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Conformal Mappings between Canonical Multiply Connected Domains

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Abstract. Explicit analytical formulae for the conformal mappings from the canonical class of multiply connected circular domains to canonical classes of multiply connected slit domains are constructed. All the formulae can be expressed in terms of the Schottky-Klein prime function associated with the multiply connected circular domains.

Keywords. Conformal map, multiply connected, Green's functions.

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1. Introduction

Since the classical work of Koebe [13] several classes of multiply connected planar domains (in particular, those involving slit regions) have now become "canonical" in the sense that they are characterized by simple geometries and are uniquely determined by specifying just a few parameters (or moduli). These canonical domains are often found listed in standard texts on classical function theory and conformal mapping [16, 11, 17]. There are deep theoretical connections between the Dirichlet and Neumann problems for Laplace's equation in multiply connected domains, conformal slit mappings, potential theory and various extremal problems [13, 17, 16].

Let $M \geq 0$ be an integer. The five most important $(M+1)$-connected canonical slit domains are:

(a) the parallel slit domain consisting of $M+1$ finite-length slits aligned at some fixed angle χ to the real axis in the image domain;
(b) the circular-arc domain comprising $M+1$ finite-length circular-arc slits all centred on the origin in the image domain;
(c) the radial slit domain in which $M+1$ finite-length slits are situated on rays emanating from the origin in the image domain;

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(d) a circular disc with M enclosed circular-arc slits all centred on the centre of the disc;
(e) a concentric circular ring with $M-1$ finite-length concentric circular-arc slits between the two circumferences of the ring.

These are the canonical multiply connected domains listed in standard texts (e.g. Nehari [16]).

The purpose of this paper is as follows. In terms of practical applications, it is very useful to have at hand explicit analytical formulae for conformal mappings from some canonical class of multiply connected planar domains to the various canonical classes of slit domains just listed. The present authors have not found any such formulae documented in the literature. Indeed, while conformal mapping theory is widely used in applications, multiply connected conformal mapping formulae are conspicuous by their absence in standard catalogs of conformal mappings [12].

A *circular domain* is a multiply connected domain all of whose boundaries are circles. The class of circular domains is itself a canonical class and is uniquely determined by specifying the centres and radii of the boundary circles. It is a convenient class of multiply connected domains on which to do analysis since there exists a special transcendental function — known as the *Schottky-Klein prime function* [2] — which is naturally associated with any such circular domain. In this paper concrete analytical formulae for the conformal mapping from a given circular domain to the five classes (a)–(e) of canonical slit domains will be described. In Section 9 we discuss a number of mathematical problems, arising in practice, in which these new formulae have already found important application.

2. The modified Green's function

Schiffer [17] demonstrates some important connections between canonical slit domains and the modified Green's function associated with multiply connected planar domains. Given the modified Green's function of a planar domain there exist formulae producing conformal slit mappings from the original planar domain to many of the canonical slit domains just cited. This fact will be exploited in the sequel. We now introduce the key elements of Dirichlet calculus that will be needed in the construction of the formulae.

Let D be an arbitrary bounded and $(M+1)$-connected planar domain in a ζ-plane. Introduce a *modified Green's function* $G_0(\zeta;\alpha)$ with respect to the two points ζ and α in D in the following way: suppose D is bounded by $M+1$ smooth Jordan curves C_j, $j=0,1,\ldots,M$. The curve C_0 is taken as the outermost boundary so that C_k, $k=1,\ldots,M$, denote the M enclosed boundaries (or the boundaries of the finite set of "holes" in the domain). The modified Green's function is defined as the function $G_0(\zeta;\alpha)$ satisfying the following properties:

(i) the function
$$g_0(\zeta;\alpha) = G_0(\zeta;\alpha) + \log r_0$$
is harmonic with respect to ζ throughout the region D including at the point α. Here r_0 is
$$r_0 = |\zeta - \alpha|;$$

(ii) if $\partial G_0/\partial n$ is the normal derivative of G_0 on a curve then
$$\begin{aligned} G_0(\zeta;\alpha) &= 0, && \text{on } C_0, \\ G_0(\zeta;\alpha) &= \gamma_{0k}(\alpha), && \text{on } C_k,\ k = 1,\dots,M, \\ \oint_{C_k} \frac{\partial G_0}{\partial n}\, ds &= 0, && k = 1,\dots,M \end{aligned}$$
where ds denotes an element of arc and $\gamma_{0k}(\alpha)$, $k = 1,\dots,M$, are constants. It should be noted that the values of these constants are determined by these integral constraints on the normal derivative of G_0.

The results of Koebe [13] imply that the function $G_0(\zeta;\alpha)$ defined by conditions (i)–(ii) above exists uniquely and satisfies the reciprocity condition
$$G_0(\zeta;\alpha) = G_0(\alpha;\zeta).$$
It is clear from the definition that the boundary C_0 has a special significance with respect to the function $G_0(\zeta;\alpha)$ defined above. It is the boundary on which $G_0(\zeta;\alpha)$ is normalized to vanish and is the only choice of boundary for D for which the quantity
$$\oint_{C_0} \frac{\partial G_0}{\partial n}\, ds$$
does not vanish. The subscript of G_0 reflects this special significance of C_0. But it should be clear that there are M alternative modified Green's functions that can also be defined analogously: one simply makes the boundary component C_j the one which has the special significance afforded to C_0 in the definition of G_0. Extending the subscript notation, these additional modified Green's functions will be denoted $G_j(\zeta;\alpha)$, $j = 1,\dots,M$. In particular, $G_j(\zeta;\alpha)$ will have a logarithmic singularity at α, will be taken to vanish on C_j, while the quantities
$$\oint_{C_k} \frac{\partial G_j}{\partial n}\, ds$$
will be non-zero only for $k = j$. The results of Koebe [13] imply that the function $G_j(\zeta;\alpha)$ defined by these conditions exists uniquely.

It is important, for later use, to define the analytic extension of the modified Green's function $G_j(\zeta;\alpha)$. Let this function be denoted $\tilde{G}_j(\zeta;\alpha)$. It is given by the formula
$$\tilde{G}_j(\zeta;\alpha) = G_j(\zeta;\alpha) + iH_j(\zeta;\alpha)$$
where $H_j(\zeta;\alpha)$ is the harmonic conjugate of $G_j(\zeta;\alpha)$.

3. Circular domains

As already mentioned, a *circular domain* is defined as a finitely connected domain all of whose boundary components are circles. Let D_ζ denote a bounded circular domain in the ζ-plane. Specifically, let D_ζ be the interior of the unit ζ-disc with M smaller circular discs excised. $M = 0$ is the simply connected case. With the notation of Section 2, let the boundaries of the smaller excised circular discs be C_j, $j = 1, \dots, M$. The outer unit circle $|\zeta| = 1$ is denoted C_0.

To uniquely specify an $(M+1)$-connected D_ζ the centres and radii of the C_j, $j = 1, \dots, M$, are needed. Let $\delta_j \in \mathbb{C}$, $j = 1, \dots, M$, be the centres of these circles and let $q_j \in \mathbb{R}$, $j = 1, \dots, M$, be their radii. A definition sketch illustrating a quadruply connected circular domain is shown in Figure 1.

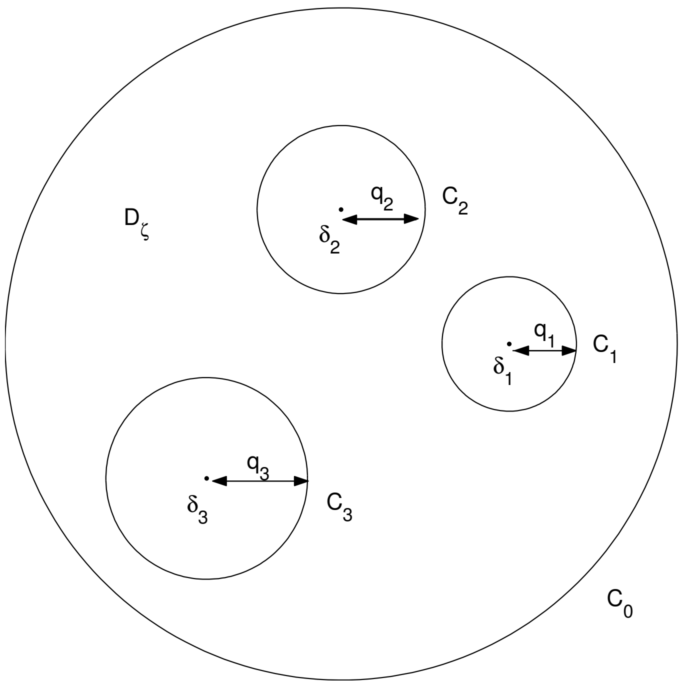

FIGURE 1. A typical circular domain. The case shown is quadruply connected.

4. Schottky groups

Now define M Möbius maps ϕ_j, $j = 1, \dots, M$, corresponding to the conjugation map for points on the circle C_j. That is, if C_j has equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \bar{\delta}_j) = {q_j}^2$$

then

$$\bar{\zeta} = \bar{\delta}_j + \frac{{q_j}^2}{\zeta - \delta_j}$$

and so

$$\phi_j(\zeta) \equiv \bar{\delta}_j + \frac{{q_j}^2}{\zeta - \delta_j}. \tag{1}$$

If ζ is a point on C_j then its complex conjugate is $\bar{\zeta} = \phi_j(\zeta)$. If ζ is on C_0, we also define $\phi_0(\zeta) \equiv \zeta^{-1}$.

Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{{q_j}^2\zeta}{1 - \bar{\delta}_j\zeta}. \tag{2}$$

Let C'_j be the circle obtained by reflection of C_j in the unit circle C_0, i.e. the circle obtained by the transformation $\zeta \mapsto \bar{\zeta}^{-1}$. It is easy to verify that the image of the circle C'_j under the transformation $\theta_j(\zeta)$ is C_j. Since the M circles C_j are non-overlapping, so are the M circles C'_j. The classical *Schottky group* Θ is defined to be the infinite free group of mappings generated by compositions of the $2M$ basic Möbius maps θ_j, $j = 1, \ldots, M$, and their inverses θ_j^{-1}, $j = 1, \ldots, M$, including the identity map. Beardon [3] contains a general discussion of such groups. An accessible discussion of Schottky groups and their mathematical properties appears in a recent monograph by Mumford, Series and Wright [15].

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles C_j and C'_j. A schematic is shown in Figure 2. This region is known as the *fundamental region* associated with the Schottky group generated by the Möbius maps θ_j, $j = 1, \ldots, M$, and their inverses. This fundamental region can be understood as having two "halves" — the half that is inside the unit circle but exterior to the circles C_j is the region we call D_ζ, the region that is outside the unit circle and exterior to the circles C'_j is the other half. The reason for it being called fundamental is that the rest of the ζ-plane is a tessellation of an infinite number of "equivalent" regions which are obtained by transformation of the fundamental region under the elements of the Schottky group. Another choice of fundamental region is given by the union of D_ζ and the domain obtained by reflection of D_ζ in any of the interior circles.

The Möbius maps introduced above have two important properties that can easily be established. The first is that

$$\theta_j^{-1}(\zeta) = \frac{1}{\phi_j(\zeta)} \qquad \text{for all } \zeta. \tag{3}$$

This can be verified using the definitions (1) and (2) (or, alternatively, by considering the geometrical effect of each map). The second property, which follows

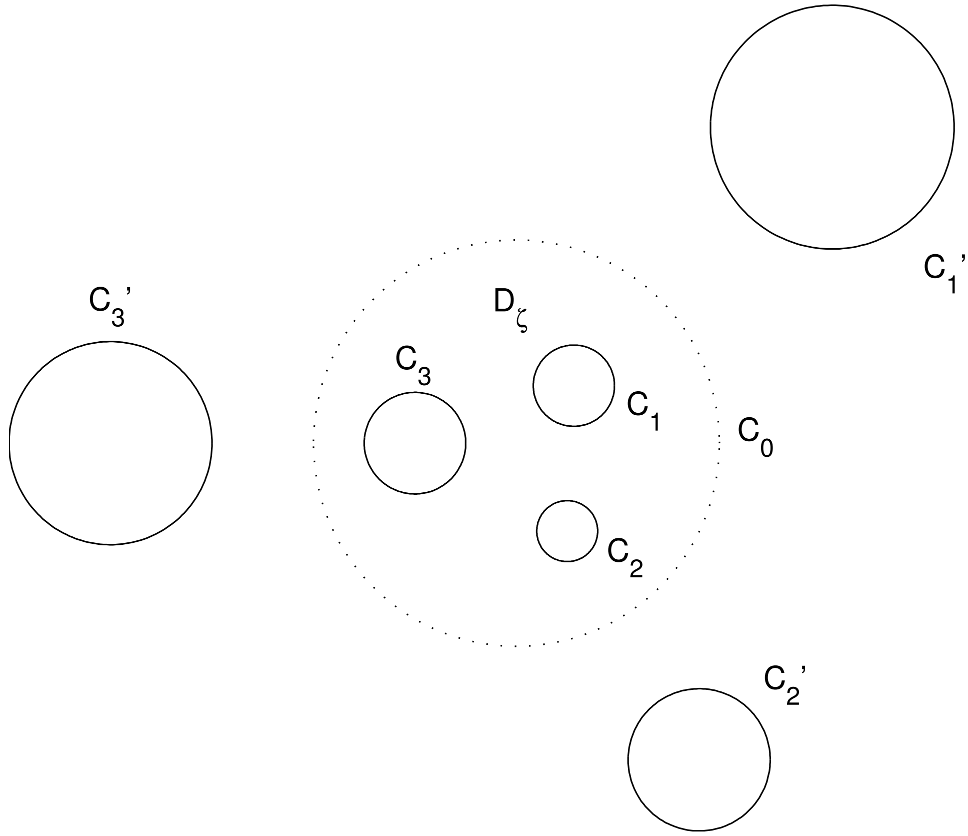

FIGURE 2. Schematic of the fundamental region associated with a typical quadruply connected circular domain. The fundamental region is the unbounded region exterior to the six Schottky circles $C_1, C_1', C_2, C_2', C_3, C_3'$.

from the first, is that

$$\theta_j^{-1}(\zeta^{-1}) = \frac{1}{\phi_j(\zeta^{-1})} = \frac{1}{\overline{\bar{\phi}_j(\bar{\zeta}^{-1})}} = \frac{1}{\overline{\theta_j(\bar{\zeta})}} = \frac{1}{\bar{\theta}_j(\zeta)} \qquad \text{for all } \zeta.$$

5. The Schottky-Klein prime function

Following Baker [2], the *Schottky-Klein prime function* is defined as

$$\omega(\zeta,\gamma) = (\zeta - \gamma)\omega'(\zeta,\gamma) \tag{4}$$

where

$$\omega'(\zeta,\gamma) = \prod_{\theta_i \in \Theta''} \frac{(\theta_i(\zeta) - \gamma)(\theta_i(\gamma) - \zeta)}{(\theta_i(\zeta) - \zeta)(\theta_i(\gamma) - \gamma)} \tag{5}$$

and where the product is over all mappings θ_i in the set Θ'' of all mappings in the group Θ excluding the identity and all inverse maps. This means that if $\theta_1\theta_2$ is included, say, then $\theta_2^{-1}\theta_1^{-1}$ (its inverse) must be excluded.

Following Baker [2], the presentation here proceeds under the assumption that the infinite product defining the Schottky-Klein prime function converges. In fact this is not always the case. To the best of our knowledge, the extent of what is known on this issue is covered by [2, Chapter 12], [4, Chapter 5] and [18, Chapter 2]. These describe results on the convergence of infinite series known as Poincaré theta series, the prime function being derivable from a particular such series [2]. The convergence properties of the product depend on the distribution of circles C_j, $j = 1, \ldots, M$, in the ζ-plane. Conditions sufficient for convergence are known; broadly speaking these state that convergence is guaranteed if the circles are sufficiently small and "well separated". However, these conditions are not especially sharp, and obtaining improved convergence criteria is an area for further research. The infinite products *do* converge, however, for a large and useful subset of the parameter space $\{q_j, \delta_j : j = 1, \ldots, M\}$. The fact that the infinite product representation (5) of the prime function does not converge for all choices of the above parameters certainly does not preclude its practical application in constructing Green's functions in broad classes of domain as is illustrated by the examples presented later — it simply precludes its universal applicability to *all* domains.

Now note that ω' can also be written as

$$\omega'(\zeta, \gamma) = \prod_{\theta_i \in \Theta''} \{\zeta, \theta_i(\zeta), \gamma, \theta_i(\gamma)\}$$

where the brace notation denotes a cross-ratio of the four arguments. This will be useful later. The function $\omega(\zeta, \gamma)$ is single-valued on the whole ζ-plane and has a simple zero at γ as well as at all points equivalent to γ under the mappings of the group Θ. The prime notation is not used here to denote differentiation.

The Schottky-Klein prime function has some important transformation properties. One such property is that it is anti-symmetric in its arguments, i.e.

$$\omega(\zeta, \gamma) = -\omega(\gamma, \zeta).$$

This is clear from inspection of (4) and (5). A second important property is given by

$$\frac{\omega(\theta_j(\zeta), \gamma_1)}{\omega(\theta_j(\zeta), \gamma_2)} = \beta_j(\gamma_1, \gamma_2)\frac{\omega(\zeta, \gamma_1)}{\omega(\zeta, \gamma_2)} \tag{6}$$

where θ_j is any one of the basic maps of the Schottky group. A detailed derivation of this result is given in [2, Chapter 12]. A formula for $\beta_j(\gamma_1, \gamma_2)$ is

$$\beta_j(\gamma_1, \gamma_2) = \prod_{\theta_k \in \Theta_j} \frac{(\gamma_1 - \theta_k(B_j))(\gamma_2 - \theta_k(A_j))}{(\gamma_1 - \theta_k(A_j))(\gamma_2 - \theta_k(B_j))}$$

where A_j and B_j are the two fixed points of the mapping θ_j satisfying

$$\theta_j(A_j) = A_j, \qquad \theta_j(B_j) = B_j,$$

and Θ_j stands for the collection of all mappings in the group which do not have any positive or negative power of θ_j at the right hand end. Note that A_j and B_j are the two solutions of a quadratic equation. It follows that

$$\frac{\theta_j(\zeta) - B_j}{\theta_j(\zeta) - A_j} = \mu_j e^{i\kappa_j} \frac{\zeta - B_j}{\zeta - A_j} \tag{7}$$

for some real parameters μ_j and κ_j. The roots A_j and B_j are ordered such that $|\mu_j| < 1$ in (7).

Finally, with the distribution of Schottky circles given by the construction described in Section 4, the prime function also satisfies the functional equation

$$\bar{\omega}(\zeta^{-1}, \gamma^{-1}) = -\frac{1}{\zeta\gamma}\omega(\zeta, \gamma). \tag{8}$$

A proof of this can be found in Crowdy and Marshall [5].

6. Explicit solution for G_j

Given a circular domain D_ζ, the associated Schottky-Klein prime function $\omega(\zeta, \gamma)$ can be constructed. As discussed earlier, there are $M + 1$ modified Green's functions that can be defined. Let $G_j(\zeta; \alpha)$ be the modified Green's function that satisfies the conditions

$$\begin{aligned} G_j(\zeta; \alpha) &= 0, && \text{on } C_j, \\ G_j(\zeta; \alpha) &= \gamma_{jk}(\alpha), && \text{on } C_k,\ k \neq j, \\ \oint_{C_k} \frac{\partial G_j}{\partial n}\, ds &= 0, && k \neq j. \end{aligned} \tag{9}$$

It is straightforward to show as in [13] that any function satisfying these requirements is unique. Indeed, if there are two functions with these properties then their difference is harmonic everywhere in D_ζ including at α, and zero on the whole boundary of D_ζ. Hence the difference is identically zero in D_ζ by the minimum and maximum principles for harmonic functions.

It will now be argued that an explicit expression for the required function is

$$G_j(\zeta; \alpha) = -\frac{1}{2}\log\left|\frac{\omega(\zeta, \alpha)\overline{\omega}(\phi_j(\zeta), \phi_j(\alpha))}{\omega(\zeta, \overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta), \bar{\alpha})}\right|. \tag{10}$$

It follows that the analytic extensions $\tilde{G}_j(\zeta, \alpha)$ of these modified Green's function are then given by

$$\tilde{G}_j(\zeta; \alpha) = -\frac{1}{2}\log\left(\frac{\omega(\zeta, \alpha)\overline{\omega}(\phi_j(\zeta), \phi_j(\alpha))}{\omega(\zeta, \overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta), \bar{\alpha})}\right). \tag{11}$$

First, define the $M+1$ functions

$$\tilde{R}_j(\zeta;\alpha) = \frac{\omega(\zeta,\alpha)\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta),\bar{\alpha})} \tag{12}$$

where $j = 0,1,\ldots,M$ so that

$$G_j(\zeta;\alpha) = -\frac{1}{2}\log|\tilde{R}_j(\zeta;\alpha)|.$$

Consider the fundamental region generated by D_ζ and the reflection of D_ζ in the j-th circle. Then since α is in the half of this fundamental region corresponding to D_ζ, $\overline{\phi_j}(\bar{\alpha})$ will be in the other half. $G_j(\zeta;\alpha)$ has a single isolated logarithmic singularity in D_ζ at $\zeta = \alpha$ as required. Since the zero of $\tilde{R}_j$ is second-order, locally $G_j(\zeta;\alpha)$ has the expansion

$$G_j(\zeta;\alpha) = -\log|\zeta-\alpha| + \mathcal{O}(1),$$

again as required.

It remains to verify that (10) satisfies the required boundary conditions on all the circles C_j, $j = 0,1,\ldots,M$. It can be shown that, on the circle C_k,

$$|\tilde{R}_j(\zeta;\alpha)| = \left|\frac{\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right|. \tag{13}$$

This formula is established in the appendix and holds for all integers j and k (between 0 and M) provided we adopt the convention that $\beta_0(\zeta,\alpha) \equiv 1$. It is immediate that, on C_j, $|\tilde{R}_j(\zeta;\alpha)| = 1$ so $G_j(\zeta;\alpha) = 0$ there. On C_k with $k \neq j$, we have

$$G_j(\zeta;\alpha) = -\frac{1}{2}\log\left|\frac{\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right| \tag{14}$$

which yields formulae for the constants $\gamma_{jk}(\alpha)$ defined in (9).

Finally, the integral constraints on the normal derivative of G_j must be verified. It is clear that

$$\begin{aligned}\oint_{C_k} d[\tilde{G}_j] &= \oint_{C_k} \frac{\partial \tilde{G}_j}{\partial s}\,ds = \oint_{C_k}\left(\frac{\partial G_j}{\partial s} + i\frac{\partial H_j}{\partial s}\right)ds \\ &= i\oint_{C_k}\frac{\partial H_j}{\partial s}\,ds = i\oint_{C_k}\frac{\partial G_j}{\partial n}\,ds\end{aligned} \tag{15}$$

where the third equality follows from the fact that G_j is constant on C_k, and the fourth equality follows from the Cauchy-Riemann relations satisfied by G_j and H_j. Note that, the integral on the left-hand side of (15) is the change in $\tilde{G}_j$ on traversing C_k. Now consider (11). In any fundamental region associated with the Schottky group, the function $\tilde{G}_j(\zeta,\alpha)$ given by (11) has precisely two logarithmic singularities of equal and opposite strength: one at α in D_ζ, the other at the point $\overline{\phi}_j(\bar{\alpha})$ in the other half of the fundamental region. A natural way to define a branch of $\tilde{G}_j$ is therefore to join by a branch cut each such pair of logarithmic singularities in the fundamental region, and in each region equivalent to it under the elements of the group. It follows that the change in $\tilde{G}_j$ on traversing C_k is zero for all k except $k = j$. Hence from (15) it follows that

$$\oint_{C_k} \frac{\partial G_j}{\partial n}\, ds$$

will be non-zero only for $k = j$, as required.

Having identified a function satisfying all the conditions required of a modified Green's function we now exploit the fact that the latter function is unique. Thus, it can be deduced that the functions we have constructed are indeed the required set of $M+1$ modified Green's functions. The special case of the function $G_0(\zeta, \alpha)$ takes the form

$$G_0(\zeta;\alpha) = -\frac{1}{2}\log\left|\frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\bar{\alpha})}\right|$$

which, on use of (8), reduces to

$$G_0(\zeta;\alpha) = -\log\left|\frac{\omega(\zeta,\alpha)}{\alpha\omega(\zeta,\bar{\alpha}^{-1})}\right|. \tag{16}$$

Its analytic extension is then simply

$$\tilde{G}_0(\zeta;\alpha) = -\log\left(\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\bar{\alpha}^{-1})}\right). \tag{17}$$

It should be noted that the present authors have previously derived formula (16) in an application of the modified Green's function to vortex dynamics [5]. The derivation given in [5] is the same as that presented here. It should also be mentioned that formulae for the first-type Green's function of a general multiply connected circular domain have been constructed using other methods [14].

7. The conformal mappings

In this section, we use the formalism just introduced to construct formulae for five of the most commonly studied types of canonical conformal mappings. These are all discussed in standard texts, for example Nehari [16, Chapter 7]. For convenience, we follow the notation of [16] as closely as possible.

7.1. The circle with concentric circular slits. The maps denoted $R_j(\zeta;\alpha)$ by Nehari [16] are those mapping to a circle with concentric circular arc slits. According to Schiffer [17] such maps are given, up to a constant factor, by

$$R_j(\zeta;\alpha) = \exp(-\tilde{G}_j(\zeta;\alpha)) \tag{18}$$

which can be seen to correspond to the functions $\tilde{R}_j(\zeta;\alpha)^{1/2}$ introduced in (12). Formulae for the required mappings are therefore given by

$$R_j(\zeta;\alpha) = \left[\frac{\omega(\zeta,\alpha)\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta),\bar{\alpha})}\right]^{1/2}. \tag{19}$$

The mappings (19) have been normalized to take the boundary circle C_j to the unit circle in the image domain. In the special case $j = 0$, (19) simplifies to

$$R_0(\zeta;\alpha) = \frac{1}{|\alpha|}\frac{\omega(\zeta,\alpha)}{\omega(\zeta,\bar{\alpha}^{-1})}.$$

7.2. The circular slit domain. Borrowing the notation from Nehari [16], let $P(\zeta;\alpha,\beta)$ be the conformal mapping to a circular slit domain where the point $\zeta = \alpha$ maps to the origin in the image plane while $\zeta = \beta$ maps to infinity. Following Schiffer [17], up to a constant factor, the required mapping is

$$P(\zeta;\alpha,\beta) = \frac{\exp(-\tilde{G}_j(\zeta;\alpha))}{\exp(-\tilde{G}_j(\zeta;\beta))} \tag{20}$$

where $\tilde{G}_j$ is the analytic extension of the j-th modified Green's function of the pre-image domain. Again, we choose to use $\tilde{G}_0$ so that, on use of (17), for the circular domains an explicit formula for the required mapping is

$$P(\zeta;\alpha,\beta) = \mathcal{A}(\alpha,\beta)\frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\beta}^{-1})}{\omega(\zeta,\beta)\omega(\zeta,\bar{\alpha}^{-1})}$$

where $\mathcal{A}(\alpha,\beta)$ is a normalization constant. Nehari [16] chooses to specify that the residue of the function (20) at β is unity, in which case,

$$\mathcal{A}(\alpha,\beta) = \frac{\omega'(\beta,\beta)\omega(\beta,\bar{\alpha}^{-1})}{\omega(\beta,\alpha)\omega(\beta,\bar{\beta}^{-1})}.$$

7.3. The circular ring with concentric circular slits. Nehari [16] introduces the notation $S_{ij}(\zeta)$ to denote a conformal mapping in which the i-th boundary component maps to the outer circumference of a circular ring domain in the image domain while the j-th component maps to the inner circumference. All other boundary components map to finite-length circular arc slits between these two bounding circumferences. Nehari [16] also points out that

$$S_{ij}(\zeta) = \mathcal{B}_{ij}(\alpha)\frac{R_i(\zeta;\alpha)}{R_j(\zeta;\alpha)} \tag{21}$$

where the functions R_i and R_j are given by (18) and $\mathcal{B}_{ij}(\alpha)$ is a normalization constant. On use of (19), explicit formulae for the mappings $S_{ij}(\zeta)$ follow immediately. Note also that on use of (13), if we require the outer circumference in the image plane to be the unit circle, the required normalization constants in (21) are given by

$$\mathcal{B}_{ij}(\alpha) = \left| \frac{\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)}{\beta_i(\overline{\phi}_j(\bar{\alpha}),\alpha)} \right|^{1/2}.$$

7.4. The parallel slit domain. Consider the conformal mapping to a parallel slit domain where each slit makes an angle χ to the real axis in the image domain. Let this conformal mapping be $\phi_\chi(\zeta;\alpha)$ where α is the point mapping to infinity. It is shown in Schiffer [17] that such a mapping is given by

$$\phi_\chi(\zeta;\alpha) = e^{i\chi}\left[\cos\chi\phi(\zeta;\alpha) - i\sin\chi\psi(\zeta;\alpha)\right] \tag{22}$$

where

$$\begin{aligned} \phi(\zeta;\alpha) &= \frac{1}{i}\frac{\partial}{\partial y_0}\tilde{G}_j(\zeta;\alpha), \\ \psi(\zeta;\alpha) &= \frac{\partial}{\partial x_0}\tilde{G}_j(\zeta;\alpha) \end{aligned} \tag{23}$$

with $\alpha = x_0 + iy_0$ and where the function $\tilde{G}_j(\zeta;\alpha)$ is the analytic extension of the modified Green's function $G_j(\zeta;\alpha)$ of the original multiply connected circular domain. Note that any of the $M+1$ functions $\tilde{G}_j(\zeta;\alpha)$ can be used in these formulae. Here we choose to use $\tilde{G}_0(\zeta;\alpha)$ since it has the simple form (17). By a simple change of variables, (23) can also be written

$$\begin{aligned} \phi(\zeta;\alpha) &= -\left(\frac{\partial}{\partial\bar{\alpha}} - \frac{\partial}{\partial\alpha}\right)\tilde{G}_0(\zeta;\alpha), \\ \psi(\zeta;\alpha) &= \left(\frac{\partial}{\partial\bar{\alpha}} + \frac{\partial}{\partial\alpha}\right)\tilde{G}_0(\zeta;\alpha) \end{aligned}$$

so that (22) takes the form

$$\phi_\chi(\zeta;\alpha) = \left[\frac{\partial}{\partial\alpha} - e^{2i\chi}\frac{\partial}{\partial\bar{\alpha}}\right]\tilde{G}_0(\zeta;\alpha).$$

From (17) it follows that

$$\begin{aligned} \frac{\partial\tilde{G}_0(\zeta;\alpha)}{\partial\alpha} &= \frac{1}{2\alpha} - \frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)}, \\ \frac{\partial\tilde{G}_0(\zeta;\alpha)}{\partial\bar{\alpha}} &= \frac{1}{2\bar{\alpha}} - \frac{1}{\bar{\alpha}^2}\frac{\omega_\alpha(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})} \end{aligned}$$

where the notation $\omega_\alpha(\zeta,\alpha)$ denotes the derivative of $\omega(\zeta,\alpha)$ with respect to its second argument. Combining all this leads to the final formula

$$\phi_\chi(\zeta;\alpha) = -e^{2i\chi}\left(\frac{1}{2\bar{\alpha}} - \frac{1}{\bar{\alpha}^2}\frac{\omega_\alpha(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})}\right) + \frac{1}{2\alpha} - \frac{\omega_\alpha(\zeta,\alpha)}{\omega(\zeta,\alpha)}.$$

7.5. The radial slit domain. Again borrowing Nehari's notation [16], let $Q(\zeta;\alpha,\beta)$ be the conformal mapping to a radial slit domain where the point $\zeta=\alpha$ maps to the origin in the image plane while $\zeta=\beta$ maps to infinity. Neither Schiffer [17] nor Nehari [16] give a simple way to construct this mapping from the modified Green's functions, but the required conformal mapping in this case turns out to be

$$Q(\zeta;\alpha,\beta)=\mathcal{C}(\alpha,\beta)\frac{\omega(\zeta,\alpha)\omega(\zeta,\bar{\alpha}^{-1})}{\omega(\zeta,\beta)\omega(\zeta,\bar{\beta}^{-1})}.$$

To establish this, the transformation formulae (6) and (8) can be used repeatedly to demonstrate that, on each boundary circle C_j, $j=0,1,\ldots,M$, the argument of $Q(\zeta;\alpha,\beta)$ is constant (this is easily done by computing $\overline{Q(\zeta;\alpha,\beta)}$ on each circle and showing that it is proportional to $Q(\zeta;\alpha,\beta)$). This shows that the images of each circle are rays emanating from the origin. Then, standard arguments based on the argument principle and following those in Nehari [16] can be used to show the univalency of the mapping using the fact that it has just one zero and one pole inside the original circular domain.

The normalization constant $\mathcal{C}(\alpha,\beta)$ can be chosen as required. Nehari [16] again elects to set the residue of $Q(\zeta;\alpha,\beta)$ at β equal to unity, in which case

$$\mathcal{C}(\alpha,\beta)=\frac{\omega'(\beta,\beta)\omega(\beta,\bar{\beta}^{-1})}{\omega(\beta,\alpha)\omega(\beta,\bar{\alpha}^{-1})}.$$

8. Examples

Since the formulae in the construction involve various infinite products it is necessary, for any numerical implementation, to truncate them in a sensible way. To do this, it is convenient to categorize all possible compositions of the basic maps according to their *level*. As an illustration, consider the case in which there are four basic maps θ_j, $j=1,2,3,4$. The identity map is considered to be the *level-zero map*. The four basic maps, together with their inverses, θ_j^{-1}, $j=1,2,3,4$, constitute the eight *level-one maps*. All possible combinations of any *two* of these eight level-one maps which do not reduce to the identity, e.g.

$$\theta_1(\theta_1(\zeta)),\ \theta_1(\theta_2(\zeta)),\ \theta_1(\theta_3(\zeta)),\ \theta_1(\theta_4(\zeta)),\ \theta_2(\theta_1(\zeta)),\ \theta_2(\theta_2(\zeta)),\ldots.$$

will be called the *level-two maps*, all possible combinations of any *three* of the eight level-one maps that do not reduce to a lower-level map will be called the *level-three maps*, and so on.

As illustrative examples of the above mapping formulae in action, Figure 3 shows the case of an arbitrarily chosen triply connected circular domain and its images under the five mappings of Subsections 7.1–7.5. Figure 4 shows the case of a quadruply connected domain. These figures were drawn by keeping all maps in the set Θ'' up to and including the level-three maps.

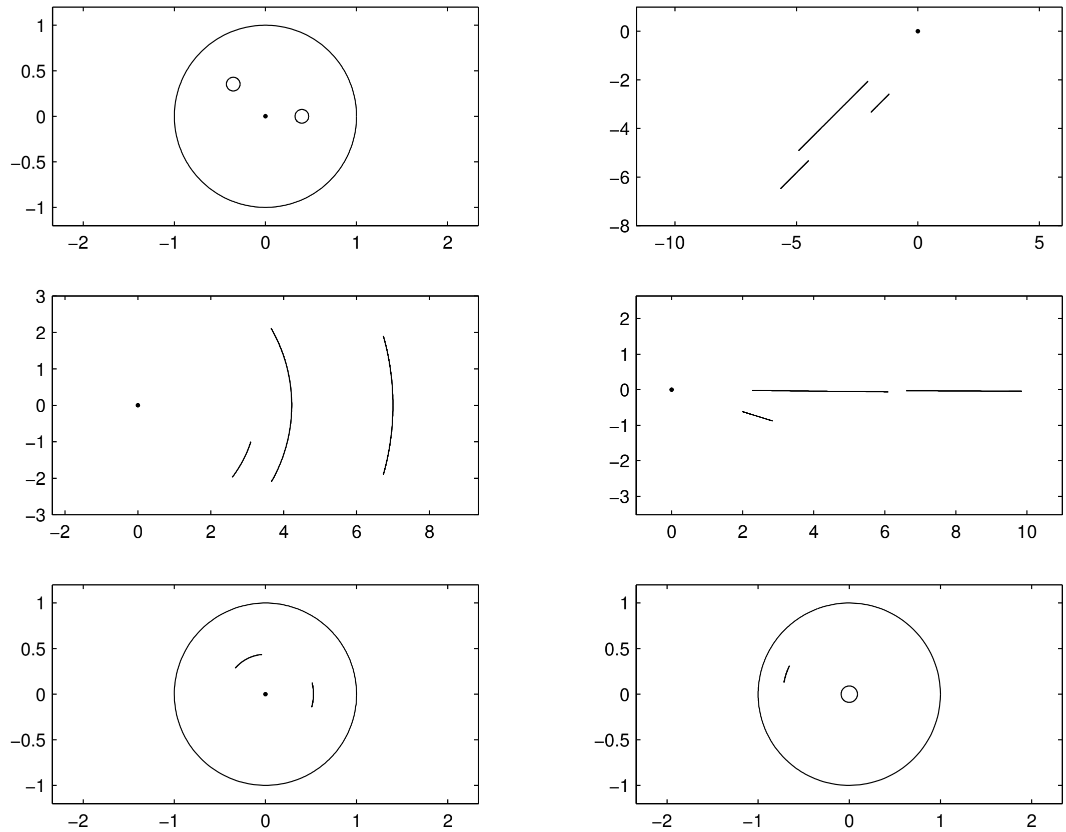

FIGURE 3. Conformal mappings from a triply connected circular domain with parameters $\delta_1 = 0.4$, $q_1 = 0.075$, $\delta_2 = 0.5e^{3\pi i/4}$, $q_2 = 0.075$ with parameters $\chi = \pi/4$, $\alpha = -0.15$ and $\beta = 0.1$ to five types of conformally equivalent slit domains.

9. Discussion

In this paper we have systematically derived analytical formulae for the conformal maps from the canonical class of multiply connected circular domains to various canonical classes of multiply connected slit domains. These should prove useful in applications, and have indeed been used already in a variety of contexts. To mention a few examples, it has recently been recognized [8] that the parallel slits maps of Subsection 7.4 can be used to provide a natural uniformization of a spectral problem associated with the integrable system known as the Benney hierarchy of moment equations [9]. On the other hand, the mappings to radial slit domains of Subsection 7.2 have formed the basis for an analytical study of the motion of point vortices through gaps in walls [6], a problem of importance in geophysical fluid dynamics. In the context of a more general problem of constructing multiply connected conformal mappings, the mappings to a circular disc with concentric circular arcs of Subsection 7.1 have provided a

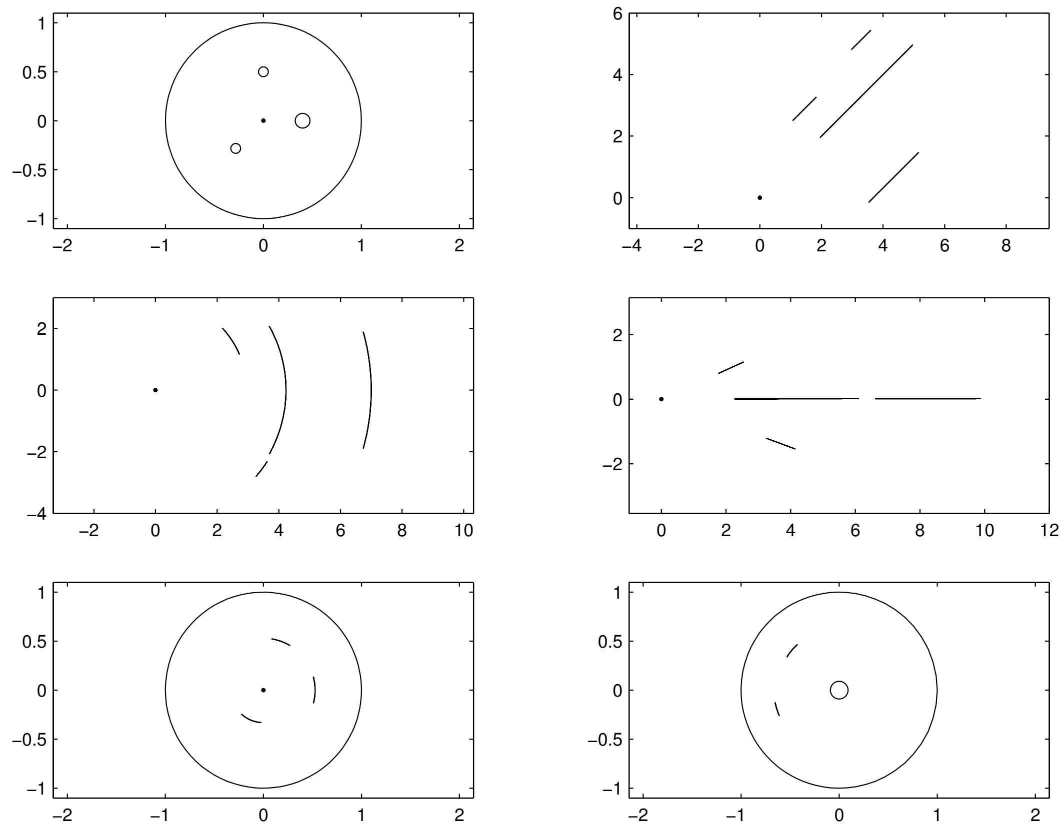

FIGURE 4. Conformal mappings from a quadruply connected circular domain with parameters $\delta_1 = 0.4$, $q_1 = 0.075$, $\delta_2 = 0.5i$, $q_2 = 0.05$, $\delta_3 = 0.4e^{5\pi i/4}$, $q_4 = 0.05$ with parameters $\chi = \pi/4$, $\alpha = -0.15$ and $\beta = 0.1$ to the five types of conformally equivalent slit domains.

crucial analytical ingredient in a recent construction of new formulae for generalized Schwarz-Christoffel mappings from circular domains to multiply connected polygonal regions [7].

Appendix: Properties of G_j on the boundary circles

In this appendix, properties of the Schottky-Klein prime function are used to establish the properties of the functions $G_j(\zeta;\alpha)$ given in (10).

First, consider the complex conjugate of $\tilde{R}_j(\zeta;\alpha)$ as defined in (12) for ζ on C_j. Then

$$\overline{\tilde{R}_j(\zeta;\alpha)} = \frac{\overline{\omega}(\bar{\zeta},\bar{\alpha})\omega(\overline{\phi}_j(\bar{\zeta}),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\bar{\zeta},\phi_j(\alpha))\omega(\overline{\phi}_j(\bar{\zeta}),\alpha)}.$$

But, for ζ on C_j,

$$\bar{\zeta} = \phi_j(\zeta)$$

so that

$$\overline{\tilde{R}_j(\zeta;\alpha)} = \frac{\overline{\omega}(\phi_j(\zeta),\bar{\alpha})\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))\omega(\zeta,\alpha)} = \frac{1}{\tilde{R}_j(\zeta;\alpha)}.$$

This confirms that

$$|\tilde{R}_j(\zeta;\alpha)| = 1 \qquad \text{on } C_j.$$

Consider next points ζ on C_0. In this case, we have

$$\overline{\tilde{R}_j(\zeta;\alpha)} = \frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})\omega(\overline{\phi}_j(\zeta^{-1}),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))\omega(\overline{\phi}_j(\zeta^{-1}),\alpha)} = \frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})\omega(\theta_j(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))\omega(\theta_j(\zeta),\alpha)} \tag{24}$$

where we have used that $\bar{\zeta} = \zeta^{-1}$ for ζ on C_0 and the identity $\overline{\phi}_j(\zeta^{-1}) = \theta_j(\zeta)$. But

$$\frac{\omega(\theta_j(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_j(\zeta),\alpha)} = \beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)\frac{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\omega(\zeta,\alpha)} \tag{25}$$

where we have used (6), and

$$\frac{\overline{\omega}(\overline{\theta}_j(\zeta^{-1}),\bar{\alpha})}{\overline{\omega}(\overline{\theta}_j(\zeta^{-1}),\phi_j(\alpha))} = \overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))}$$

which implies that

$$\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))} = \frac{1}{\overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))}\frac{\overline{\omega}(\phi_j(\zeta),\bar{\alpha})}{\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}. \tag{26}$$

On use of (25) and (26) in (24), we get

$$\overline{\tilde{R}_j(\zeta;\alpha)} = \frac{\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)}{\overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))}\frac{1}{\tilde{R}_j(\zeta;\alpha)}$$

or

$$|\tilde{R}_j(\zeta;\alpha)| = |\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)|.$$

Finally, consider points ζ on C_k where $k \neq j$. Then, we have

$$\begin{aligned}
\overline{\tilde{R}_j(\zeta;\alpha)} &= \frac{\overline{\omega}(\phi_k(\zeta),\bar{\alpha})\omega(\overline{\phi}_j(\phi_k(\zeta)),\overline{\phi}_j(\bar{\alpha}))}{\overline{\omega}(\phi_k(\zeta),\phi_j(\alpha))\omega(\overline{\phi}_j(\phi_k(\zeta)),\alpha)} \\
&= \frac{\overline{\omega}(\overline{\theta}_k(\zeta^{-1}),\bar{\alpha})}{\overline{\omega}(\overline{\theta}_k(\zeta^{-1}),\phi_j(\alpha))}\frac{\omega(\theta_j((\phi_k(\zeta))^{-1}),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_j((\phi_k(\zeta))^{-1}),\alpha)} \\
&= \beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha))\overline{\beta}_k(\bar{\alpha},\phi_j(\alpha))\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))}\frac{\omega((\phi_k(\zeta))^{-1},\overline{\phi}_j(\bar{\alpha}))}{\omega((\phi_k(\zeta))^{-1},\alpha)} \\
&= \beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha))\overline{\beta}_k(\bar{\alpha},\phi_j(\alpha))\frac{\overline{\omega}(\zeta^{-1},\bar{\alpha})}{\overline{\omega}(\zeta^{-1},\phi_j(\alpha))}\frac{\omega(\theta_k^{-1}(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_k^{-1}(\zeta),\alpha)}
\end{aligned} \tag{27}$$

where, in the last equation, we have made use of (3). But,

$$\frac{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))}{\omega(\zeta,\alpha)} = \beta_k(\overline{\phi}_j(\bar{\alpha}),\alpha)\frac{\omega(\theta_k^{-1}(\zeta),\overline{\phi}_j(\bar{\alpha}))}{\omega(\theta_k^{-1}(\zeta),\alpha)}. \tag{28}$$

On use of (26) and (28) in (27)

$$\overline{\tilde{R}_j(\zeta;\alpha)} = \frac{\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)\overline{\beta}_k(\bar{\alpha},\phi_j(\alpha))}{\overline{\beta}_j(\bar{\alpha},\phi_j(\alpha))\beta_k(\overline{\phi}_j(\bar{\alpha}),\alpha)}\frac{1}{\tilde{R}_j(\zeta;\alpha)},$$

or

$$|\tilde{R}_j(\zeta;\alpha)| = \left|\frac{\beta_j(\overline{\phi}_j(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi}_j(\bar{\alpha}),\alpha)}\right|.$$

To summarize all the above results, we conclude that

$$|\tilde{R}_j(\zeta;\alpha)| = \left|\frac{\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right| \qquad \text{on } C_k.$$

This formula holds for all j and k provided we adopt the convention $\beta_0(\zeta,\alpha)\equiv 1$.

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