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## Conformal Invariance of Nonunitary 2d-Models.

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**Abstract.** – Finite-size effects for the free energy as well as the spectrum of the transfer matrix are modified in a specific way for nonunitary conformal invariant 2d-systems. We illustrate this behaviour in the case of the Lee-Yang edge singularity.

The central charge  $c$  of the Virasoro algebra of conformal generators is a major characteristic of two-dimensional critical models in statistical mechanics and field theory as was discussed by BELAVIN, POLYAKOV and ZAMOŁODCHIKOV [1]. For the so-called minimal degenerate models, *i.e.* those including a finite number of fundamental covariant conformal operators, these authors have obtained the values

$$c = 1 - \frac{6(p-p')^2}{pp'}, \quad (1)$$

where  $p$  and  $p'$  are a pair of coprime integers and for definiteness  $p > p'$ . Moreover, FRIEDAN, QIU and SHENKER [2] showed that unitary representations occur only for

$$p = m + 1, \quad p' = m, \quad m \text{ integer } \geq 3. \quad (2)$$

The physical interpretation of  $c$  emerges in restricted geometries. An argument given and tested by BLÖTE, CARDY and NIGHTINGALE, and AFFLECK [3] associates  $c$  to a kind of «Casimir effect», *i.e.* a width-dependent shift in the free energy per unit length of a strip of width  $L$  and length  $T \rightarrow \infty$ . For periodic boundary conditions across the strip they obtain, with  $Z(T, L)$  the partition function

$$\lim_{T \rightarrow \infty} \frac{\ln Z(T, L)}{T} = f_0 L + c \frac{\pi}{6} \frac{1}{L}, \quad (3)$$

where  $f_0$  is a constant, the free energy per unit area at criticality in the infinite plane. In the above relation  $c$  is dimensionless and characteristic of the model. For instance  $c = \frac{1}{2}$  for the Ising model ( $m = 3$ ),  $c = \frac{4}{5}$  for the 3-state Potts model ( $m = 5$ ).

In this letter, we would like to discuss the case of nonunitary models and to show how eq. (3) is affected. The conformal dimensions of the primary operators in the degenerate representations are given by Kac formula [4]:

$$h_{rs} = \frac{1}{4pp'} [(rp - sp')^2 - (p - p')^2] \quad (4)$$

with  $r, s$  positive integers. BPZ show that one may restrict oneself to a finite set of such operators labelled by

$$1 \leq r \leq p' - 1, \quad 1 \leq s \leq p - 1. \quad (5)$$

For  $p, p'$  coprimes,  $p - p' > 1$ , there always exist two integers  $r_0, s_0$  satisfying (5) such that  $r_0 p - s_0 p' = 1$ ,  $(r_0, s_0) \neq (1, 1)$ . The corresponding operator has a *negative* dimension, hence a correlation function increasing with distance, and this mere fact makes the calculation of the free energy per unit length in (3) slightly questionable. As an alternative, we consider the conformal theory on a torus, *i.e.* on a parallelogram with periodic boundary conditions, and take the limit of an infinitely long strip. Let the sides of the parallelogram be described by two complex numbers,  $\omega_1, \omega_2$ ,  $\text{Im } \omega_2/\omega_1 > 0$ , and let  $q = \exp [2\pi i \omega_2/\omega_1]$ . In a rectangular geometry ( $\omega_2/\omega_1 = i(T/L)$ ), the partition function is  $Z = \text{Tr } \exp [i\omega_2 H]$ , where the generator  $H$  of translations along the  $\omega_2$  direction may be obtained from the dilatation operator  $L_0 + \bar{L}_0$  in the infinite plane by a conformal mapping

$$H = \frac{2\pi}{\omega_1} \left( L_0 + \bar{L}_0 - \frac{c}{12} \right). \quad (6a)$$

In a normalization corresponding to  $f_0 = 0$

$$Z = \text{Tr } q^{(L_0 + \bar{L}_0 - c/12)}. \quad (6b)$$

In an arbitrary parallelogram,

$$Z = \text{Tr} (q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}) \quad (7)$$

may be expanded in characters of the tensor product of the two Virasoro algebras [5]

$$Z = (q\bar{q})^{-c/24} \sum N_{[r,s],[r',s']} \chi_{[r,s]}(q) \overline{\chi_{[r',s']}(q)}. \quad (8)$$

Here  $\chi_{[r,s]}(q)$  stands for the character pertaining to the highest-weight irreducible representation of conformal dimension  $h_{rs}$ , and the positive integers  $N_{[rs],[r',s']}$  count the number of times each representation  $(r, s) \otimes (r', s')$  occurs. ROCHA-CARIDI [6] has given explicit expressions for the  $\chi$ 's:

$$\chi_{[r,s]}(q) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)} \sum_{n=-\infty}^{\infty} \{ q^{(2npp' + rp - sp')^2 - (p-p')^2/4pp' - (s \rightarrow -s)} \}. \quad (9)$$

In the limit of an infinitely long strip:  $\omega_2/\omega_1 = i(T/L) \rightarrow i\infty$ ,  $q \rightarrow 0$  and

$$\chi_{[r,s]}(q) \sim q^{h_{rs}}. \quad (10)$$

Modular invariance of the partition function, *i.e.* its independence with respect to the choice of periods  $\omega_1, \omega_2$  chosen to describe the torus, strongly restricts the possible values of  $N_{[rs], [r',s']}$ . It may be shown [5, 7] that

$$Z = (q\bar{q})^{-c/24} \sum_{(r,s)} |\chi_{[r,s]}(q)|^2, \quad (11)$$

where the sum runs over all  $(r, s)$  satisfying (5) is always a modular invariant. Were the identity operator ( $h_{11} = 0$ ) to give the leading behaviour as  $q \rightarrow 0$ ,—as in unitary cases—, one would get  $Z \sim (q\bar{q})^{-c/24}$  in agreement with (3). However, the operator of negative dimension

$$h_{r_0 s_0} = -\frac{(p-p')^2 - 1}{4pp'} \quad (12)$$

dominates. This results in a modified behaviour of  $Z$ :

$$Z \underset{q \sim 0}{\sim} (q\bar{q})^{-c'/24}, \quad c' = c - 24h_{r_0 s_0}, \quad c' = 1 - 6/pp' \quad (13)$$

and in (3),  $c'$  must be substituted for  $c$ .

Let us illustrate this in the case of the Lee-Yang edge singularity [8]. As discussed by CARDY [9], this singularity corresponds to the minimal conformal theory with  $p = 5$ ,  $p' = 2$ ,  $c = -22/5$ . There are only two primary fields, the identity  $I$  with  $h_{11} = 0$  and the scalar field  $\varphi$  with  $h_{12} = -1/5$ . The physical dimension [1] of  $\varphi$  is  $d_\varphi = \gamma/2 = 2h_{12} = -2/5$ . The only modular invariant partition function built out of these two fields is

$$\begin{aligned} Z &= (q\bar{q})^{11/60} \{ |\chi_{[1,1]}(q)|^2 + |\chi_{[1,2]}(q)|^2 \} = \\ &= (q\bar{q})^{-1/60} \{ |1 + q + q^2 + q^3 + 2q^4 + \dots|^2 + (q\bar{q})^{1/5} |1 + q^2 + q^3 + q^4 + \dots|^2 \}. \end{aligned} \quad (14)$$

We have studied numerically this singularity for the Ising model with Hamiltonian

$$\beta\mathcal{H} = -\beta \sum_{(i,j)} S_i S_j - H \sum_i S_i. \quad (15)$$

At any temperature above the critical temperature, *i.e.* for  $\beta < \beta_c$ , the model (15) is critical for a purely imaginary magnetic field  $H_c(\beta) = \pm i\hat{H}_c(\beta)$ . As  $H \rightarrow H_c$ , the correlation length diverges as  $|H - H_c|^{-\nu}$  with  $\nu = 1/(2 - d_\varphi)$  [8], while for  $H = H_c$ , the spin-spin correlation increases like (distance) $^{-2d_\varphi}$ .

We consider a typical high-temperature point  $\beta = \beta_c/3$  in a strip of width  $L$  with a standard transfer matrix technique. One can study the critical properties using finite-size scaling methods [10]. The free energy per site  $f_L$  is given by  $f_L = L^{-1} \log \Lambda_1$  and the first gap by  $m_L = \log \Lambda_1/\Lambda_2$ , where  $\Lambda_1$  and  $\Lambda_2$  are the two largest eigenvalues of the transfer matrix. At a fixed value of  $\beta$ , one gets estimates of the critical field by solving the phenomenological renormalization equations [10]

$$Lm_L [i\hat{H}_c^{(L)}] = (L-1)m_{L-1} [i\hat{H}_c^{(L)}], \quad (16)$$

while the derivatives of the gap give the exponent  $\nu$  by

$$1/\nu^{(L)} = \frac{\log \left( \frac{dm_L}{d\hat{H}} (\hat{H}_c^{(L)}) / \frac{dm_{L-1}}{d\hat{H}} (\hat{H}_c^{(L)}) \right)}{\log L/(L-1)} + 1. \quad (17)$$

The values of  $\hat{H}_c^{(L)}$  and  $\nu^{(L)}$  obtained in this way are given in the first two columns of table I. The  $\nu^{(L)}$  converge rapidly to the exact value  $\nu = 5/12$ . The phenomenological renormalization for the Lee-Yang edge singularity has also been applied in the «Hamiltonian» version of (15) with very similar results [11]. For two consecutive widths, we can then form the quantities

$$c^{(L)} = \frac{6}{\pi} \frac{f_L[\hat{H}_c^{(L)}] - f_{L-1}[\hat{H}_c^{(L-1)}]}{L^{-2} - (L-1)^{-2}}. \quad (18)$$

TABLE I. — In this table we give the results of our numerical study for the Lee-Yang edge singularity at  $\beta = \beta_c/3$ . The expected values are explained in the text.

$L$	$\hat{H}_c^{(L)}$	$\nu^{(L)}$	$c^{(L)}$	$Lm_L/2\pi$
2	0.4560	0.3254	0.5959	0.2580
3	0.4116	0.3605	0.5787	0.3300
4	0.4041	0.3821	0.5178	0.3594
5	0.4022	0.3940	0.4769	0.3735
6	0.4016	0.4007	0.4529	0.3813
7	0.4014	0.4049	0.4384	0.3861
8	0.4013	0.4076	0.4293	0.3894
9	0.4012	0.4094	0.4231	0.3916
10	0.4012	0.4109	0.4192	0.3933
Expected values		$\frac{5}{12} = 0.4167$	$\frac{2}{5} = 0.40$	$\frac{2}{5} = 0.40$

These  $c^{(L)}$  are given in the third column of table I. One observes a rapid convergence to  $c' = 2/5$  (see (13)) instead of  $c = -22/5$  as would be expected from formula (3).

The complete spectrum of the transfer matrix is also of interest. By the arguments of Cardy [5, 12] one would expect the first gap to be given by  $m_L = 2\pi d_\varphi/L$ . This is not possible here since  $d_\varphi$  is negative, while  $m_L$  is—by definition—positive. The correct form of  $m_L$  derives from (14). Since the dominant behaviour of the free energy is given by factorizing  $(q\bar{q})^{h_{\eta_0} - c/24}$  the first gap is in fact given by  $m_L = -2\pi d_\varphi/L$ . This can be checked in the fourth column of table I. Thus conformal invariance allows the determination of the exponent  $\eta$  even when  $\eta$  is negative, which corresponds to correlation functions increasing with the distance. Such a behaviour occurs also in models which look more physical like for example the collapsed phase of  $2d$  polymers [13].

The complete spectrum of the transfer matrix is very similar to what would be the spectrum of a unitary model with only two primary fields, the identity and a scalar field of physical dimension  $d = -d_\varphi$ . However, there are slight differences which allow one to distinguish between the two cases. Let us consider for example the mass spectrum at zero momentum, which is related to the dimensions of the scalar operators. In units of  $2\pi/L$ , this spectrum for the unitary theory would be given (for  $L \rightarrow \infty$ ) by the following series:

$$d, d+2, 4, d+4, 6, d+6, 8, d+8, \dots \quad (19)$$

The series  $d + 2n$  corresponds to the block of the scalar field and the series  $2n'$  ( $n' \geq 2$ ) to the block of the identity. The dimension 2, which would correspond to the operator  $L_{-1}\bar{L}_{-1}I$  is not present, since  $L_{-1}I = \bar{L}_{-1}I = 0$ .

TABLE II. - Conformal dimensions obtained with the Kac formula in the case  $c = -3/5$ :  $h_{r,s} = ((5r - 3s)^2 - 4)/60$ . The  $h$ 's in the solid box form the finite set of formula (5). The infinite set  $h_{p,1}$  indicated by the dotted lines is the thermal series of the Potts model with  $Q = 4 \cos^2(2\pi/5)$ .

r	s					
	1	2	3	4	5	6
1	0	- 1/20	1/5	3/4	8/5	11/4
2	3/4	1/5	- 1/20	0	7/20	1
3	7/3	77/60	8/15	1/12	- 1/15	1/12
4	19/4	16/5	39/20	1	7/20	$h_{r+3,s+5} = h_{r,s}$
5	8	119/20	21/5	11/4	8/5	
	⋮					

Because of the factorization of the term  $(q\bar{q})^{h_{r,s}}$ , the zero momentum spectrum for the Lee-Yang edge singularity is given, according to (14), by

$$d, 2, 4, d + 4, 6, d + 6, 8, d + 8, \dots \tag{20}$$

and this can be easily checked numerically. Contrary to (19), the term  $d + 2$  is now absent, while 2 is present. This shows that in the absence of any information on the nature of the theory, the numerical study of the transfer matrix spectrum allows one to detect the presence of a negative dimension. One can then obtain the correct value of the central charge using (13). (This argument can be easily generalized to problems with several negative dimensions.)

We remark, however, that one can build critical models which correspond also to nonunitary theories but show a very different behaviour. Consider for instance the  $Q$ -state Potts model for continuous  $Q$ , which can be defined using the high-temperature expansion of the partition function [14]. Parametrizing  $Q$  by  $Q = 4 \cos^2(\pi/m + 1)$  ( $m \geq 1$ ), it has been conjectured by DOTSENKO and FATEEV [15] that the corresponding central charge is  $C = 1 - 6/m(m + 1)$ . The value  $C = -22/5$  does not correspond to a Potts model, so we have studied another nonunitary case with  $p = 5$ ,  $p' = 3$  which gives  $Q = 4 \cos^2(2\pi/5)$  and  $C = -3/5$ . Using the transfer matrix of Blöte *et al.* [16], we have calculated free energies at criticality for this special value of  $Q$ . In contrast to what happened in the Lee-Yang case, these free energies have a behaviour given by formula (3), which means that there is no negative dimension in this case. However, the complete spectrum shows a much richer structure than previously.

One observes for example a large number of scalar thermal operators whose dimensions appear in the first column of the conformal grid (see table II). Although this is difficult to check numerically, it seems probable that this thermal series is in fact infinite and contains all the operators  $\varphi_{p,1}$  ( $p \geq 1$ ). One also observes several scalar magnetic operators, whose dimensions do not appear in table II, but are given by the Kac formula with rational indices,  $\varphi_{p+1/4,5/4}$  ( $p \geq 1$ ). These results agree with conjectures of Dotsenko and Fateev [15]<sup>(1)</sup>. Such rational indices have also been observed in a different context [17].

The problem of constructing an invariant partition function analogous to (14) remains open.

<sup>(1)</sup> It must be remarked here that these authors use  $\Delta_{rs}$  for what is denoted here by  $h_{rs}$ .

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