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CONFORMAL MOTION OF CONTACT MANIFOLDS WITH CHARACTERISTIC VECTOR FIELD IN THE k-NULLITY DISTRIBUTION

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Dedicated to the memory of Professor Kentaro Yano

1. Introduction

It is known (see for example, [17]) that if an m-dimensional Riemannian manifold admits a maximal, i.e., an $(m+1)(m+2)/2$-parameter group of conformal motions, then it is conformally flat. It is also known [9] that a conformally flat Sasakian (normal contact metric) manifold is of constant curvature 1. This shows that the existence of maximal conformal group places a severe restriction on the Sasakian manifold. Thus one is led to examine the effect of the existence of a single 1-parameter group of conformal motions on a Sasakian manifold. All the transformations considered in this paper are infinitesimal. Okumura [10] proved that a non-isometric conformal motion of a Sasakian manifold M of dimension $2n+1$ $(n > 1)$ is special concircular and hence if, in addition, M is complete and connected then it is isometric to a unit sphere. The proof is based on Obata's theorem [8]: "Let M be a complete connected Riemannian manifold of dimension $m > 1$. In order for M to admit a non-trivial solution ρ of the system of partial differential equations $\nabla\nabla\rho = -c^2\rho g$ (c = a constant > 0), it is necessary and sufficient that M be isometric to a unit sphere of radius $1/c$." The purpose of this paper is (i) to extend Okumura's result to dimension 3 and (ii) to study conformal motion of the more general class of contact metric manifolds (M, η, ξ, ϕ, g) satisfying the condition that the characteristic vector field ξ belongs to the k-nullity distribution $N(k)$: $p \to N_p(k) = \{Z \text{ in } T_pM\colon R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)$ for any X, Y in T_pM and a real number $k\}$ (see Tanno [15]). For $k = 1$, M is Sasakian. For $k = 0$, M is flat in dimension 3 and in dimension $2n+1 > 3$, it is locally the Riemannian product $E^{n+1} \times S^n(4)$ (see Blair [3]). We say that a vector field v on M is an infinitesimal contact transformation [12] if $£_v\eta = f\eta$ for some function f where £ denotes the Lie-derivative operator. We also say that a vector field v on M is an automorphism of the contact metric structure if v leaves all the structure tensors η, ξ, ϕ, g invariant (see [13]).

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THEOREM 1. Let v be a non-isometric conformal motion on a Sasakian 3-manifold M.

(A) If the scalar curvature of M is constant, then M is of constant curvature 1 and v is special concircular.
(B) If v is an infinitesimal contact transformation, then v is special concircular.

Hence in either case, if in addition, M is complete and connected, then it is isometric to a unit sphere.

In view of Watanabe's result [16] that a Sasakian 3-manifold is locally ϕ-symmetric if and only if its scalar curvature is constant and Theorem 1 we obtain the following corollary.

COROLLARY. Among all complete and simply connected ϕ-symmetric Sasakian 3-manifolds only the unit 3-sphere admits a non-isometric conformal motion.

Remark. For a Sasakian 3-manifold we know that the scalar curvature $R = 4 + 2H$, where H is the ϕ-sectional curvature (i.e., the sectional curvature of plane section orthogonal to ξ). So in Theorem 1 (part (A)) we could equivalently assume H constant instead of R constant.

THEOREM 2. Let M^{2n+1} be a contact metric manifold with ξ in $N(k)$ and v a conformal motion on M^{2n+1}. For $n > 1$, M is either Sasakian or v is Killing. In the second case v is an automorphism of the contact metric structure except when $k = 0$. Further for $k = 0$, a Killing vector field orthogonal to ξ cannot be an infinitesimal automorphism of the associated contact metric structure. For $n = 1$, M is either flat or Sasakian or v is an automorphism of the contact metric structure.

Remark. Theorem 2 shows that the existence of a non-isometric conformal motion on contact metric manifolds M with ξ in $N(k)$ singles out those with $k = 1$, i.e., Sasakian manifolds.

COROLLARY. Let M be a contact metric manifold of dimension ≥ 5 with ξ in $N(k)$, $k \neq 0$. If M admits a vector field leaving the Riemann curvature tensor of type $(1, 3)$ invariant then v is an automorphism of the contact metric structure.

2. Preliminaries

A differentiable $(2n+1)$-dimensional manifold M is called a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well known that given η there exists a unique vector field ξ (called the characteristic vector field) such that $(d\eta)(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric g and a $(1, 1)$-tensor field ϕ such that

$$(2.1) \qquad (d\eta)(X, Y) = g(\phi X, Y), \eta(X) = g(X, \xi) \text{ and } \phi^2 = -I + \eta \otimes \xi.$$

g is called an associated metric of η and (ϕ, η, ξ, g) a contact metric structure (see [2] as a general reference). Following [3] we denote the tensor $(1/2)\pounds_\xi\phi$ by h. h is self-adjoint and satisfies

$$(2.2) \qquad h\xi = 0, \qquad \operatorname{Tr} h = 0, \qquad \operatorname{Tr}(h\phi) = 0, \qquad h\phi = -\phi h.$$

The contact metric structure is called a K-contact structure if ξ is Killing. A contact metric structure is K-contact if and only if $h = 0$. For a contact metric manifold

$$(2.3) \qquad \nabla_X \xi = \phi X - \phi h X$$

where ∇ is the Riemannian connection of g. The contact structure on M is said to be normal if the almost complex structure on $M \times R$ defined by $J(X, f d/dt) = (\phi X - f\xi, \eta(X)d/dt)$, where f is a real-valued function, is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian manifolds are K-contact and 3-dimensional K-contact manifolds are Sasakian. If $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y)$ for a function k on a contact metric manifold then k is constant (see [11]). This generalizes Schur's theorem on contact Riemannian manifolds. We let (x^a) be a local coordinate system on M. For a contact metric manifold (see Lemma 2.1 in [15]) $\nabla^a\nabla_a\eta^b = R_a{}^b\eta^a - 4n\eta^b$, $R_a{}^b$ denoting the Ricci operator. Since the deRham Laplacian $\Delta_{\mathrm{dR}} = d\delta + \delta d$ (δ denoting codifferential operator) acts on a vector field X as $\Delta_{\mathrm{dR}} X^a = -\nabla^b\nabla_b X^a + R_b{}^a X^b$, we obtain $\Delta_{\mathrm{dR}}\xi = 4n\xi$.

PROPOSITION. The characteristic vector field of a contact metric manifold is an eigenvector of the associated deRham Laplacian with eigenvalue $4n$.

Next if ξ lies in the k-nullity distribution $N(k)$ then

$$(2.4) \qquad \eta_d R_{cba}{}^d = k(\eta_c g_{ba} - \eta_b g_{ca}).$$

For such manifolds we know [5] that $k \leq 1$, $h^2 = (k-1)\phi^2$ and

$$(2.5) \qquad R_a{}^b\phi_c{}^a - \phi_a{}^b R_c{}^a = 4(n-1)\phi_a{}^b h_c{}^a.$$

Furthermore following [5] and using (2.5) one can show that

$$(2.6) \qquad \phi_{ac} R_b{}^c + (1/2)\phi^{cd} R_{cdab} = (2n - 2 + k)\phi_{ab} + 2(n-1)\phi_{cb} h_a{}^c.$$

For $k = 1$, M is Sasakian. For $k < 1$ (see [5]) we have

$$(2.7) \qquad R_{ab} = 2(n-1)(g_{ab} + h_{ab}) + 2(nk + 1 - n)\eta_a\eta_b.$$

(2.8) $$R = 2n(2n - 2 + k).$$

In dimension 3 ($n = 1$) the condition (2.4) is equivalent (see [6]) to

(2.9) $$R_{ab} = (1/2)\{(R - 2k)g_{ab} + (6k - R)\eta_a\eta_b\}.$$

Moreover contact metric 3-manifold satisfying (2.4) or (2.9) is Sasakian ($k = 1$), flat ($k = 0$), or of constant non-zero ξ-sectional curvature $k < 1$ and constant ϕ-sectional curvature $-k$ (see [6]). (By ξ-sectional curvature we mean sectional curvature of a plane section containing ξ and by ϕ-sectional curvature the sectional curvature of a plane section spanned by a vector X and ϕX where X is orthogonal to ξ). In the last case M is locally isometric to a left-invariant metric on the Lie-group $SU(2)$ for $k > 0$ and $SL(2, R)$ for $k < 0$ (see [4]).

A vector field v in an m-dimensional Riemannian manifold (M, g) is called a conformal motion if there is a smooth scalar function ρ such that

(2.10) $$£_v g = 2\rho g.$$

A conformal motion defined by (2.10) satisfies (see [17])

(2.11) $$£_v R_{cba}{}^d = \delta_b{}^d \nabla_c \rho_a - \delta_c{}^d \nabla_b \rho_a + g_{ca}\nabla_b \rho^d - g_{ba}\nabla_c \rho^d,$$

(2.12) $$£_v R_{ab} = -(m - 2)\nabla_a \rho_b + (\Delta\rho)g_{ab},$$

(2.13) $$£_v R = -2\rho R + 2(m - 1)\Delta\rho,$$

where$\rho_a = \nabla_a \rho$, $\Delta\rho = -\nabla_a \rho^a$ and R is the scalar curvature. v is said to be concircular if $\nabla_a \rho_b = \sigma g_{ab}$ and special concircular if $\sigma = c_1\rho + c_2$ where c's are constants.

3. Auxiliary Results

LEMMA 3.1 (OKUMURA [10]). For a conformal motion v and its associated function ρ on a contact metric manifold, $\eta^a £_v \eta_a = \rho$.

LEMMA 3.2 (TANNO [12]). If a conformal motion v on a contact metric manifold leaves η invariant then v is an infinitesimal automorphism of the contact metric structure.

LEMMA 3.3. If on a contact metric manifold with ξ in $N(k)$, there exist scalar functions ρ, σ and τ satisfying $\nabla_a \rho_b = \sigma g_{ab} + \tau\eta_a\eta_b$, then $\tau = 0$.

Proof. Differentiating the equation in the hypothesis gives

$$\nabla_c \nabla_b \rho_a = (\nabla_c \sigma)g_{ab} + (\nabla_c \tau)\eta_a\eta_b + \tau[(\phi_{cb} - \phi_{db}h_c{}^d)\eta_a + (\phi_{ca} - \phi_{da}h_c{}^d)\eta_b].$$

Transvecting it with ϕ^{cb}, using Ricci identities and transvecting with η^a gives $\phi^{cb} R_{cbd}{}^a \eta_a \rho^d = 4n\tau$. Using (2.6) in the last equation gives $\tau = 0$, completing the proof. □

4. Proofs of the theorems

Before proving the theorems we first find some integrability conditions for (2.10) using (2.4), (2.10) and (2.11). Lie-differentiating (2.4) along v and using (2.11) gives

$$(4.1)\quad R_{cba}{}^{d}\pounds_v\eta_d = (k\pounds_v\eta_c + 2\rho k\eta_c + \eta_d\nabla_c\rho^d)g_{ab} - (k\pounds_v\eta_b + 2k\rho\eta_b + \eta_d\nabla_b\rho^d)g_{ca} + \eta_c\nabla_b\rho_a - \eta_b\nabla_c\rho_a.$$

Transvecting it with η^c and using (2.4) we have

$$(4.2)\quad \nabla_b\rho_a = -(2k\rho + \eta^c\eta^d\nabla_c\rho_d)g_{ab} + 2k\rho\eta_a\eta_b + \eta_c(\eta_b\nabla_a\rho^c + \eta_a\nabla_b\rho^c).$$

Transvecting this with g^{ba} gives

$$(4.3)\quad \Delta\rho = 4kn\rho + (2n-1)\eta^a\eta^b\nabla_b\rho_a.$$

Using (4.3) in (4.2),

$$(4.4)\quad \nabla_b\rho_a = \{1/(2n-1)\}(2k\rho - \Delta\rho)g_{ab} + 2k\eta_a\eta_b + \eta_c(\eta_b\nabla_a\rho^c + \eta_a\nabla_b\rho^c).$$

Next, transvecting (4.1) with ϕ^{cb} we obtain

$$(4.5)\quad (\phi^{cb}R_{cba}{}^{d} + 2k\phi_a{}^{d})\pounds_v\eta_d = -2\eta^d\phi_a{}^{c}\nabla_c\rho_d.$$

Proof of Theorem 1. Here $n = 1$, $k = 1$ and $h = 0$. Use of (2.6) and (2.9) in (4.5) yields

$$(4.6)\quad (R-6)\pounds_v\eta_a = 2\eta_c\nabla_a\rho^c + (\rho R - 6\rho - 2\eta^d\eta^c\nabla_c\rho_d)\eta_a.$$

Now applying ∇_c to (4.4), transvecting with ϕ^{cb}, using the Ricci identities, equations (2.6), (4.3) and (2.9) we obtain

$$(4.7)\quad \eta_c\nabla_a\rho^c = (1/6)\{R\rho_c - 2\nabla_c(\Delta\rho)\}\phi_a{}^{c} - (4\rho - \Delta\rho)\eta_a.$$

At this point, using the hypothesis $R =$ constant in (2.13) we have $\rho R = 2\Delta\rho$ which, on differentiation, gives $R\rho_a = 2\nabla_a(\Delta\rho)$. So (4.7) yields $\eta_c\nabla_a\rho^c = (\rho/2)(R-8)\eta_a$ and (4.4) reduces to

$$(4.8)\quad \nabla_b\rho_a = (\rho/2)(4-R)g_{ba} + \rho(R-6)\eta_a\eta_b.$$

This shows by virtue of Lemma 3.3 that $R = 6$ and hence equation (2.9) provides $R_{ab} = 2g_{ab}$. Hence M is Einstein and, being 3-dimensional, is of constant curvature 1. Finally, v is special concircular from (4.8). This proves part (A). For part (B) since $\pounds_v\eta_a = f\eta_a$ and $f = \rho$, from Lemma 3.1, we have from (4.6) that $\eta_c\nabla_a\rho^c = (\eta^d\eta^c\nabla_c\rho_d)\eta_a$. Using this and (4.3) in (4.4) we obtain $\nabla_b\rho_a = (2\rho - \Delta\rho)g_{ab} + 2(\Delta\rho - 3\rho)\eta_a\eta_b$. Applying Lemma 3.3 immediately gives $\nabla_b\rho_a = -\rho g_{ab}$, proving part (B). Thus in either case v is special concircular. And hence, if in addition, M is complete and connected then by Obata's theorem M is isometric to a unit sphere, completing the proof. □

Proof Of Theorem 2. First we consider $n > 1$. If $k = 1$ then M is Sasakian and hence v is special concircular by Okumura's theorem. So let $k < 1$. Using (2.8) in (2.13) we find

$$\Delta\rho = (2n - 2 + k)\rho. \tag{4.9}$$

Transvecting (4.2) with h^{ab} gives

$$h^{ab}\nabla_a\rho_b = 0. \tag{4.10}$$

As $\pounds_v\, g_{ab} = 2\rho g_{ab}$ we have $\pounds_v\, g^{ab} = -2\rho g^{ab}$ and hence

$$\pounds_v R^{ab} = (\pounds_v R_{cd})g^{ca}g^{db} - 4\rho R^{ab}. \tag{4.11}$$

Now using (2.7) we compute

$$R_{ab}R^{ab} = 4n[2(n-1)^2(2-k) + nk^2]. \tag{4.12}$$

Hence from (4.11) we have

$$\begin{aligned} 0 &= \pounds_v(R_{ab}R^{ab}) = (\pounds_v R_{ab})R^{ab} + R_{ab}(\pounds_v R^{ab}) \\ &= 2(\pounds_v R_{ab})R^{ab} - 4\rho R_{ab}R^{ab}. \end{aligned}$$

Use of (2.12) and (4.12) in the above and simplification yields

$$(n-1)[\Delta\rho - \{2(n-1)(2-k) + k\}\rho] = 0.$$

As $n > 1$, $\Delta\rho = (2(n-1)(2-k)+k)\rho$. Comparing with (4.9) gives $(n-1)(1-k)\rho = 0$. Since $n > 1$ and $k < 1$ we conclude $\rho = 0$, showing that v is Killing. Hence $\eta^a\pounds_v\eta_a = 0 = \eta_a\pounds_v\eta^a$. That is, $\pounds_v\eta^a$ is orthogonal to η^a. Thus taking the Lie-derivative of (2.7) along v, transvecting with η^b (since v is Killing) we have

$$(n-1)(\pounds_v h_{ab})\eta^b + (nk + 1 - n)\pounds_v\eta_a = 0. \tag{4.13}$$

Since $h_{ab}\eta^b = 0$ we have $(\pounds_v\, h_{ab})\eta^b = -h_{ab}\,\pounds_v\eta^b$. Hence (4.13) becomes

$$hX = ((nk + 1 - n)/(n-1))X \tag{4.14}$$

where X is given by $X^a = \pounds_v\eta^a$. If $X = 0$ on M then v is an automorphism of the contact metric structure. If $X \neq 0$ in some open neighborhood of a point p of M then (4.14) says that X is an eigenvector of h with eigenvalue $(nk + 1 - n)/(n-1)$ in that neighborhood. But it is well-known [15] that the eigenvalues of h for eigenvectors orthogonal to ξ are $\pm(1-k)^{1/2}$. So $(nk + 1 - n)/(n-1) = \pm(1-k)^{1/2}$. This simplifies to $k(kn^2 - n^2 + 1) = 0$. Hence either $k = 0$ or $1 - n^{-2}$. However the second possibility can be ruled out as follows: Lie-differentiating (2.4) along v and using $\pounds_v R_{abc}{}^d = 0$ (as v is Killing) we have

$$R_{cbad}\pounds_v\eta^d = k(g_{ab}\pounds_v\eta_c - g_{ca}\pounds_v\eta_b).$$

This shows $£_v\eta^a$ lies in $N(k)$. But for $k \neq 0$ and < 1 it is known [1] that $N(k)$ is the linear span of ξ. It therefore follows that $£_v\eta^a = f\ \eta^a$. Since $f = (£_v\eta^a)\eta_a = 0$ (as v is Killing) we conclude that $£_v\eta^a = 0$, a contradiction. So the only case when the Killing v may not be an automorphism of the contact metric structure is $k = 0$ for which we know that $N(k)$ is the tangent bundle of the factor E^{n+1} of the sphere bundle $E^{n+1} \times S^n(4)$ (see [1], [3]). Let us examine it more closely. For $k = 0$, M is locally $E^{n+1} \times S^n(4)$ and hence admits Killing vector fields orthogonal to ξ (note that ξ is tangential to E^{n+1}). Now h has eigenvalues 0 corresponding to eigenvector ξ, 1 corresponding to n-dimensional eigenspace $\{1\}$ and -1 corresponding to n-dimensional eigenspace $\{-1\}$. If X is an eigenvector of h with eigenvalue 1 then ϕX is also an eigenvector of h with eigenvalue -1. The eigenspace $\{-1\}$ and ξ span an integrable distribution $\xi \oplus \{-1\}$ that is tangent to E^{n+1}; and $\{1\}$ is tangent to $S^n(4)$. Let X be an arbitrary vector field in $\{1\}$. Then $g(X, \xi) = 0$ whence $g(£_vX, \xi) + g(X, £_v\xi) = 0$. But (4.14) says $£_v\xi$ is in $\{-1\}$, giving $g(£_vX, \xi) = 0$. As v is orthogonal to ξ it follows that $g(X, \nabla_v\xi) = g(\nabla_X\xi, v)$. Using (2.3) and the fact that ϕh is self-adjoint, we obtain $g(\phi X, v) = 0$ which shows that v is in $\{1\}$. Next $£_v\xi = \nabla_v\xi - \nabla_\xi v = -2\phi v - \nabla_\xi v$. As $£_v\xi$ and ϕv both lie in $\{-1\}$ so does $\nabla_\xi v$. Therefore $h(\nabla_\xi v) = -\nabla_\xi v$; i.e., $\nabla_\xi(hv) - (\nabla_\xi h)v = -\nabla_\xi v$. But $hv = v$ and since $\nabla_\xi h = 0$ (see [1]) we get $\nabla_\xi v = 0$. Therefore $£_v\xi = -2\phi v$. This shows $£_v\xi \neq 0$ otherwise v would vanish. Hence v can not be an automorphism of the contact metric structure on M.

Next we turn our attention to the case $n = 1$. If $k = 0$, M is flat and for $k = 1$, M is Sasakian which has been discussed in Theorem 1. So we consider $k < 1$ and $\neq 0$. In this case too, equations (4.1) through (4.5) hold. As stated in Section 2 we have

$$R_{ab} = 2k\eta_a\eta_b \text{ and } R = 2k.$$

Therefore from (2.13) and (4.3),

$$\Delta\rho = k\rho \text{ and } \eta^a\eta^b\nabla_a\rho_b = -3k\rho.$$

Now (4.4) becomes

$$\nabla_b\rho_a = k\rho(g_{ab} + 2\eta_a\eta_b) + \eta_d(\eta_b\nabla_a\rho^d + \eta_a\nabla_b\rho^d), \tag{4.15}$$

and hence $h^{ab}\nabla_b\rho_a = 0$. Applying ∇_c on (4.15) and transvecting with ϕ^{cb}, we get

$$\begin{aligned}\phi^{cb}\nabla_c\nabla_b\rho_a &= k\rho_c\phi^c{}_a + 2k\rho\eta_a\phi^{cb}\nabla_c\eta_b + \phi^{cb}(\nabla_c\eta_d)(\nabla_b\rho^d)\eta_a \\ &\quad + \eta_d[(\phi^{cb}\nabla_c\eta_b)\nabla_a\rho^d + \phi^{cb}(\nabla_c\eta_a)\nabla_b\rho^d + (\phi^{cb}\nabla_c\nabla_b\rho^d)\eta_a].\end{aligned} \tag{4.16}$$

Using the Ricci identities and skew-symmetry of ϕ^{cb}, we have $\phi^{cb}\nabla_c\nabla_b\rho_a = -(1/2)\phi^{cb}R_{cba}{}^d\rho_d$. Using (2.3), $h\phi = -\phi h$ and (2.6) we get

$$\begin{aligned}\rho^d[\phi_{ae}R_d{}^e - k\phi_{ad}] &= k\rho_c\phi^c{}_a + 9k\rho\eta_a + 3\eta^d\nabla_a\rho_d + \eta^d h_a{}^b\nabla_b\rho_d \\ &\quad + (\phi_{de}R_b{}^e - k\phi_{db})\eta_a\eta^d\rho^b,\end{aligned}$$

but $R_{ab} = 2k\eta_a\eta_b$ and hence

$$9k\rho\eta_a + 3\eta^d\nabla_a\rho_d + h_a{}^b\eta^d\nabla_b\rho_d = 0. \tag{4.17}$$

Transvecting with $h_c{}^a$ and using $h^2 = (k-1)\phi^2$,

$$(1-k)\{\eta^d\nabla_c\rho_d - (\eta^b\eta^d\nabla_b\rho_d)\eta_c\} + 3h_c{}^b\eta^d\nabla_b\rho_d = 0.$$

Using (4.17) and simplifying,

$$(k+8)(\eta^d\nabla_c\rho_d + 3k\rho\eta_c) = 0.$$

Thus $k = -8$ or $\eta^d\nabla_c\rho_d = -3k\rho\eta_c$. In the second case, (4.15) reduces to $\nabla_b\rho_a = k\rho(g_{ab} - 4\eta_a\eta_b)$ and hence Lemma 3.3 gives $k\rho = 0$ and in turn $\rho = 0$. So, in the second case v is Killing and hence $\pounds_v R_{ab} = 0$. Using this in $R_{ab} = 2k\eta_a\eta_b$ gives

$$(\pounds_v\eta_a)\eta_b + \eta_a(\pounds_v\eta_b) = 0.$$

Transvecting with η^b gives $\pounds_v\eta_a = 0$, because $\eta^b\pounds_v\eta_b = \rho = 0$. This shows, by virtue of Lemma 3.2, that v is an infinitesimal automorphism of the contact metric structure. Now the first case seems obviously unnatural. In order to dispose of this case we use the Lie-group theoretic approach. Let (e_1, e_2, e_3) be an orthonormal basis of the Lie-algebra of vector fields on M defined by (we refer to [4] for details):

$$[e_1, e_2] = (1+\lambda)e_3, [e_3, e_1] = (1-\lambda)e_2, [e_2, e_3] = 2e_1,$$

where $e_1 = \xi$, e_2 is a unit eigenvector of h corresponding to eigenvalue λ and $e_3 = \phi e_2$. In our case $k = 1-\lambda^2 < 1$ and $\neq 0$. Following Milnor's classification [7] of 3-dimensional manifolds admitting the Lie-algebra defined above we see that the universal covering space of M is either $SU(2)$ for $k > 0$ or $SL(2, R)$ for $k < 0$. The case $k = -8$ corresponds to $\lambda = \pm 3$. As $g(e_a, e_b) = \delta_{ab}$, we have $(\pounds_v g)(e_a, e_b) = g([e_a, v], e_b) + g(e_a, [e_b, v])$. Setting $v = v^a e_a$ we have

$$(\pounds_v\eta)(e_1) = e_1v^1, (\pounds_v\eta)(e_2) = e_2v^1 + 2v^3, (\pounds_v\eta)(e_3) = e_3v^1 - 2v^2. \tag{4.18}$$

Since $\pounds_v g = 2\rho g$ we get

$$\begin{aligned} &e_1v^1 = e_2v^2 = e_3v^3 = \rho,\\ &e_1v^2 + e_2v^1 + (\lambda+1)v^3 = 0,\\ &e_1v^3 + e_3v^1 + (\lambda-1)v^2 = 0,\\ &e_2v^3 + e_3v^2 - 2\lambda v^1 = 0. \end{aligned} \tag{4.19}$$

Introduce auxiliary functions a_1, a_2, a_3 by

$$\begin{aligned} e_2v^1 &= a_1 - ((\lambda+1)/2)v^3, e_3v^2 = a_2 + \lambda v^1,\\ e_1v^3 &= a_3 - ((\lambda-1)/2)v^2. \end{aligned} \tag{4.20}$$

Then

$$(4.21)\qquad \begin{aligned} e_3v^1 &= -a_3-(1/2)(\lambda-1)v^2, e_1v^2=-a_1-(1/2)(\lambda+1)v^3,\\ e_2v^3 &= -a_2+\lambda v^1. \end{aligned}$$

Now

$$\begin{aligned} 0 &= ([e_1,e_2]-(1+\lambda)e_3)v^1\\ &= e_1(a_1-((\lambda+1)/2)v^3)-e_2\rho+(1+\lambda)(a_3+((\lambda-1)/2)v^2) \end{aligned}$$

whence

$$e_1a_1-e_2\rho+((1+\lambda)/2)a_3+(3/4)(\lambda^2-1)v^2=0.$$

Similarly,

$$\begin{aligned} &e_2a_1+e_1\rho-(3/2)(1+\lambda)a_2-(\lambda/2)(1+\lambda)v^1=0,\\ &e_2a_3+e_1a_2+(1/2)(3-\lambda)\rho=0,\\ &e_2a_3+e_3a_1+\rho=0,\\ &e_2a_2-e_3\rho+(2+\lambda)a_1+(1/2)(2+\lambda-\lambda^2)v^3=0,\\ &e_3a_2+e_2\rho+(\lambda-2)a_3+(1/2)(\lambda^2+\lambda-2)v^2=0,\\ &e_1a_3+e_3\rho+(1/2)(\lambda-1)a_1+(3/4)(1-\lambda^2)v^3=0,\\ &e_3a_1+e_1a_2+(1/2)(3+\lambda)\rho=0,\\ &e_3a_3-e_1\rho+(3/2)(1-\lambda)a_2-(\lambda/2)(1-\lambda)v^1=0. \end{aligned}$$

Solving them and setting $b_a=e_a\rho$, we get

$$\begin{aligned} &e_1a_1=b_2-(1/2)(\lambda+1)a_3-(3/4)(\lambda^2-1)v^2,\\ &e_2a_1=-b_1+(3/2)(1+\lambda)a_2+(\lambda/2)(\lambda+1)v^1,\\ &e_3a_1=-(1/2)(\lambda+1)\rho,\\ &e_1a_2=-\rho,\\ &e_2a_2=b_3-(2+\lambda)a_1-(1/2)(2+\lambda-\lambda^2)v^3,\\ &e_3a_2=-b_2-(1/2)(\lambda^2+\lambda-2)v^2-(\lambda-2)a_3,\\ &e_1a_3=-b_3-(1/2)(\lambda-1)a_1-(3/4)(1-\lambda^2)v^3,\\ &e_2a_3=(1/2)(\lambda-1)\rho,\\ &e_3a_3=b_1+(3/2)(\lambda-1)a_2-(\lambda/2)(\lambda-1)v^1. \end{aligned}$$

Their integrability conditions are

$$\begin{aligned} &e_1b_2-e_2b_1=(1+\lambda)b_3,\\ &e_2b_3-e_3b_2=2b_1,\\ &e_3b_1-e_1b_3=(1-\lambda)b_2,\\ &e_1b_1+e_2b_2=2(\lambda^2-1)\rho\\ &e_3b_1=(1-\lambda)b_2+2a_3(1-\lambda^2)+(3+\lambda-3\lambda^2-\lambda^3)v^3,\\ &e_3b_2=(\lambda-1)b_1,\\ &e_1b_3=2(1-\lambda^2)a_3+(3+\lambda-3\lambda^2-\lambda^3)v^2, \end{aligned}$$

$$e_2b_2 + e_3b_3 = 2\rho(1-\lambda^2),$$
$$e_1b_2 = 2(\lambda^2-1)a_1 + (\lambda - 3 + 3\lambda^2 - \lambda^3)v^3,$$
$$e_2b_3 = (1+\lambda)b_1,$$

$$e_2b_1 = 2(\lambda^2-1)a_1 - (1+\lambda)b_3 + (\lambda - 3 + 3\lambda^2 - \lambda^3)v^3 \tag{4.22}$$

$$e_1b_1 + e_3b_3 = 2(\lambda^2-1)\rho.$$

Therefore $e_1b_1 = -3(e_2b_2) = -3(e_3b_3) = 3(\lambda^2-1)\rho$. Next applying $e_1e_2 - e_2e_1 - (1+\lambda)e_3 = 0$ and two other Lie-algebra equations to any two of b_1, b_2, b_3 and using above equations we obtain $\rho = 0$, $a_2 = -v^1$, $2a_3 + (\lambda+3)v^2 = 0$ and for $\lambda \neq 3$, $2a_1 - (\lambda - 3)v^3 = 0$. As $\rho = 0$, $b_a = 0$. Going back to equations (4.19), (4.20) and (4.21), (4.18) shows $£_v\eta = 0$. Hence v is an infinitesimal automorphism of the contact metric structure. For case $\lambda = 3$, i.e., $k = -8$, we have $\rho = 0$, $a_2 + v^1 = 0$, $a_3 + 3v^2 = 0$ and $b_a = 0$, but no information on a_1. However appealing to (4.22) we obtain $a_1 = 0$. Again equations (4.18) through (4.21) show $£_v\eta = 0$. This completes the proof. □

Proof of the Corollary to Theorem 2. In [11] it was proved that if M is a contact metric manifold with non-vanishing $K(\xi, X)$ and $K(\xi, X) = K(\xi, \phi X)$ everywhere and for all X orthogonal to ξ, then a vector field v satisfying $£_v R_{abc}{}^d = 0$ is homothetic. Now we have equation (2.4) which implies that the ξ-sectional curvature $K(\xi, X) = k$. By hypothesis $k \neq 0$. Thus v is homothetic. Obviously $R_{ab}\eta^a\eta^b = 2nk$ and $£_v R_{ab} = 0$ and hence $R_{ab}\eta^a £_v\eta^b = 0$. But $R_{ab}\eta^a = 2nk\eta_b$ and so $(£_v\eta^b)\eta_b = 0$, since $k \neq 0$. Lie-differentiating $g_{ab}\eta^a\eta^b = 1$ gives $(£_v g_{ab})\eta^a\eta^b = -2(£_v\eta^b)\eta_b = 0$. Now $£_v g_{ab} = cg_{ab}$ (c constant) gives $c = (£_v g_{ab})\eta^a\eta^b = 0$. Thus v is Killing and hence from Theorem 2, v is an automorphism of the contact metric structure. □

Concluding Remark. Motivated by the result (see [14]) that conformally flat K-contact manifolds are Sasakian manifolds of constant curvature we pose this question: "Are there K-contact manifolds that admit a conformal motion and are not Sasakian?"

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