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CONFORMAL MAPPING OF THE HALFPLANE ONTO A STRIP WITH VARIABLE WIDTH

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1. INTRODUCTION

Many applications of conformal mapping in the solution of two-dimensional potential field problems require to find a conformal mapping of a complex halfplane into an infinitely long strip whose one boundary is a straight line while the other is a polygonal line. Let us have the complex plane $w = \xi + i\eta$. Our aim is to find a conformal mapping of the half plane $\text{Im}(w) > 0$ into a polygon in the complex plane $U = y + iz$, this polygon being of the shape of an infinitely long strip whose one boundary is the straight line $z = 0$, $y \in (-\infty, +\infty)$ while the other is the following polygonal line:

$$z = h_1, \quad y \in (-\infty, 0\rangle,$$
$$z = h_1 + by, \quad y \in (0, D),$$
$$z = h_2, \quad y \in \langle D, +\infty),$$

$b = \tan \varphi$, $D = (h_2 - h_1)/\tan \varphi$, φ is the slope angle of the oblique part of the polygonal line, $h_1 < h_2$. This problem has been until now solved only for two cases of the slope angle $\varphi = \pi/2$, $\varphi = \pi/4$, [1], [2]. We shall consider more general cases of the angles: $\varphi = m\pi/r$ where m, r are positive integers, $m < r$.

2. EXPRESSION OF THE MAPPING

In order to obtain the required conformal mapping we shall use Schwarz-Christoffel theorem which states for the differential $\mathrm{d}U$ of the interior of the polygon:

$$\mathrm{d}U = A(w - \xi_1)^{\alpha_1/\pi - 1} (w - \xi_2)^{\alpha_2/\pi - 1} \dots (w - \xi_n)^{\alpha_n/\pi - 1} \,\mathrm{d}w \tag{1}$$

where $\xi_1, \xi_2, \dots, \xi_n$ are the points of the real axis of the w-plane corresponding to the vertices of the given polygon, while $\alpha_1, \alpha_2, \dots, \alpha_n$ are the interior angles of the polygon and A is a scale constant. Following the technique given e.g. in [3], [4] we can arrange the following pairs of points from the planes w and U corresponding to each other as well as the angles α_k belonging to them.

w plane	U plane	angle α_k
$\xi_1 \to 0$	$U_1 = -\infty + i\|_{h_1}^{0}$	$\alpha_1 = 0$
$\xi_2 \to -1$	$U_2 = ih_1$	$\alpha_2 = \pi(1 + m/r)$
$\xi_3 \to -c$	$U_3 = D + ih_2$	$\alpha_3 = \pi(1 - m/r)$
$\xi_4 \to -\infty$	$U_4 = +\infty + ih_2$	$\alpha_4 = 0$

Tab. 1.

The situation in the planes w and U is illustrated in fig. 1 where the path orientation is also indicated. The real number c introduced in Tab. 1 will be determined later; it will be shown that it depends on some dimensional parameters of the strip. Since $\xi_4 = -\infty$ and $\alpha_4 = 0$, the term $(w - \xi_4)^{\alpha_4/\pi - 1}$ will not occur in the expression (1), [5]. So we get

$$\mathrm{d}U = Aw^{-1}(w + 1)^{m/r}\,(w + c)^{-m/r}\,\mathrm{d}w \tag{2}$$

$$U = A\int w^{-1}[(w + 1)/(w + c)]^{m/r}\,\mathrm{d}w + B\,. \tag{3}$$

Now we shall determine the constants A and c by investigation of the properties of our conformal mapping at some points of the plane w. Passing in the w plane along the real axis from $+\infty$ to 0, we pass in the plane U along the real axis y from $+\infty$ to $-\infty$. Now let us pass along a small semicircle $w = \tau \exp(i\theta)$ with its centre at

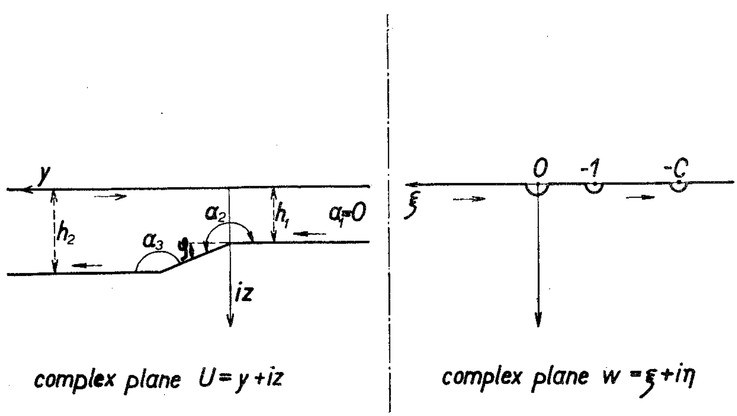

Fig. 1

$w = 0(\tau \ll 1,\ \theta \in \langle 0, \pi \rangle)$. In the U plane the corresponding change is $\int \mathrm{d}U = ih_1$. Thus we get:

$$ih_1 = \lim_{\tau \to 0} A \int_0^{\pi} \frac{i\tau \exp(i\,\theta)}{\tau \exp(i\,\theta)} \left[\frac{\tau \exp(i\theta) + 1}{\tau \exp(i\theta) + c}\right]^{m/r} \mathrm{d}\theta = Ai\pi(1/c)^{m/r}\,.$$

So we have obtained

$$A\pi(1/c)^{m/r} = h_1\,. \tag{4}$$

If we pass in the w plane along a large semicircle $w = R \exp(i\,\theta)$, $\theta \in \langle \pi, 0 \rangle$, $R \gg c$, then we pass in the U plane at $y \to +\infty$ from the lower boundary to the upper one, so the imaginary part of U changes by $-ih_2$. Thus it is:

$$-ih_2 = \lim_{R \to \infty} A \int_{\pi}^{0} \frac{iR \exp(i\theta)}{R \exp(i\theta)} \left[\frac{R \exp(i\theta) + 1}{R \exp(i\theta) + c}\right]^{m/r} \mathrm{d}\theta = -Ai\pi\,.$$

From this we obtain

$$A = h_2/\pi\,. \tag{5}$$

Substituting into (4) we get

$$c = (h_2/h_1)^{r/m}\,. \tag{6}$$

Since we considered $h_2 > h_1$ it will be $c > 1$.

The constant of integration B can be determined on the basis of the properties of the transformation (3) at the points $w = -1, -c$. To this purpose we have to perform the integration indicated in (3). We need to calculate the integral

$$J = \int w^{-1}[(w + 1)/(w + c)]^{m/r}\,\mathrm{d}w\,. \tag{7}$$

We introduce a new variable p by the substitution

$$p = [(w + 1)/(w + c)]^{1/r}\,. \tag{8}$$

Thus we can write

$$J = r \int \frac{(c - 1)\,p^{m+r-1}}{(cp^r - 1)(1 - p^r)}\,\mathrm{d}p = r \int \frac{p^{m-1}}{1 - p^r}\,\mathrm{d}p - r \int \frac{p^{m-1}}{1 - cp^r}\,\mathrm{d}p\,. \tag{9}$$

We can express both integrals in (9) by means of elementary functions, distiguishing the cases of even or odd r. According to [6] we get:

a) for r even; $r = 2n$:

$$\begin{aligned} 2n \int [p^{m-1}/(1 - p^{2n})]\,\mathrm{d}p &= (-1)^{m+1} \ln(1 + p) - \ln(1 - p) - \\ &- \sum_{k=1}^{n-1} \cos(km\pi/n) \ln(1 - 2p \cos(k\pi/n) + p^2) + \\ &+ 2 \sum_{k=1}^{n-1} \sin(km\pi/n) \arctan[(p - \cos(k\pi/n))/\sin(k\pi/n)]\,, \end{aligned} \tag{10}$$

b) for r odd; $r = 2n + 1$:

$$(11) \qquad (2n + 1)\int[p^{m-1}/(1 - p^{2n+1})]\,dp =$$

$$= -\ln(1 - p) + (-1)^{m+1}\sum_{k=1}^{n}\cos[m\pi(2k - 1)/(2n + 1)]\,.$$

$$.\ln[1 + 2p\cos((2k - 1)\pi/(2n + 1)) + p^2] +$$

$$+ 2(-1)^{m+1}\sum_{k=1}^{n}\sin[m\pi(2k - 1)/(2n + 1)]\,.$$

$$.\arctan[(p + \cos((2k - 1)\pi/(2n + 1)))/\sin(\pi(2k - 1)/(2n + 1))]\,.$$

As it is convenient to reduce the fraction m/r if possible, it is evident that for even r we have m odd only. So in (10) we can write $(-1)^{m+1}\ln(1 + p) - \ln(1 - p) = \ln[(1 + p)/(1 - p)]$. Using (10) and (11) we can easily calculate also the second integral in (9), obtaining explicit expressions of the mapping:

a) for r even, $r = 2n$

$$(12) \qquad U = \frac{h_2}{\pi}\left\{\ln\left(\frac{1 + p}{1 - p}\right) - \sum_{k=1}^{n-1}\cos(km\pi/n)\ln[1 - 2p\cos(k\pi/n) + p^2] +\right.$$

$$+ 2\sum_{k=1}^{n-1}\sin(km\pi/n)\arctan[(p - \cos(k\pi/n))/\sin(k\pi/n)] + \frac{h_1}{h_2}\left[\ln\left(\frac{c_1p - 1}{c_1p + 1}\right) +\right.$$

$$+ \sum_{k=1}^{n-1}\cos(km\pi/n)\ln[1 - 2c_1p\cos(k\pi/n) + c_1^2p^2] -$$

$$\left.\left. - 2\sum_{k=1}^{n-1}\sin(km\pi/n)\arctan[(c_1p - \cos(k\pi/n))/\sin(k\pi/n)]\right]\right\} + Q\,,$$

b) for r odd; $r = 2n + 1$:

$$(13) \qquad U = h_2/\pi\,\{-\ln(1 - p) + (-1)^{m+1}\sum_{k=1}^{n}\cos(m\pi(2k - 1)/r)\,.$$

$$.\ln[1 + 2p\cos((2k - 1)\pi/r) + p^2] + (-1)^{m+1}\,2\sum_{k=1}^{n}\sin(m\pi(2k - 1)/r)\,.$$

$$.\arctan[(p + \cos((2k - 1)\pi/r))/\sin((2k - 1)\pi/r)] -$$

$$- h_1/h_2[-\ln(c_1p - 1) + (-1)^{m+1}\sum_{k=1}^{n}\cos(m\pi(2k - 1)/r)\,.$$

$$.\ln[1 + c_1p\cos((2k - 1)\pi/r) + c_1^2p^2] + (-1)^{m+1}\,2\sum_{k=1}^{n}\sin(m\pi(2k - 1)/r)\,.$$

$$.\arctan[(c_1p + \cos((2k - 1)\pi/r))/\sin((2k - 1)\pi/r)]]\} + K\,.$$

In these expressions we put $c_1 = (h_2/h_1)^{1/m} = c^{1/r}$, Q is a new constant of integration for r even; it contains B and the purely imaginary term ih_1 arising by the change of sign in the argument of $\ln(1 - c_1p)$. K is another constant of integration for r odd, it contains B and the purely imaginary term ih_1 arising similarly as above. In view of different transformation expressions for r even or odd we can expect that Q and K will be different too. We shall determine them by using the corresponding relations required by Tab. 1. First of all we arrange (12) and (13) into the form convenient for the calculation in the vicinity of the point $w = -c$ for which $p \to \infty$. Investigating the terms of the sum occuring in (12): $\sum_{k=1}^{n-1} \cos(km\pi/n) \ln[1 - 2p\cos(k\pi/n) + p^2]$ m odd, we find that there is either zero (for $m/r = \frac{1}{2}$) or an even number of terms, and that it is possible to group together the k-th and the $(n-k)$-th term because $\cos(km\pi/n) = -\cos[(n-k)m\pi/n]$. Thus it follows that:

$$\sum_{k=1}^{n-1} \cos(km\pi/n) \ln[1 - 2p\cos(k\pi/n) + p^2] = \tag{14}$$

$$= \sum_{k=1}^{N} \cos(km\pi/n) \ln[1 - 2p\cos(k\pi/n) + p^2)/(1 + 2p\cos(k\pi/n) + p^2)]$$

where $N = [(n-1)/2]$, $N \leqq (n-1)/2$. In a similar way it is possible to arrange also the next sum in (12) in which instead of p there occurs $c_1 p$. So we get the following expression of the transformation for r even:

$$\begin{aligned} U = h_2/\pi\{&\ln[(1+p)/(1-p)] - \sum_{k=1}^{N} \cos(km\pi/n) \,. \\ &.\,[\ln[1 - 2p\cos(k\pi/n) + p^2)/(1 + 2p\cos(k\pi/n) + p^2)] - \\ &- (h_1/h_2)\ln[(1 - 2c_1 p\cos(k\pi/n) + c_1^2p^2)/(1 + 2pc_1\cos(k\pi/n) + p^2c_1^2)]] + \\ &+ (h_1/h_2)\ln[(c_1p - 1)/(c_1p + 1)] + \\ &+ 2\sum_{k=1}^{n-1} \sin(km\pi/n)[\arctan[(p\cos(k\pi/n))/\sin(k\pi/n)] - \\ &- (h_1/h_2)\arctan[(c_1p - \cos(k\pi/n))/\sin(k\pi/n)]]\} + Q\,. \end{aligned} \tag{15}$$

In order to arrange the expression (13) we use the following property of trigonometric functions:

$$(-1)^{m+1}\, 2\sum_{k=1}^{n} \cos[m\pi(2k-1)/(2n+1)] = 1\,, \quad m < 2n+1\,. \tag{16}$$

This formula can be easily proved by using Euler formula for cosine and by summing geometrical succesions. Then we can write:

$$\begin{aligned} &-\ln(1-p) + (-1)^{m+1}\sum_{k=1}^{n} \cos[m\pi(2k-1)/(2n+1)]\,. \\ &.\,\ln[1 + 2p\cos((2k-1)\,\pi/(2n+1)) + p^2] = \\ &= 2(-1)^{m+1}\sum_{k=1}^{n} \cos(m\pi(2k-1)/(2n+1))\,. \\ &.\,\ln[(1 + 2p\cos((2k-1)\,\pi/(2n+1)) + p^2)^{1/2}/(1-p)]\,. \end{aligned} \tag{17}$$

The relation (13) can be transcribed in the form:

$$(18) \qquad U = 2h_2/\pi\{(-1)^{m+1} \sum_{k=1}^{n} \cos(m\pi(2k-1)/r) \,.$$
$$. [\ln [(1 + 2p \cos((2k-1)\pi/r) + p^2)^{1/2}/(1-p)] -$$
$$- (h_1/h_2) \ln [(1 + 2c_1 p \cos((2k-1)\pi/r) + c_1^2 p^2)^{1/2}/(c_1 p - 1)]] +$$
$$+ \sin(m\pi(2k-1)/r) [\arctan [(p + \cos((2k-1)\pi/r))/\sin((2k-1)\pi/r)] -$$
$$- (h_1/h_2) \arctan [(c_1 p + \cos((2k-1)\pi/r))/\sin((2k-1)\pi/r)]]\} + K \,,$$
$$r = 2n + 1 \,.$$

Now we can determine the integration constants Q and K. According to Tab. 1 and (8):

(19) for $w = -1$ there is $p = 0$ and it should be $U = ih_1$,

(20) for $w = -c$ there is $p \to \infty$ and it should be $U = D + ih_2$.

Let r be even, $r = 2n$. We put $p = 0$ in (15). According to (19) it should be

$$ih_1 = h_2 \pi\{2(1 - h_1/h_2) \sum_{k=1}^{n-1} \sin(km\pi/n) \arctan(-\text{cotan}(k\pi/n)) +$$
$$+ h_1/h_2 \ln(-1)\} + Q \,.$$

Considering the principal value of arctan it can be easily shown that the sum in the last term yields zero and because of $\ln(-1) = i\pi$, we get

$$(21) \qquad Q = 0 \,.$$

With such $Q = 0$ also the second relation (20) must be valid. Passing to the limit of the right-hand side of (15) for $p \to \infty$, it should be

$$D + ih_2 = h_2/\pi\{\ln(-1) + \pi(1 - h_1/h_2) \sum_{k=1}^{n-1} \sin(km\pi/n)\} \,.$$

The imaginary part of this equality is evidently fulfilled. The real parts of the right-hand and left-hand sides of it are also equal as the validity of the following relation can be proved (in a similar way as in (16)):

$$(22) \qquad 1/\tan(m\pi/2n) = \sum_{k=1}^{n-1} \sin(km\pi/n) \,; \quad m \text{ odd} \,, \quad m < 2n \,.$$

Now we determine the integration constant K in the case of r odd, $r = 2n + 1$. Let us pass to the limit of the right-hand side of (18) for $p \to \infty$. According to (20) it should be:

$$(23) \qquad D + ih_2 = h_2/\pi\{(-1)^{m+1}\, 2 \sum_{k=1}^{n} \cos(m\pi(2k-1)/(2n+1)) +$$
$$+ \pi(1 - h_1/h_2)(-1)^{m+1} \sum_{k=1}^{n} \sin(m\pi(2k-1)/(2n+1))\} + K \,.$$

The first termin the curly brackets in view of (16) yields $\ln(-1)$ so that the imaginary part of K will be zero. Then K will be a real number:

$$K = D - (-1)^{m+1} (h_2 - h_1) [1 + (-1)^{m+1} \cos (m\pi/(2n + 1))] \,. \, [2 \sin (m\pi/(2n + 1))]^{-1} \,, \tag{24}$$

because, in a similar way as in (16), it can be proved that

$$\sum_{k=1}^{n} \sin [m\pi(2k - 1)/(2n + 1)] = = [1 + (-1)^{m+1} \cos (m\pi/(2n + 1))]/[2 \sin (m\pi/(2n + 1))] \,. \tag{25}$$

We can see that in the case of r odd we obtained a non zero integration constant.

Its verification can be made by using (19). Performing the limit process $p \to 0$ in (13) we find that in order that (19) be fulfilled it must be:

$$K + 2/\pi(-1)^{m+1} (h_2 - h_1) \sum_{k=1}^{n} \sin [m\pi(2k - 1)/(2n + 1)] \,. \, \arctan \{\cotan [(2k - 1) \pi/(2n + 1)]\} = 0 \,.$$

This relation can be used for testing numerical calculation.

Now when we have determined the integration constants Q and K we know completely the analytical form of the required conformal mapping. It can be easily shown that the mappings known until now for the slopes $\varphi = \pi/2, \pi/4$ given in [1] and [2] are special cases of that of ours (it is sufficient to put in (15) $m/r = \frac{1}{2}, \frac{1}{4}$ respectively). By means of the conformal mapping obtained it is possible to calculate e.g. the stationary electric current through a conductor which is bounded on the upper side by a plane and on the lower side by a piecewise planar boundary. Another application is possible in the investigation of the flow of an incompressible fluid. In such applications it is possible to proceed analogously as it is suggested e.g. in [7].

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Súhrn

KONFORMNÉ ZOBRAZENIE POLROVINY NA PÁS PREMENLIVEJ VÝŠKY

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Pomocou Schwarz-Christoffelovej vety je odvodené konformné zobrazenie polroviny na nekonečne dlhý pás, ktorého jedna hranica je priamka a druhá lomená čiara skladajúca sa z dvoch polpriamok rovnobežných s prvou hranicou a spojených šikmou úsečkou s uhlom sklonu rovným racionálnemu násobku π. Toto zobrazenie je vyjadrené pomocou elementárnych funkcií, pri rozlišovaní prípadov kedy je π delené párnym a kedy nepárnym celým číslom; sú ukázané niektoré jeho dôležité vlastnosti.

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