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CONFORMAL METRICS AND RICCI TENSORS IN THE PSEUDO-EUCLIDEAN SPACE

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ABSTRACT. We consider constant symmetric tensors T on R^n, $n \geq 3$, and we study the problem of finding metrics $\bar{g}$ conformal to the pseudo-Euclidean metric g such that $\mathrm{Ric}\,\bar{g} = T$. We show that such tensors are determined by the diagonal elements and we obtain explicitly the metrics $\bar{g}$. As a consequence of these results we get solutions globally defined on R^n for the equation $-\varphi\Delta_g\varphi + n||\nabla_g\varphi||^2/2 + \lambda\varphi^2 = 0$. Moreover, we show that for certain unbounded functions $\overline{K}$ defined on R^n, there are metrics conformal to the pseudo-Euclidean metric with scalar curvature $\overline{K}$.

1. Introduction

Over the last few years several authors have considered the following problem:

(P) Given a symmetric tensor of order two T defined on a manifold M^n, is there a Riemannian metric g such that Ric $g = T$?

Finding solutions to this problem is equivalent to solving a nonlinear system of second-order partial differential equations. Deturck showed in [D1] that if $n \geq 3$, problem (P) has a local solution, when the given tensor T is nonsingular. Results on the existence and uniqueness of solutions for the problem (P), whenever M^n is a bi-dimensional manifold, can be found in [D2] and [CD1]. For compact manifolds, some results can be found in [DK], [H] and [X].

Cao and Deturck [CD2] studied the existence and uniqueness of global solutions in R^n and S^n for rotationally symmetric and nonsingular tensors. In this case, they showed that problem (P) has a unique solution (up to homothety) and that for certain tensors in R^n, there is a complete metric g, globally defined on R^n, such that Ric $g = T$. On the sphere S^n, they proved some non-existence results and they found necessary conditions on a given tensor T, for the existence of a metric g on S^n satisfying Ric $g = T$.

There are two reasons for considering only nonsigular tensors T in [CD2]. First, uniqueness may fail. In the nonrotationally-symmetric context, there are examples where the solution of Ric $g = T$ is not unique (see [DK]). The second reason is that even local existence may fail (see [D2]).

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Our main purpose in this work is to study problem (P) in R^n, $n \geq 3$, for constant symmetric tensors of the form

$$T = \sum_{i,j} \varepsilon_j c_{ij} dx_i \otimes dx_j \quad \text{with} \quad c_{ij} \in R, \tag{1}$$

requiring the metric to be conformal to the pseudo-Euclidean metric.

More precisely, we consider (R^n, g) with $n \geq 3$, $g_{ij} = \delta_{ij}\varepsilon_i$, $\varepsilon_i = \pm 1$, for all $1 \leq i, j \leq n$, where at least one eigenvalue ε_i is positive. We want to find metrics $\bar{g}$ such that

$$\begin{cases} \bar{g} = \dfrac{1}{\varphi^2} g, \\ \text{Ric } \bar{g} = T. \end{cases} \tag{2}$$

In Theorems 1.1 and 1.2, we treat the case of non-diagonal constant symmetric tensors T. In Theorem 1.1, we assume $\sum_i c_{ii} \neq 0$ and we give a necessary and sufficent condition for the existence of a metric $\bar{g}$ satisfying (2). We show that such tensors are determined by its diagonal elements c_{ii}, $1 \leq i \leq n$, which belong to a subset of R^n. This is a non-empty set obtained as the intersection of half spaces. For each such n-tuple $c = (c_{11}, ..., c_{nn})$, there are at least 2 and generically 2^{n-1} tensors T for which there exists, up to homothety, two metrics $\bar{g}$ satisfying Ric $\bar{g} = T$. In Theorem 1.2 we consider non-diagonal tensors T which satisfy $\sum_i c_{ii} = 0$. In Theorem 1.3 we treat the existence of metrics $\bar{g}$ satsfying (2) for non-zero diagonal tensors T. The case $T \equiv 0$ is treated in Theorem 1.4. Moreover, in each of these theorems, the metrics are given explicitly and most of them are globally defined on R^n. However, we show that there are no complete metrics $\bar{g}$, conformal to g, such that Ric $\bar{g} = T$.

As a consequence of Theorems 1.1, 1.2 and 1.4 we find infinitely many explicit solutions of C^∞ class, defined on R^n for the equation

$$-\varphi \Delta_g \varphi + \frac{n}{2} ||\nabla_g \varphi||^2 + \lambda \varphi^2 = 0$$

where Δ_g and ∇_g are the laplacian and gradient in the metric g respectively and the constant $\lambda \leq 0$ whenever g is the Euclidean metric and $\lambda \in R$ when g is the pseudo-Euclidean metric.

Finally, we show that for certain functions $\overline{K}$ defined on R^n, there are metrics $\bar{g}$, conformal to g, with scalar curvature $\overline{K}$. These provide examples of unbounded functions which have positive answers to the following problem: Given a smooth function $\overline{K} : M \to R$ on a manifold (M, g) is there a metric $\bar{g}$ conformal to g whose scalar curvature is $\overline{K}$?

This problem has been studied by various authors. Particularly, when $\overline{K}$ is a constant it is known as the Yamabe Problem. If $M^n = R^n$ with $n \geq 3$ and g is the Euclidean metric, various results can be found in [B], [K], [CN], [N], [DN], [LN] and in their references.

In order to state the results obtained in this paper, we need to introduce some notation. For a fixed pseudo-Euclidean metric $g_{ij} = \delta_{ij}\varepsilon_i$, $\varepsilon_i = \pm 1$, we consider the linear functions β_i, $1 \leq i \leq n$, defined for each $x = (x_1, ..., x_n) \in R^n$ by

$$\beta_i(x) = (n-1)x_i - \sum_{k=1}^{n} x_k. \tag{3}$$

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We consider the following subsets of R^n:

$$D = \{x \in R^n;\ \varepsilon_j \beta_j(x) \geq 0\ \forall j,\ 1 \leq j \leq n\}, \tag{4}$$

$$L = \{x \in R^n;\ \varepsilon_j \beta_j(x) \leq 0\ \forall j,\ 1 \leq j \leq n\} \tag{5}$$

and the hyperplanes

$$\pi_i = \{x \in R^n;\ \beta_i(x) = 0\}, \quad 1 \leq i \leq n. \tag{6}$$

D and L are nonempty subsets of R^n, obtained as the intersection of half-spaces of R^n, whose boundary is the union of the hyperplanes π_i. With this notation we can now state our results.

Theorem 1.1. *Let (R^n, g) be a pseudo-Euclidean space and let T be a non-diagonal symmetric tensor as in (1) such that $\sum_i c_{ii} \neq 0$. Then there is a metric $\bar{g} = g/\varphi^2$ such that Ric $\bar{g} = T$, if and only if, $c = (c_{11}, \ldots, c_{nn}) \in D \setminus \{\pi_\ell \cup \pi_k\}$ for some $\ell \neq k$ and*

$$c_{ij} = \frac{\varepsilon_j \gamma_i \gamma_j}{n-1} \sqrt{\varepsilon_i \varepsilon_j \beta_i \beta_j}(c) \quad \forall i \neq j \tag{7}$$

where $\gamma_j = \pm 1$ for $1 \leq j \leq n$. Moreover, for any such fixed tensor T, the solutions are given by

$$\varphi(x) = k \exp\left(\frac{\delta}{\sqrt{(n-2)(n-1)}} \left(\sum_j \gamma_j \sqrt{\varepsilon_j \beta_j(c)}\, x_j \right) \right) \tag{8}$$

where k is a non-zero constant and $\delta = \pm 1$.

In Theorem 1.1, for each $c \in D \setminus \{\pi_\ell \cup \pi_k\}$, the expressions in (7) define at least two and generically 2^{n-1} tensors T.

Theorem 1.2. *Let (R^n, g) be a pseudo-Euclidean space and let T be a non-diagonal symmetric tensor as in (1) such that $\sum_i c_{ii} = 0$. Then there is a metric $\bar{g} = g/\varphi^2$ such that Ric $\bar{g} = T$, if and only if $c = (c_{11}, \ldots, c_{nn}) \in (D \cup L) \setminus \{\pi_\ell \cup \pi_k\}$ for some $\ell \neq k$ and*

$$c_{ij} = \begin{cases} \varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}} & \forall i \neq j \quad \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\}, \\ -\varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}} & \forall i \neq j \quad \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\} \end{cases} \tag{9}$$

where $\gamma_j = \pm 1$ for $1 \leq j \leq n$. Moreover, for any such fixed tensor T, the function φ is constant if g is the Euclidean metric and otherwise it is given by

$$\varphi(x) = \begin{cases} k_1 \exp\left(\sum_j h_j(x_j)\right) + k_2 \exp\left(-\sum_j h_j(x_j)\right) & \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\}, \\ k_1 \cos\left(\sum_j h_j(x_j)\right) + k_2 \sin\left(\sum_j h_j(x_j)\right) & \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\} \end{cases} \tag{10}$$

and

$$h_j(x_j) = \begin{cases} \sqrt{\frac{\varepsilon_j c_{jj}}{n-2}}\gamma_j x_j & \text{if } c \in D \setminus \{\pi_\ell \cup \pi_k\}, \\ \sqrt{\frac{-\varepsilon_j c_{jj}}{n-2}}\gamma_j x_j & \text{if } c \in L \setminus \{\pi_\ell \cup \pi_k\}. \end{cases} \tag{11}$$

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Theorem 1.3. *Let (R^n, g) be a pseudo-Euclidean space and let $T = \sum_{i=1}^n \varepsilon_i c_{ii} dx_i^2$ be a non-zero diagonal tensor. Then there exists $\bar{g} = g/\varphi^2$ such that Ric $\bar{g} = T$, if and only if $T = T_k$ for some k, $1 \leq k \leq n$, where $T_k = b \sum_{i \neq k} \varepsilon_i dx_i^2$ with $b\varepsilon_k < 0$. In this case,*

$$\bar{g}_{ij} = \delta_{ij} \varepsilon_i \exp\left(a - 2\delta \sqrt{\frac{-b\varepsilon_k}{n-2}} x_k \right)$$

where $\delta = \pm 1$ and $a \in R$.

The tensors T_k considered in this theorem are singular. However, in contrast with the results of [D2], they admit metrics $\bar{g}$, globally defined on R^n such that Ric $\bar{g} = T_k$.

Theorem 1.4. *Let (R^n, g) be a pseudo-Euclidean space. Then there exists $\bar{g} = g/\varphi^2$ such that Ric $\bar{g} = 0$, if and only if*

$$\varphi = \sum_{j=1}^n (A\varepsilon_j x_j^2 + B_j x_j + C_j) \qquad \textit{where} \qquad 4A \sum_j C_j - \sum_j \varepsilon_j B_j^2 = 0 \tag{12}$$

and the constants $A, C_j, B_j \in R$.

As a consequence of the above theorems we obtain:

Corollary 1.5. *Let (R^n, g) be a pseudo-Euclidean space. For each $\lambda \in R$ ($\lambda \leq 0$ if g is the Euclidean metric), the equation*

$$-\varphi \Delta_g \varphi + \frac{n}{2} ||\nabla_g \varphi||^2 + \lambda \varphi^2 = 0 \tag{13}$$

has infinitely many solutions, of C^∞ class, globally defined on R^n:

- **a)** *If $\lambda \neq 0$, then the functions given by (8) satisfy (13), whenever $c = (c_{11}, \ldots, c_{nn}) \in D \setminus \{\pi_\ell \cup \pi_k\}$ is chosen such that $\lambda = \sum_{i=1}^n c_{ii}/2(n-1)$.*
- **b)** *If $\lambda = 0$, the functions given by (12) and (10), where $c \in (D \cup L) \setminus \{\pi_\ell \cup \pi_k\}$ is chosen such that $\sum_{i=1}^n c_{ii} = 0$, are solutions of (13). In particular, the solutions given by (10) satisfy $||\nabla_g \varphi|| = \Delta_g \varphi = 0$.*

Corollary 1.6. *Let (R^n, g) be a pseudo-Euclidean space. For each n-tuple $c = (c_{11}, \ldots, c_{nn}) \in D \setminus \{\pi_i \cup \pi_j\}$, such that $\sum_i c_{ii} \neq 0$, let $\beta_j(c)$ be the constants defined by (3). Consider the function $\overline{K} : R^n \to R$ given by*

$$\overline{K}(x) = \sum_i c_{ii} \exp\left(\frac{2\delta}{\sqrt{(n-2)(n-1)}} \left(\sum_j \gamma_j \sqrt{\varepsilon_j \beta_j(c)}\, x_j \right) \right) \tag{14}$$

where $\delta = \pm 1$, $\gamma_j = \pm 1$ for $1 \leq j \leq n$. Then the metric $\bar{g} = g/\varphi^2$, where φ is given by (8), has scalar curvature $\overline{K}$. In particular, if (R^n, g) is the Euclidean space, then $\overline{K} < 0$.

Corollary 1.7. *Let (R^n, g) be a pseudo-Euclidean space. The metrics $\bar{g} = g/\varphi^2$, where φ is given by (10) and (12), have flat scalar curvature $\bar{K}$. In particular, in the latter case the metrics have flat seccional curvature.*

Corollary 1.8. *Let (R^n, g) be a pseudo-Euclidean space. For any constant symmetric tensor T, there are no complete metrics $\bar{g}$, conformal and non-homothetic to g, such that Ric $\bar{g} = T$.*

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We conclude this section by observing that the metrics $\bar{g}$, obtained in Theorems 1.1-1.3, satisfy the relation

$$\text{Ric } \bar{g} - \text{Ric } g = C.g \quad \text{where} \quad C = (c_{ij}) \quad \text{with} \quad c_{ij} = \frac{\varepsilon_j}{\varepsilon_i} c_{ji} \in R.$$

In [KR], Khünel and Rademacher studied conformal metrics $\bar{g} = g/\varphi^2$ in semi-Riemannian manifolds (M, g) satisfying the relation Ric $\bar{g}$ − Ric $g = (n-1)\lambda g$ where $\lambda \in R$. They showed that if M is a pseudo-Euclidean space, then $\lambda = 0$ and φ is constant. Consequently g and $\bar{g}$ are homothetic. Moreover, they showed that if (M, g) is a semi-Riemannian complete manifold and $\bar{g} = g/\varphi^2$ is globally defined, then M is necessarily a Riemannian manifold. Theorems 1.1, 1.2 and 1.3 show that this result does not hold for matrices C which are not multiple of the identity matrix.

2. Proof of the main results

We will start with some lemmas which will be used in the proof of Theorems 1.1-1.4.

Lemma 2.1. *Solving problem* (2) *is equivalent to studying the following system of equations:*

$$\text{(15)} \qquad \begin{cases} \varphi_{x_i x_i} = \varepsilon_i \left(\lambda_i \varphi + \dfrac{||\nabla_g \varphi||^2}{2\varphi} \right), \\ \varphi_{x_i x_j} = \dfrac{\varepsilon_j c_{ij}}{n-2} \varphi, \end{cases} \qquad 1 \leq i \neq j \leq n,$$

where

$$\text{(16)} \qquad \lambda_i = \frac{2(n-1)c_{ii} - \sum_{\ell} c_{\ell\ell}}{2(n-1)(n-2)}.$$

Proof. We know (see for example [E], [KR]) that if (M, g) is a semi-Riemannian manifold and $\bar{g} = g/\varphi^2$, then the Ricci tensors satisfy the relation

$$\text{(17)} \qquad \text{Ric } \bar{g} - \text{Ric } g = \frac{1}{\varphi^2} \left\{ (n-2)\varphi \, \text{Hess}_g(\varphi) + (\varphi \Delta_g \varphi - (n-1)||\nabla_g \varphi||^2) g \right\}.$$

Since Ric $g = 0$, using (17) we obtain that (2) is equivalent to studying the following system of equations:

$$\text{(18)} \qquad \frac{1}{\varphi^2} \{ (n-2)\varphi \text{Hess}_g(\varphi)_{ij} + \left(\varphi \Delta_g \varphi - (n-1)||\nabla_g \varphi||^2 \right) g_{ij} \} = \varepsilon_j c_{ij}$$

where, for each $1 \leq i, j \leq n$,

$$(\text{Hess}_g \varphi)_{ij} = \varphi_{x_i x_j}, \quad \Delta_g \varphi = \sum_i \varepsilon_i \varphi_{x_i x_i}, \quad ||\nabla_g \varphi||^2 = \sum_i \varepsilon_i (\varphi_{x_i})^2.$$

The system of equations (18) is given by

$$\text{(19)} \qquad \begin{cases} \dfrac{1}{\varphi^2} \left\{ (n-2)\varphi \varphi_{x_i x_i} + \left(\varphi \Delta_g \varphi - (n-1)||\nabla_g \varphi||^2 \right) \varepsilon_i \right\} = \varepsilon_i c_{ii}, \\ \varphi_{x_i x_j} = \dfrac{\varepsilon_j c_{ij} \varphi}{n-2}, \qquad 1 \leq i \neq j \leq n. \end{cases}$$

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Substituing $\Delta_g\varphi$ in the first n equations of (19) we have

$$\sum_{j\neq i}\varepsilon_j\varphi_{x_jx_j}+\varepsilon_i(n-1)\varphi_{x_ix_i}=c_{ii}\varphi+\frac{(n-1)||\nabla_g\varphi||^2}{\varphi}\qquad\forall\, i,\ 1\leq i\leq n. \tag{20}$$

For a fixed i, multiplying equation (20) by $(2n-3)$ and adding with the $(n-1)$ remaining equations we obtain

$$\varphi_{x_ix_i}=\varepsilon_i\left(\lambda_i\varphi+\frac{||\nabla_g\varphi||^2}{2\varphi}\right),$$

where λ_i is given by (16). The proof of the lemma follows from (19) and (20). □

Remark 2.2. For future use, considering $c=(c_{11},...,c_{nn})$, we point out the following relations:

$$(n-2)\lambda_i-\sum_k\lambda_k=\frac{\beta_i}{n-1},\qquad\forall 1\leq i\leq n, \tag{21}$$

$$\sum_i\frac{\beta_i}{n-1}=-2\sum_i\lambda_i=-\sum_i\frac{c_{ii}}{n-1}, \tag{22}$$

which follow from a straightforward computation by using (3) and (16).

Lemma 2.3. *If $\varphi:R^n\to R$ is a solution of the system of equations* (15)*, then the first derivates of φ are related by*

$$c_{ji}\varphi_{x_i}=\frac{\beta_i}{n-1}\varphi_{x_j}\quad\forall\, i\neq j. \tag{23}$$

Proof. Since φ satisfies the system (15), it follows from the comutativity of the third order derivates that the following equations hold:

$$(\lambda_i+\lambda_j)\varphi_{x_j}+\sum_{\substack{\ell\neq j\\ \ell\neq i}}\frac{c_{j\ell}}{n-2}\varphi_{x_\ell}=0,\qquad 1\leq i\neq j\leq n. \tag{24}$$

If $n=3$, then equation (24) is given by

$$(\lambda_i+\lambda_j)\varphi_{x_j}+c_{j\ell}\varphi_{x_\ell}=0,\qquad 1\leq i\neq j\neq\ell\leq 3,$$

which reduces to (23) as a consequence of (21). If $n\geq 4$, for a fixed pair (i,j), multiplying equation (24) by $-(n-3)$ and adding with the $n-2$ equations (24) given by the pairs (k,j), with $k\neq i$ and $k\neq j$, we obtain equation

$$c_{ji}\varphi_{x_i}=\left((n-2)\lambda_i-\sum_k\lambda_k\right)\varphi_{x_j}\quad\forall\, i\neq j.$$

Equation (23) follows from (21). □

Our next lemma shows that the symmetric tensors T given by (1), for which the system of equations (2) has a solution, are necessarily determined by its diagonal elements.

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Lemma 2.4. *Let (R^n, g) be a pseudo-Euclidean space and let $T = \sum_{i,j} \varepsilon_j c_{ij} dx_i \otimes dx_j$ be a non-diagonal symmetric constant tensor. If there exists a metric $\bar{g} = g/\varphi^2$ such that Ric $\bar{g} = T$, then*

$$\frac{||\nabla_g \varphi||^2}{2\varphi} = -\sum_{k=1}^{n} \frac{\lambda_k \varphi}{n-2} \tag{25}$$

and the components of the tensor T are such that $c = (c_{11}, \dots, c_{nn}) \in (D \cup L) \setminus \{\pi_r \cup \pi_\ell\}$ for some pair $(r, \ell), 1 \le r \ne \ell \le n$, D, L and π_r are given by (4), (5) *and* (6) *and*

$$c_{ij} = \pm \frac{\sqrt{\varepsilon_i \varepsilon_j \beta_i \beta_j}}{n-1}(c_{11}, \dots, c_{nn}), \qquad i \ne j, \tag{26}$$

where $\beta_i(c_{11}, ..., c_{nn})$ is given by (3) *and λ_i by* (16).

Proof. Taking the derivative of (23) with respect to the variable x_j and using the system (15) we obtain

$$\frac{\beta_i}{n-1}\left(\varepsilon_j\left(\lambda_j\varphi + \frac{||\nabla_g\varphi||^2}{2\varphi}\right)\right) = \frac{c_{ji}^2}{n-2}\varepsilon_i\varphi. \tag{27}$$

Now taking the derivative of (23) with respect to the variable x_i we have

$$c_{ji}\left(\lambda_i\varphi + \frac{||\nabla_g\varphi||^2}{2\varphi}\right) = \frac{\beta_i}{(n-1)(n-2)}c_{ji}\varphi. \tag{28}$$

Since, there exists at least one term $c_{ji} \ne 0$ for $j \ne i$, we obtain equation (25) directly from (28) and (21). Now, substituing (25) in (27) and using again (21) for all j, we obtain (26). In order to have the non-diagonal terms well defined, we need to have $\varepsilon_i \varepsilon_j \beta_i \beta_j \ge 0$ for all $i \ne j$. Moreover, since T is a non-diagonal tensor, we conclude that $\beta_r^2 + \beta_\ell^2 \ne 0$, for some $r \ne \ell$. Therefore, $(c_{11}, \dots, c_{nn})$ belongs to $D \cup L \setminus \{\pi_r \cup \pi_\ell\}$. □

Remark 2.5. Let (R^n, g), $n \ge 3$, be a pseudo-Euclidean space, i.e. $g_{ij} = \delta_{ij}\varepsilon_i$, $\varepsilon_i = \pm 1$. For any fixed pair $k \ne s$, $D \setminus \{\pi_k \cup \pi_s\}$ (resp. $L \setminus \{\pi_k \cup \pi_s\}$) is a non-empty subset of R^n. In fact, let $a_j \in R$ be such that $a_j \le 0$ (resp. $a_j \ge 0$) for $1 \le j \le n$ and $a_k a_s \ne 0$. We consider $x_\ell = \sum_{j \ne \ell} a_j \varepsilon_j$. Then we have $\beta_j = -(n-1)a_j\varepsilon_j$ and hence $(x_1, ..., x_n)$ belongs to $D \setminus \{\pi_k \cup \pi_s\}$ (resp. $L \setminus \{\pi_k \cup \pi_s\}$).

We consider the map $S:\ R^n \longrightarrow R$ that for each $x = (x_1, ..., x_n) \in R^n$ associates $S(x) = \sum_i x_i$. Let (R^n, g) be a pseudo-Euclidean space. For any fixed pair $k \ne s$, let S_1 (resp. S_2) be the restriction of the function S to $D \setminus \{\pi_k \cup \pi_s\}$ (resp. $L \setminus \{\pi_k \cup \pi_s\}$). Then, one can easily see that if g is the Euclidean metric, then the image of S_1 (resp. S_2) is $(-\infty, 0)$ (resp. $(0, \infty)$). Otherwise, the image of S_1 and S_2 is the whole real line.

Proof of Theorem 1.1. From Lemma 2.1 solving the system (2) is equivalent to obtaining a nonvanishing solution φ of (15). It follows from Lemma 2.4 and (21), that if φ satisfies the system (15), then $c = (c_{11}, \dots, c_{nn}) \in (D \cup L) \setminus \{\pi_r \cup \pi_s\}$ for some $r \ne s$ and φ is a solution of

$$\begin{cases} \varphi_{x_i x_i} = \alpha_i \varphi, \\ \varphi_{x_i x_j} = \dfrac{\varepsilon_j c_{ij}}{n-2}\varphi \end{cases} \tag{29}$$

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where

$$\alpha_i = \frac{\varepsilon_i \beta_i(c)}{(n-1)(n-2)}, \qquad c_{ij} = \pm \frac{1}{n-1} \sqrt{\varepsilon_i \varepsilon_j \beta_i \beta_j}(c) \tag{30}$$

and β_i is given by (3). Moreover, φ satisfies (25).

Assume that $(c_{11}, \dots, c_{nn}) \in L \setminus \{\pi_r \cup \pi_s\}$. Then $\alpha_s < 0$ and consequently the solutions of (29) are given by

$$\varphi(x_1, \dots, x_n) = f(\hat{x}_s)\cos(\sqrt{-\alpha_s} x_s) + g(\hat{x}_s)\sin(\sqrt{-\alpha_s} x_s) \tag{31}$$

where f and g are smooth functions of $\hat{x}_s = (x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_n)$.

Since $\varphi_{x_s x_j} = \dfrac{\varepsilon_j c_{sj}}{n-2}\varphi$ for all $j \neq s$ we obtain that

$$(f)_{x_j} = \frac{-\varepsilon_j c_{sj}}{(n-2)\sqrt{-\alpha_s}} g \quad \text{and} \quad (g)_{x_j} = \frac{\varepsilon_j c_{sj}}{(n-2)\sqrt{-\alpha_s}} f$$

for all $j \neq s$. Therefore, we have

$$||\nabla_g \varphi||^2 \quad = -\sum_{i=1}^{n} \frac{\beta_i}{(n-1)(n-2)} \left(f \sin(\sqrt{-\alpha_s} x_s) - g \cos(\sqrt{-\alpha_s} x_s) \right)^2.$$

On the other hand, from (25) we have

$$||\nabla_g \varphi||^2 = \frac{-2}{n-2} \sum_i \lambda_i \left(f \cos(\sqrt{-\alpha_s} x_s) + g \sin(\sqrt{-\alpha_s} x_s) \right)^2.$$

Comparing those relations we get, as a consequence of (22), that

$$\left(f^2(\hat{x}) + g^2(\hat{x}) \right) \sum_{i=1}^{n} c_{ii} = 0.$$

Since $\sum_{i=1}^n c_{ii} \neq 0$, we conclude that if $c = (c_{11}, \dots, c_{nn}) \in L \setminus \{\pi_s \cup \pi_r\}$, then the system (29) does not admit non-zero solution.

If $c \in D \setminus \{\pi_s \cup \pi_r\}$, then $\alpha_j \geq 0$ for all j. Let

$$\Im = \{j, 1 \leq j \leq n, ; \alpha_j > 0\}.$$

It follows from the first equation of (29) that

$$\varphi(x) = \tilde{f} \exp\left(\sum_{j \in \Im} \gamma_j \sqrt{\alpha_j} x_j \right) + \tilde{g} \exp\left(-\sum_{j \in \Im} \gamma_j \sqrt{\alpha_j} x_j \right) \tag{32}$$

where $\gamma_j = \pm 1$, $\tilde{f}$ and $\tilde{g}$ are functions of the variables x_i with $i \notin \Im$. From the second equation of (29), we have that

$$\varphi_{x_i x_j} = 0 \quad \text{for all} \quad j \in \Im \text{ and } i \notin \Im.$$

Therefore, we conclude that the functions $\tilde{f}$ and $\tilde{g}$ are constant.

If $i, j \in \Im$, it follows from (32), (30) and the second equation of (29) that

$$c_{ij} = \frac{\varepsilon_j \gamma_i \gamma_j}{n-1} \sqrt{\varepsilon_i \varepsilon_j \beta_i \beta_j}.$$

Finally, using the relation (25) one concludes that $\tilde{f} = 0$ or $\tilde{g} = 0$.

Conversely, for each $c = (c_{11}, ..., c_{nn}) \in D \setminus \{\pi_\ell \cup \pi_k\}$, such that $\sum_i c_{ii} \neq 0$, let c_{ij} for $i \neq j$ be defined by (7), where we have chosen $\gamma_j = \pm 1$ for $1 \leq j \leq n$. The tensor T is fixed with any such choice. Then the two functions φ defined by (8) satisfy (15) and therefore provide metrics $\bar{g} = g/\varphi^2$ for which Ric $\bar{g} = T$. ☐

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Proof of Theorem 1.2. By hypothesis T is a non-diagonal tensor such that $\sum_i c_{ii} = 0$. Therefore, it follows from Lemma 2.4, (22), (3), (16), (25) and (26) that the system of equations (15) is given by

$$\varphi_{x_i x_i} = \frac{\varepsilon_i c_{ii}}{n-2}\varphi, \tag{33}$$

$$\varphi_{x_i x_j} = \frac{\varepsilon_j c_{ij}}{n-2}\varphi, \tag{34}$$

where $c = (c_{11}, ..., c_{nn}) \in (L \cup D) \setminus (\pi_\ell \cup \pi_k)$,

$$c_{ij} = \pm\sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}},$$

$$\begin{array}{rcl} D & = & \{(x_1, ..., x_n) \in R^n; \varepsilon_j x_j \geq 0 \ \forall j\}, \\ L & = & \{(x_1, ..., x_n) \in R^n; \varepsilon_j x_j \leq 0 \ \forall j\}, \\ \pi_j & = & \{(x_1, ..., x_n) \in R^n; x_j = 0\}. \end{array}$$

Moreover, $||\nabla_g \varphi||^2 = 0$. Therefore, if g is the Euclidean metric we conclude that φ is necessarily constant.

If $c = (c_{11}, ..., c_{nn}) \in D \setminus (\pi_\ell \cup \pi_k)$, then $\varepsilon_i c_{ii} \geq 0$ for all i. Let $\Im$ be the set of indices i such that $c_{ii} \neq 0$. Then it follows from (33) that

$$\varphi(x) = k_1 \exp(\sum_{j\in\Im} h_j(x_j)) + k_2 \exp(-\sum_{j\in\Im} h_j(x_j))$$

where h_j is defined by (11) and k_1, k_2 are functions which depend on x_i for $i \notin \Im$. From equation (34) we conclude that k_1 and k_2 are constants and

$$c_{ij} = \varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}}.$$

If $c \in L \setminus (\pi_\ell \cup \pi_k)$, then $\varepsilon_i c_{ii} \leq 0$ for all i. Let $\Im$ be the set of indices i such that $c_{ii} \neq 0$. Then it follows from (33) that

$$\varphi(x) = k_1 \cos(\sum_{j\in\Im} h_j(x_j)) + k_2 \sin(-\sum_{j\in\Im} h_j(x_j))$$

where k_1 and k_2 are functions which depend on x_i for $i \notin \Im$. From equation (34) we conclude that k_1 and k_2 are constants and

$$c_{ij} = -\varepsilon_j \gamma_i \gamma_j \sqrt{\varepsilon_i \varepsilon_j c_{ii} c_{jj}}.$$

The converse of this theorem is a straightforward computation. □

Proof of Theorem 1.3. Since $T = \sum_i \varepsilon_i c_{ii} dx_i^2$ is a non-zero tensor, it follows from Lemma 2.3 that if φ satisfies the system of equations (15), then φ is not constant and

$$0 = \beta_i(c)\varphi_{x_j} \qquad \forall i \neq j.$$

Let k be such that $\varphi_{x_k} \neq 0$. Since $n \geq 3$, there exists $i_1 \neq i_2$ distinct from k such that $\beta_{i_1} = \beta_{i_2} = 0$. It follows from (3) that $c_{i_1 i_1} = c_{i_2 i_2}$. Hence, for all $i \neq k$, $c_{ii} = b$ and therefore $\sum_j c_{jj} = (n-1)b + c_{kk}$. Since for all $i \neq k$ we have $\beta_i = 0$, we conclude from (3) that $c_{kk} = 0$. It follows that φ does not depend on more than one variable. In fact, otherwise $c_{ii} = 0$ for all i which is a contradiction since T is a non-zero tensor.

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Therefore we have that $\varphi = \varphi(x_k)$ for some $k; 1 \leq k \leq n$. Moreover, $c_{ii} = b \neq 0$ for all $i \neq k$ and $c_{kk} = 0$, i.e. $T = T_k$. In this case, the system (15) is given by

$$\begin{cases} \dfrac{\varepsilon_k(\varphi'(x_k))^2}{\varphi} + \dfrac{b\varphi}{n-2} = 0, \\ 2\varepsilon_k\varphi''(x_k) + \dfrac{b\varphi}{n-2} - \dfrac{\varepsilon_k(\varphi'(x_k))^2}{\varphi} = 0. \end{cases} \tag{35}$$

It follows from the first equation of (35) that $b\varepsilon_k < 0$ and

$$\varphi(x_k) = \frac{1}{A} \exp\left(\delta \sqrt{\frac{-b\varepsilon_k}{n-2}}\, x_k\right) \tag{36}$$

where $A \neq 0$ is a real constant and $\delta = \pm 1$. The second equation of (35) is satisfied by φ. Therefore, $\bar{g}_{ij} = \delta_{ij}\varepsilon_i \exp\left(a - 2\delta\sqrt{\frac{-\varepsilon_k b}{n-2}}\, x_k\right)$ satisfies $\mathrm{Ric}\ \bar{g} = T_k$.

Conversely, for $T = T_k$ the functions φ given by (36) define metrics $\bar{g}$ for which $\mathrm{Ric}\ \bar{g} = T_k$. □

Proof of Theorem 1.4. When $T \equiv 0$, it follows from Lemma 2.1 that φ satisfies the system of equations (15), where $\lambda_i = 0$ for all i. Therefore, it is easy to see that φ is given by (12). □

Proof of Corollary 1.5. From Theorems 1.1, 1.2 and 1.4, we have that the functions φ, given by (8), (10) and (12), are solutions of the system (15). In particular, from (18) φ also satisfies the equations

$$\frac{1}{\varphi^2}\left\{\varepsilon_i(n-2)\varphi\, \varphi_{x_i x_i} + \varphi\Delta_g\varphi - (n-1)||\nabla_g\varphi||^2\right\} = c_{ii}$$

where for all $1 \leq i \leq n$.

Equation (13) is obtained by adding these equations on i. If $\lambda \neq 0$, there are infinitely many ways to obtain $\lambda = \sum_{i=1}^{n} c_{ii}/2(n-1)$ with $c = (c_{11}, \dots, c_{nn}) \in D \setminus \{\pi_k \cup \pi_s\}$. We conclude that equation (13) has infinitely many solutions given by (8). If (R^n, g) is the Euclidean space, it is easy to see that, if $c \in D$, then $c_{ii} \leq 0$ for all i; hence $\lambda \in (-\infty, 0]$.

Similarly, when $\lambda = 0$, there are infinitely many n-tuples $c \in (D \cup L) \setminus (\pi_\ell \cup \pi_k)$ such that $\sum_i c_{ii} = 0$. Hence, the functions φ given by (10) are solutions of (13). Moreover, the family of functions φ given by (12) are also solutions of (13) when $\lambda = 0$. □

Proof of Corollary 1.6. It follows from the relation (17) that, if (R^n, g) with $n \geq 3$ is the pseudo-Euclidean space and $\overline{K} : R^n \to R$ is a smooth function, to find $\bar{g} = g/\varphi^2$ with scalar curvature $\overline{K}$ is equivalent to solving the following differential equation

$$-\varphi\Delta_g\varphi + \frac{n}{2}||\nabla_g\varphi||^2 + \frac{\overline{K}}{2(n-1)} = 0. \tag{37}$$

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Since $\overline{K} = \sum_{ij} \bar{g}^{ij} \bar{R}_{ij}$, if

$$\overline{K} = \lambda \, \exp\left(\frac{2\delta}{\sqrt{(n-2)(n-1)}} \left(\sum_j \gamma_j \sqrt{\varepsilon_j \beta_j} \;\; x_j \right) \right),$$

where β_j is given by (3), it follows from Corollary 1.4 that the functions given in (8) are solutions of the equation (36), showing that there exist metrics $\bar{g} = g/\varphi^2$ with scalar curvature $\overline{K}$. If (R^n, g) is the Euclidean space we have that $\sum c_{ii} < 0$ and consequently $\overline{K} < 0$. □

Proof of Corollary 1.7. It is a straightforward computation which follows from Theorems 1.2 and 1.4. □

Proof of Corollary 1.8. For each fixed tensor T as in Theorems 1.1 and 1.3, there exist two semi-Riemannian metrics (given by $\delta = \pm 1$) in the same conformal class which have pointwise the same Ricci tensor. Since they are not homothetic to each other, it follows from the results of [F] and [KR, Corollary 2] that they are not complete. A similar argument applies to the metrics obtained in Theorem 1.2 when $c \in D \setminus \{\pi_\ell \cup \pi_k\}$. In the remaining cases, the metric $\bar{g} = g/\varphi^2$ has singularity points. □

We conclude observing partial results were obtained in [P]. A similar theory in the hyperbolic space $H^n(-1)$ will be treated in another paper. Moreover, the techniques introduced in this paper were also used to obtain our results in [PT], on the problem of finding metrics g, conformal to the pseudo-euclidean metric, satisfying the equation Ric $g - Kg/2 = T$, where K is the scalar curvature of g and T is a constant symmetric tensor.

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