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Conformal Invariance, the Central Charge, and Universal Finite-Size Amplitudes at Criticality

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We show that for conformally invariant two-dimensional systems, the amplitude of the finite-size corrections to the free energy of an infinitely long strip of width L at criticality is linearly related to the conformal anomaly number c , for various boundary conditions. The result is confirmed by renormalization-group arguments and numerical calculations. It is also related to the magnitude of the Casimir effect in an interacting one-dimensional field theory, and to the low-temperature specific heat in quantum chains.

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The principle of conformal invariance at a critical point has been shown to be remarkably powerful, especially in two dimensions.^{1,2} Universality classes appear to be characterized by a single dimensionless number c , the conformal anomaly or the value of the central charge of the Virasoro algebra.³ It was shown by Friedan, Qiu, and Shenker² that unitarity constrains those values of c less than unity to be quantized. For such theories, the critical exponents are given by the Kac formula,⁴ and the correlation functions are determined.^{1,5} For various models, c has been determined indirectly by use of exact information on exponents and correlation functions obtained by other means.^{1,2,5} In this Letter we give a simple means of determining c .

The free energy (measured in units of $k_B T$) per unit length of an infinitely long strip of width L at criticality has the finite-size scaling form $F = fL + f^x + \Delta/L + \dots$, where f is the bulk free energy per unit area, and $\frac{1}{2}f^x$ is the surface free energy, which vanishes in the case of periodic boundary conditions. It has been argued, from the assumption that L^{-1} is a scaling field which does not require the introduction of a metric factor,^{6,7} that Δ is universal. We find that

$$\Delta = \begin{cases} -\pi c/6, & \text{periodic boundary conditions,} \\ -\pi c/24, & \end{cases} \quad (1)$$

$$\text{free or fixed boundary conditions,} \quad (2)$$

where, in the last case, the order parameter is fixed to the *same* value on either side of the strip.

These results have several other interesting physical interpretations. Since F corresponds to the ground-state energy of a (1+1)-dimensional quantum field

theory in a finite volume, Eq. (2) also gives the magnitude of the Casimir effect⁸ in such a theory. The partition function of a classical system of finite width with periodic boundary conditions may also be interpreted as the Feynman path integral for an infinitely long quantum chain at finite temperature $T \propto L^{-1}$. In that case Eq. (1) gives the leading $T \rightarrow 0$ correction to the free energy, from which may be deduced the specific heat C . In fact the conformal result applies only if the two-dimensional classical system is rotationally invariant at large distances. This is equivalent to the requirement that the dispersion relation for gapless excitations of the quantum chain is of the form $\omega \sim vk$ with $v=1$. The case $v \neq 1$ can be accommodated by a suitable rescaling of time versus length for the quantum chain. The result is $C \sim \pi ck_B^2 T / 3\hbar v$. This is confirmed by exact results for the spin- $\frac{1}{2}$ XXZ chain⁹ ($c=1$) and for the anisotropic spin- $\frac{1}{2}$ XY model in a critical transverse field¹⁰ ($c=\frac{1}{2}$). In three dimensions, the analog of Δ is the interaction energy (in units of $k_B T$) per unit area of two plates immersed in a critical system.¹¹ Universality in this case was verified by Monte Carlo techniques.¹² The same constant also plays a role in determining the thickness of gravity-thinned, critical wetting layers.¹³ Two-dimensional analogs of these systems, which would allow an experimental determination of c , are conceivable.

A system at a critical point is governed by a reduced fixed-point Hamiltonian¹⁴ \mathcal{H}^* . Under a coarse graining in which lengths are rescaled uniformly, the form of the Hamiltonian is invariant. For short-range interactions, the Hamiltonian remains at the fixed point also under conformal transformations, which corre-

spond to a *nonuniform* rescaling and rotation. Transformations with a shear component, however, modify the Hamiltonian. The response of \mathcal{H} to such an infinitesimal transformation of the form $x^\mu \rightarrow x^\mu + \alpha^\mu$ is

$$\delta \mathcal{H} = -\frac{1}{2\pi} \int \frac{\partial \alpha^\mu}{\partial x_\nu} T_{\mu\nu} d^2x. \quad (3)$$

This defines¹⁵ the stress tensor $T_{\mu\nu}$. In complex coordinates (z, \bar{z}) the only nonzero components are $T_{zz} = T(z)$ and $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$. The conformal anomaly number c may then be defined by^{1,2}

$$\langle T(z)T(z') \rangle_c = (c/2)(z-z')^{-4}. \quad (4)$$

Even if T is subtracted so that $\langle T \rangle = 0$ in the infinite plane, it is nonzero in the strip. As we now show, its value is related to Δ . Consider the nonconformal transformation $u' = u(1-\lambda)$, $v' = v(1+\lambda)$, where $\lambda \ll 1$, and (u, v) measure distances along and across the strip, respectively. According to Eq. (3)

$$\begin{aligned} \delta \langle \mathcal{H} \rangle &= -(\lambda/2\pi) \int_{-\infty}^{\infty} du \int_0^L dv \langle -T_{uu} + T_{vv} \rangle \\ &= (\lambda L/\pi) \int_{-\infty}^{\infty} (\langle T \rangle + \langle \bar{T} \rangle) du \end{aligned} \quad (5)$$

for the translationally invariant case of periodic boundary conditions. Invariance of the partition function implies that this is compensated by a change in F , which is $-2\lambda\Delta/L$. Hence we find $\Delta = (L^2/\pi)\langle T \rangle$, since $\langle T \rangle = \langle \bar{T} \rangle$ by symmetry. Now the response of $\langle T \rangle$ to $\delta \mathcal{H}$ is

$$\begin{aligned} \delta \langle T(0,0) \rangle &= -(\lambda/\pi) \int_{-\infty}^{\infty} du \int_0^L dv \langle T(0,0)T(u,v) \rangle_c, \end{aligned} \quad (6)$$

using $\langle T\bar{T} \rangle_c = 0$. The connected correlation function $\langle TT \rangle_c$ in the strip may be found¹⁶ by conformal transformation of the infinite-plane result Eq. (4) using the transformation $w = u + iv = (L/2\pi)\ln z$. The result is

$$\langle T(0)T(w) \rangle_c = (c/2)(\pi/L)^4 [\sinh(\pi w/L)]^{-4}.$$

The integral is divergent as $w \rightarrow 0$, but the final result is independent of the particular method of regularization. The integrals over u and v in Eq. (6) are then elementary, and one obtains $\delta \langle T \rangle = \lambda \pi^2 c / 3L^2$. This is to be compared with $\delta \langle T \rangle = \delta(\pi \Delta / L^2) = -2\pi \Delta \lambda / L^2$, and the result in Eq. (1) follows.

In the case of free or fixed boundary conditions, the correlation function in the strip is found¹⁶ with use of the transformation $w = (L/\pi)\ln z$ from the upper half plane. In the latter geometry, the $\langle TT \rangle$ correlation function is as in Eq. (4), while¹⁷ $\langle T(z_1)\bar{T}(\bar{z}_2) \rangle = (c/2)(z_1 - \bar{z}_2)^{-4}$. However, this term does not contribute to Δ . Thus the only difference between the two cases is that L is replaced by $2L$. This accounts for the factor of 4 difference between Eqs. (1) and (2).

The results in Eqs. (1) and (2) agree with exact results for the Gaussian model¹⁸ ($c=1$) and the Ising model¹⁹ ($c=1/2$). Equation (1) has also been verified²⁰ for all the theories in the unitary classification of Friedan, Qiu, and Shenker.² In fact, it is possible to calculate the free energy in an arbitrarily shaped parallelogram with periodic boundary conditions,²⁰ of which the infinitely long strip is only a special case.

The result in Eq. (1) can be verified in a modified Gaussian model with reduced Hamiltonian

$$\mathcal{H} = \frac{1}{2} K \sum_{k=1}^m \sum_{l=1}^L [(\phi_{k,l} - \phi_{k+1,l})^2 + (\phi_{k,l} - \phi_{k,l+1})^2] + i\alpha \sum_{k=1}^m (\phi_{k,l} - \phi_{k+1,l}), \quad (7)$$

where the $\phi_{k,l} \in R$ are located at the sites of a simple square lattice on a cylinder, i.e., $\phi_{k,1} = \phi_{k,L+1}$ for $k=1, \dots, m$, subject to the constraints $\phi_{1,1} = \phi_{1,s}$ and $\phi_{m,1} = \phi_{m,s}$ for $s=2, \dots, L$. The second term in Eq. (7) represents a defect line. A duality transformation changes $K \rightarrow K^{-1}$ in the above Hamiltonian, while the last term becomes $(\alpha/K) \sum_{k=1}^m (\phi_{k,1} - \phi_{k,L})$. This term may be eliminated by a shift $\phi_{k,l} \rightarrow \phi_{k,l} + \alpha l/L$, which adds a constant $-\alpha^2(L-1)/2KL$ to the free energy per unit length F . This modifies Δ to

$$\Delta = -\frac{\pi}{6} + \frac{\alpha^2}{2K}. \quad (8)$$

The defect line is equivalent to charges $\pm \alpha$ at $k=1$ and $k=m$, respectively. As $m \rightarrow \infty$, this is equivalent to a charge -2α at infinity, as considered by Dotsenko and Fateev.⁵ They found $c=1-24\alpha^2$, when

$K=1/8\pi$, in agreement with Eqs. (1) and (8).

From (8) we derive the value of Δ for the q -state Potts model ($0 \leq q \leq 4$) as follows. The critical Potts model can be represented as an F model.²¹ With the usual labeling of the vertices (see Fig. 2 of Ref. 22), the vertex weights are $(\omega_1, \dots, \omega_6) = (1, 1, 1, 1, z^\pi + z^{-\pi}, z^\pi + z^{-\pi})$, where $q^{1/2} = 2 \cosh \theta$ and $z = e^{\theta/2}$. As noted in Ref. 21, cylindrical boundary conditions lead to a seam of vertices with modified weights $\omega'_3 = e^{2\theta}$ and $\omega'_4 = e^{-2\theta}$. In the body-centered solid-on-solid representation of the F model,²³ the weight of the modified vertices is $e^{2\theta(n_2 - n_4)}$, where n_2 and n_4 are the column heights at next-nearest-neighbor sites straddling the seam. This corresponds to $\alpha = 2i\theta$ in Eq. (7). The presence of "external" sites²¹ leads to the restriction of constant column height at $k=1, m$, as introduced above. Under renormalization, this body-

TABLE I. Numerical results for Δ as a function of the four-spin interaction K_4 of the Baxter model, obtained from data for $L = 4, 6, \dots, 16$. The exact result in this case is $-\pi/6 \approx -0.523\ 599$.

K_4	Δ	K_4	Δ
-1.0	-0.525	0.1	-0.523 604
-0.8	-0.524	0.2	-0.523 604
-0.6	-0.524	0.3	-0.523 602
-0.4	-0.5239	0.4	-0.523 590
-0.3	-0.523 69	0.6	-0.524
-0.2	-0.523 608	0.8	-0.525
-0.1	-0.523 604	1.0	-0.57
0.0	-0.523 604		

centered solid-on-solid model flows to the Gaussian model with^{22, 24, 25} $K = \pi(2-y)$, where $q^{1/2} = 2 \times \cos(\pi y/2)$ and $0 \leq y \leq 2$. Topological objects, such as charges, remain unrenormalized. From Eq. (8) we therefore find

$$\Delta = -\frac{\pi}{6} + \frac{\pi y^2}{2(2-y)}. \quad (9)$$

The q dependence of c that we then infer from Eq. (2) agrees with that derived by Kadanoff, and quoted in Ref. 2, and with that conjectured by Dotsenko and Fateev.⁵

The same argument can be used for the $O(n)$ model on a hexagonal lattice introduced by Nienhuis,²⁶ which can be mapped onto a six-vertex model on a Kagomé lattice, which in turn may be represented by a solid-on-solid model. Again a defect line has to be introduced to obtain the correct weights for cylindrical boundary conditions, and the argument proceeds in

TABLE II. Numerical results for Δ for the q -state Potts model compared with exact results derived in the text; free-energy data of Blöte and Nightingale (Ref. 28) for $L = 2, \dots, 11$.

q	Δ	Exact
$\frac{1}{64}$	0.869 148	0.869 154
$\frac{1}{16}$	0.708 251	0.708 256
$\frac{1}{2}$	0.233 420	0.233 438
0.95	0.018 4268	0.018 4267
1.05	-0.017 6778	-0.017 779
2	-0.261 796	-0.261 799
3	-0.418 92	-0.418 88
4	-0.525 30	-0.523 599

the same way as above. Renormalization maps the $O(n)$ model onto a Gaussian model with interaction²⁴ $K = \pi(2-y)$, where $n = 2 \cos(\pi y/2)$, and $-2 \leq y \leq 0$. The value of Δ agrees with Eq. (1) if the n dependence of c conjectured by Dotsenko and Fateev⁵ is used.

The F model and the critical Baxter model²⁷ both renormalize onto a Gaussian model with no defect line, and so we expect $\Delta = -\pi/6$ universally for these cases, in agreement with the idea² that models with continuously varying exponents have $c = 1$.

Finally, we present numerical results supporting the expressions derived for Δ for periodic boundary conditions. The results were obtained from the free energy per site of infinitely long strips of increasing width L , by standard extrapolation techniques.²⁸ The models in the universality class of the $O(n)$ model that we studied are the continuous n -component cubic model defined by Blöte and Nightingale,²⁹ in the two cases considered there: $L' = 0$ and $A = 0$ ($e^{-L'} = \cosh K$), where L' and K are the coefficients of the quadratic and quartic terms in the Hamiltonian.

For the Baxter model in the Ising spin representation³⁰ we varied the four-spin and the two equal next-nearest-neighbor interactions K_4 and K_2 , along the critical line. As shown in Tables I-III, the results agree very well with the theory in all cases, particularly for those values of the parameters where also in previ-

TABLE III. Numerical results for Δ as a function of n for two special cases ($L' = 0$ and $A = 0$) of the n -component cubic model (Ref. 29) extrapolated from data for $L = 2, \dots, 8$, compared with exact results derived in the text.

n	$\Delta(A=0)$	$\Delta(L'=0)$	Exact
-1	0.312		0.3142
$-\frac{1}{2}$	0.146		0.1461
$-\frac{1}{4}$	0.0711		0.0711
$-\frac{1}{8}$	0.0352		0.0351
$-\frac{1}{16}$	0.0175		0.01746
$-\frac{1}{32}$	0.0087		0.00870
$-\frac{1}{64}$	0.0044		0.00435
$\frac{1}{64}$	-0.0043	-0.00434	-0.00433
$\frac{1}{32}$	-0.0087	-0.00867	-0.00866
$\frac{1}{16}$	-0.0173	-0.00173	-0.001727
$\frac{1}{8}$	-0.0344	-0.0344	-0.0344
$\frac{1}{4}$	-0.0682	-0.0682	-0.0681
$\frac{1}{2}$	-0.1343	-0.134	-0.1340
1	-0.262	-0.262	-0.2618
2	-0.523	-0.525	-0.5236

ous calculations^{6,29,31} the asymptotic behavior was observed to set in for those system sizes considered here.

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Note added.—After this paper was submitted for publication, we learned that Affleck³² has also obtained the result in Eq. (1).

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