{d}r + r^2 a\,(t)^2\, \mathrm{d}\theta \otimes \mathrm{d}\theta \\ + r^2 a\,(t)^2 \sin\,(\theta)^2\, \mathrm{d}\phi \otimes \mathrm{d}\phi + \left(-kr^2+1\right) \mathrm{d}\psi \otimes \mathrm{d}\psi \end{aligned} \tag{2}$$

The connection components are required to understand the curvature, which exists in the manifold. The Riemann tensors are computed for each component of the metric, with the help of the above connections. The Sagemath is applied to calculate Riemann tensor. Riemann module of Sagemath calculates the curvature tensor of the metric g. Riemannian tensor field of type (1,4) on the 5-dimensional Lorentzian manifold M [27] is computed.

Even for the negative curvatures, the curvature tensor behave as

$$R^{\psi}{}_{r\ r\ \psi} = \frac{-1}{r^2-1} \tag{3}$$

For the negative and positive curvatures, it is observed that the resultant curvature remain positive for both. Also, it states that the smaller scale factors produce high curvatures, which seems to be local curvature in the late time evolution of the universe. Once the curvature reaches maximum scale factor value, then there will the conformal evolution of the universe in higher dimensions instead of big bounce. (The conformal evolution of the universe is the idea, that is proposed by Penrose [6] [28] [29]. In this model, he suggests that the universe evolves as conformal cycles. The initial and final singularities can be replaced by topological smooth surfaces.) Similarly, the higher curvature values lead to the exit of the universe into higher dimensions. Those results are confirmed by the equation 92. Like Riemann tensor, the Ricci tensor also does not change for variable curvatures.

Hence, the curvature prolongs to quantum scales. The curvature and scale factor are replaced by quantized curvature variables, which are driven by loop quantum gravity [30]. Hence, conformal evolution of a higher dimensional baby universe is plausible. Very high curvature consequences big bounce for the universe during its evolution [31] [32]. Similarly, from the Ricci scalar calculated above, it is also observed that the fourth component of the Ricci scalar increases when the scale factor is increased. The curvature attains maximum magnitude for the maximum scale factor as suggested from equation 91. The field is calculated, which is in the form of

$$\mathbb{R}\left(t, r, \theta, \phi, \psi\right) \rightarrow \frac{6\left(\left(\dot{a}\right)^2 + a\left(t\right)\frac{\partial^2 a}{\partial t^2} + 2\,k\right)}{a\left(t\right)^2} \tag{4}$$

Stress energy Field of symmetric bilinear forms on the 5-dimensional Lorentzian manifold M is derived as,

$$\begin{aligned} T = \rho\left(t\right) \mathrm{d}t \otimes \mathrm{d}t \\ + \left(-\frac{4\,a\left(t\right)^2}{kr^2 - 1}\right) \mathrm{d}r \otimes \mathrm{d}r + 4\,r^2 a\left(t\right)^2 \mathrm{d}\theta \otimes \\ \mathrm{d}\theta + 4\,r^2 a\left(t\right)^2 \sin\left(\theta\right)^2 \mathrm{d}\phi \otimes \mathrm{d}\phi + \left(-4\,kr^2 + 4\right) \mathrm{d}\psi \otimes \mathrm{d}\psi \end{aligned} \tag{5}$$

First Friedmann equation is computed as

$$-8\,\pi G\rho\left(t\right) - \Lambda + \frac{3\,\dot{a}\left(t\right)^2}{a\left(t\right)^2} + \frac{6\,k}{a\left(t\right)^2} = 0 \tag{6}$$

For $k = 0$ the solution behaves like ordinary Friedman solution. For $k = \pm 1$ the solution behaves differently than the previously obtained solutions.

Second Friedmann equation is computed as

$$-4\,\pi G\rho\left(t\right) - 64\,\pi G + \Lambda - \frac{3\,\frac{\partial^2}{(\partial t)^2} a\left(t\right)}{a\left(t\right)} = 0 \tag{7}$$

Similarly Second Friedman equation is also verified by the Higher dimensional interpretation.

3 Einstein equation with Kaluza Klein reduction

The Einstein equation is derived from Kaluza Klein solutions. Calculation of Connection, Riemann tensor and Ricci scalar are reported in the Appendix.

A scalar field introduced in the action. It is calculated as

$$S = \int \sqrt{-g}\ (\frac{1}{2}\kappa^{-1}(R - 2\Lambda) + \mathcal{L}_{\mathrm{mat}} + \phi(x))\ \mathrm{d}x \tag{8}$$

The right hand side of the equation is included with matter Lagrangian.

$$S = \int (\frac{1}{2}\sqrt{-g}\kappa^{-1}R - \sqrt{-g}\kappa^{-1}\Lambda + \sqrt{-g}\mathcal{L}_{\mathrm{mat}} + \sqrt{-g}\phi(x))\ \mathrm{d}x \tag{9}$$

Variation in metric is defined as

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{10}$$

The energy-momentum tensor and matter Lagrangian invariant is calculated as

$$\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\mathcal{L}_{\mathrm{mat}} - 2\sqrt{-g}\delta\mathcal{L}_{\mathrm{mat}} = \sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} \tag{11}$$

The covariant derivative is calculated below.

$$\partial_\sigma \delta\Gamma^\mu{}_{\nu\rho} = -\Gamma^\mu{}_{\sigma\alpha}\delta\Gamma^\alpha{}_{\nu\rho} + \Gamma^\alpha{}_{\sigma\nu}\delta\Gamma^\mu{}_{\alpha\rho} + \Gamma^\alpha{}_{\sigma\rho}\delta\Gamma^\mu{}_{\nu\alpha} + \nabla_\sigma\delta\Gamma^\mu{}_{\nu\rho} \tag{12}$$

Variation in Riemann tensor is calculated as

$$\begin{aligned}\delta R^\rho{}_{\sigma\mu\nu} = -\Gamma^\rho{}_{\mu\alpha}\delta\Gamma^\alpha{}_{\nu\sigma} + \Gamma^\alpha{}_{\mu\nu}\delta\Gamma^\rho{}_{\alpha\sigma} + \Gamma^\alpha{}_{\mu\sigma}\delta\Gamma^\rho{}_{\nu\alpha} + \nabla_\mu\delta\Gamma^\rho{}_{\nu\sigma} + \Gamma^\rho{}_{\nu\alpha}\delta\Gamma^\alpha{}_{\mu\sigma}\\ -\Gamma^\alpha{}_{\nu\mu}\delta\Gamma^\rho{}_{\alpha\sigma} - \Gamma^\alpha{}_{\nu\sigma}\delta\Gamma^\rho{}_{\mu\alpha} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\sigma} + \Gamma^\lambda{}_{\nu\sigma}\delta\Gamma^\rho{}_{\mu\lambda}\\ +\Gamma^\rho{}_{\mu\lambda}\delta\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\lambda{}_{\mu\sigma}\delta\Gamma^\rho{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\lambda}\delta\Gamma^\lambda{}_{\mu\sigma}\end{aligned} \tag{13}$$

By simplifying the above relation, it is obtained as

$$\delta R^\rho{}_{\sigma\mu\nu} = \Gamma^\alpha{}_{\mu\nu}\delta\Gamma^\rho{}_{\alpha\sigma} + \nabla_\mu\delta\Gamma^\rho{}_{\nu\sigma} - \Gamma^\alpha{}_{\nu\mu}\delta\Gamma^\rho{}_{\alpha\sigma} - \nabla_\nu\delta\Gamma^\rho{}_{\mu\sigma} \tag{14}$$

Variation in Ricci tensor is calculated as

$$\delta R_{\sigma\nu} = \delta R^\rho{}_{\sigma\rho\nu} \tag{15}$$

$$\delta R_{\sigma\nu} = \nabla_\rho\delta\Gamma^\rho{}_{\nu\sigma} - \nabla_\nu\delta\Gamma^\rho{}_{\rho\sigma} \tag{16}$$

$$\tag{17}$$

Variation on scalar curvature is calculated as

$$\delta R = \nabla_\nu(\delta\Gamma^{\nu}{}_{\rho\sigma} g^{\rho\sigma}) - \nabla_\nu(\delta\Gamma^{\rho}{}_{\rho\sigma} g^{\nu\sigma}) + R_{\nu\sigma}\delta g^{\nu\sigma} \tag{18}$$

Substituting the action into $\delta g_{\mu\nu}$, matter Lagrangian invariant and scalar curvature, then the action becomes

$$\begin{aligned}\delta S = \int (\frac{1}{2}\delta\sqrt{-g}\kappa^{-1}R + \frac{1}{2}\sqrt{-g}\kappa^{-1}\delta R - \delta\sqrt{-g}\kappa^{-1}\Lambda \\ +\delta\sqrt{-g}\mathcal{L}_{\mathrm{mat}} + \sqrt{-g}\delta\mathcal{L}_{\mathrm{mat}} + \delta\sqrt{-g}\phi(x) \\ +\sqrt{-g}\delta\phi(x))\ \mathrm{d}x\end{aligned} \tag{19}$$

The action and Einstein field equations are obtained as

$$2\kappa(-\frac{1}{4}g_{\mu\nu}\kappa^{-1}R + \frac{1}{2}\kappa^{-1}R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\kappa^{-1}\Lambda - \frac{1}{2}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\phi(x)) = 0 \tag{20}$$

$$-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu} + g_{\mu\nu}\Lambda - \kappa g_{\mu\nu}\phi(x) = \kappa T_{\mu\nu} \tag{21}$$

$$G_{\mu\nu} + g_{\mu\nu}\Lambda - \kappa g_{\mu\nu}\phi(x) = \kappa T_{\mu\nu} \tag{22}$$

To obtain higher dimensional reduction via Kaluza-Klein solutions, the following terms to be calculated.

$$X = g_{m_1\ m}R^{m_1}{}_{4\ n\ 4} + g_{4\ m}R^{4}{}_{4\ n\ 4} \tag{23}$$

In terms of field strength, the Kaluza-Klein solution can be discussed as,

$$\begin{aligned}X = -\frac{1}{4}\partial_m\phi\partial_n\phi\phi^{-1} + \frac{1}{4}\partial_p\phi\partial_n h_{mq}h^{pq} \\ -\frac{1}{2}\partial_{mn}\phi + \frac{1}{4}F_{mp}F_{nq}\phi^3 h^{pq} + \frac{1}{4}\partial_p\phi\partial_q\phi\phi^{-1}h_{mn}h^{pq} \\ -\frac{1}{4}\partial_p\phi\partial_q h_{mn}h^{pq} + \frac{1}{4}\partial_p\phi\partial_m h_{nq}h^{pq}\end{aligned} \tag{24}$$

Such a resolution can be combined with the late time universe. For $\omega < -1$ the universe appears to have phantom dominance in the latter times of its evolution [33]. As per the classical evolution, such universe will face the big rip singularity in future. But there exists a possibility of avoidance of big rip singularity, which is driven by the loop quantum cosmology. LQG resolves the singularities exist in the evolution of the universe. It suggests that the universe will bounce back once the density parameter attains the critical density $\rho \sim \rho_{crit}$. The late time universe can attain such densities due to increase in the phantom energy density. Hence, the future universe will bounce back instead of stopping its evolution by facing the big rip singularity. The resolution of the big rip also shows some glimpse on higher dimensional evolution of the universe. They are all discussed in the next section.

4 Late time universe and higher dimensions

from Six bidimensional dark energy parameterisations are studied and tested with SNeIa and BAO data [34]. Results obtained from such data are in favor of the LCDM model.The dark energy can be generalized as

$$\rho_{de} = \rho_{0(de)} f(z), \tag{25}$$

where,

$$f(z) = exp\left[3\int_0^z \frac{1+\omega(\tilde{z})}{1+\tilde{z}} d\tilde{z}\right] \tag{26}$$

For quintessence (ω=constant), the solution of $f(z)$ is

$$f(z) = (1+z)^{1+\omega} \tag{27}$$

For cosmological constant $\omega = -1$ and $f = 1$

from By having only dark matter and cosmological constant late time phantom like accelleration will be a consequence on the evolution of the universe [35]. Though such model does not require any phantom dominated scenereo, presence of quintessence will lead to avoid phantom dividing line $\omega = 1$. Based on the absence of cold dark matter ($\Omega_m \simeq 0$). The Raychoudhuri equation becomes [36],

$$2\dot{H} = -(1+\omega_{eff})\rho_{eff} \tag{28}$$

The phantom energy effects in higher dimensions too. This effect consequences the value of the cosmological constant might be larger than the observational limit predicted by the LCDM cases.

The cosmological constant can be replaced with quintessence in LDGP cases [37].

For $\omega < -1$ the dark energy crosses the phantom divide line, it behaves like phantom energy. The modified scale factor of EiBI late time universe as suggested form [1]

$$a = \left(\frac{3}{4}\rho_c t^2 + 1\right)^{\frac{1}{3}} [1 + \tan^2(\left(\frac{3}{4}\rho_c t^2 + 1\right)^{\frac{1}{3}})\eta] \tag{29}$$

When scale factor reaches such magnitudes, instead of attaining the big rip the universe will face future bounce. The bounce will occur in higher dimensions. Modified Friedman equation is obtained for the EiBi model as

$$\begin{aligned} H^2 = \frac{1}{3k}\left[k\rho - 1 + \frac{1}{3\sqrt{3}}\sqrt{(k\rho+1)(3-k\rho)^3}\right] \\ \times \left[\frac{(k\rho+1)(3-k\rho)^3}{(3-k^2\rho^2)^2}\right] \end{aligned} \tag{30}$$

Comparing equation 6 and 30 leads to the Friedmann solution,

$$\begin{aligned} \frac{1}{3k}\left[k\rho - 1 + \frac{1}{3\sqrt{3}}\sqrt{(k\rho+1)(3-k\rho)^3}\right] \\ \times \left[\frac{(k\rho+1)(3-k\rho)^3}{(3-k^2\rho^2)^2}\right] \\ = \frac{8\,\pi G\rho\,(t)}{3} + \frac{\Lambda}{3} - \frac{2\,k}{a\,(t)^2} \end{aligned} \tag{31}$$

Hence, the scale factor becomes

$$\left(\frac{2k}{\frac{1}{3k}\left[k\rho - 1 + \frac{1}{3\sqrt{3}}\sqrt{(k\rho+1)(3-k\rho)^3}\right] \times \left[\frac{(k\rho+1)(3-k\rho)^3}{(3-k^2\rho^2)^2}\right] - \frac{8\,\pi G\rho(t)}{3} - \frac{\Lambda}{3}}\right)^{\frac{1}{2}} = a\,(t) \tag{32}$$

On higher dimensions, the Friedmann equation 6 behaves as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{V_{N-3}\ \rho\,(t)}{3} + \frac{\Lambda}{3m_{pl}^2} - \frac{2\,k}{a\,(t)^2} \tag{33}$$

When the scale factor approaches a_{mb} the critical density will be $\rho \to \rho_{crit}$. From the Loop Quantum Gravity, the Ricci scalar is found as

$$\begin{aligned} R = 6\Bigg(\frac{4\pi G}{3}(\rho + 3p) + \frac{k}{a^2} + \\ \frac{8\pi G}{3}(\frac{\rho}{\rho_{crit}} + k\chi)(\rho + 3p) \\ + \frac{k\chi}{\gamma^2\Delta} - \frac{2\xi_k}{\gamma^2\Delta}(\frac{\rho}{\rho_{crit}} + k\chi - \frac{1}{\rho})\Bigg) \end{aligned} \tag{34}$$

Equations 34 and 92 are compared to obtain curvature in higher dimensions. The obtained result is a combined solution of higher dimensional approach and loop quantum formalism.

$$R = -\frac{2\ a(t)^2}{k^2r^2-1}\Bigg(\frac{4\pi G}{3}(\rho+3p)+\frac{k}{a^2}+ \\ \frac{8\pi G}{3}(\frac{\rho}{\rho_{crit}}+k\chi)(\rho+3p) \\ +\frac{k\chi}{\gamma^2\Delta}-\frac{2\xi_k}{\gamma^2\Delta}(\frac{\rho}{\rho_{crit}}+k\chi-\frac{1}{\rho})\Bigg) \tag{35}$$

The scale factor which is derived by supersymmetric cosmology is

$$a(t) = e^{2t_0}+\left(3\left(\sqrt{\frac{\kappa^2\dot{\varphi_0^2}e^{4t_0}}{6}+\frac{\kappa^4}{32}}\right)(t-t_0)\right)^{\frac{1}{3}} \tag{36}$$

In equation 36 the scale factor is derived from supersymmetric cosmology. Equation 29 tells the scale factor from the Eddington inspired Born infield model. The scale factor relation tells the phantom dominated late time universe can be modified with the EiBI model. The equation 32 comes with scale factor, that consists of the solutions from the Friedmann equation of EiBI model. The equations 29, 32 and 36 provide a scale factor for various scenarios.During the very late time of evolution, these scale factors can be distinguished in their natural manner.

From higher dimensional solutions one can understand that the fundamental constraints can be written as following way.

$$8\pi G = \frac{V_{N-3}}{m_{pl}^2} \tag{37}$$

Here V_{N-3} is the volume of the extra-dimensional space. Hence, the action is modified for $N = 5$.

$$S = \frac{m_{pl}^2}{V_2}\int d^5x\sqrt{g}R_5 \tag{38}$$

In higher dimensional brane world cosmology, the action can be expressed as ([38], [39])

$$S = \epsilon M_{pl}^3\left[\int_{Bulk}(\mathcal{R}-2\Lambda_b)-2\int_{Brane}K\right]+\int_{Brane}(m_{pl}^2R-2\sigma)+\int_{Brane}L(h_{ab}+\phi) \tag{39}$$

Here $\mathcal{R}$ is scalar curvature of the metric g_{ab} in five dimensional bulk. R is scalar curvature. $h_{ab} = g_{ab} - n_a n_b$ in the Brane. n^a is a vector normal to the plane. $K = K_{ab}h^{ab}$ is the trace of metric tensor of extrinsic curvature $K_{ab} = h_a^c T_c n_b$ of the Brane. The symbol $L(h_{ab}, \phi)$ is Lagrangian density of four dimensional matter field ϕ. The integrals are taken over bulk and Brane with volume elements $\sqrt{-g}d^5x$ and $\sqrt{-h}d^4x$ respectively. g, h are the determinant of corresponding metrics. M_{pl} and m_{pl} are the five dimensional and four dimensional Planck mass. Λ_b and σ are bulk and Brane cosmological constants respectively.

As from [40] ,

$$L^2\rho_\Lambda \sim \frac{(N-1)A_{N-1}L^{N-2}m_{pl}^2}{2V_{N-3}} \tag{40}$$

By implementing the solution 40 the time-varying energy density can be rewritten as

$$\rho_\Lambda(t) = \frac{C^2\xi(t)(N-1)A_{N-1}L^{N-5}m_{pl}^2}{2V_{N-3}} \tag{41}$$

Here $\xi(t)$ is a time varying parameter as suggested for interaction solutions [1].

For $N = 4$ sphere , Kaluza Klein solution gives

$$\rho_\Lambda\xi(t) = 3C^2\pi^2L^2\xi(t) \tag{42}$$

Many theories confirm that the particles can be treated as black holes [41]. The radius of the apparent horizon is derived as

$$r_a = \frac{1}{H} = r_H = L \tag{43}$$

Mass of such particles produced in higher dimensions are calculated as

$$M = \frac{N-1\ A_{N-1}\ < r_H >^{N-2}\ m_{pl}^2}{2\hat{V}_{N-3}} \tag{44}$$

Here $< r_H >$ is the expectation value for the particle horizon. $\hat{V}$ is the volume operator given by loop quantum gravity [42].

There are eigenstates of volume operator $\hat{V}$ can be written as

$$V_j = (\gamma l_{pl}^2)^{\frac{3}{2}}\sqrt{\frac{1}{27}j(j+\frac{1}{2})(j+1)} \tag{45}$$

When a black hole has a Schwarzschild radius equal to Compton wavelength, it is refereed as Planck particles [43].

Mass of such particles are

$$m = \sqrt{\frac{h\ c}{2G}} \tag{46}$$

In higher dimensions using Kaluza Klein solution, the Compton wavelength can be written as

$$r = \left(\frac{1}{m_{pl}} \sqrt{\frac{V_{N-3}\hbar}{2c^3}} \right)^{N-2} \tag{47}$$

Similarly, as like particle production, the universe in late times can be exited into higher dimensions, in which it might be embedded as a baby universe. In recent days many theories confirm that the universe can be embedded on a supermassive black hole [44].

4.1 Comparative solutions

For any arbitrary time t the equation of state can be written as

$$\omega(t) = -\frac{(1-2q) + \frac{kc^2}{(H^2a^2)}}{3[1 + \frac{kc^2}{(H^2a^2)}]} \tag{48}$$

With $q(t) = -\frac{1}{a}\ddot{a}[\frac{1}{a}\dot{a}]^{-2}$ and $k = 0, \pm 1$ for flat, open and closed universe respectively. H^2 in Equation 48 can be modified with equation 30.

The results can be compared with quintessence model. Quintessence is a time-varying spatially inhomogeneous and negative pressure component of the cosmic fluid [45][46][47]. The energy density and pressure of quintessence are time-dependent components. The EoS for quintessence is $-1 < \omega < 0$, whereas for cosmological constant $\omega = -1$ and for phantom $\omega < -1$. In Quintessence plus Cold Dark Matter model (QCDM) the matter density is described as $\omega_m = 1 - \omega_q$. The energy density can be constructed as

$$\rho(a) = \omega_q \rho_{crit} \exp(3 \int_a^1 [1 + \omega(a)] d \ln a) \tag{49}$$

This relation can be applied to the modified Friedmann equation of LQC (equation 77).

$$H_0^2 = \frac{8\pi G(\rho(a))^2}{3} \left(1 - \frac{\rho(a)}{\rho_{crit}} \right) \tag{50}$$

This substitution results the modified FRW equaion in terms of LQC and quintessence cosmology. Quintessence and phantom fields can be combined as quintom. In this model, dominations by quintessence $\omega > -1$ in the earlier phases of the expansion and by phantom $\omega < -1$ [48]. Such phase transitions are explained as

$$H^2 = \frac{4\pi}{3}[\dot{\phi}^2 - \dot{\sigma}^2 + 2v(\phi, \sigma)2\rho_m] \tag{51}$$

The state finder parameter of dark energy can be modified with loop quantum cosmology. That opens a new window for LCDM [49]. One can relate both LCDM and quintessence models as braneworld models. the relativistic parameter is defined as

$$E(z) = \frac{H}{H_0} \tag{52}$$

Hence, the avoidance of big rip is possible as a consequence of H, $\dot{H} \to 0$ as $a \to \infty$ [50]. Consider, the late time universe is filled with generalized Chaplygin gas. The scalar curvature for $k = 0, \pm 1$ is,

$$R = 6(\dot{H} + 2H^2 + \frac{k}{a^2}) \tag{53}$$

This relation can be compared with 35. As suggested from the reference [51], the curvature can be obtained as,

$$R = 12D[A + \frac{B}{4a^3}(1+\alpha)(A + \frac{B}{a^3(1+\alpha)})]^{-\frac{\alpha}{1+\alpha}} \tag{54}$$

where $D = \frac{8\pi G}{3}$. For phantom field

$$a(t) = [a_0^{\frac{3\beta}{2}} + \frac{3\beta}{2}C^{\frac{1}{2}}(t - t_0)]^{\frac{2}{3\beta}} \tag{55}$$

Here, β is constant and $C = \frac{8\pi G}{\tilde{A}}$ with $\tilde{A}$ is an integration constant. For $\beta < 0$ the energy density grows instead of decreasing. Hence,

$$B = (\rho_0^{\alpha+1} - A)a_0^{3(\alpha+1)} \tag{56}$$

The Chaplygin gas can be considered as a perfect fluid [52]. The phantom energy can be generalized by Chaplygin gas ($\beta < 0$ and $\alpha < -1$).

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left[A + \frac{B}{a^{3(\alpha+1)}}\right]^{\frac{1}{1+\alpha}} \tag{57}$$

Hence, the Friedmann equation can be modified in terms of quintessence and phantom models.

5 Non-interaction solutions

On the late time universe, the dominance of phantom energy determines the evolution of the universe. Fro the universe to avoid the big rip and to reach the big bounce, non-interacting scenario between dark energy and dark matter must occur.

Effective equation of state is determined as

$$\omega_\Lambda = \omega_\Lambda^{eff} - \frac{\gamma}{4H\rho_\Lambda} \tag{58}$$

As the interaction is nullified, $\gamma = 0$ then,

$$\omega_\Lambda = \omega_\Lambda^{eff} \tag{59}$$

Where

$$\omega_\Lambda^{eff} = -\left(1 + \frac{\dot{H}}{2H^2}\right) \tag{60}$$

The effective equation of state can be modified by the solutions obtained from [53] as dust solutions.

$$\omega_\Lambda^{eff} = -\frac{\epsilon(2+p) + (p+1)^2 \times 3m_{pl}^2}{3(-\epsilon + p)} \tag{61}$$

As suggested from [10],

$$\omega_\Lambda = -\frac{1}{2} + \frac{\ddot{a}a}{2\dot{a}^2} - 3b\frac{\Omega_\Lambda - \Omega_m}{4\Omega_\Lambda} \tag{62}$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} \tag{63}$$

When $\rho_\Lambda \sim \rho_c$, $\Omega_\Lambda = 1$. Then $\Omega_M \to 0$. Hence,

$$\omega_\Lambda = -\frac{1}{2} + \frac{\ddot{a}a}{2\dot{a}^2} - \frac{3b}{4} \tag{64}$$

Setting $b = 0$,

$$\omega_\Lambda = -\frac{1}{2} + \frac{\ddot{a}a}{2\dot{a}^2} \tag{65}$$

Equation 65 conforms modified equation of state for non-interacting dark matter and dark energy. The higher curvature due to phantom density will be a local kind of quantized curvature.

As per the LQC theory, the scale factor and curvature can be quantized as quantum variables. Triads can be written as desensitized form. That the densitized triads conjugate extrinsic curvature coefficient.

$$k_i^a = K_{ab} e_i^b \tag{66}$$

$$\{k_a^i(x), E_j^b(y)\} = 8\pi G \delta_a^b \delta_j^i \delta(x, y) \tag{67}$$

The curvature is replaced with Ashteaker connections.

$$A_a^i = \Gamma_a^i + \gamma k_a^i \tag{68}$$

Ashteaker connections conjugate to triads will be

$$\{A_a^i(x), E_j^b(y)\} = 8\pi G \delta_a^b \delta_j^i \delta(x, y) \tag{69}$$

Hence, the spin connection will be

$$\Gamma_a^i = \epsilon^{ijk} e_j^b (\partial_{[ae_e^k]} + \frac{1}{2} e_k^c e_a^i \partial_{[ce_b^i]}) \tag{70}$$

The spatial geometry is obtained from densitized triads.

$$E_i^a E_i^b = q^{ab} det\ q \tag{71}$$

Equation 67 can be rewritten as

$$\{k_a^i(x), E_j^b(y)\} = \frac{\hat{V}}{m_{pl}^2} \delta_a^b \delta_j^i \delta(x, y) \tag{72}$$

Such quantization can be done in extra dimensions.

In terms of LQG, the action can be modified as the function of triads and connections.

$$S[e, \omega] = \frac{1}{2} \int e \wedge e \wedge F^* + \frac{1}{\gamma} e \wedge e \wedge F \tag{73}$$

F reprecents the curvature as

$$R^\sigma_{\mu\nu\rho} = e_\rho^j e^{\sigma k} F_{\mu\nu jk} \tag{74}$$

e_ρ^j , $e^{\sigma k}$ are triads. More generally

$$S = -\int e_{\sigma l} F^l_{\mu\nu} \tilde{\epsilon}^{\sigma\mu\nu} d^3x \tag{75}$$

More precisely the action can be written as

$$S = -\frac{1}{16} \frac{c_1^2}{c_2} \int F_{\mu\nu j} F^j_{\rho\sigma} g^{\rho\sigma} g^{\sigma\nu} \sqrt{g} d^3x \tag{76}$$

Equation 76 can be applied to any Brane-world solutions as suggested from 39. The action is modified with loop quantum gravitational solutions.

6 Discussion

In higher dimensions the LQC formalism can be applied, that is conformed by equation 72. Modified Friedmann equation which is derived from LQC as [54]

$$H^2 = \frac{\dot{V}^2}{9V^2} = \frac{8\pi G}{3}\rho(1 - \frac{\rho}{\rho_{crit}}) \tag{77}$$

Similarly modified Friedmann equation from LQC can be rewritten for higher dimensions as

$$H^2 = \frac{V_{N-3}}{3m_{pl}^2}\rho(1 - \frac{\rho}{\rho_{crit}}) \tag{78}$$

or

$$H^2 = \frac{V_{N-3}}{V_{pl}^2\rho_{pl}^2}\rho(1 - \frac{\rho}{\rho_{crit}}) \tag{79}$$

Modified Friedman equation powered by Loop Quantum Gravity in higher dimensions is defined in 72. The equation can be applied to calculate the evolution of the universe in higher dimensions.

Increasing energy density is the cause of the local domain to pass into higher dimensions. As suggested from previous work [1] higher dimensional exit might produce locally high curved domains due to the increment in phantom energy density. Such localized curvatures can be discussed as Kaluza Klein higher dimensional solutions. Also due to the negative pressure in such scales, a consequence of the local high curvature produces high energy particles in higher dimensions.

Bounce is higher dimensions is predicted by the equation 92. When the scale factor approaches very small, then the curvature will be maximum. This resolution is also predicted by the loop quantum cosmology. As suggested by the loop quantum gravity, space is discreet in nature at quantum levels. As the critical density approaches, the universe bounces back, but in higher dimensions for the case of phantom dominated final stages.

The Friedmann equation can be modified for Chaplygin gas dominated solution, that is explained in equation 57. It is observed that the Hubble parameter will be modified by the presence of energy density and pressure. It is also understood that the loop quantum action that consequences the evolution of the quantum geometry, also plays the key role in the quantum mechanical evolution of the universe. The effective equation of the state is obtained as a form of dust solutions. The equation contains the Planck mass term.

Scale factor solutions for the late time and cyclic cosmological can be implemented to its evolution. [55] Thus the oscillating scale factor solutions confirm the evolution will be in a conformal manner. At the late time during the evolution, the scale factor and the equation of state of the ghost dark energy can be compared [56]. Similarly, the scale factor solutions discussed in this work also rely on the same kind of analysis. Likewise, the avoidance of big rip singularity is also made possible from our solutions.

At the very late time, instead of facing the big rip, the universe will continue its evolution to next Aeon. The scenario confirms the credibility of Conformal Cyclic Cosmology and Loop Quantum Cosmology. As like the big bang is replaced by the big bounce in Loop Quantum Cosmology and big bang singularity is replaced by the smooth topological surface in Conformal Cyclic Cosmology, the big rip singularity also can be dealt with the same kind of approaches. If the Conformal Cosmological model is applied to analyze the big rip singularity, it can be understood that the evolution of the universe will not be perturbed by the big rip and the universe will evolve conformally. Similarly, it can be found that the curvature increment due to the increment of the energy density will consequence the universe to exit into higher dimensions. While the Kaluza Klein action is compared with the action obtained by triad formalism, one can observe that the quantum gemoerical consequences would play in the determination of the final stages of the universe. The curvature components are discussed in the equations 19-40. Similarly, the Friedman equation can be modified to obtain the behaviour of the late time universe. The modified Friedman solutions are reported in equation 31. Evolution of the universe that is dominated by the phantom energy can be derived from this solution.

The no interaction between phantom energy and dark matter will avoid future big rip. Hence, the non-interaction is also confirmed using non-interacting solutions. The effective equation of state obtained by the equation 65 fits exclusively with the scale factor and equation of statecomparision plot which is defined from [57]. The plot is given in figure 1

7 Conclusion

The local higher curvature and higher dimensions in late time universe consequence the evolution of the baby universe, that emerge out of bouncing initial conditions. The conformal cyclic cosmology [6] predicts the existence of such possibilities. As a result, the universe evolves conformally without facing an end at the big rip. Hence, an immediate conclusion can be convinced as our universe might have initiated from such kind of past lower-dimensional

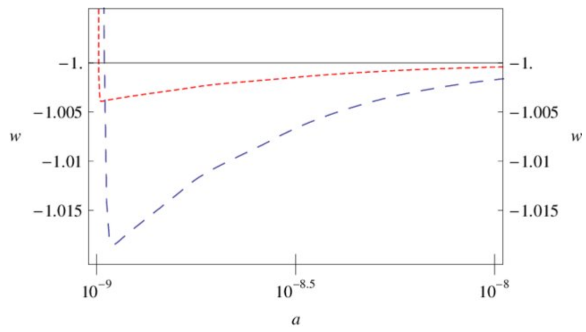

Figure 1: The scale factor is compared and plotted with equation of state. The plot is obtained from [57]

initial conditions. This conclusion indicates that our universe might be the conformal evolution of past big rip universe. Also if such universe at the late time was populated by supermassive black holes, another conclusion can arrive that the universe could be embedded in the higher dimensional black hole [58] [59] [60]. Here the predicted solutions support for the conformal evolution of the universe, without facing any kind of singularities.

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Appendices

Non-redundant components of Riemann tensor are reduced as,

$$\mathrm{R}^{\phi}{}_{\psi\phi\psi} = -\frac{k^2r^2-k}{a\,(t)^2} \tag{80}$$

$$\mathrm{R}^{\psi}{}_{tr\psi} = -\frac{kr\dot{a}}{(kr^2-1)a\,(t)} \tag{81}$$

$$\mathrm{R}^{\psi}{}_{rt\psi} = -\frac{kr\dot{a}}{(kr^2-1)a\,(t)} \tag{82}$$

$$\mathrm{R}^{\psi}{}_{rr\psi} = \frac{k}{kr^2-1} \tag{83}$$

$$\mathrm{R}^{\psi}{}_{\theta\theta\psi} = -kr^2 \tag{84}$$

$$\mathrm{R}^{\psi}{}_{\phi\phi\psi} = -kr^2\sin(\theta)^2 \tag{85}$$

Curvature in extra dimension can be obtained from equations 80 to 85. From equation 83 it is observed that the curvature in higher dimension is a pure effect of curvature constant k. Equation 82 predicts the curvature tensor in extra dimension.

Consistency of the result is observed for the equations 80 to 85.

Ricci tensor within the manifold can be identified as.

$$\mathrm{R}_{tt} = -\frac{3\,\frac{\partial^2 a}{\partial t^2}}{a\,(t)} \tag{86}$$

$$\mathrm{R}_{tr} = \frac{kr\dot{a}}{(kr^2-1)a\,(t)} \tag{87}$$

$$\mathrm{R}_{rt} = \frac{kr\dot{a}}{(kr^2-1)a\,(t)} \tag{88}$$

$$\mathrm{R}_{rr} = -\frac{2\,(\dot{a})^2 + a\,(t)\,\frac{\partial^2 a}{\partial t^2} + 3\,k}{kr^2-1} \tag{89}$$

$$\mathrm{R}_{\theta\theta} = 2\,r^2\,(\dot{a})^2 + r^2 a\,(t)\,\frac{\partial^2 a}{\partial t^2} + 3\,kr^2 \tag{90}$$

$$\mathrm{R}_{\phi\phi} = \left(2\,r^2\,(\dot{a})^2 + r^2 a\,(t)\,\frac{\partial^2 a}{\partial t^2} + 3\,kr^2\right)\sin(\theta)^2 \tag{91}$$

$$\mathrm{R}_{\psi\psi} = -\frac{3\,(k^2r^2-k)}{a\,(t)^2} \tag{92}$$

From equation 92 it is observed that, when the local curvature tends to maximum, then the localized scale factor approaches to a minimum.

The connection is evaluated as

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}) \tag{93}$$

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \tag{94}$$

$$R_{\sigma\nu} = R^{\rho}{}_{\sigma\rho\nu} \tag{95}$$

$$R = R_{\sigma\nu} g^{\sigma\nu} \tag{96}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{97}$$

The Ricci component of the metric is computed. It is a field of symmetric bilinear forms Ric(g) on the 5-dimensional Lorentzian manifold M.

$$\begin{aligned}
\mathrm{Ric}\,(g) = & -\frac{3\,\frac{\partial^2 a}{\partial t^2}}{a\,(t)} \mathrm{d}t \otimes \mathrm{d}t + \frac{kr\dot{a}}{(kr^2-1)a\,(t)} \mathrm{d}t \otimes \mathrm{d}r + \frac{kr\dot{a}}{(kr^2-1)a\,(t)} \mathrm{d}r \otimes \mathrm{d}t \\
& + \left(-\frac{2\,(\dot{a})^2 + a\,(t)\,\frac{\partial^2 a}{\partial t^2} + 3\,k}{kr^2-1} \right) \mathrm{d}r \otimes \mathrm{d}r \\
& + \left(2\,r^2\,(\dot{a})^2 + r^2 a\,(t)\,\frac{\partial^2 a}{\partial t^2} + 3\,kr^2 \right) \mathrm{d}\theta \otimes \mathrm{d}\theta \\
& + \left(2\,r^2\,(\dot{a})^2 + r^2 a\,(t)\,\frac{\partial^2 a}{\partial t^2} + 3\,kr^2 \right) \sin{(\theta)}^2\, \mathrm{d}\phi \otimes \mathrm{d}\phi \\
& - \frac{3\,(k^2r^2 - k)}{a\,(t)^2} \mathrm{d}\psi \otimes \mathrm{d}\psi
\end{aligned} \tag{98}$$

A Modified result for Klein Kaluza reduction

In this section, the Kaluza-Klein reduction of the R_{m4n4} component of the Riemann tensor is computed. Such components pass the Riemann tensor to metrics. The metric is written in terms of the Kaluza-Klein gauge field A_μ and the scalar ϕ. Substitutions are done for $g_{44} \to \phi$

$g_{m4} \to \phi A_m$,

$g_{4m} \to \phi A_m$

$g_{mn} \to \phi**{-1} h_{mn} + \phi A_m A_n$,

$g^{44} \to \phi**{-1} + \phi A_m h^{mn} A_n$

$g^{m4} \to -\phi h^{mn} A_n$,

$g^{4m} \to -\phi h^{mn} A_n$
, $g^{mn} \to \phi h^{mn}$.

The term $1 - kr^2$ is introduced in g_{44} and g^{44} components. The results are obtained below. Setting symmetries for Riemann tensor and connections. That leads to

$$R^{\lambda}{}_{\mu\nu\kappa} \to -\partial_{\kappa}\Gamma^{\lambda}{}_{\mu\nu} + \partial_{\nu}\Gamma^{\lambda}{}_{\mu\kappa} - \Gamma^{\eta}{}_{\mu\nu}\Gamma^{\lambda}{}_{\kappa\eta} + \Gamma^{\eta}{}_{\mu\kappa}\Gamma^{\lambda}{}_{\nu\eta} \tag{99}$$

$$\Gamma^{\lambda}{}_{\mu\nu} \to \frac{1}{2} g^{\lambda\kappa}(\partial_{\nu} g_{\kappa\mu} + \partial_{\mu} g_{\kappa\nu} - \partial_{\kappa} g_{\mu\nu}) \tag{100}$$

Here the Kaluza-Klein reduction of the R_{m4n4} component of the Riemann tensor is calculated as follows. R_{m4n4} component provides the Riemann tensor to metrics. Products over sums are also distributed in this procedure. To distribute derivatives over factors in a product, the product rule is applied. [61]. Setting

$$X = g_{m1m} R^{m1}{}_{4n4} + g_{4m} R^{4}{}_{4n4} \tag{101}$$

By substituting equations 99 , 100 in 101 and distribution is made. Using *split_index* module from Cadabra [61], fix $\mu, m1, 4$, and substituting $\partial_4 A \to 0$, $\partial_{4m} A \to 0$, $\partial_{m4} A \to 0$ .Writing the metric in terms of the Kaluza-Klein gauge field A_μ, the scalar ϕ and $1 - k * r^2$. Substitutions are done as

$$g_{44} \to 1 - k * r * r \tag{102}$$
$$g_{m4} \to \phi A_m \tag{103}$$
$$g_{4m} \to \phi A_m \tag{104}$$
$$g_{mn} \to \phi * * {-1} h_{mn} + \phi A_m A_n \tag{105}$$
$$g^{44} \to 1 - k * r * r * * {-1} + 1 - k * r * r A_m h^{mn} A_n \tag{106}$$
$$g^{m4} \to -\phi h^{mn} A_n \tag{107}$$
$$g^{4m} \to -\phi h^{mn} A_n \tag{108}$$
$$g^{mn} \to \phi h^{mn} \tag{109}$$

Derivatives over sums are distributed and the product rule is applied. The derivatives of the inverse metric are converted to derivatives of the metric. The substitutions are made as

$$\partial_p h^{nm} h_{qm} \to -\partial_p h_{qm} h^{nm} \tag{110}$$
$$h_{m1m2} h^{m3m2} \to \delta^{m3}_{m1} \tag{111}$$

Using *canonicalise* module of cadabra [62], the solution is rewritten.

Further simplification it in terms of the field strength can be written by substituting $\partial_n A_m \to 1/2 * \partial_n A_m + 1/2 * F_{nm} + 1/2 * \partial_m A_n$.