Conformal Regge theory

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Abstract

In this thesis, we explore the connection between Mellin amplitudes and scattering amplitudes to analyze the Regge limit for correlation functions in conformal field theories. The main novelty in this analysis is the relation between OPE coefficients of leading twist operators and the pomeron residue present in the Regge limit of a four point function.

We study the applications of Regge theory in $\mathcal{N} = 4$ SYM, which is a superconformal quantum field theory. In this example, we have obtained a complete agreement between dimensions and OPE coefficients and pomeron pole and residue.

We have also computed strong coupling correction to the supergravity result of the correlation function involving four Lagrangians in $\mathcal{N} = 4$ SYM using the flat space limit.

Resumo

Nesta tese estudamos a relação entre amplitudes de Mellin e de difusão para estudar o limite de Regge em funções de correlação em teorias de campo com invariância conforme. A principal novidade desta analise é a relação entre coeficients de OPE de operadores com twist mais baixo e o residuo do pomeron obtido a partir da função de quatro pontos depois de tirar o limite de Regge.

Estudamos a aplicação do limite de Regge na teoria $\mathcal{N} = 4$ SYM, que é uma teoria conforme e super-simétrica. Neste exemplo, obtivemos uma confirmação das relações entre dimensões de operadores, coeficientes de OPE coefficients e o pole e residuo do pomeron.

Ainda nesta teoria, obtivemos, usando o limite de espaço plano, a correção aos resultados de supergravidade da função de correlação a quatro pontos de quatro Lagrangianos.

Publication List

This thesis is based in four publications:

- **Conformal Regge theory**
  Miguel S. Costa , Vasco Gonçalves , João Penedones
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- **The role of leading twist operators in the Regge and Lorentzian OPE limits**
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- **Factorization of Mellin amplitudes**
  Vasco Gonçalves , João Penedones , Emilio Trevisani
  hep-th/1410.4185

- **Four point function of $\mathcal{N}=4$ stress-tensor multiplet at strong coupling**
  Vasco Gonçalves
  Submitted to JHEP. hep-th/1411.1675

This thesis does not contain the paper:

- **Spinning AdS Propagators**
  Miguel S. Costa , Vasco Gonçalves , João Penedones
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Chapter 1

Introduction

1.1 Conformal field theory

Renormalization group (RG) is an important example of the interplay of ideas between quantum field theory (QFT) and statistical physics. The problem of infinities that comes from the high energy modes or short distances was a serious issue in the early days of QFT and a satisfactory answer came from the connection with statistical mechanics. One solution to this problem was addressed by imposing a cut-off in momentum space. One of Wilson's ideas was to divide the momentum space into slices and then integrate over the degrees of freedom close to the cut-off. This procedure shows that the high energy modes when integrated change the coupling constants of the lower degrees of freedom, generating a iterative procedure

$$g_{n+1} = f(g_n). \tag{1.1}$$

where g_n should be thought as the set of possible coupling constants of the theory. This led to the notion of renormalization group flows or in other words to the idea that couplings change with the energy scale. The action of RG is naturally defined in the space of possible Lagrangians where it induces this flow. The short distance degrees of freedom are smeared out as the the system flows from the high to the low energy regime along the renormalization group. This introduces another important concept, the operator product expansion (OPE). Consider two operators at two separate points x_1 and x_2. Then consider taking the renormalization group action such that all degrees of freedom bellow a certain cutoff L, with $|x_1 - x_2| < L$, are smeared out. Then the two operators in the renormalized theory sit on the same point and thus cannot be distinguished from each other. So the product of these two operators can be written as sum over operators defined in this smeared space. More concretely we have,

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k B_{12k}(x_2 - x_1)\mathcal{O}_k(x_2). \tag{1.2}$$

As suggested in the previous paragraph the OPE is present in every quantum field theory but it is not garanteed to have a finite radius of convergence. Notice that this was just a schematic motivation of the OPE, we shall derive this property more rigorously in the specific case of a conformal field theory.

A fixed point is one possible end to the renormalization group flow. At this site the theory is scale invariant and generically, in QFT, this symmetry is enhanced to conformal symmetry, *i.e.* has one additional symmetry called special conformal transformation[1, 2, 3, 4]. A theory with both these symmetries is called a conformal field theory (CFT). In this sense, CFT's are stop signs in the space of possible QFTs.

The absence of a length scale in a CFT makes ill-defined the concept of an S-matrix and so correlation functions are one of the most interesting observables that can be studied in a CFT.

One consequence of scale and special conformal symmetry is that two and three point correlation functions are completely determined by symmetry up to a few numbers, one simple argument is that it is impossible to construct, using just two or three points, a variable that is invariant for both scale, Poincaré symmetries and inversions. For instance the correlation function of two scalar primary operators with dimensions Δ and Δ'(where the dimension controls how an operator transform under scale transformation) is,

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_{\Delta'}(x_2)\rangle = \frac{\delta_{\Delta,\Delta'}}{x_{12}^{2\Delta}}, \tag{1.3}$$

where we used the notation $x_{ij}^2 = (x_i - x_j)^2$. For correlation function involving three scalar primary operators we have,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_1+\Delta_3-\Delta_2}x_{23}^{\Delta_2+\Delta_3-\Delta_1}}, \tag{1.4}$$

where the constant c_{123} is called OPE coefficient and is determine by the dynamics of the theory.

The dimensions and the OPE coefficients of the operators specify all correlation functions of a CFT. So these two sets of numbers define a quantum field theory. This becomes more interesting since it is possible to add an operator that induces a flow from the CFT. Conformal perturbation theory is an interesting application of this idea. Basically one defines a QFT by saying that it can be approximated by perturbing it from a given CFT. Thus, this offers an alternative to the usual perturbation theory.

Another reason to study conformal field theories is that some of them are secretly theories of quantum gravity by the AdS/CFT correspondence. So we can use the knowledge acquired from a CFT to define non-perturbatively strings in AdS. The prime example of this duality is between a CFT called $\mathcal{N} = 4\, SYM$ in $d = 4$ and type IIB string theory in $AdS_5 \times S^5$.

Conformal field theories are also present in condensed matter systems where for example the $3d$ Ising model is used to describe the critical opalescence of water. Again we might hope to have a chance to model strongly interacting condensed matter systems with CFTs.

1.2 Conformal transformations

A coordinate transformation $x^\mu \to \widetilde{x}^\mu$ is a conformal transformation if the metric in the new coordinates is given by

$$\widetilde{g}_{\mu\nu} = \Omega(x) g_{\mu\nu}. \tag{1.5}$$

Transformations generated by the Poincaré group are conformal since the metric stays invariant under these changes. Scale transformations, *i.e.* $x^\mu \to \widetilde{x}^\mu = \lambda x^\mu$ with $\lambda \in \mathbb{R}$ are also of this form. There is another transformation that falls in this class

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2} = x^\mu + 2x \cdot b x^\mu - b^\mu x^2 + O(b^2) \tag{1.6}$$

which is called special conformal transformation and where b^μ is any vector. This transformation can be obtained from an inversion, translation and another inversion

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu - x^2 b^\mu}{x^2} \to \frac{x^\mu - x^2 b^\mu}{1 - 2x \cdot b + (x^2) b^2}. \tag{1.7}$$

The conformal algebra can be obtained form these transformations, for example the generator associated with dilatations can be computed from its action on a test function

$$e^{i\lambda D} f(x) = f((1+\lambda)x) \Leftrightarrow iD = x \cdot \partial. \tag{1.8}$$

Using the same reasoning it is possible to obtain the other generators

$$iP^\mu = \partial^\mu\,, \quad iM^{\mu\nu} = x^\mu \partial^\nu - x^\nu \partial^\mu\,, \quad iK^\mu = 2x^\mu x \cdot \partial - x^2 \partial^\mu \tag{1.9}$$

where K^μ generate special conformal transformations. The commutations relations between the generators are

$$\begin{aligned} &\big[M_{\mu\nu}, M_{\rho\sigma}\big] = -i(\delta_{\mu\rho} M_{\nu\sigma} + (-1)^\pi \text{permutations}), \qquad \big[M_{\mu\nu}, X_\rho\big] = i(\delta_{\nu\rho} X_\mu - \delta_{\mu\rho} X_\nu) \\ &\big[D, P_\mu\big] = -iP_\mu, \qquad \big[D, K_\mu\big] = iK_\mu, \qquad \big[P_\mu, K_\nu\big] = 2i(\delta_{\mu\nu} D - M_{\mu\nu}) \end{aligned} \tag{1.10}$$

where K_μ and D are the generators of special conformal transformations and dilatations respectively, $(-1)^\pi$ represents the sign of each permutation and X^μ represents either P^μ or K^μ. The first two commutation relations are just telling how a tensor and a vector transform under Lorentz transformations. The commutator between dilatation and translation means that we are just making a scale transformation to a vector. The commutator between the dilatation and special conformal transformation can be understood by thinking about the latter as being

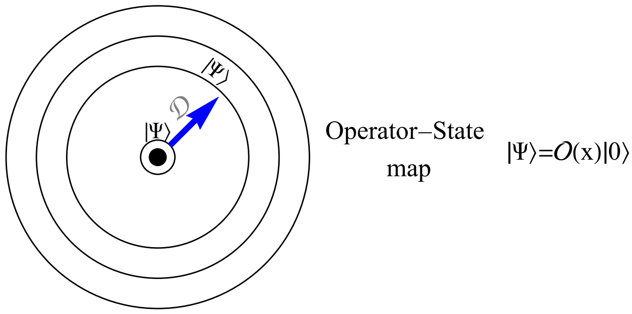


Figure 1.1: In conformal field theories it is possible to quantize the theory radially. This leads directly to the state operator map.

translations conjugated with inversions. Thus, it should have the opposite effect to translations. The conformal group is isomorphic to $SO(d+1,1)$, this can be seen more explicitly by rewriting the generators as

$$J_{\mu\nu} = M_{\mu\nu}\,, \quad \tfrac{1}{2}\big(J_{\mu 0} + iJ_{\mu\, D+1}\big) = P_\mu\,, \quad \tfrac{1}{2}\big(J_{\mu 0} - iJ_{\mu\, D+1}\big) = K_\mu\,, \quad J_{0\, D+1} = D \tag{1.11}$$

which satisfies the usual commutation relation,

$$\big[J_{MN}, J_{RS}\big] = -i\big(\eta_{MR}J_{NS} + (-1)^\pi \text{permutations}\big). \tag{1.12}$$

where $M\,, N = 0\,, \ldots D+1$.

1.3 State-operator map, primaries and descendants

The state-operator map is certainly among the most important features of conformal field theories. This is just the statement that to every state of the system there is a corresponding operator and vice-versa. The standard way to quantize a theory in QFT consists in choosing a specific foliation of the space and define in each leaf an Hilbert space. Then different leafs can be connected by evolving the system with one generator of the theory and generally in QFT one chooses surfaces of equal time and evolve them with the Hamiltonian. The same procedure can be repeated in a conformal field theory but there is a better alternative since there is scale symmetry. It is possible to foliate the system radially and then use the generator of the scale symmetry to evolve the system between different spherical leafs. Thus this relates every possible state in the CFT with a state defined near the origin, which defines an operator. Let us remark that this is possible because the system has a scale symmetry and so it is not valid in every QFT. This leads directly to the operator product expansion, consider the product of two operators acting on the vacuum, this defines a state but from the state-operator map it also corresponds to an operator defined in the vicinity of the origin, thus we can express the product of the two operators in terms of one operator. More concretely consider the product of two operators that are at positions x_1 and x_2 and let us say that we are quantizing our theory around the point y

$$\mathcal{O}(x_1)\mathcal{O}(x_2)\big|0\rangle = \big|\Psi\rangle = \sum_k B(x_1-y, x_2-y)\mathcal{O}_k(y)\big|0\rangle. \tag{1.13}$$

In general the state $\big|\Psi\rangle$ is not an eigenstate of the dilation operator but can be express as a linear combination of such eigenstates defined by $\mathcal{O}_k(y)\big|0\rangle$. The function B is related to the OPE coefficients C_{12k}, to see this notice that acting on the left hand side of (1.13) with the state $\langle\mathcal{O}_k\big|$ defines a three point function, while the on the right hand side we just have the two point function of the primary operator $\mathcal{O}_k$. The expansion (1.13) has a finite radius of convergence as was recently proven in [5].

Notice that P_μ and K_μ can be seen as raising and lowering operators with respect to D, according to (1.10). Imposing that there are no states with infinity negative dimension Δ implies that associated to each state there is one state such that it is killed by the action of K^μ, such a state is usually called primary

$$K^\mu\big|\Psi\big\rangle = 0. \tag{1.14}$$

Otherwise we could create states with an arbitrarily low dimension just by applying the K^μ operator. A state that is not a primary is called a descendant. It follows that in a CFT there are just two types of operators, primaries and descendants.

Notice that the contribution of descendants to the OPE (1.13) can be computed from the primary, since they are obtained by applying total derivatives. More important, the finite radius of convergence allows the successive use of (1.2), this reduces a n-point correlation function up to a two or three point correlation function. So any correlation function of local operators is completely determined by specifying two sets of numbers the dimension of operators Δ and the OPE coefficients c_{123}. It is usual to denote these numbers collectively as CFT data.

This gives the opportunity to define a quantum field theory without using a Lagrangian, which is interesting in the context of strongly coupled systems where Lagrangians are not so useful.

1.4 Higher point functions

As mentioned, a CFT is defined by two sets of numbers, the dimension of operators and their OPE coefficients. The first natural question to ask is: *Can these numbers, featuring in the CFT data, be arbitrary?* The answer is no. For example in QFT there is a probabilistic interpretation of transition amplitudes which implies that the theory should be unitary. Imposing positivity of the two point function of a primary operator and its descendants implies the following bound on the dimension of operators

$$\Delta \geq \frac{d-2}{2} \ \ \text{(scalar operators)}, \qquad \Delta \geq J+d-2 \ \ \text{(spin } J \text{ sym. and traceless op.)} \tag{1.15}$$

where d is the space-time dimension.

Another natural question which arises is: *Is it important to study higher point function?* Before answering this question let us just review the structure of a four point correlation function. It is possible to construct variables out of the four positions that are invariant under translations, rotations, scale transformations and inversions[1]

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}). \tag{1.16}$$

This means that in a conformal field theory the four point function of scalars is a non-trivial function of the cross ratios u and v,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{\mathcal{A}(u,v)}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}}\,, \tag{1.17}$$

where we used $\Delta_{ij} = \Delta_i - \Delta_j$. We know, from the OPE, that this four point function can be written in terms of three point functions, thus it organizes several two and three point function in a non-trivial function of two variables. This has been used to constraint the allowed space of dimensions of operators and OPE coefficients[6, 7, 8, 9]. One example is given by the bootstrap equations

$$\langle \mathcal{O}_1\overbracket{(x_1)\mathcal{O}_2}(x_2)\mathcal{O}_3\overbracket{(x_3)\mathcal{O}_4}(x_4)\rangle = \langle \mathcal{O}_1\overbracket{(x_1)\mathcal{O}_3}(x_3)\mathcal{O}_2\overbracket{(x_2)\mathcal{O}_4}(x_4)\rangle, \tag{1.18}$$

where the braces over the operators represent the OPE. For the particular case of equal external operators this can be phrased as

$$g(u,v) - \left(\tfrac{u}{v}\right)^{\Delta_{\mathcal{O}}} g(v,u) = 0\,, \tag{1.19}$$

$$g(u,v) = \sum_{\mathcal{O}_k \in \mathcal{O}\times\mathcal{O}} C_{12\mathcal{O}_k}^2 G_{\Delta_k, J_k}(u,v) \tag{1.20}$$

where $G_\Delta(u,v)$ is the conformal block which will be defined in the next section. This is an infinite set of equations(since it should be valid for every u and v) that are generically hard to solve. Recently, it was possible to use this equations to bound the space of all possible CFTs [6].

1.5 Outline of thesis

One of the most interesting observables in conformal field theories is the correlation function of local observables. The four point function is the first non-trivial object, in terms of the dependence on the positions, that can be studied.

The main goal of this thesis is to study and relate some singularities of a four point function.

[1] To check the invariance under inversion it might be useful to notice the identity $x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$ under inversions.

We shall study the Lorentzian OPE, flat space limit and the Regge limit, with emphasis on the latter.

There are several tools that are useful for this analysis. In chapters 2 and 3 we review the tools used in the rest of the thesis.

In chapter 4 we start by reviewing Regge theory for scattering amplitudes and then generalize to correlation functions in conformal field theories.

In chapter 5 we apply the Regge limit for a specific correlation function in $\mathcal{N} = 4$ SYM. The last section of this chapter shows how it is possible to constraint a four point function at strong coupling using the flat space limit. In chapter 6 we conclude and summarize the questions that remain open. Technical details are left for several appendixes.

Chapter 2

CFT technology

In this thesis we study kinematical limits of four point correlation functions in CFTs. For this analysis we need to introduce conformal blocks, embedding space formalism and Mellin amplitudes. The goal of this and the next chapters is to review these concepts. Below we will show how it is possible to obtain conformal blocks, analyze briefly the singularity structure of a four point correlation function and the embedding space formalism. For clarity we have decided to reserve a chapter to Mellin amplitudes.

2.1 Conformal blocks

The OPE of two scalar primary operators only contains totally symmetric and traceless tensors. The expression (1.13) for the OPE can be written as

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k \frac{C_{12k}}{(x^2)^{\frac{1}{2}(\Delta_1+\Delta_2-\Delta)}} \left[\frac{x_{\mu_1}\dots x_{\mu_J}}{(x^2)^{\frac{J}{2}}}\, \mathcal{O}_k^{\mu_1\dots\mu_J}(0) + \text{descendants} \right], \tag{2.1}$$

$$= \sum_k \frac{C_{12k} B^{\mu_1\dots\mu_J}(x,\partial_y)}{(x^2)^{\frac{\Delta_1+\Delta_2-\Delta}{2}}} \mathcal{O}_k^{\mu_1\dots\mu_J}(y)\Big|_{y=0} \tag{2.2}$$

where Δ and J are respectively the dimension and spin of the operator $\mathcal{O}_k$, and all operators are normalized to have two-point function

$$\langle \mathcal{O}_{\mu_1\dots\mu_J}(x)\mathcal{O}_{\nu_1\dots\nu_J}(0)\rangle = \frac{1}{J!}\sum_{\text{perm }\sigma} \frac{I_{\mu_1\nu_{\sigma(1)}}\, I_{\mu_J\nu_{\sigma(J)}}}{(x^2)^{\Delta}} - \text{traces}\,, \tag{2.3}$$

with

$$I_{\mu\nu} = \eta_{\mu\nu} - 2\,\frac{x_\mu x_\nu}{x^2}\,. \tag{2.4}$$

In this sum we just consider primary operators, the contribution of descendants is of kinematical nature and is encoded in the function $B^{\mu_1\cdots\mu_J}$. The OPE can be used to express a four point function in terms of two and three point functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_k \frac{C_{12k}C_{34k}B^{\mu_1\cdots\mu_J}(x_{12},\partial_{x_2})B^{\nu_1\cdots\nu_J}(x_{34},\partial_{x_4})}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4-\Delta}{2}}}\langle \mathcal{O}_k^{\mu_1\cdots\mu_J}(x_2)\mathcal{O}_k^{\mu_1\cdots\mu_J}(x_4)\rangle$$
$$= \sum_k \frac{C_{12k}C_{34k}G_{\Delta_k,J_k}(u,v)}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} \tag{2.5}$$

where the function $G_{\Delta,J}$ is called a conformal block. This function captures the contribution of a given primary operator and all its conformal descendants to the four point function.

There are two at least two methods to obtain conformal blocks[10, 11], one can act with the functions $B^{\mu_1\cdots\mu_J}(x,\partial_x)$ on the two point function or we can use the fact that the conformal block is the solution of the Casimir equation

$$\left[(J_1+J_2)^2 - C_{\Delta,J}\right]\frac{G_{\Delta,J}(u,v)}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} = 0 \tag{2.6}$$

or just in terms of a differential operator acting on the cross ratios [11]

$$\mathcal{D}\, G_{\Delta,J}(u,v) = \frac{1}{2}\, C_{\Delta,J}\, G_{\Delta,J}(u,v)\,, \tag{2.7}$$

where

$$\mathcal{D} = (1-u-v)\frac{\partial}{\partial v}\left(v\frac{\partial}{\partial v}+\frac{\Delta_{34}-\Delta_{12}}{2}\right)+u\frac{\partial}{\partial u}\left(2u\frac{\partial}{\partial u}-d\right)$$
$$-(1+u-v)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}-\frac{\Delta_{12}}{2}\right)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+\frac{\Delta_{34}}{2}\right), \tag{2.8}$$

$C_{\Delta,J} = \Delta(\Delta-d)+J(J+d-2)$ is the conformal quadratic Casimir and J_1 , J_2 are generators of conformal transformations. As explained in [10], the conformal blocks have a series expansion of the form

$$G_{\Delta,J}(u,v) = u^{\frac{\Delta-J}{2}}\sum_{m=0}^{\infty}u^m g_m(v)\,, \tag{2.9}$$

where the first term reads

$$g_0(v) = \left(\frac{v-1}{2}\right)^J {}_2F_1\left(\frac{\Delta+J-\Delta_{12}}{2},\frac{\Delta+J+\Delta_{34}}{2},\Delta+J,1-v\right). \tag{2.10}$$

When applied to the power series (2.9) this partial differential equation turns into the following (differential) recursion relation for the functions $g_m(v)$,

$$\begin{aligned}
&4v(v-1)^2 g_m'' - 2(v-1)g_m'\big(2v(J-\Delta-2m-1)+2\big) + g_m\Big(4m(m(v+1)-2h) + J^2(v-1) \\
&-2J\big((\Delta+2m)(v+1)-2\big) + 4\Delta m(v+1) + \Delta^2(v-1)\Big) = 4v(v+1)g_{m-1}'' \qquad (2.11)\\
&+2g_{m-1}'\big(2v(\Delta-J+2m-1)+2\big) + g_{m-1}\left(J-\Delta-2m+2\right)\left(J-\Delta-2m+2\right)
\end{aligned}$$

where derivatives in g_m are taken in order of v and we have set $\Delta_{12} = \Delta_{34} = 0$ for simplicity. This is an ugly equation which the reader should not look at in detail. Nevertheless, it is not hard to check that $g_0(v)$ given by (2.10) solves the $m = 0$ equation. This representation for the conformal blocks packages the contribution of operators with the same twist which is defined as $\tau = \Delta - J$.

In even space time dimensions there are explicit formulas for the conformal blocks, for example in $d = 4$ they are given by,

$$G_{\Delta,J}(u,v) = \left(\tfrac{-1}{2}\right)^J \frac{z\bar{z}}{z-\bar{z}}\left[\zeta_{\Delta+J}(z)\zeta_{\Delta-J-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right] \qquad (2.12)$$

$$\zeta_\beta(x) = x^{\beta/2}{}_2F_1\left(\frac{\beta-\Delta_{12}}{2}, \frac{\beta+\Delta_{34}}{2}, \beta, x\right), \quad u = z\bar{z}\,, \quad v = (1-z)(1-\bar{z}). \qquad (2.13)$$

Notice that in the conformal block decomposition (2.5) the conformal block receives as an input the dimension of the operator. It is convenient to introduce another representation that separates completely the dynamical and kinematical data

$$\mathcal{A}(u,v) = \sum_{J=0}^{\infty}\int_{-\infty}^{\infty} b_J(\nu^2)F_{\nu,J}(u,v) \qquad (2.14)$$

where $F_{\nu,J}(u,v)$ is given by the sum of two conformal blocks with dimensions $h+i\nu$ and $h-i\nu$ and the conformal partial wave coefficient should have physical poles

$$b_J(\nu) \approx C_{12k}C_{34k}\,\frac{K_{\Delta,J}}{\nu^2+(\Delta-h)^2}. \qquad (2.15)$$

In appendix C we show how it is possible to invert (2.14) to obtain $b_J(\nu)$ in terms of $\mathcal{A}(u,v)$. In particular we compute $b_J(\nu)$ for a specific for point function. More precisely, one can write [1]

$$F_{\nu,J}(u,v) = \kappa_{\nu,J}\,G_{h+i\nu,J}(u,v) + \kappa_{-\nu,J}\,G_{h-i\nu,J}(u,v)\,, \qquad (2.16)$$

where the normalization constant

$$\kappa_{\nu,J} = \frac{i\nu}{2\pi K_{h+i\nu,J}} \qquad (2.17)$$

[1] A similar equation can be found in [12] where the function $F_{\nu,J}(u,v)$ was defined by the integral of the product of the 3-point function of the operators $\mathcal{O}_1$, $\mathcal{O}_2$ and an operator of spin J and dimension $h+i\nu$, times the 3-point function of the operators $\mathcal{O}_3$, $\mathcal{O}_4$ and an operator of spin J and dimension $h-i\nu$.

and

$$K_{\Delta,J} = \frac{\Gamma(\Delta+J)\,\Gamma(\Delta-h+1)\,(\Delta-1)_J}{4^{J-1}\Gamma\left(\frac{\Delta+J+\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta+J-\Delta_{12}}{2}\right)\Gamma\left(\frac{\Delta+J+\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta+J-\Delta_{34}}{2}\right)} \tag{2.18}$$
$$\frac{1}{\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta+J}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-\Delta+J}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta+J-d}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4+\Delta+J-d}{2}\right)}\,.$$

Notice that the two conformal blocks in (2.16) satisfy the same differential equation (2.7) because they have the same Casimir $C_{h+i\nu,J} = C_{h-i\nu,J}$. The second conformal block is usually called the shadow of the first (see for example [13] for details). Inserting (2.16) in (2.14), one can write

$$\mathcal{A}(u,v) = 2\sum_{J=0}^{\infty}\int_{-\infty}^{\infty} d\nu\, b_J(\nu^2)\,\kappa_{\nu,J}\, G_{h+i\nu,J}(u,v)\,, \tag{2.19}$$

which can be easily converted into the usual conformal block decomposition (2.5) by deforming the ν-contour into the lower-half plane and picking the contribution from all poles of the integrand with negative imaginary part of ν. Notice that the contribution from infinity vanishes because the conformal block $G_{h+i\nu,J}(u,v)$ decays exponentially for $\mathrm{Im}(\nu) \to -\infty$.

2.2 Singularity structure of correlation function

In this thesis we will study some kinematical limits of four point correlation functions. In this section we will review these limits. The non-trivial dependence on the positions of a four point function is encoded in the sum of conformal blocks. Thus, in the following we will analyze the how the conformal blocks behave in three situations[2]: Euclidean OPE limit, Lorentzian OPE limit and Regge limit.

2.2.1 Euclidean OPE limit

The Euclidean OPE limit consists in the limit in which the point x_2 is approaching the point x_1 or in terms of cross ratios, $u \to 0$ and $v \to 1$ the conformal block reduces to

$$G_{\Delta,J}(u,v) \to \frac{J!}{2^J(h-1)_J} u^{\frac{\Delta}{2}} C_J^{h-1}\left(\frac{v-1}{2\sqrt{u}}\right) \tag{2.20}$$

where $C_J^{h-1}(x)$ is a Gegenbauer polynomial. Notice that in this limit the operators that dominate are the ones with smallest dimension, so in this case the conformal block is just the contribution of the primary operator since all descendants have higher dimension. In this case we just take into account the first term in OPE (2.1).

[2]There is another interesting limit which we do not study and it is related with the contribution of large spin operators [14].

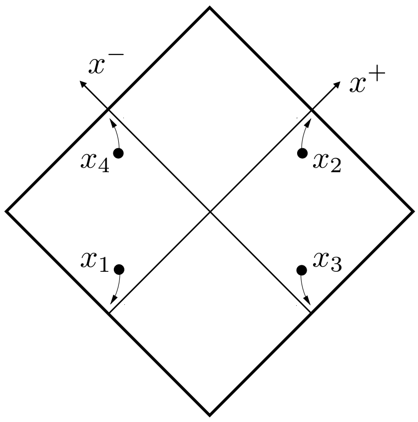


Figure 2.1: Conformal compactification of the light cone plane. In the Regge limit the positions of the operators x_i go to null infinity.

2.2.2 Lorentzian OPE limit

Another limit is the one in which the point x_2 approaches the light-cone of the point x_1, *i.e.* $|x_{12}| \to 0$. In terms of cross ratios this corresponds to the limit $u \to 0$ with v kept fixed. The leading contribution to the four point function is given by $g_0(v)$ in the series expansion (2.9).

There are more operators contributing to the leading contribution in this limit than the Euclidean OPE limit. For example, operators of this type

$$\mathcal{O}\,, \qquad \partial_\mu \mathcal{O}\,, \quad \partial_\mu \partial_\nu \mathcal{O}\,, \qquad \partial_\mu \partial_\nu \partial \mathcal{O} \ldots \tag{2.21}$$

give contribute to $g_0(v)$.

2.2.3 Regge limit

The main goal of this thesis is to study another limit which is associated with the disposition of the points shown in figure 2.1. This limit can be defined by $x_1^+ \to \lambda x_1^+$, $x_2^+ \to \lambda x_2^+$, $x_3^- \to \lambda x_3^-$, $x_4^- \to \lambda x_4^-$ and $\lambda \to \infty$, keeping the causal relations $x_{14}^2, x_{23}^2 < 0$ and all the other $x_{ij}^2 > 0$. This requires an analytic continuation from the Euclidean to the Lorentzian setting which amounts to the usual wick rotation

$$x_i^0 \to x_i^0 e^{i\pi\frac{\alpha}{2}} \tag{2.22}$$

and varying α from 0 to 1 where x_i^0 is the temporal component of x_i. This variation draws a path in the complex z and $\bar{z}$ plane as shown in 2.2. The Regge limit corresponds, in terms of cross ratios, to the limit $u \to 0$ and $v \to 1$ with $\frac{\sqrt{u}}{v-1}$ fixed. Notice that even if this looks similar to the Euclidean OPE limit it has a very different physical content due to the different causal relations. The leading term of the conformal block can be obtained from the analytic continuation 2.2 [15, 16]. Let us analyze the conformal blocks (2.12).The hypergeometric function has branch

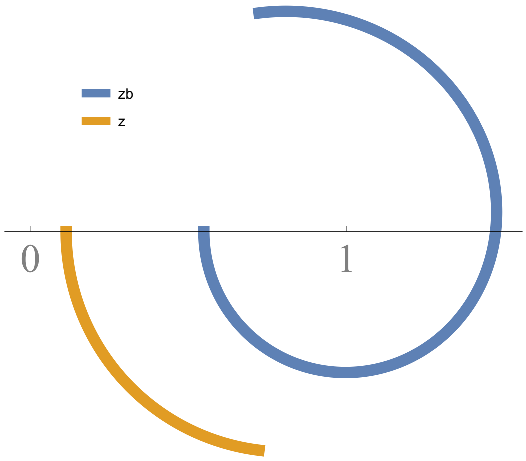


Figure 2.2: The Wick rotation defined in (2.22) draws a path in the complex z and $\bar{z}$ plane. Here is an example of the path corresponding to the kinematics of the Regge limit.

points at 0 and ∞ with the discontinuity across the branch cut given by

$$ {}_2F_1\left(\tfrac{\Delta+J}{2},\tfrac{\Delta+J}{2},\Delta+J,\bar{z}\right) \to {}_2F_1\left(\tfrac{\Delta+J}{2},\tfrac{\Delta+J}{2},\Delta+J,\bar{z}\right) - \frac{2i\pi\Gamma(J+\Delta)\,{}_2F_1\left(\frac{J+\Delta}{2},\frac{J+\Delta}{2};1;1-\bar{z}\right)}{\Gamma\left(\frac{J+\Delta}{2}\right)^2}. \quad (2.23) $$

Taking the Regge limit, *i.e.* $z\,,\bar{z} \to 0$ with $\frac{z}{\bar{z}}$ fixed we obtain[3] ,

$$ G_{\Delta,J}(z,\bar{z}) \approx \left(-\tfrac{1}{2}\right)^J \frac{i2\pi^2\left(\frac{z}{\bar{z}}\right)^{\Delta-1}\bar{z}^2(z\bar{z})^{\frac{1-J}{2}}\Gamma(J+\Delta)}{(\bar{z}^2-z^2)\Gamma(2-J-\Delta)\Gamma\left(\frac{J+\Delta}{2}\right)^4\sin(\pi(\Delta+J))}. \quad (2.28) $$

We shall analyze this limit more rigorously in chapter 4. One important difference is that in this limit the dominant operators are those with highest spin. So we need to sum the contribution of all spins before we take this limit in a four point function. This same phenomena happens for scattering amplitudes. The Regge limit for a 2 to 2 scattering process is defined by the large $s=-(p_1+p_3)^2$ and fixed $t=-(p_1+p_2)^2$ where s and t are the Mandelstam. This limit can be

[3] Notice that the result is not invariant under the exchange of z and $\bar{z}$, however the four point function is symmetric under this exchange. The four point function is a single valued function on the Euclidean domain, this means that the monodromy around, for example $z=1$ is equal to the monodromy around $\bar{z}=1$ but going in the opposite direction,

$$ M_1\mathcal{A}(z,\bar{z}) = \overline{M}_1\mathcal{A}(z,\bar{z}). \quad (2.24) $$

The claim is that the analytic continuation of $\mathcal{A}(z,\bar{z})$ is symmetric under the exchange of z and $\bar{z}$. This is the same to say that

$$ \begin{aligned} (M_1\mathcal{A})(\bar{z},z) &= \overline{M}_1\mathcal{A}(\bar{z},z)\,, \quad \text{(by definition)} && (2.25)\\ &= \overline{M}_1\mathcal{A}(z,\bar{z})\,, \quad \text{(by symmetry of } \mathcal{A}) && (2.26)\\ &= M_1\mathcal{A}(z,\bar{z})\,, \quad \text{(by single valuedness).} && (2.27) \end{aligned} $$

On the other hand, the conformal partial wave (2.16) is symmetric under $z \leftrightarrow \bar{z}$.

obtained by applying to the momenta the same transformation represented in 2.1 $p_1^+ \to \lambda p_1^+$, $p_2^+ \to \lambda p_2^+$, $p_3^- \to \lambda p_3^-$, $p_4^- \to \lambda p_4^-$ and $\lambda \to \infty$. Physically this just means that we are applying a boost in the plus light-cone direction to the particles at x_1 and x_2 and the same for the particles at positions x_3 and x_4 but in the minus light-cone direction.

2.3 Embedding space formalism

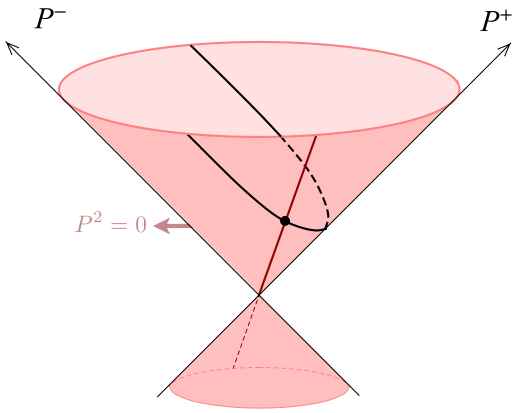


Figure 2.3: Physical points correspond to light-rays in the embedding space. As an example we show the Poincaré section.

In the introduction we derived the conformal algebra in d dimensions and proved that it is isomorphic to the group $SO(d+1,1)$. This suggests that the action of the conformal group is linearized by lifting $\mathbb{R}^d$ to $\mathbb{M}^{d+2}$[17, 18, 19, 20]. This is accomplished by associating a point x in $\mathbb{R}^d$ to null rays in $\mathbb{M}^{d+2}$ via

$$P^A = \lambda(1, x^2, x^a) = (P^+, P^-, P^a)\,, \qquad \lambda \in \mathbb{R}^+ \tag{2.29}$$

with the scalar product in $\mathbb{M}^{2+d}$ is defined by

$$P \cdot P = -P^+P^- + \delta_{ab}P^aP^b. \tag{2.30}$$

Example of translations and special conformal transformations

For example, we can recover translations and special conformal transformations (1.6)

$$P'^A = \Lambda(b)P = e^{i\omega^{CD}L_{CD}}P^A = P^A + i2\omega^{CD}P_C\delta^A_D. \tag{2.31}$$

Special conformal transformations and translations correspond to $\omega^{\nu-} = \lambda b^\nu$ and $\omega^{\nu+} = \lambda b^\nu$, respectively with λ a being a small parameter. Thus we obtain,

$$P'^A = P^A + i2\lambda b^\nu \left(P_\pm \delta^A_\nu - P_\nu \delta^A_\pm\right) \tag{2.32}$$

where the plus sign is used for special conformal transformations and minus for translations. The vector can be projected to physical space by multiplying it by[4]

$$\frac{\partial P^A}{\partial x^\mu} = (0, 2x_\mu, \delta^a_\mu). \tag{2.33}$$

Thus, we obtain

$$\frac{\partial P}{\partial x^\mu} \cdot P' = \begin{cases} x_\mu + 2i\lambda(b_\mu - x\cdot b)\,, & \text{for plus sign} \\ x_\mu + 2i\lambda(x^2 b^\mu - x\cdot b x^\mu), & \text{for minus sign} \end{cases} . \tag{2.34}$$

2.3.1 Fields in embedding space formalism

The embedding space is particularly interesting to study correlation function of local operators. First we need to establish how the operators are lifted from the physical space to the embedding. A symmetric and traceless tensor operator $\mathcal{O}_{A_1\ldots A_J}(P)$ has the following properties:

- It is defined on the cone $P^2 = 0$;
- Homogenoues of degree $-\Delta$, *i.e.* $\mathcal{O}_{A_1\ldots A_J}(\lambda P) = \lambda^{-\Delta}\mathcal{O}_{A_1\ldots A_J}(P)$;
- It is transverse *i.e.* $P^{A_1}\mathcal{O}_{A_1\ldots A_J}(P) = 0$.

The tensor can be projected to the physical space $\mathbb{R}^d$ by

$$\mathcal{O}_{a_1\ldots a_J}(x) = \frac{\partial P^{A_1}}{\partial x^{a_1}} \cdots \frac{\partial P^{A_J}}{\partial x^{a_J}} \mathcal{O}_{A_1\ldots A_J}(P). \tag{2.35}$$

To encode the symmetric and traceless properties of tensor operators, we will use an auxiliary variable. More concretely, a symmetric and traceless tensor operator of spin J can be written using a null vector Z by

$$\mathcal{O}_J(P,Z) \equiv Z^{A_1}\ldots Z^{A_J}\mathcal{O}_{A_1\ldots A_J}(P), \tag{2.36}$$

with $Z^2 = 0$. The indices can be recovered using Todorov differential operator via

$$\mathcal{O}_{A_1\ldots A_J}(P) = \frac{1}{J!(h-1)_J} D_{A_1}\ldots D_{A_J}\mathcal{O}_J(P,Z) \tag{2.37}$$

[4]This projector maps a vector on the embedding to a specific section. We could have chosen another projector that would map to another section. These sections are related by a conformal transformation.

with D_A defined

$$D_A = \left(h - 1 + Z \cdot \frac{\partial}{\partial Z}\right) \frac{\partial}{\partial Z^A} - \frac{Z_A}{2} \frac{\partial^2}{\partial Z \cdot \partial Z}. \tag{2.38}$$

Two and three point correlation function involving tensor operators can now be cast in an elegant form using this formalism

$$\langle \mathcal{O}_{\Delta,J}(P_1, Z_1) \mathcal{O}_{\Delta,J}(P_2, Z_2) \rangle = \frac{H_{12}^J}{P_{12}^{\Delta+J}}, \tag{2.39}$$

$$\langle \mathcal{O}_{\Delta_1}(P_1) \mathcal{O}_{\Delta_2}(P_2) \mathcal{O}_{\Delta_3,J}(P_3, Z_3) \rangle = C_{123} \frac{2^J (Z_3 \cdot P_1 P_2 \cdot P_3 - Z_3 \cdot P_2 P_1 \cdot P_3)^J}{P_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3+J}{2}} P_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2+J}{2}} P_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+J}{2}}}, \tag{2.40}$$

with

$$H_{ij} = (Z_i \cdot Z_2)(P_i \cdot P_j) - (P_1 \cdot Z_2)(P_2 \cdot Z_1), \qquad P_{ij} = -2 P_i \cdot P_j. \tag{2.41}$$

Notice that this is the unique structure satisfying

$$\mathcal{O}(\lambda P, \alpha Z) = \lambda^{-\Delta} \alpha^J \mathcal{O}(P, Z) \ , \qquad \forall \alpha, \lambda \in \mathbb{R} \tag{2.42}$$

and

$$\mathcal{O}(P, Z + \beta P) = \mathcal{O}(P, Z) \ , \qquad \forall \beta \in \mathbb{R} \ . \tag{2.43}$$

Chapter 3

Mellin amplitudes

Regge theory is usually defined for scattering amplitudes as will be reviewed in the following chapter. These amplitudes are expressed in terms of momenta of the incoming particles. On the other hand, correlation functions in CFT are usually defined in terms of positions. The main goal of this chapter is to introduce a Mellin representation for correlation functions in a CFT. This is particularly interesting as it has several similarities with scattering amplitudes.

Define a Mellin amplitude $M(\gamma_{ij})$ associated with a n-point correlation function of scalar primary operators with dimension Δ_i by

$$\langle \mathcal{O}_1 (x_1) \dots \mathcal{O}_n (x_n) \rangle = \int [d\gamma] \, M(\gamma_{ij}) \prod_{1 \le i < j \le n} \Gamma(\gamma_{ij}) \, (x_{ij})^{-\gamma_{ij}} \, , \tag{3.1}$$

where the integration $[d\gamma]$ is subject to the constraints [1]

$$\sum_{i=1}^{n} \gamma_{ij} = 0 \, , \qquad \gamma_{ij} = \gamma_{ji} \, , \qquad \gamma_{ii} = -\Delta_i \, , \tag{3.2}$$

ensuring that the correlation function transforms appropriately under conformal transformations. The integration contours for the independent γ_{ij} variables run parallel to the imaginary axis. The Mellin amplitude M depends on the variables γ_{ij} subject to the constraints (3.2) but we shall often keep this dependence implicit to simplify our formulas.

The relation between Mellin and scattering amplitudes was noticed after studying the OPE structure of correlation functions in CFTs [21, 22, 23]. In this context it is common to introduce fictitious momenta p_i

$$p_i \cdot p_j = \gamma_{ij}. \tag{3.3}$$

Mellin amplitudes have other interesting features, for example it simplifies the computation of

[1] The notation $[d\gamma]$ includes the a factor of $\frac{1}{2\pi i}$ for each one of the $n(n-3)/2$ independent γ_{ij} variables.

Witten diagrams [23, 24, 25] and the analysis of the flat space limit which will be reviewed later on [23, 24].

We will begin by defining a representation for Mellin amplitudes for operators with spin and we will review the factorization properties of a four point function using the technology of the previous chapter. Then we analyze the factorization of higher point functions but restricted to the scalar and vector case. The last section is devoted to the flat space limit of correlation functions.

3.1 Mellin amplitudes for tensor operators

In the following we will define two possible representations for Mellin amplitudes associated with correlation functions with one tensor operator[26]. It is convenient to use the embedding space formalism introduced in section 2.3. Let us begin with the simplest case, a vector primary operator $\mathcal{O}(P,Z)$ and k scalars operators $\mathcal{O}_i(P_i)$.

$$\langle \mathcal{O}(P,Z)\mathcal{O}_1(P_1)\dots\mathcal{O}_k(P_k)\rangle \ . \tag{3.4}$$

A Mellin representation for this correlator is

$$\sum_{a=1}^{k}(Z\cdot P_a)\int[d\gamma]\,M^a\prod_{\substack{i,j=1\\ i<j}}^{k}\frac{\Gamma(\gamma_{ij})}{(-2P_i\cdot P_j)^{\gamma_{ij}}}\prod_{i=1}^{k}\frac{\Gamma(\gamma_i+\delta_i^a)}{(-2P_i\cdot P)^{\gamma_i+\delta_i^a}}\ , \tag{3.5}$$

where δ_i^a is the Kronecker-delta and

$$\gamma_i=-\sum_{j=1}^{k}\gamma_{ij}\ , \qquad \gamma_{ij}=\gamma_{ji}\ , \qquad \gamma_{ii}=-\Delta_i\ , \tag{3.6}$$

as required by (2.42) applied to each scalar operator. Imposing (2.42) for the vector operator, we obtain the final constraint

$$\sum_{i,j=1}^{k}\gamma_{ij}=1-\Delta\ . \tag{3.7}$$

In this case, it is convenient to think of γ_{ij} for $1\le i<j\le k$ as the independent Mellin variables subject to the single constraint (3.7) (recall that $\gamma_{ii}=-\Delta_i$). From (2.43), we conclude that the Mellin amplitudes M^a are constrained by

$$\sum_{a=1}^{k}\gamma_a M^a=0\ . \tag{3.8}$$

Another possible Mellin representation for the correlator (3.4) is

$$\sum_{a=1}^{k} D_a \int [d\gamma]\, \check{M}^a \prod_{\substack{i,j=1\\ i<j}}^{k} \frac{\Gamma(\gamma_{ij})}{(-2P_i \cdot P_j)^{\gamma_{ij}}} \prod_{i=1}^{k} \frac{\Gamma(\gamma_i + \delta_i^a)}{(-2P_i \cdot P)^{\gamma_i+\delta_i^a}} \ , \tag{3.9}$$

where the D_a is the following differential operator,

$$D_a = (P \cdot P_a)(Z \cdot \partial_P) - (Z \cdot P_a)(P \cdot \partial_P - Z \cdot \partial_Z) \ . \tag{3.10}$$

This was suggested in [25] from the study of Witten diagrams. In this representation, the Mellin variables obey the same constraints (3.6) and (3.7). Acting with the differential operator, it is not hard to see that the two representations are related through

$$M^a = \sum_{b=1}^{k} \gamma_b \left(\check{M}^a - \check{M}^b \right) \ . \tag{3.11}$$

Notice that the constraint (3.8) on M^a is automatic in terms of $\check{M}^a$. On the other hand, the second description $\check{M}^a$ is redundant because the shift

$$\check{M}^a \to \check{M}^a + \Lambda \tag{3.12}$$

leaves M^a invariant for any function Λ of the Mellin variables γ_{ij}. Since these two representations are equivalent, we shall use them according to convenience. For example, M^a seems to be more useful to formulate factorization and to impose conservation, while $\check{M}^a$ leads to a simpler formula for the flat space limit.

3.1.1 Tensor operator

Let us now generalize the Mellin representation for the correlator (3.4) involving one primary operator $\mathcal{O}(P, Z)$ with spin J. The first representation is

$$\sum_{a_1,\dots,a_J=1}^{k} \left(\prod_{\ell=1}^{J} (Z \cdot P_{a_\ell}) \right) \int [d\gamma]\, M^{\{a\}} \prod_{\substack{i,j=1\\ i<j}}^{k} \frac{\Gamma(\gamma_{ij})}{(-2P_i \cdot P_j)^{\gamma_{ij}}} \prod_{i=1}^{k} \frac{\Gamma(\gamma_i + \{a\}_i)}{(-2P_i \cdot P)^{\gamma_i+\{a\}_i}} \tag{3.13}$$

where $\{a\}$ stands for the set $a_1, \dots, a_J$ and $\{a\}_i$ counts the number of occurrences of i in the list $a_1, \dots, a_J$, *i.e.*

$$\{a\}_i = \delta_i^{a_1} + \dots + \delta_i^{a_J} \ . \tag{3.14}$$

The constraints on the Mellin variables are

$$\gamma_i = -\sum_{j=1}^{k} \gamma_{ij} \ , \qquad \gamma_{ij} = \gamma_{ji} \ , \qquad \gamma_{ii} = -\Delta_i \ , \qquad \sum_{i,j=1}^{k} \gamma_{ij} = J - \Delta \ , \tag{3.15}$$

and the Mellin amplitude is symmetric under permutations of the indices $a_1, \ldots, a_J$ and obeys

$$\sum_{a_1=1}^{k} (\gamma_{a_1} + \delta_{a_1}^{a_2} + \delta_{a_1}^{a_3} + \cdots + \delta_{a_1}^{a_J}) M^{a_1 a_2 \ldots a_J} = 0 \ . \tag{3.16}$$

The generalization of the second representation is

$$\sum_{a_1,\ldots,a_J=1}^{k} \left(\prod_{\ell=1}^{J} D_{a_\ell} \right) \int [d\gamma]\, \check{M}^{\{a\}} \prod_{1\le i<j\le k} \frac{\Gamma(\gamma_{ij})}{(-2P_i \cdot P_j)^{\gamma_{ij}}} \prod_{1\le i\le k} \frac{\Gamma(\gamma_i + \{a\}_i)}{(-2P_i \cdot P)^{\gamma_i + \{a\}_i}} \ . \tag{3.17}$$

Since $[D_a, D_b] = 0$, we can choose $\check{M}^{a_1 \ldots a_J}$ invariant under permutation of the indices a_l. Moreover, from the identity

$$\sum_{a_1=1}^{k} D_{a_1} \prod_{1\le i\le k} \frac{\Gamma(\gamma_i + \{a\}_i)}{(-2P_i \cdot P)^{\gamma_i + \{a\}_i}} = 0 \tag{3.18}$$

we conclude that the correlator is invariant under

$$\check{M}^{a_1 \ldots a_J} \to \check{M}^{a_1 \ldots a_J} + \sum_{m=1}^{J} \Lambda^{a_1 \ldots a_{m-1} a_{m+1} \ldots a_J} \ , \tag{3.19}$$

where $\Lambda^{a_2 \ldots a_J}$ is any function of the Mellin variables that depends on one less index.

3.2 Four point function

We will analyze the factorization of Mellin amplitudes for a four point function of scalars[27]. This case exhibits the pole structure implied by the OPE, some similarities with scattering amplitudes and does not require the definition of Mellin amplitudes for operators with spin. There are two independent integration variables which are the analogues of the Mandelstam invariants [2]

$$t = -(p_1 + p_2)^2 = \Delta_1 + \Delta_2 - 2\gamma_{12} \,, \qquad s = -(p_1 + p_3)^2 - \Delta_1 - \Delta_4 = \Delta_3 - \Delta_4 - 2\gamma_{13} \,. \tag{3.20}$$

The reduced correlator $\mathcal{A}(u, v)$ (1.17) has the following Mellin representation

$$\begin{aligned} \mathcal{A}(u,v) = \int_{-i\infty}^{i\infty} \frac{dt ds}{(4\pi i)^2}\, M(s,t)\, u^{t/2} v^{-(s+t)/2}\, \Gamma\!\left(\frac{\Delta_1 + \Delta_2 - t}{2}\right) \Gamma\!\left(\frac{\Delta_3 + \Delta_4 - t}{2}\right) \\ \Gamma\!\left(\frac{\Delta_{34} - s}{2}\right) \Gamma\!\left(\frac{-\Delta_{12} - s}{2}\right) \Gamma\!\left(\frac{t + s}{2}\right) \Gamma\!\left(\frac{t + s + \Delta_{12} - \Delta_{34}}{2}\right) , \end{aligned} \tag{3.21}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$. The integration contours run parallel to the imaginary axis and should be placed such that the infinite series of poles produced by each Γ-function stays entirely to one

[2]The shift in the definition of s is convenient to simplify the formulas for the Mack polynomials given below. In any case, this shift is irrelevant in the Regge limit of large s.

side of the contour. The same requirement applies to the poles of the Mellin amplitude $M(s,t)$ itself, which are described in equation (3.22) below. The Mellin amplitude must have poles in the variable t to reproduce the power law behavior of $\mathcal{A}$ at small u predicted by the OPE or conformal block decomposition (2.9). More precisely,

$$M(s,t) \approx \frac{C_{12k}C_{34k}\,\mathcal{Q}_{J,m}(s)}{t-\Delta+J-2m}\,, \qquad m=0,1,2,\dots\,, \tag{3.22}$$

where, as before, Δ and J are the dimension and spin of an operator $\mathcal{O}_k$ that appears in both OPEs $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$. It is convenient to write $\mathcal{Q}_{J,m}(s)$ in terms of new polynomials $Q_{J,m}(s)$ defined by

$$\begin{aligned}\mathcal{Q}_{J,m}(s) = -\,&\frac{2\Gamma(\Delta+J)(\Delta-1)_J}{4^J\Gamma\big(\frac{\Delta+J+\Delta_{12}}{2}\big)\,\Gamma\big(\frac{\Delta+J-\Delta_{12}}{2}\big)\,\Gamma\big(\frac{\Delta+J+\Delta_{34}}{2}\big)\,\Gamma\big(\frac{\Delta+J-\Delta_{34}}{2}\big)}\\ &\frac{Q_{J,m}(s)}{m!(\Delta-h+1)_m\,\Gamma\big(\frac{\Delta_1+\Delta_2-\Delta+J}{2}-m\big)\,\Gamma\big(\frac{\Delta_3+\Delta_4-\Delta+J}{2}-m\big)}\,,\end{aligned} \tag{3.23}$$

where we used the Pochhammer symbol

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1)\dots(a+m-1)\,. \tag{3.24}$$

The integer m that labels the poles in (3.22) corresponds precisely to the label m in (2.9). This shows that the $m>0$ poles correspond to conformal descendant operators with twist greater than $\Delta-J$. The residues of the poles are the kinematical polynomials $\mathcal{Q}_{J,m}(s)$ given in (3.23). To determine these polynomials we require that the contribution of the series of poles (3.22) reproduces the conformal block $G_{\Delta,J}(u,v)$. Picking the poles (3.22) in the integral (3.21) one obtains a series of the form (2.9) with

$$\begin{aligned}g_m(v) = &\frac{2\Gamma(\Delta+J)(\Delta-1)_J}{4^J m!(\Delta-h+1)_m\,\Gamma\big(\frac{\Delta+J+\Delta_{12}}{2}\big)\,\Gamma\big(\frac{\Delta+J-\Delta_{12}}{2}\big)\,\Gamma\big(\frac{\Delta+J+\Delta_{34}}{2}\big)\,\Gamma\big(\frac{\Delta+J-\Delta_{34}}{2}\big)}\\ &\int_{-i\infty}^{i\infty}\frac{ds}{8\pi i}\,v^{-(s+\tau)/2}Q_{J,m}(s)\,\Gamma\Big(\frac{\Delta_{34}-s}{2}\Big)\,\Gamma\Big(\frac{-\Delta_{12}-s}{2}\Big)\,\Gamma\Big(\frac{\tau+s}{2}\Big)\,\Gamma\Big(\frac{\tau+s+\Delta_{12}-\Delta_{34}}{2}\Big)\,,\end{aligned} \tag{3.25}$$

where $\tau=\Delta-J+2m$. This explains the position of the poles (3.22) of the Mellin amplitude.

Let us consider first the $m=0$ case. Expanding (3.25) in powers of $(1-v)$ and using the explicit expression of $g_0(v)$ in (2.10), we obtain the following set of equations

$$\begin{aligned}&\frac{(-2)^J\Gamma\big(\frac{\Delta-J-\Delta_{12}}{2}+n\big)\,\Gamma\big(\frac{\Delta-J+\Delta_{34}}{2}+n\big)\,\Gamma\big(\frac{\Delta+J+\Delta_{12}}{2}\big)\,\Gamma\big(\frac{\Delta+J-\Delta_{34}}{2}\big)\,n!}{\Gamma(\Delta+n)(\Delta-1)_J(n-J)!} =\\ &=\int_{-i\infty}^{i\infty}\frac{ds}{4\pi i}\,Q_{J,0}(s)\,\Gamma\Big(\frac{\Delta_{34}-s}{2}\Big)\,\Gamma\Big(\frac{-\Delta_{12}-s}{2}\Big)\,\Gamma\Big(\frac{\tau+s}{2}+n\Big)\,\Gamma\Big(\frac{\tau+s+\Delta_{12}-\Delta_{34}}{2}\Big)\,.\end{aligned} \tag{3.26}$$

For $n < J$ the LHS vanishes and this equation can be written as follows

$$\begin{aligned} 0 = \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i}\, Q_{J,0}(s) \left(\frac{\tau+s}{2}\right)_n \\ \Gamma\!\left(\frac{\Delta_{34}-s}{2}\right)\Gamma\!\left(\frac{-\Delta_{12}-s}{2}\right)\Gamma\!\left(\frac{\tau+s}{2}\right)\Gamma\!\left(\frac{\tau+s+\Delta_{12}-\Delta_{34}}{2}\right). \end{aligned} \tag{3.27}$$

Taking linear combinations of this equation with $n < J$, we conclude that it defines an inner product under which $Q_{J,0}(s)$ is orthogonal to all polynomials of s with degree less than J. In other words, the polynomials $Q_{J,0}(s)$ must satisfy

$$\begin{aligned} \delta_{J,J'} \propto \int_{-i\infty}^{i\infty} \frac{ds}{4\pi i}\, Q_{J,0}(s)\, Q_{J',0}(s) \\ \Gamma\!\left(\frac{\Delta_{34}-s}{2}\right)\Gamma\!\left(\frac{-\Delta_{12}-s}{2}\right)\Gamma\!\left(\frac{\tau+s}{2}\right)\Gamma\!\left(\frac{\tau+s+\Delta_{12}-\Delta_{34}}{2}\right). \end{aligned} \tag{3.28}$$

This fixes the polynomials $Q_{J,0}(s)$ uniquely, up to normalization. The normalization can be fixed by imposing (3.26) for any $n \geq J$.

The orthogonality of the polynomials $Q_{J,0}(s)$ suggests that they are the solutions of a Sturm-Liouville problem. Indeed, the difference operator $\mathcal{D}_s$, defined by

$$\mathcal{D}_s Q(s) = (s+\tau+\Delta_{12}-\Delta_{34})\Big[(s+\tau)Q(s+2) - 2sQ(s)\Big] + (s+\Delta_{12})(s-\Delta_{34})Q(s-2)\,, \tag{3.29}$$

is self-adjoint with respect to the inner product above. Therefore, eigenfunctions of $\mathcal{D}_s$ with different eigenvalues are automatically orthogonal. By construction, the action of $\mathcal{D}_s$ on a polynomial of s of degree J produces another polynomial of s of degree J. Thus, we can look for polynomial eigenfunctions of $\mathcal{D}_s$,

$$\mathcal{D}_s\, Q_{J,0}(s) = \lambda_J\, Q_{J,0}(s)\,. \tag{3.30}$$

The eigenvalue λ_J is fixed by comparing the coefficient of the highest degree term s^J, with the result

$$\lambda_J = 4J^2 + 4J(\tau-1) + (\tau+\Delta_{12})(\tau-\Delta_{34})\,. \tag{3.31}$$

Finally, the solution can be written in terms of hypergeometric functions [3]

$$Q_{J,0}(s) = \frac{2^J \left(\frac{\Delta_{12}+\tau}{2}\right)_J \left(\frac{\Delta_{34}+\tau}{2}\right)_J}{(\tau+J-1)_J}\, {}_3F_2\!\left(-J, J+\tau-1, \frac{\Delta_{34}-s}{2}; \frac{\tau+\Delta_{12}}{2}, \frac{\tau+\Delta_{34}}{2}; 1\right). \tag{3.32}$$

Consider now the case $m > 0$. Most importantly, if we replace $g_m(v)$ given by (3.25) in equation

[3] Interestingly, these polynomials already appeared in the QCD Pomeron literature [28]. We thank Gregory Korchemsky for informing us that these polynomials are known in the mathematical literature as continuous Hahn polynomials (see http://aw.twi.tudelft.nl/ koekoek/askey/ch1/par4/par4.html).

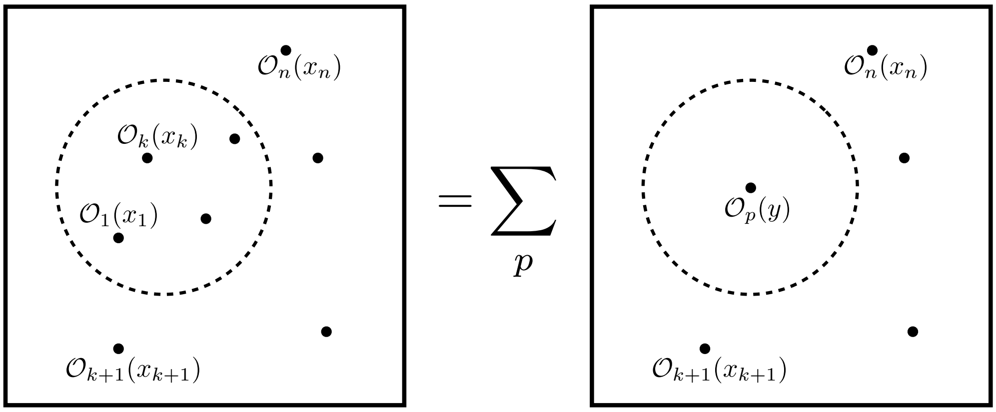


Figure 3.1: In a CFT correlation function, one can replace multiple operators inside a sphere by a (infinite) sum of local operators inserted at the center of the sphere.

(2.11), we obtain a set of recursion relations for the polynomials $Q_{J,m}(s)$,

$$(\mathcal{D}_s - \lambda_J)\, Q_{J,m}(s) = 4m(h - \Delta - m)\big(2Q_{J,m}(s) - Q_{J,m-1}(s+2) - Q_{J,m-1}(s)\big)\,, \tag{3.33}$$

where $\mathcal{D}_s$ and λ_J are respectively given by (3.29) and (3.31) with $\tau = \Delta - J + 2m$. This equation, plus the boundary condition $Q_{J,0}(s) = s^J + O(s^{J-1})$ which follows from (3.32), determine the polynomials $Q_{J,m}(s)$ for all $m \geq 0$. In particular, it is clear that the leading behavior is given by $Q_{J,m} = s^J + O(s^{J-1})$ for all m. This follows from the fact that the LHS of (3.33) is automatically a polynomial of degree $(J-1)$, if we assume that $Q_{J,m}(s)$ is a polynomial of degree J. Imposing the same condition to the RHS implies that $Q_{J,m}(s)$ and $Q_{J,m-1}(s)$ have the same leading behavior. In order to obtain the conformal block $G_{\Delta,J}(u,v)$ we only kept the contribution from the series of poles (3.22) in the integral (3.21). However, the integrand in (3.21) has more poles in the variable t. These poles occur at $t = \Delta_1 + \Delta_2 + 2m$ and $t = \Delta_3 + \Delta_4 + 2m$, which is the twist of the composite operators $\mathcal{O}_1\partial_{\mu_1}\dots\partial_{\mu_J}\partial^{2m}\mathcal{O}_2$ and $\mathcal{O}_3\partial_{\mu_1}\dots\partial_{\mu_J}\partial^{2m}\mathcal{O}_4$, in the limit where the external operators $\mathcal{O}_i$ interact weakly.

3.3 Factorization of higher point functions

The goal of this section is to analyze factorization of Mellin amplitudes associated with higher point correlation functions. The four point function was a good warm up to understand the concepts that are involved. In the previous example we used expression for the conformal block to determine the factorization for the four point function. There is no known analogous expression for higher point correlation functions. There are at least two ways to fully determine the factorization: shadow formalism and the Casimir equation. We will use the shadow formalism in the main text and present the approach based on Casimir equation in the appendix E. In this thesis we just present the computations for the scalar and vector factorization to show the general method. In [26] we have computed the factorization for the spin two case.

The idea of factorization is simple so let us motivate and briefly mention the main results of this section. Consider the OPE of k-local operators as depicted in 3.1

$$\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)=\sum_p C_p\left(x_1,\dots,x_k,y,\partial_y\right)\mathcal{O}_p\left(y\right), \tag{3.34}$$

where p runs over all primary local operators. Thus a correlation function of n local primary operators can be written as

$$\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle=\sum_p C_p\left(x_1,\dots,x_k,y,\partial_y\right)\langle\mathcal{O}_p(y)\mathcal{O}_{k+1}(x_{k+1})\dots\mathcal{O}_n(x_n)\rangle. \tag{3.35}$$

The function $C_p\left(x_1,\dots,x_k,y,\partial_y\right)$ is associated with a correlation function of $k+1$ local operators by

$$\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_k(x_k)\mathcal{O}_p\left(x_{k+1}\right)\rangle=C_p\left(x_1,\dots,x_k,y,\partial_y\right)\frac{1}{(y-x_{k+1})^{\Delta_p}} \tag{3.36}$$

So the Mellin amplitude associated with n local operators can be be naturally decomposed into a left and right part. As explained in [24] (section 2.1), and below, for each primary operator $\mathcal{O}_p$ with dimension Δ and spin J that appears in the OPE (3.34), the Mellin amplitude has an infinite sequence of poles

$$M\approx\frac{\mathcal{Q}_m}{\gamma_{LR}-(\Delta-J+2m)}\,,\qquad m=0,1,2,\dots\,, \tag{3.37}$$

in the variable

$$\gamma_{LR}=\sum_{a=1}^{k}\sum_{i=k+1}^{n}\gamma_{ai} \tag{3.38}$$

where the m just labels the twist of the descendant operator associated with the primary of spin J and dimension Δ as in (3.22). The residue Q_m has a part that depends on the Mellin amplitude associated with the correlation function (3.36), a analogous Mellin amplitude for the right correlation function and a kinematical polynomial. The factorization is simpler for four point correlation functions because the left and right correlation functions are just the OPE coefficients.

3.3.1 Factorization from the shadow operator formalism

3.3.1.1 Factorization for a scalar operator

Using the multiple OPE (3.34), one can write a CFT n-point function as a sum over the contribution of each primary operator (and its descendants). As explained in [24], each term in this sum gives rise to a series of poles in the Mellin amplitude. In this section, our strategy to obtain these poles is to use the *projector* [29, 13]

$$|\mathcal{O}| = \frac{1}{\mathcal{N}_\Delta} \int d^d y d^d z |\mathcal{O}(y)\rangle \frac{\Gamma(d-\Delta)}{(y-z)^{2(d-\Delta)}} \langle \mathcal{O}(z)| \tag{3.39}$$

inside the correlation function. The conformal integral

$$\frac{1}{\mathcal{N}_\Delta} \int d^d y d^d z \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \mathcal{O}(y)\rangle \frac{\Gamma(d-\Delta)}{(y-z)^{2(d-\Delta)}} \langle \mathcal{O}(z) \mathcal{O}_{k+1}(x_{k+1}) \dots \mathcal{O}_n(x_n)\rangle \ , \tag{3.40}$$

gives the contribution of the operator $\mathcal{O}$ in the multiple OPE of $\mathcal{O}_1 \dots \mathcal{O}_k$, to the n-point function $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n)\rangle$. In fact, this integral includes an extra shadow contribution, which can be removed by doing an appropriate monodromy projection [13]. Fortunately, if we are only interested in the poles of the Mellin amplitude, this monodromy projection is very simple to perform in Mellin space. The reason is that the Mellin amplitude of (3.40) has poles associated with the operator $\mathcal{O}$ and other poles associated with its shadow. Therefore, the monodromy projection amounts to focusing on the first set of poles. This follows from the fact that each series of poles in Mellin space gives rise to a power series expansion in position space with different monodromies.

Let us start by considering the case where the exchanged operator $\mathcal{O}$ is a scalar[26]. If we normalize the operator to have unit two point function $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = |x|^{-2\Delta}$, the normalization constant $\mathcal{N}_\Delta$ is given by

$$\mathcal{N}_\Delta = \frac{\pi^d \, \Gamma\left(\Delta - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} \ , \tag{3.41}$$

as we show in appendix D.3. We start by writing the correlation functions that appear in (3.40) in the Mellin representation,

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \mathcal{O}(y)\rangle = \int [d\lambda] \, M_L \prod_{1 \le a < b \le k} \frac{\Gamma(\lambda_{ab})}{(x_{ab}^2)^{\lambda_{ab}}} \prod_{1 \le a \le k} \frac{\Gamma(\lambda_a)}{(x_a - y)^{2\lambda_a}} \tag{3.42}$$

where

$$\lambda_a = -\sum_{b=1}^{k} \lambda_{ab} \ , \qquad \lambda_{ab} = \lambda_{ba} \ , \qquad \lambda_{aa} = -\Delta_a \ , \qquad \sum_{a,b=1}^{k} \lambda_{ab} = -\Delta \ . \tag{3.43}$$

The integration measure $[d\lambda]$ denotes $(k-2)(k+1)/2$ integrals running parallel to the imaginary axis, over the independent variables λ_{ab} that remain after solving the constraints (3.43). Similarly, the second correlator in (3.40) reads

$$\langle \mathcal{O}(z) \mathcal{O}_{k+1}(x_{k+1}) \dots \mathcal{O}_n(x_n)\rangle = \int [d\rho] \, M_R \prod_{k < i < j \le n} \frac{\Gamma(\rho_{ij})}{(x_{ij}^2)^{\rho_{ij}}} \prod_{k < i \le n} \frac{\Gamma(\rho_i)}{(x_i - z)^{2\rho_i}} \tag{3.44}$$

where

$$\rho_i = -\sum_{j=k+1}^{n} \rho_{ij} \ , \qquad \rho_{ij} = \rho_{ji} \ , \qquad \rho_{ii} = -\Delta_i \ , \qquad \sum_{i,j=k+1}^{n} \rho_{ij} = -\Delta \ . \tag{3.45}$$

We use a, b to label the first k points of the n-point function and i, j to denote the other $n-k$ points. In appendix D.1, we insert (3.42) and (3.44) in (3.40) and simplify the resulting integral in Mellin space, until we arrive at

$$M_{\mathcal{O}} = \frac{\pi^d}{\mathcal{N}_\Delta} \frac{\Gamma\left(\frac{2\Delta-d}{2}\right)\Gamma\left(\frac{d-\Delta-\gamma_{LR}}{2}\right)}{\Gamma\left(\frac{\Delta-\gamma_{LR}}{2}\right)} \, F_L \times F_R \tag{3.46}$$

where $\gamma_{LR} = \sum_{a=1}^{k}\sum_{i=k+1}^{n} \gamma_{ai}$ and

$$F_L = \int [d\lambda] \, M_L(\lambda_{ab}) \prod_{1\le a<b\le k} \frac{\Gamma(\lambda_{ab})\Gamma(\gamma_{ab}-\lambda_{ab})}{\Gamma(\gamma_{ab})} \tag{3.47}$$

$$F_R = \int [d\rho] \, M_R(\rho_{ij}) \prod_{k<i<j\le n} \frac{\Gamma(\rho_{ij})\Gamma(\gamma_{ij}-\rho_{ij})}{\Gamma(\gamma_{ij})} \tag{3.48}$$

Expression (3.46) is the final result for the Mellin amplitude $M_{\mathcal{O}}$ of the conformal integral (3.40). As expected, the Mellin amplitude $M_{\mathcal{O}}$ has poles at $\gamma_{LR} = d-\Delta+2m$ for $m = 0, 1, 2, \dots$, which are associated with the shadow of $\mathcal{O}$. We are not interested in these poles because they are not poles of the Mellin amplitude M of the physical n-point function. On the other hand, $M_{\mathcal{O}}$ has poles at $\gamma_{LR} = \Delta + 2m$ for $m = 0, 1, 2, \dots$, which are also present in the n-point Mellin amplitude M with exactly the same residues. Our goal is to compute these residues. From (3.46), we conclude that both F_L and F_R must have simple poles at $\gamma_{LR} = \Delta + 2m$. In appendix D.1, we deform the integration contours in (3.47) and arrive at the following formula for the residues of F_L,

$$F_L \approx \frac{-2(-1)^m}{\gamma_{LR} - \Delta - 2m} \sum_{\substack{n_{ab}\ge 0\\ \sum n_{ab}=m}} M_L(\gamma_{ab}+n_{ab}) \prod_{1\le a<b\le k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \ . \tag{3.49}$$

We can now return to (3.46) and conclude that the poles of the Mellin amplitude associated with the exchange of a scalar operator $\mathcal{O}$ (of dimension Δ) between the first k operators and the other $(n-k)$ operators of a n-point function, is given by equations (3.37).

3.3.1.2 Factorization on a vector operator

With this method, one can find similar factorization formulas for the residues of poles associated with tensor operators. One just needs to generalize the projector (3.39) for tensor operators and perform several conformal integrals using (a generalized) Symanzik's formula. We describe this calculation in appendix D.2 for the case of vector operators. The result is that the residues $\mathcal{Q}_m$

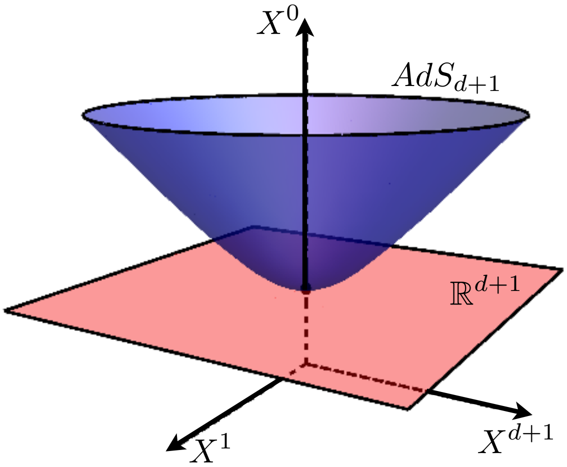


Figure 3.2: Euclidean AdS$_{d+1}$ embedded in Minkowski space $\mathbb{M}^{d+2}$. The tangent space $\mathbb{R}^{d+1}$ is a good local approximation to AdS in a region smaller than the AdS radius of curvature.

in equation (3.37) are given by

$$\mathcal{Q}_m = \frac{\Delta\Gamma(\Delta-1)m!}{\left(1+\Delta-\frac{d}{2}\right)_m}\sum_{a=1}^{k}\sum_{i=k+1}^{n} L_m^a\, R_m^i \left[\gamma_{ai} + \frac{d-2\Delta}{2m(\Delta-d+1)}\sum_{b=1}^{k}\gamma_{ab}\sum_{j=k+1}^{n}\gamma_{ij}\right] \tag{3.50}$$

where

$$L_m^a = \sum_{\substack{n_{ab}\geq 0\\ \sum n_{ab}=m}} M_L^a(\gamma_{ab}+n_{ab}) \prod_{1\leq a<b\leq k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \tag{3.51}$$

is constructed in terms of the Mellin amplitude M_L^a of the correlator of the first k scalar operators and the exchanged vector operator, as defined in (3.5) and similarly for the right factor R_m^i. Notice that the second term vanishes for $m=0$ due to the transversality constraint (3.8). This leads to a particularly simple formula for the residue of the first pole

$$\mathcal{Q}_0 = \Delta\Gamma(\Delta-1)\sum_{a=1}^{k}\sum_{i=k+1}^{n}\gamma_{ai} M_L^a\, M_R^i\ . \tag{3.52}$$

However, the calculations quickly become very lengthy as spin increases.

3.4 Flat space limit

The flat space limit amounts to approximating AdS by its tangent space $\mathbb{R}^{d+1}$ at the point $X^A=(R,0,\ldots,0)$, as shown in figure 3.2. From the physical point of view, the flat space limit requires the radius of curvature of AdS to be much larger than any intrinsic length ℓ_s of the bulk theory. [4] In the dual CFT, the dimensionless ratio $R/\ell_s \equiv \theta$ is a coupling constant that

[4] In string theory, the intrinsic length ℓ_s can be the string length or the Planck length.

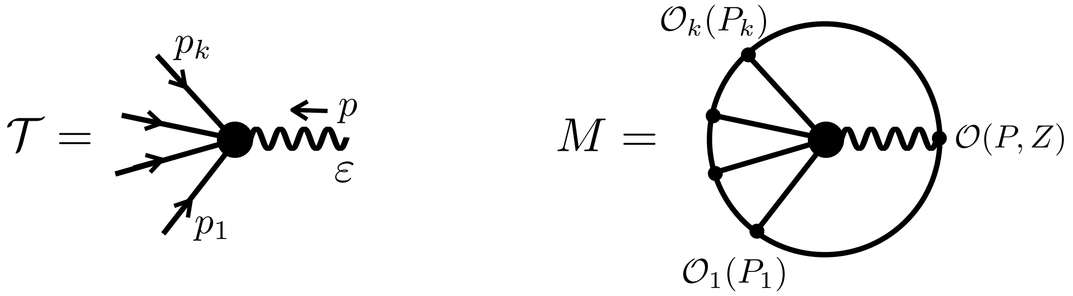


Figure 3.3: (Left) Scattering amplitude associated with a tree level Feynman diagram with a single interaction vertex. (Right) Mellin amplitude associated with tree level Witten diagram using the same interaction vertex but now in AdS. The flat space limit of AdS leads to the general relation (3.84) between the two amplitudes.

parametrizes a family of theories. However, not all observables of a CFT with $\theta \gg 1$ correspond to flat space observables of the dual bulk theory. For example, states of the CFT on the cylinder $S^{d-1} \times \mathbb{R}$ with energy Δ of order one, are dual to wave functions that spread over the scale R and can not be described in flat space. On the other hand, states with large energy such that $\lim_{\theta \to \infty} \Delta/\theta = \alpha$ correspond to states with mass $M = \alpha/\ell_s$ in flat space.

It is natural to ask what happens to other observables under this flat space limit. In particular, in this section we will be interested in obtaining scattering amplitudes on $\mathbb{R}^{d+1}$ as a limit of the Mellin amplitudes of the same theory in AdS. The Mellin amplitudes depend on the Mellin variables γ_{ij} and on the CFT coupling constant θ. As explained in [23, 30], the flat space limit corresponds to

$$\theta \to \infty \,, \qquad \gamma_{ij} \to \infty \,, \qquad \text{with } \frac{\gamma_{ij}}{\theta^2} \text{ fixed}\,. \tag{3.53}$$

We shall consider a local interaction vertex between a set of fields (scalar, vector, etc) and compare the associated tree level scattering amplitude in flat space with the corresponding Mellin amplitude of the CFT correlation function obtained by computing the tree-level Witten diagram using the same vertex in AdS (see figure 3.3). By considering an infinite class of local interactions we will be able to derive a general relation between the scattering amplitude and the Mellin amplitude, as it was done in [23, 30] for scalar fields. To make the exposition pedagogical we will start with a simple interaction vertex for a vector particle and k scalar particles and then generalize to interactions involving a spin J particle.

Let us introduce our notation for scattering amplitudes since we will use them to compare with the Mellin amplitude. A scattering amplitude of a massive vector boson and k scalars is a function of the polarization vector ε and the momenta p and p_i (see figure 3.3). It can be written as

$$\mathcal{T}(\varepsilon, p, p_i) = \sum_{a=1}^{k} \varepsilon \cdot p_a \, \mathcal{T}^a(p_i \cdot p_j) \tag{3.54}$$

where the $\mathcal{T}^a$ are functions of the Lorentz invariants $p_i \cdot p_j$. Notice that momentum conservation

$$p + \sum_{a=1}^{k} p_a = 0 \tag{3.55}$$

can be used to write $p \cdot p_i$ in terms of $p_i \cdot p_j$. Moreover, the condition $\varepsilon \cdot p = 0$ leads to the redundancy

$$\mathcal{T}^a(p_i \cdot p_j) \to \mathcal{T}^a(p_i \cdot p_j) + \Lambda(p_i \cdot p_j) \tag{3.56}$$

for any function Λ of the Lorentz invariants $p_i \cdot p_j$. If the vector particle is massless (photon), then we have gauge invariance

$$\mathcal{T}(\varepsilon + \lambda p, p, p_i) = \mathcal{T}(\varepsilon, p, p_i) \ , \qquad \forall \lambda \in \mathbb{R} \ , \tag{3.57}$$

which leads to the constraint

$$\sum_{a=1}^{k} p \cdot p_a \, \mathcal{T}^a(p_i \cdot p_j) = 0 \ . \tag{3.58}$$

Consider now the scattering amplitude of a massive spin J boson and k scalars. In this case the polarization vector is a symmetric and traceless tensor with J indices, which we shall encode with a null polarization vector with just one index ε^μ. In this way we can write the scattering amplitude as a function of ε and the momenta p and p_i as follows

$$\mathcal{T}(\varepsilon, p, p_i) = \sum_{a_1, \ldots, a_J = 1}^{k} \left(\prod_{\ell=1}^{J} \varepsilon \cdot p_{a_\ell} \right) \mathcal{T}^{a_1 \ldots a_J}(p_i \cdot p_j) \tag{3.59}$$

where $\mathcal{T}^{a_1 \ldots a_J}(p_i \cdot p_j)$ are functions of the Lorentz invariants $p_i \cdot p_j$, totally symmetric under permutations of the indices $a_1 \ldots a_J$. The condition $\varepsilon \cdot p = 0$ leads to the redundancy

$$\mathcal{T}^{a_1 \ldots a_J}(p_i \cdot p_j) \to \mathcal{T}^{a_1 \ldots a_J}(p_i \cdot p_j) + \sum_{m=1}^{J} \Lambda^{a_1 \ldots a_{m-1} a_{m+1} \ldots a_J}(p_i \cdot p_j) \tag{3.60}$$

where $\Lambda^{a_1 \ldots a_{m-1} a_{m+1} \ldots a_J}(p_i \cdot p_j)$ is any function of $p_i \cdot p_j$ that depends on one less index and it is symmetric under permutations of its indices.

When the spin J field is massless, the gauge invariance $\mathcal{T}(\varepsilon + \lambda p, p, p_i) = \mathcal{T}(\varepsilon, p, p_i)$ leads to the constraint

$$\sum_{a_1=1}^{k} p \cdot p_{a_1} \, \mathcal{T}^{a_1 \ldots a_J}(p_i \cdot p_j) = 0. \tag{3.61}$$

3.4.1 Scattering amplitude with a vector particle

Let us start with the simplest example involving a massive vector field A_μ. Consider the local interaction vertex [5]

$$gA_\mu(\nabla^\mu\phi_1)\phi_2\dots\phi_k \tag{3.62}$$

with scalar fields ϕ_i and coupling constant g. The associated scattering amplitude is

$$\mathcal{T} = g\,\varepsilon\cdot p_1 \tag{3.63}$$

where ε is the polarization vector of the massive vector boson and p_i is the momentum of the particle ϕ_i. The polarization vector obeys $\varepsilon\cdot p = 0$ where p is the momentum of the vector boson. Comparing with equation (3.54), we conclude that all $\mathcal{T}^l$ vanish except $\mathcal{T}^1 = g$.

The basic ingredients to evaluate the Mellin amplitude associated with the tree-level Witten diagram using the same interaction vertex in AdS are the bulk to boundary propagators of a scalar field, [6]

$$\Pi_{\Delta,0}(X,P) = \frac{\mathcal{C}_{\Delta,0}}{(-2P\cdot X)^\Delta}\ , \tag{3.64}$$

and of a vector field,

$$\Pi^A_{\Delta,1}(X,P;Z) = \frac{\mathcal{C}_{\Delta,1}}{\Delta}\left[P^A(Z\cdot\partial_P) - Z^A(P\cdot\partial_P)\right]\frac{1}{(-2P\cdot X)^\Delta}\ , \tag{3.65}$$

where X is a point in AdS$_{d+1}$ embedded in $\mathbb{M}^{d+2}$ (*i.e.* $X^2=-1$), A is an embedding AdS index, and [31]

$$\mathcal{C}_{\Delta,J} = \frac{(J+\Delta-1)\Gamma(\Delta)}{2\pi^{\frac{d}{2}}(\Delta-1)\Gamma\left(\Delta+1-\frac{d}{2}\right)}\ . \tag{3.66}$$

The correlation function of k scalars and one vector operator associated with the interaction (3.62) is

$$\begin{aligned}\langle\mathcal{O}(P,Z)\mathcal{O}_1(P_1)\dots\mathcal{O}_k(P_k)\rangle &= g\int_{\text{AdS}} dX\,\Pi^A_{\Delta,1}(X,P;Z)\nabla_A\Pi_{\Delta_1,0}(X,P_1)\prod_{i=2}^k\Pi_{\Delta_i,0}(X,P_i)\\ &= g\frac{2\Delta_1\mathcal{C}_{\Delta,1}\mathcal{C}_{\Delta_1,0}}{\Delta}D_1\int_{\text{AdS}}\frac{dX}{(-2P\cdot X)^\Delta(-2P_1\cdot X)^{\Delta_1+1}}\prod_{i=2}^k\Pi_{\Delta_i,0}(X,P_i)\end{aligned} \tag{3.67}$$

where D_1 is precisely the differential operator given in equation (3.10) that was used to define the Mellin amplitudes $\check{M}$. The AdS covariant derivative ∇_A can be easily computed using the embedding formalism. As explained in [31], it amounts to projecting embedding space partial

[5] Generically, the coupling constant g is dimensionful and therefore it defines an intrinsic length scale ℓ_s of the bulk theory, as discussed above.

[6] We set the AdS curvature radius $R=1$. We shall reintroduce R at the end, in the final formulae.

derivatives $\frac{\partial}{\partial X^A}$ with the projector

$$U_A^B = \delta_A^B + X_A X^B. \tag{3.68}$$

Performing the integral over the bulk position X, [7] we conclude that the correlation function can be written as

$$g\pi^{\frac{d}{2}} \frac{\mathcal{C}_{\Delta,1}\Gamma\left(\frac{\sum_i \Delta_i+\Delta+1-d}{2}\right)}{\Gamma(\Delta+1)} \prod_{i=1}^{k} \frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)} D_1 \int [d\gamma] \prod_{i<j}^{k} \frac{\Gamma(\gamma_{ij})}{(-2P_i\cdot P_j)^{\gamma_{ij}}} \prod_{i=1}^{k} \frac{\Gamma(\gamma_i+\delta_i^1)}{(-2P\cdot P_i)^{\gamma_i+\delta_i^1}}$$

where the Mellin variables γ_{ij} obey the constraints (3.6) and (3.7). Comparing with the Mellin representation (3.9), we obtain

$$\check{M}^1 = g\pi^{\frac{d}{2}}\Gamma\Big(\frac{\sum_{i=1}^k \Delta_i+\Delta+1-d}{2}\Big) \frac{\mathcal{C}_{\Delta,1}}{\Gamma(\Delta+1)} \prod_{i=1}^{k} \frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)} \tag{3.70}$$

and $\check{M}^l = 0$ for $l = 2, 3, \ldots, k$. We conclude that for this simple interaction both the scattering amplitudes $\mathcal{T}^l$ and the Mellin amplitudes $\check{M}^l$ are constants. In addition, they are proportional to each other $\check{M}^l \propto \mathcal{T}^l$. This suggests, as already observed in [25], that the representation $\check{M}^l$ is more suitable to study the flat space limit of AdS than the representation M^l introduced in the beginning of this chapter.

3.4.2 Scattering amplitude for a vector particle - generic interaction

The interaction (3.62) is the simplest local vertex for one vector and k scalar operators. The generalization to other interactions follows essentially the same steps with minor modifications. Take the following local interaction vertex

$$g\nabla\ldots\nabla A_\mu \nabla\ldots\nabla(\nabla^\mu\phi_1)\nabla\ldots\nabla\phi_2\ldots\nabla\ldots\nabla\phi_k \tag{3.71}$$

where there are α_{ij} derivatives acting on the field ϕ_i contracted with derivatives acting on the field ϕ_j and α_i derivatives acting on A_μ contracted with derivatives acting on ϕ_i. In total, the vertex contains $1 + 2\sum_{i=1}^k \alpha_i + 2\sum_{i<j}^k \alpha_{ij} \equiv 1 + 2N$ derivatives. The scattering amplitude associated to this interaction is given by

$$\mathcal{T}(\varepsilon, p, p_i) = g\,\varepsilon\cdot p_1 \prod_{i=1}^{k}(-p\cdot p_i)^{\alpha_i} \prod_{i<j}^{k}(-p_i\cdot p_j)^{\alpha_{ij}} = \sum_{l=1}^{k} \varepsilon\cdot p_l\, \mathcal{T}^l(p_i\cdot p_j). \tag{3.72}$$

[7] This type of integral can be done using the (generalized) Symanzik formula [32],

$$\int_{AdS} dX \prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{(-2P_i\cdot X)^{\Delta_i}} = \frac{1}{2}\pi^{\frac{d}{2}}\Gamma\left(\frac{\sum_{i=1}^n \Delta_i - d}{2}\right) \int [d\gamma] \prod_{i<j}^{n} \frac{\Gamma(\gamma_{ij})}{(-2P_i\cdot P_j)^{\gamma_{ij}}}\ , \tag{3.69}$$

where the integration variables satisfy $\sum_{i=1}^n \gamma_{ij} = 0$ with $\gamma_{ii} = -\Delta_i$.

where the last equality defines the partial amplitudes. Notice that, as in the previous example, only $\mathcal{T}^1$ is non-zero.

Consider now the correlation function of k scalars and a vector operator associated with the Witten diagram using the same interaction vertex but now in AdS. One should replace the fields in (3.71) by bulk-to-boundary propagators, compute their covariant derivatives, contract the indices and integrate over the interaction point X in AdS. As we shall argue below, in the flat space limit one can replace covariant derivatives by simple partial derivatives in the embedding space (*i.e.*, we drop the second term in the projector (3.68)). Using this simplifying assumption, the Witten diagram is given by

$$2g(-2)^N \frac{\mathcal{C}_{\Delta,1}}{\Gamma(\Delta+1)} \prod_{i=1}^{k} \frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)} D_1 \left[\prod_{i<j}^{k} (-2P_i \cdot P_j)^{\alpha_{ij}} \prod_{i=1}^{k} (-2P_i \cdot P)^{\alpha_i} \right. \tag{3.73}$$
$$\left. \int dX \frac{\Gamma(\Delta + \sum_i \alpha_i)}{(-2P \cdot X)^{\Delta+\sum_i \alpha_i}} \prod_{i=1}^{k} \frac{\Gamma(\Delta_i + \delta_i^1 + \alpha_i + \sum_{j\neq i}^{k} \alpha_{ij})}{(-2P_i \cdot X)^{\Delta_i+\delta_i^1+\alpha_i+\sum_{j\neq i}^{k}\alpha_{ij}}} \right] .$$

The integral over X is again of Symanzik type and can be done using (3.69). After shifting the integration variables γ_{ij} in (3.69) to bring the result to the standard form (3.9), we obtain

$$\check{M}^1 = g(-2)^N \pi^{\frac{d}{2}} \Gamma\left(\tfrac{\sum_{i=1}^{k} \Delta_i+\Delta+2N+1-d}{2}\right) \frac{\mathcal{C}_{\Delta,1}}{\Gamma(\Delta+1)} \prod_{i=1}^{k} \frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)} \prod_{i<j}^{k} (\gamma_{ij})_{\alpha_{ij}} \prod_{i=1}^{k} (\gamma_i + \delta_i^1)_{\alpha_i} \tag{3.74}$$

and $\check{M}^l = 0$ for $l > 1$. We conclude that the Mellin amplitude is a polynomial of degree N. Moreover, its leading behavior at large γ_{ij} is $\check{M}^1 \propto \prod_{i<j}^{k} \gamma_{ij}^{\alpha_{ij}} \prod_{i=1}^{k} \gamma_i^{\alpha_i}$. Notice that this is exactly the form of the scattering amplitude (3.72) if we identify $\gamma_{ij} \leftrightarrow p_i \cdot p_j$ (which implies $\gamma_i \leftrightarrow p \cdot p_i$). In fact, we can write a general formula for the relation between the Mellin amplitude at large γ_{ij} and dual scattering amplitude,

$$\check{M}^l(\gamma_{ij}) \approx \pi^{\frac{d}{2}} \frac{\mathcal{C}_{\Delta,1}}{\Gamma(\Delta+1)} \prod_{i=1}^{k} \frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)} \int_0^\infty d\beta \beta^{\frac{\sum_i \Delta_i+\Delta-1-d}{2}} e^{-\beta} \mathcal{T}^l(p_i \cdot p_j = 2\beta\gamma_{ij}) , \tag{3.75}$$

where the goal of the β-integral is to create the first Γ-function in (3.74), which has information about the number of derivatives in the interaction vertex.

Let us return to the approximation used to compute AdS covariant derivatives. Notice that the term we neglected in the projector (3.68) would contribute to (3.73) with similar expressions but with lower powers of $(-2P_i \cdot P_j)$. In other words, the effect of the neglected terms can be thought of as an interaction with smaller number of derivatives. Therefore, they give rise to subleading contributions in the large γ_{ij} limit of the Mellin amplitude $\check{M}^l$.

3.4.3 Scattering amplitude with a spin J particle

Consider a local interaction of the form

$$g(\nabla \dots \nabla h_{A_1 \dots A_J})(\nabla \dots \nabla \phi_1) \dots (\nabla \dots \nabla \phi_k). \tag{3.76}$$

with a total of $2N + J$ derivatives distributed in the following way: there are α_{ij} derivatives acting on ϕ_i contracted with derivatives acting on ϕ_j; there are α_i derivatives acting on $h_{A_1 \dots A_J}$ contracted with derivatives acting on ϕ_i; there are β_i derivatives acting on ϕ_i contracted with indices of the spin J field. The scattering amplitude associated to this interaction is given by

$$\mathcal{T}(\varepsilon, p, p_i) = g \prod_{i=1}^{k} (\varepsilon \cdot p_i)^{\beta_i} \prod_{i=1}^{k} (-p \cdot p_i)^{\alpha_i} \prod_{i<j}^{k} (-p_i \cdot p_j)^{\alpha_{ij}} \ . \tag{3.77}$$

Comparing with the representation (3.59), we conclude that the only non-zero components of $\mathcal{T}^{a_1 \dots a_J}$ are the ones with β_i indices equal to i,

$$\mathcal{T}^{\overbrace{1\dots 1}^{\beta_1}\overbrace{2\dots 2}^{\beta_2}\dots\overbrace{k\dots k}^{\beta_k}} = g \prod_{i=1}^{k} (-p \cdot p_i)^{\alpha_i} \prod_{i<j}^{k} (-p_i \cdot p_j)^{\alpha_{ij}} \ . \tag{3.78}$$

Consider now the correlation function of k scalars and a tensor operator associated with the Witten diagram using the same interaction vertex but now in AdS. One should replace the fields in (3.71) by bulk-to-boundary propagators, compute their covariant derivatives, contract the indices and integrate over the interaction point X in AdS. For the same reason that was explained in the last subsection, in the flat space limit, we can replace AdS covariant derivatives by the corresponding embedding partial derivatives. The main difference from the spin one example is that the bulk-to-boundary propagator for a spin J field has more indices. It is convenient to write the spin J bulk-to-boundary propagator as

$$\begin{aligned} \Pi_{\Delta,J}(X, P; W, Z) &= \mathcal{C}_{\Delta,J} \frac{\big((-2P \cdot X)(W \cdot Z) + 2(W \cdot P)(Z \cdot X)\big)^J}{(-2P \cdot X)^{\Delta+J}} \\ &= \frac{\mathcal{C}_{\Delta,J}}{(\Delta)_J} \big((P \cdot W)(Z \cdot \partial_P) - (Z \cdot W)(P \cdot \partial_P - Z \cdot \partial_Z)\big)^J \frac{1}{(-2P \cdot X)^{\Delta}} \end{aligned} \tag{3.79}$$

where the normalization constant $\mathcal{C}_{\Delta,J}$ is given by (3.66) and the vector W is null, to encode the property that the field is symmetric and traceless. Notice that the vector W is just an artifact to hide bulk indices and for that reason it will not appear in the final formula. In fact these indices should be contracted with J derivatives that act on the remaining fields. Let us focus on these contractions since they are the only difference compared to the previous case,

$$
\begin{aligned}
&h_{A_1\dots A_{\beta_1}B_1\dots B_{\beta_2}\dots}(\nabla^{A_1}\dots\nabla^{A_{\beta_1}}\phi_1)(\nabla^{B_1}\dots\nabla^{B_{\beta_2}}\phi_2)\dots\to\\
&\Pi_{\Delta,J}(X,P;\vec{W},Z)\prod_{i=1}^{k}(W\cdot\partial_{P_i})^{\beta_i}\Pi_{\Delta_i,0}(X,P_i)
\end{aligned}
\tag{3.80}
$$

where the notation $\vec{W}$ denotes that we should expand the expression and use

$$
\vec{W}^{A_1}\dots\vec{W}^{A_J}W^{B_1}\dots W^{B_J}=\mathcal{P}^{A_1\dots A_J,B_1\dots B_J}\,, \tag{3.81}
$$

where $\mathcal{P}$ is a projector onto symmetric and traceless tensors. After taking the partial embedding derivatives and performing the index contractions encoded in $\vec{W}$, we obtain [8]

$$
\Pi_{\Delta,J}(X,P;\vec{W},Z)\prod_{i=1}^{k}(W\cdot\partial_{P_i})^{\beta_i}\Pi_{\Delta_i,0}(X,P_i)=\frac{2^J}{(\Delta)_J}\left(\prod_{i=1}^{k}\frac{\mathcal{C}_{\Delta_i,0}(\Delta_i)_{\beta_i}}{(-2P_i\cdot X)^{\Delta_i+\beta_i}}D_i^{\beta_i}\right)\frac{\mathcal{C}_{\Delta,J}}{(-2P\cdot X)^{\Delta}}\,.
$$

Acting with the remaining $2N$ derivatives, we conclude that the Witten diagram associated with (3.76), in the flat space limit, is given by

$$
\begin{aligned}
g\,2^J(-2)^N\frac{\mathcal{C}_{\Delta,J}}{\Gamma(\Delta+J)}\left(\prod_{i=1}^{k}\frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)}D_i^{\beta_i}\right)&\left[\prod_{i<j}^{k}(-2P_i\cdot P_j)^{\alpha_{ij}}\prod_{i=1}^{k}(-2P_i\cdot P)^{\alpha_i}\right.\\
&\left.\int dX\frac{\Gamma(\Delta+\sum_i\alpha_i)}{(-2P\cdot X)^{\Delta+\sum_i\alpha_i}}\prod_{i=1}^{k}\frac{\Gamma(\Delta_i+\beta_i+\alpha_i+\sum_{j\neq i}^{k}\alpha_{ij})}{(-2P_i\cdot X)^{\Delta_i+\beta_i+\alpha_i+\sum_{j\neq i}^{k}\alpha_{ij}}}\right].
\end{aligned}
\tag{3.82}
$$

The integral over X is again of Symanzik type and can be done using (3.69). After shifting the integration variables γ_{ij} in (3.69) to bring the result to the standard form (3.9), we obtain

$$
\begin{aligned}
\check{M}^{\overbrace{1\dots1}^{\beta_1}\overbrace{2\dots2}^{\beta_2}\dots\overbrace{k\dots k}^{\beta_k}}&=g(-2)^N2^{J-1}\pi^{\frac{d}{2}}\Gamma\Big(\tfrac{\sum_{i=1}^{k}\Delta_i+\Delta+2N+J-d}{2}\Big)\\
&\times\frac{\mathcal{C}_{\Delta,J}}{\Gamma(\Delta+J)}\prod_{i=1}^{k}\frac{\mathcal{C}_{\Delta_i,0}}{\Gamma(\Delta_i)}\prod_{i<j}^{k}(\gamma_{ij})_{\alpha_{ij}}\prod_{i=1}^{k}(\gamma_i+\beta_i)_{\alpha_i}
\end{aligned}
\tag{3.83}
$$

and all other components of $\check{M}^{a_1\dots a_J}$ are zero. We conclude that the Mellin amplitude is a polynomial of degree N. Moreover, its leading behavior at large γ_{ij} is proportional to the scattering amplitude (3.77) if we identify $\gamma_{ij}\leftrightarrow p_i\cdot p_j$. In fact, we can write a general formula

[8] Notice that the differential operator in equation (3.79) can be written as $W^A\mathcal{D}_A$ where $\mathcal{D}_A$ is null, *i.e.* $\mathcal{D}_A\mathcal{D}^A=0$ on the null cone ($P^2=P\cdot Z=Z^2=0$). This implies that $\mathcal{D}_{A_1}\dots\mathcal{D}_{A_J}=\mathcal{P}^{A_1\dots A_J,B_1\dots B_J}\mathcal{D}_{B_1}\dots\mathcal{D}_{B_J}$, which greatly simplifies the computation.

for the relation between the Mellin amplitude at large γ_{ij} and dual scattering amplitude,

$$\check{M}^{a_1 \dots a_J} \approx \mathcal{N} R^{\frac{(k+1)(1-d)}{2}+d+1-J} \int_0^\infty \frac{d\beta}{\beta} \beta^{\frac{\sum_i \Delta_i + \Delta - d + J}{2}} e^{-\beta} \mathcal{T}^{a_1 \dots a_J} \left(p_i \cdot p_j = \frac{2\beta}{R^2} \gamma_{ij} \right) , \tag{3.84}$$

where we reintroduced the AdS radius R and

$$\mathcal{N} = \pi^{\frac{d}{2}} 2^{J-1} \frac{\sqrt{\mathcal{C}_{\Delta,J}}}{\Gamma\big(\Delta + J\big)} \prod_{i=1}^{k} \frac{\sqrt{\mathcal{C}_{\Delta_i,0}}}{\Gamma(\Delta_i)} \ . \tag{3.85}$$

In the last equation, we have converted to the standard CFT normalization of operators, which corresponds to $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle = |x|^{-2\Delta}$ for scalar operators and (D.24) for tensor operators. This differs from the natural AdS normalization by $\mathcal{O}_{\text{AdS}}(x) = \sqrt{\mathcal{C}_{\Delta,J}}\, \mathcal{O}_{\text{CFT}}(x)$.

The inverted form of equation (3.84),

$$\mathcal{T}^{a_1 \dots a_J}(p_i \cdot p_j) = \lim_{R \to \infty} \frac{1}{\mathcal{N}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \alpha^{\frac{d - \sum_i \Delta_i - \Delta - J}{2}} e^{\alpha} \, \frac{\check{M}^{a_1 \dots a_J} \left(\gamma_{ij} = \frac{R^2}{2\alpha}\, p_i \cdot p_j \right)}{R^{\frac{(k+1)(1-d)}{2}+d+1-J}} , \tag{3.86}$$

realizes the flat space limit intuition that the Mellin amplitude can be used to define the scattering amplitude. The final formulae (3.84) and (3.86) were derived based on the interaction vertex (3.76). However, this vertex is a basis for all possible interactions, so we expect the final formulae to be valid in general.

Chapter 4

Conformal Regge theory

4.1 Review of Regge theory

The goal of this section is to review the main concepts involved in Regge theory for the S-matrix. We will introduce the partial wave decomposition and use the dispersion representation for the scattering amplitude to access the asymptotics of the partial wave coefficients. Then we introduce the Sommerfeld-Watson transform and analyze the Regge limit. This is an important section as we will try to generalize Regge theory techniques to correlation functions in conformal field theories. So, later on we will try to make contact with Regge theory for scattering amplitudes. This review section will follow the references [33, 34]. The main object of this section is a scattering amplitude $\mathcal{T}(s,t)$ for a 2 to 2 equal mass process. The scattering amplitude is a function of the Mandelstam invariants

$$s=-(p_1+p_3)^2, \quad t=-(p_1+p_2)^2, \qquad u=-(p_1+p_4)^2, \quad s+t+u=4m^2 \tag{4.1}$$

where p_i are the momentum of the particles and m is the mass of the particles. One of the postulates of S matrix theory is that the scattering amplitudes are analytic functions of the Mandelstam invariants, regarded as complex variables, with just the singularities required by unitarity. The singularities can be poles, which are associated to the exchanged of a physical

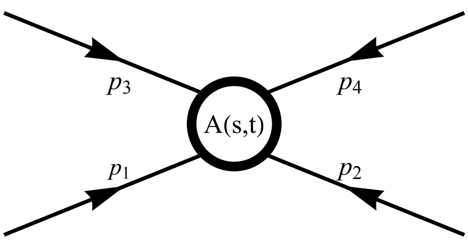


Figure 4.1: The scattering amplitude for a 2-2 process depends on two Mandelstam variables s and t.

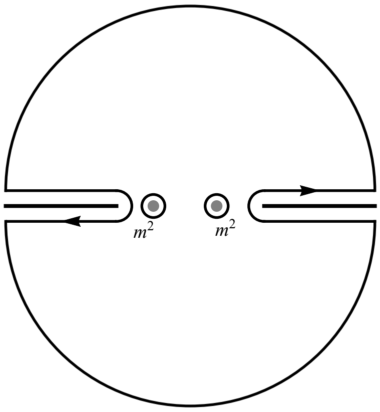


Figure 4.2: This is the integration contour for Cauchy theorem.

particle or they can be branch point singularities which are associated with the exchange of two or more particles.

It is convenient to express the scattering amplitude using the Cauchy formula as

$$\mathcal{T}(s,t,u) = \frac{1}{2\pi i}\oint \frac{ds'}{s'-s}\mathcal{T}(s',t,u') \tag{4.2}$$

where the contour runs anticlock wise around a closed curve with $\mathcal{T}(s',t,u')$ being regular inside it.

Under the assumption that the scattering amplitude $\mathcal{T}(s,t,u)$ vanishes at least as $|s|^{-\epsilon}$, for large s and positive ϵ, we can deform the contour such that it will encircle all the poles and branch cuts

$$\mathcal{T}(s,t,u) = \frac{r_s(t)}{m^2-s} + \frac{r_u(t)}{m^2-u} + \frac{1}{\pi}\int_{s_t}^{\infty}\frac{D_s(s',t)}{s'-s}ds' + \frac{1}{\pi}\int_{u_t}^{\infty}\frac{D_u(u',t)}{u'-u}du' \tag{4.3}$$

where $D_s(s,t)$ and $D_u(u',t)$ denote the s and u channels discontinuities of the scattering amplitude $\mathcal{T}$

$$D_s(s,t) = \frac{1}{2i}(\mathcal{T}(s+i0,t,u) - \mathcal{T}(s-i0,t,u))\,, \tag{4.4}$$

$$D_u(u,t) = \frac{1}{2i}(\mathcal{T}(s,t,u+i0) - \mathcal{T}(s,t,u-i0)). \tag{4.5}$$

and s_t and u_t the location of the branch point singularities. In a case where the scattering amplitude does not vanish for large s but instead grows as $|s|^{N-\epsilon}$ we can apply the Cauchy formula to $\frac{\mathcal{T}(s,t,u)}{\prod_{i=1}^{N}(s-a_i)}$. The representation (4.3) for the scattering amplitude will be slightly modified, including an undetermined polynomial of degree $N-1$.

One of the main goals of Regge theory is to understand how the high energy regime in the s-channel is obtained from the physics of the t-channel. For this purpose it is usual to decompose the scattering amplitude into partial waves in the t-channel

$$\mathcal{T}(s,t) = \sum_{J=0}^{\infty} a_J(t) P_J(z_t) \tag{4.6}$$

where $P_J(z)$ is a Legendre polynomial and $z_t = 1 + \frac{2s}{t-4m^2}$ is related to the angle of the incident and outgoing particle. The partial wave amplitudes $a_J(t)$ can be obtained using the orthogonality relation of the Legendre polynomials,[1]

$$a_J(t) = \frac{(2J+1)!}{2^{J+1}J!^2} \int_{-1}^{1} dz_t \, \mathcal{T}(s(z_t,t),t) P_J(z_t). \tag{4.8}$$

Inserting the dispersion representation (4.3) for the scattering amplitude

$$\begin{aligned} a_J(t) = & \frac{2r_s(t)Q_J(z_t(m^2,t))}{t-4m^2} + \frac{2r_u(t)Q_J(-z_t(m^2,t))}{t-4m^2} \\ & \frac{2}{\pi}\int_{z_t(s_t)}^{\infty} \frac{D_s(s'(z_t',t),t)Q_J(z_t')}{t-4m^2} dz_t + \frac{2}{\pi}\int_{z_t(u_t)}^{\infty} \frac{D_u(u'(z_t',t),t)Q_J(z_t')}{t-4m^2} dz_t' \end{aligned} \tag{4.9}$$

where $Q_J(z)$ is a Legendre polynomial of second kind defined as

$$Q_J(z) = \frac{(2J+1)!}{2^{J+1}J!^2} \int_{-1}^{1} \frac{dz'}{z-z'} P_J(z') \tag{4.10}$$

and where we used $s-s' = \frac{1}{2}(z_t - z_t')(t-4m^2)$. This representation for the partial wave coefficients $a_J(t)$ is valid as long as the integral over z_t in the second line of (4.10) is convergent. For large z the function $Q_J(z)$ behaves as $Q_J(z) \sim z^{-(J+1)} \frac{2^J J!^2}{(2J+1)!}$, so this representation for the partial waves is valid for $J > N$ assuming that the discontinuities grows as $D_s(z,t) \sim |z|^N$ for large z.

Notice that each term in the sum (4.6) is an entire function of s, which in particular does not reproduce the s-channel singularities.

4.1.1 Regge limit

The Regge limit corresponds to the physical regime where s is very large while t is kept fixed, *i.e.* large center of mass energy with fixed transfered momentum. One of the goals of Regge theory is to explain how this regime can be obtained from the partial wave decomposition (4.6) which is associated with the particles that are exchanged in the t-channel.

This limit cannot be obtained from the naive application to each term in the partial wave decompositon (4.6). Notice that the dependence in s enters just in the partial wave function

[1] The orthogonality relation for the Legendre functions is given by

$$\int_{-1}^{1} dz P_J(z) P_{J'}(z) = \frac{2^{J+1}J!^2 \delta_{J,J'}}{(2J+1)!}. \tag{4.7}$$

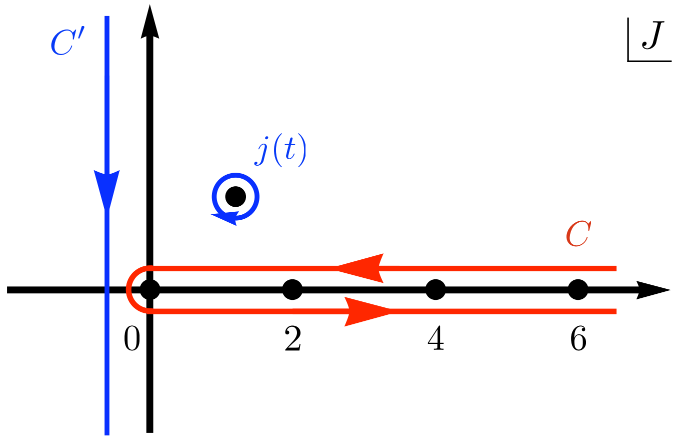


Figure 4.3: The contour C encircles the positive integers. The original sum is recovered by closing the contour and picking the contribution of the poles of the sin function.

$P_J(z_t)$ that satisfies

$$\lim_{s\to\infty} P_J(z_t) \approx s^J. \tag{4.11}$$

Thus, this shows that the sum does not converge in this limit. One way to solve this is to perform first the sum in J and then take the limit.

4.1.1.1 Sommerfeld-Watson transform

The sum over J can be done via a Sommerfeld-Watson transform. However there are some subtleties, such as the need for scattering amplitudes of definite signature, that we will justify later on. Let us define the scattering amplitude of definite signature $\mathcal{T}^{\pm}(s,t)$ as

$$\mathcal{T}(s,t) = \mathcal{T}^{+}(s,t) + \mathcal{T}^{-}(s,t) = \sum_{J=0}^{\infty} a_J^{+}(t)\frac{P_J(z_t)+P_J(-z_t)}{2} + a_J^{-}(t)\frac{P_J(z_t)-P_J(-z_t)}{2} \tag{4.12}$$

where the partial wave coefficients $a_J^{+}(t) = a_J(t)$ for even J and $a_J^{-}(t) = a_J(t)$ for odd values of J and we have used $P_J(z) = (-1)^J P_J(-z)$. Then we define the Sommerfeld-Watson transform of scattering amplitudes with definite signature as

$$\mathcal{T}^{\pm}(s,t) = \frac{1}{2i}\int_{C_1} a_J^{\pm}(t)\frac{P_J(z_t)\pm P_J(-z_t)}{2\sin(\pi J)}dJ. \tag{4.13}$$

The contour C_1 runs along the positive integers such that it just encircles the poles coming from the $\sin(\pi J)$. The partial wave coefficient $a_J(t)$ in (4.6) was defined for integer J while the Sommerfeld-Watson representation requires the analytic continuation to complex J. This analytic continuation is not unique, notice that the partial wave coefficient could be multiplied by $\cos(2\pi J)$ and it would give the same result for integer J. A theorem by Carlson says that there is a unique analytic continuation for a function defined on integers provided it falls exponentially fast in the complex plane[2].

The large J behavior can be obtained from the asymptotic behavior of Legendre functions of second kind

$$Q_J(z) \underset{J\to\infty}{\sim} \frac{e^{-(J+\frac{1}{2})\zeta(z)}}{J^{\frac{1}{2}}}, \qquad \zeta(z) = \ln(z+\sqrt{z^2-1}) \tag{4.15}$$

so the leading contribution of the right hand cut to the partial wave coefficient is given by

$$f(t)\frac{e^{-J\zeta(z_0)}}{J^{\frac{1}{2}}} \tag{4.16}$$

where z_0 is the branch point singularity represented in fig. 4.2 and $f(t)$ is some function where we have neglected the contribution of the poles[3]. The contribution from the left hand branch point is different since $z < -1$

$$f(t)e^{-J\zeta(|z_0|)}e^{-i\pi J}. \tag{4.17}$$

Follows from the above that the partial wave coefficient $a_J(t)$ grows exponentially for $J \to i\infty$ and thus does not have a unique analytic continuation. However $a_J^{\pm}(t)$ does not have this problem as can be checked from

$$\begin{aligned} a_J^{\pm}(t) =& \frac{2r_s(t)Q_J(z_t(m^2,t))}{t-4m^2} \pm \frac{2r_u(t)Q_J(z_t(m^2,t))}{t-4m^2} \\ &+\frac{2}{\pi}\int_{z_t(s_t)}^{\infty}\frac{D_s(s'(z_t',t),t)Q_J(z_t')}{t-4m^2}dz_t \pm \frac{2}{\pi}\int_{z_t(u_t)}^{\infty}\frac{D_u(u'(z_t',t),t)Q_J(|z_t'|)}{t-4m^2}dz_t' \end{aligned} \tag{4.18}$$

4.1.1.2 Deforming the contour

We can deform the contour to C_2 and neglect the contribution from the integral over the semi-circle at infinity since the integrand decays at large J. Notice that (4.10) and (4.19) is valid as long as $J > N$, assuming D_s, $D_u \sim z^N$. To deform further the contour we need to make one more assumption, which is the partial wave coefficient $a_J(t)$ should only have isolated singularities in the J plane. For example, let us consider that the first singularity after $\text{Re}(J) = N$ is just a

[2]The precise statement is:
Theorem: Let $f(J)$ be a function satisfying

$$\begin{aligned} &f(J) = 0 \;\text{ with } J \text{ a non-negative integer} \\ &\text{and} \quad |f(z)| < Ce^{k|z|} \quad \text{where } k<\pi, \text{ for } Re(z) \geq 0 \end{aligned} \tag{4.14}$$

then $f(J)$ is unique.

[3]The contribution of the pole term $\frac{2r_s(t)Q_J(z_t(m^2,t))}{t-4m^2}$ is exponentially suppressed by $e^{-J\zeta(z_t(m^2,t))}$.

simple pole,

$$a_J^{\pm}(t) = \frac{\beta^{\pm}(t)}{J - j(t)} \tag{4.19}$$

then the scattering amplitude $\mathcal{T}^{\pm}(s,t)$ can be written as,

$$\mathcal{T}^{\pm}(s,t) = -\frac{1}{2i}\int_{C_2'} a_J^{\pm}(t)\frac{P_J(z) \pm P_J(-z)}{2\sin(\pi J)}dJ - \pi\beta^{\pm}(t)\frac{P_{j(t)}(z_t) \pm P_{j(t)}(-z_t)}{2\sin(\pi j(t))}. \tag{4.20}$$

In the limit of s large the contribution of the contour integral C_2' is sub dominant compared to the contribution coming from the pole of the partial wave coefficient. So to leading order in large s the scattering amplitude $\mathcal{T}(s,t)$ is well approximated just by the pole term,

$$\mathcal{T}^{\pm}(s,t) \approx -\beta^{\pm}(t)\frac{z^{j(t)} \pm (-z)^{j(t)}}{2\sin(\pi j(t))}. \tag{4.21}$$

4.1.2 Regge theory of 4 dilaton scattering

The goal of this section is to apply the ideas of previous section to the 4 dilaton scattering in type II superstring theory given by Virasoro Shapiro amplitude

$$\mathcal{T}(s,t) = 8\pi G_N\left(\frac{tu}{s} + \frac{su}{t} + \frac{st}{u}\right)\frac{\Gamma\left(1 - \frac{\alpha' s}{4}\right)\Gamma\left(1 - \frac{\alpha' u}{4}\right)\Gamma\left(1 - \frac{\alpha' t}{4}\right)}{\Gamma\left(1 + \frac{\alpha' s}{4}\right)\Gamma\left(1 + \frac{\alpha' u}{4}\right)\Gamma\left(1 + \frac{\alpha' t}{4}\right)}, \tag{4.22}$$

where G_N is the 10-dimensional Newton constant, α' is the square of the string length. As the dilaton is massless the Mandelstam invariants satisfy $s+t+u=0$. This amplitude has an infinite number of poles that correspond to the exchange of an infinite number of particles, which can be organized in Regge trajectories as shown in figure 4.4 and it has no branch points singularities. In this example the Legendre polynomials in the partial wave decomposition (4.6) are replaced by the 10-dimensional partial wave functions, i.e. a Gegenbauer polynomial. More precisely, the partial waves are just Gegenbauer polynomials[4] (with $D = 10$ in our case),

$$P_J(z) = \frac{J!\,\Gamma\left(\frac{D-3}{2}\right)}{2^J\Gamma\left(J + \frac{D-3}{2}\right)}\,C_J^{\left(\frac{D-3}{2}\right)}(z)\;, \tag{4.24}$$

which we normalized such that the highest degree term has unit coefficient, $P_J(z) = z^J + O(z^{J-1})$. In the present example only even spins contribute because the initial particles are identical scalars. To determine the spectrum of exchanged particles we use the fact that each exchanged particle

[4]The partial wave in D dimensions satisfy the orthogonality relation

$$\int_{-1}^{1} dz P_J(z)P_{J'}(z)(1-z^2)^{\frac{D-4}{2}} = \delta_{J,J'}\frac{\pi J! 2^{5-D-2J}\Gamma(D+J-3)}{(D+2J-3)\Gamma^2\left(\frac{D-3+2J}{2}\right)}. \tag{4.23}$$

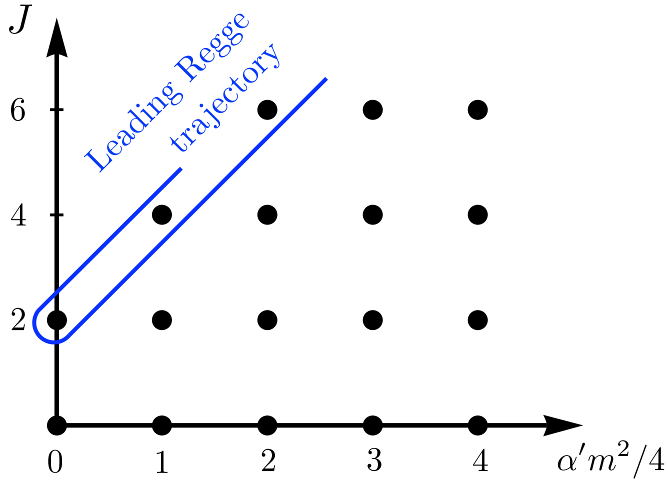


Figure 4.4: Chew-Frautschi plot of the spectrum of exchanged particles in the Virasoro-Shapiro amplitude.

of mass m and spin J gives rise to a pole of $a_J(t)$ at $t = m^2$. The full scattering amplitude has poles at $t = 4n/\alpha'$, for $n = 0, 1, 2, \dots$, as can be seen in figure 4.4. Computing the residues of equations (4.22) and (4.6), we obtain

$$\sum_{J=0}^{\infty} P_J(z)\, \mathrm{Res}_{t=\frac{4n}{\alpha'}} a_J(t) = \mathrm{Res}_{t=\frac{4n}{\alpha'}} \mathcal{T}(s,t) = -\frac{128\pi G_N}{(\alpha' n!)^2} \left(\frac{nz}{2}\right)^{2+2n} + O(z^{2n})\,, \tag{4.25}$$

where the RHS is a polynomial of degree $2n+2$ in z, whose leading term we wrote explicitly. In fact, this equation is satisfied with a finite sum over J because this is just an equality between polynomials of z. More precisely, it tells us that $a_J(t)$ has poles at $t = 4n/\alpha'$ for $n = \frac{J}{2} - 1, \frac{J}{2}, \frac{J}{2}+1, \dots$. The first pole in this series gives

$$a_J(t) \approx \frac{r(J)}{t - m^2(J)}\,, \tag{4.26}$$

where

$$m^2(J) = \frac{2}{\alpha'}(J-2)\,, \qquad r(J) = -\frac{128\pi G_N}{\alpha'^2 \Gamma^2(J/2)} \left(\frac{J-2}{4}\right)^J . \tag{4.27}$$

This pole describes the leading Regge trajectory, i.e. the lightest exchanged particle for each spin J. The residue of the pole encodes the cubic couplings between the external particles and the exchanged particles in the leading Regge trajectory.

The goal of Regge theory is to describe the high energy limit of scattering processes. We shall think of the amplitude (4.22) describing elastic scattering of the initial particles 1 and 3 to the final particles 2 and 4, respectively. Thus, the Regge regime is defined by large s and fixed t

given by (4.1). In this limit the amplitude (4.22) simplifies to [5]

$$\mathcal{T}(s \pm i\epsilon, t) \approx \frac{32\pi G_N}{\alpha'}\, e^{\mp \frac{i\pi\alpha' t}{4}}\, \frac{\Gamma\big(-\frac{\alpha' t}{4}\big)}{\Gamma\big(1+\frac{\alpha' t}{4}\big)} \left(\frac{\alpha' s}{4}\right)^{2+\frac{\alpha' t}{2}} . \tag{4.28}$$

In this example, it was trivial to obtain the Regge limit of the scattering amplitude because we knew the exact amplitude (4.22). However, the achievement of Regge theory is to derive the behavior of the amplitude in the Regge limit without knowing the full result. To understand how this works, it is instructive to stick to this example and ask the question: what is the minimal amount of information that we need to fix the amplitude in the Regge limit? The answer is the spectrum of particles in the leading Regge trajectory and their cubic couplings to the external particles. Let us review how this works.

For completeness we will repeat the main steps of the previous subsection, the first step is to analytically continue the partial waves as a function of the spin J, and then transform the sum (4.6) into a contour integral in the J-plane, [6]

$$\mathcal{T}(s,t) = \int_{\mathrm{C}} \frac{dJ}{2\pi i}\frac{\pi}{2\sin(\pi J)}\, a_J(t)\big(P_J(z)+P_J(-z)\big) , \tag{4.29}$$

where the contour C is shown in figure 4.3. The symmetry property $P_J(-z) = (-1)^J P_J(z)$ ensures that we are only summing over even spins. The final step is to continuously deform the integration contour C to the contour C' also shown in figure 4.3. This is possible because of the large J behavior of the partial wave

$$P_J(\cos\theta) \approx \frac{\cos\big(J\theta + \frac{7}{2}\theta + \frac{\pi}{4}\big)}{2^{J+\frac{5}{2}}\sin^{\frac{7}{2}}(\theta)} , \tag{4.30}$$

and because the analytically continued partial amplitude $a_J(t)$ does not increase exponentially in any direction in the right half of the complex J–plane [35]. In the contour deformation process, one picks up contributions from poles of $a_J(t)$ with $\mathrm{Re}(J) > 0$. These are Regge poles and are directly related to the physical poles of the scattering amplitude. In particular, the pole with largest $\mathrm{Re}(J)$ follows from the leading Regge trajectory (4.26) and can be written as

$$a_J(t) \approx -\frac{j'(t)\, r\big(j(t)\big)}{J - j(t)} , \tag{4.31}$$

[5] This expression is valid for $|s|$ large in any direction of the complex plane, except along the real axis where the amplitude has an infinite series of poles in both directions. If we take s large and almost real, the amplitude has a different phase depending if we go slightly above or below the real axis. This is encoded in the $i\epsilon$ prescription.

[6] Usually, this step requires more care because even and odd spin partial waves must be analytically continued separately [35]. In our case, there are only even spins and one analytic continuation is sufficient.

where $j(t) = 2 + \alpha' t/2$. The contribution of this Regge pole for the scattering amplitude reads

$$j'(t)\, r\big(j(t)\big)\, \frac{\pi}{2\sin\big(\pi j(t)\big)} \left(P_{j(t)}(z) + P_{j(t)}(-z)\right) , \tag{4.32}$$

and therefore in the Regge limit of large s we obtain

$$\mathcal{T}(s,t) \approx -\frac{32\pi^2 G_N}{\alpha' \Gamma^2\big(j(t)/2\big)} \left(\frac{\alpha'}{4}\right)^{j(t)} \frac{s^{j(t)} + (-s)^{j(t)}}{\sin\big(\pi j(t)\big)} . \tag{4.33}$$

This gives exactly the Regge limit (4.28) of the Virasoro-Shapiro amplitude. This precise matching follows from the fact that the other Regge poles have smaller $\mathrm{Re}(J)$ and therefore are subdominant in the Regge limit.

4.2 Conformal Regge theory

The extension of Regge theory for correlation functions in conformal field theories follows essentially the same steps, although it is technically more challenging. The analogue of the partial wave decomposition is precisely the conformal block decomposition presented in (2.14). We shall perform this analysis using Mellin amplitudes, given the similarities to scattering amplitudes.

4.2.1 Conformal partial waves in Mellin space

The first step to study Regge theory is to write down a partial wave expansion. For our purposes, the best starting point is the partial wave expansion described in [21], which is the Mellin space version of [36]. We write

$$M(s,t) = \sum_{J=0}^{\infty} \int_{-\infty}^{\infty} d\nu\, b_J(\nu^2)\, M_{\nu,J}(s,t) , \tag{4.34}$$

with the partial waves $M_{\nu,J}(s,t) = M_{-\nu,J}(s,t)$ given by

$$M_{\nu,J}(s,t) = \omega_{\nu,J}(t)\, P_{\nu,J}(s,t) , \tag{4.35}$$

where

$$\omega_{\nu,J}(t) = \frac{\Gamma\left(\frac{\Delta_1+\Delta_2+J+i\nu-h}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4+J+i\nu-h}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+J-i\nu-h}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4+J-i\nu-h}{2}\right)}{8\pi\Gamma(i\nu)\Gamma(-i\nu)} \frac{\Gamma\left(\frac{h+i\nu-J-t}{2}\right)\Gamma\left(\frac{h-i\nu-J-t}{2}\right)}{\Gamma\left(\frac{\Delta_1+\Delta_2-t}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-t}{2}\right)} , \tag{4.36}$$

and $P_{\nu,J}(s,t)$ is a Mack polynomial of degree J in both variables s and t. We normalized these polynomials such that they obey $P_{\nu,J}(s,t) = s^J + O(s^{J-1})$. The precise definition is given in appendix A.3. This is just the Mellin transform of the representation (2.14)

$$\mathcal{A}(u,v) = \sum_{J=0}^{\infty} \int_{-\infty}^{\infty} d\nu \, b_J(\nu^2) \, F_{\nu,J}(u,v) \,, \tag{4.37}$$

where $F_{\nu,J}(u,v)$ is the transform (3.21) of a single partial wave $M_{\nu,J}(s,t)$,

$$\begin{aligned} F_{\nu,J}(u,v) = \int_{-i\infty}^{i\infty} \frac{dtds}{(4\pi i)^2} \, M_{\nu,J}(s,t) \, u^{t/2} v^{-(s+t)/2} \, \Gamma\left(\frac{\Delta_1+\Delta_2-t}{2}\right) \Gamma\left(\frac{\Delta_3+\Delta_4-t}{2}\right) \\ \Gamma\left(\frac{\Delta_{34}-s}{2}\right) \Gamma\left(\frac{-\Delta_{12}-s}{2}\right) \Gamma\left(\frac{t+s}{2}\right) \Gamma\left(\frac{t+s+\Delta_{12}-\Delta_{34}}{2}\right) . \end{aligned} \tag{4.38}$$

Thus, in order to reproduce the contribution of a single-trace operator $\mathcal{O}_k$ of dimension Δ and spin J that appears in both OPEs $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$, the partial amplitude $b_J(\nu^2)$ must have poles of the form

$$b_J(\nu^2) \approx C_{12k} C_{34k} \, \frac{K_{\Delta,J}}{\nu^2 + (\Delta-h)^2} \,. \tag{4.39}$$

We rederive this result in appendix A.2.2 where we discuss the analytic structure of the partial amplitude $b_J(\nu^2)$ more systematically.

We are ready to generalize Regge theory to Mellin amplitudes. To avoid cluttering of the formulae we shall restrict to the case $\Delta_{12} = \Delta_{34} = 0$. It is convenient to change from the variable s to the variable $z = 1 + \frac{2s}{t}$. We define

$$\mathcal{P}_{\nu,J}(z,t) = P_{\nu,J}\left(\frac{t(z-1)}{2}, t\right) . \tag{4.40}$$

Then, for integer spin J, we have the symmetry

$$\mathcal{P}_{\nu,J}(-z,t) = (-1)^J \mathcal{P}_{\nu,J}(z,t) \,. \tag{4.41}$$

4.2.2 Regge theory

The starting point for Regge theory is the conformal partial wave expansion (4.34). The construction is now analogous to that reviewed in section 4.1. Firstly, we analytically continue the partial amplitudes $b_J(\nu^2)$ to complex values of J. Even and odd spins give rise to different analytic continuations $b_J^+(\nu^2)$ and $b_J^-(\nu^2)$, respectively. Secondly, we perform a Sommerfeld-Watson transform in (4.34),

$$M(s,t) = M^+(s,t) + M^-(s,t) \,, \tag{4.42}$$

with

$$M^{\pm}(s,t) = \int_{-\infty}^{\infty} d\nu \int_C \frac{dJ}{2\pi i} \frac{\pi}{2\sin(\pi J)} \, b_J^{\pm}(\nu^2) \, \omega_{\nu,J}(t) \left(\mathcal{P}_{\nu,J}(-z,t) \pm \mathcal{P}_{\nu,J}(z,t)\right) . \tag{4.43}$$

In the case of Regge theory for scattering amplitudes, after the Sommerfeld-Watson transformation we deform the contour to C_2 as shown in fig 4.3 . For scattering amplitudes this was possible because the asymptotics of both partial wave function and partial wave coefficients were under control.

To access if the contribution of the integral at infinity is negligible or not, one needs to study the asymptotics of both conformal partial waves and the conformal partial wave coefficient. However the expression for $b_J(\nu)$ seems to be more involved than for scattering amplitudes space analogue, check for instance how the conformal partial wave coefficient is written in of the amplitude $\mathcal{A}(u,v)$ (C.17).

The next step in Regge theory is to deform the J-contour and pick up the pole with maximal real part of J, i.e. the leading Regge pole. Before doing this we need to consider the poles (A.28) of the partial amplitude $b_J(\nu^2)$. We will be mostly interested in the poles associated to the leading Regge trajectory $\Delta(J)$ for $J = 2, 4, 6, \dots$. These are the operators of lowest dimension for each even spin. This means that

$$b_J^+(\nu^2) \approx \frac{r(J)}{\nu^2 + \big(\Delta(J) - h\big)^2}\,, \tag{4.44}$$

where the residue

$$r(J) = C_{12J} C_{34J}\, K_{\Delta(J),J}\,, \tag{4.45}$$

is determined by the OPE coefficients of the operators in the leading Regge trajectory that appear in the OPEs of the external operators (see equation (4.39)). After analytic continuation in J this pole becomes a pole in J, more precisely

$$b_J^+(\nu^2) \approx -\frac{j'(\nu)\, r(j(\nu))}{2\nu\big(J - j(\nu)\big)}\,, \tag{4.46}$$

where $j(\nu)$ is essentially the inverse function of $\Delta(J)$ defined by

$$\nu^2 + \big(\Delta(j(\nu)) - h\big)^2 = 0\,. \tag{4.47}$$

The contribution of this Regge pole is then

$$\int_{-\infty}^{\infty} d\nu\, \frac{\pi}{2\sin(\pi j(\nu))}\, \frac{j'(\nu)\, r(j(\nu))}{2\nu}\, \omega_{\nu,j(\nu)}(t)\, \big(\mathcal{P}_{\nu,j(\nu)}(-z,t) + \mathcal{P}_{\nu,j(\nu)}(z,t)\big)\,. \tag{4.48}$$

In the Regge limit ($s \to \infty$) this pole dominates and we obtain the result

$$M(s,t) \approx \int d\nu\, \beta(\nu)\, \omega_{\nu,j(\nu)}(t)\, \frac{s^{j(\nu)} + (-s)^{j(\nu)}}{\sin\big(\pi j(\nu)\big)}\,, \tag{4.49}$$

where

$$\beta(\nu) = \frac{\pi}{4\nu}\, j'(\nu)\, r\big(j(\nu)\big) = \frac{\pi}{4\nu}\, j'(\nu)\, K_{h\pm i\nu, j(\nu)}\, C_{12j(\nu)} C_{34j(\nu)}\,. \tag{4.50}$$

Equation (4.49) is our main result. It encodes the contribution of a Regge trajectory to the Mellin amplitude, which is fixed by conformal symmetry up to the dynamical observables $j(\nu)$ and $\beta(\nu)$. The Reggeon spin $j(\nu)$ is determined by the dimensions $\Delta(J)$ of the physical operators in the leading Regge trajectory through (4.47). The residue $\beta(\nu)$ is controlled by the OPE coefficients of the leading twist operators in the OPE of the external operators.

4.2.3 Regge limit in position space

The definition of the Regge limit of the Mellin amplitude (large s and fixed t) corresponds to the Regge limit defined in position space in [16, 37]. This is shown in detail in appendix A.4. For the sake of clarity, here we just state how to relate the result (4.49) to the Regge limit of the correlator in position space, leaving the details to the appendix. First one needs to consider a specific Lorentzian kinematical limit where all the points are taken to null infinity. In such limit, it is convenient to introduce the variables σ and ρ that are related to the cross ratios u and v defined in (1.16) by

$$u = \sigma^2 , \qquad v = (1 - \sigma e^{\rho})(1 - \sigma e^{-\rho}) \approx 1 - 2\sigma \cosh \rho \,. \tag{4.51}$$

The Regge limit corresponds to $\sigma \to 0$ with fixed ρ. The position space version of equation (4.49) is then [7]

$$\mathcal{A}(\sigma, \rho) \approx 2\pi i \int d\nu \, \alpha(\nu) \, \sigma^{1-j(\nu)} \Omega_{i\nu}(\rho) \,, \tag{4.52}$$

where $\Omega_{i\nu}(\rho)$ is a harmonic function on $(2h-1)$-dimensional hyperbolic space. In appendix A.4, we show that the residues $\alpha(\nu)$ in (4.52) and $\beta(\nu)$ in (4.49) are related by

$$\alpha(\nu) = - \frac{\pi^{h-1} 2^{j(\nu)-1} e^{i\pi j(\nu)/2}}{\sin\left(\frac{\pi j(\nu)}{2}\right)} \, \gamma(\nu)\gamma(-\nu)\beta(\nu) \,, \tag{4.53}$$

where

$$\gamma(\nu) = \Gamma\left(\frac{2\Delta_1 + j(\nu) + i\nu - h}{2}\right) \Gamma\left(\frac{2\Delta_3 + j(\nu) + i\nu - h}{2}\right) . \tag{4.54}$$

The form (4.52) was first derived in [16] applying Regge theory to the conformal partial wave expansion. Here, we have improved the result because we related the functions $\alpha(\nu)$ and $\beta(\nu)$ to the product of OPE coefficients (see equation (4.50)).

As we emphasized in the introduction every correlation function of local operators can be determined in terms of the CFT data, *i.e.* the dimension and OPE of operators. In the Regge limit the relevant dynamic data in a four point function is the pomeron pole and residue as is expressed in (4.52). These can be determined without actually computing a four point function. So we can check the relations (4.47), (4.50) and (4.53) given by conformal Regge theory. On the

[7] We remark that the definition of $\alpha(\nu)$ in this thesis differs from that in [16, 37] by a factor of $4^{1-j(\nu)} e^{i\pi j(\nu)}$.

other hand we can also determine these using an explicit four point function computation and check the validity of conformal Regge theory.

Let us conclude with a comparison between the Regge theory for scattering amplitudes and correlation functions in CFTs

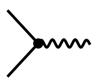
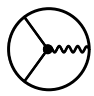
Strings in flat spacetime	$\textbf{CFT}_d$ **or Strings in** $\textbf{AdS}_{d+1}$
Scattering amplitude $\mathcal{T}(s,t)$	Correlation function or Mellin amplitude $M(s,t)$
Tree-level: $g_s \to 0$	Planar level: $N \to \infty$
Finite string length $l_s = \sqrt{\alpha'}$	Finite 't Hooft coupling $g^2 \sim g_{YM}^2 N = \dfrac{R^4}{\alpha'^2}$
Partial wave expansion $\mathcal{T}(s,t) = \sum_J a_J(t) \underbrace{P_J(\cos\theta)}_{\text{partial wave}}$	Conformal partial wave expansion $M(s,t) = \sum_J \int d\nu\, b_J(\nu^2) \underbrace{M_{\nu,J}(s,t)}_{\text{partial wave}}$
On-shell poles $a_J(t) \sim \dfrac{C^2(J)}{t - m^2(J)}$	On-shell poles $b_J(\nu^2) \sim \dfrac{C^2(J)}{\nu^2 + \big(\Delta(J) - \frac{d}{2}\big)^2}$
Leading Regge trajectory $m^2(J) = \dfrac{2}{\alpha'}(J-2)$	Leading twist operators $\Delta(J) = d - 2 + J + \underbrace{\gamma(J,g^2)}_{\substack{\text{anomalous}\\ \text{dimension}}}$
Cubic couplings $C(J) \sim$ 	3-pt functions or OPE coefficients $C(J) \sim$ 
Regge limit: $s \to \infty$ with fixed t $P_J(\cos\theta) \approx \left(\frac{2s}{t}\right)^J$ $T(s,t) \approx \beta(t)\, s^{j(t)}$	Regge limit: $s \to \infty$ with fixed t $M_{\nu,J}(s,t) \approx \omega_{\nu,J}(t)\, s^J$ $M(s,t) \approx \int d\nu\, \omega_{\nu,j(\nu)}(t)\, \beta(\nu)\, s^{j(\nu)}$
Regge pole and residue $t - m^2(J) = 0 \;\Rightarrow\; J = j(t)$ $\beta(t) \sim C^2\big(j(t)\big)$	Regge pole and residue $\big(\Delta(J) - \frac{d}{2}\big)^2 + \nu^2 = 0 \;\Rightarrow\; J = j(\nu)$ $\beta(\nu) \sim C^2\big(j(\nu)\big)$

Table 4.1: Analogy between standard Regge theory for scattering amplitudes in flat spacetime and conformal Regge theory. The notation will be explained later, but the analogy should already be clear for readers familiarized with Regge theory (and AdS/CFT).

Chapter 5

Applications to $\mathcal{N} = 4$ SYM

The goal of this section is to the test the validity of the results of conformal Regge theory in $\mathcal{N} = 4$ SYM. This is a four dimensional conformal field theory that describes the interaction between 6 real scalars, gluons and sixteen component Majorana spinors given by the Lagrangian

$$\mathcal{L} = \frac{1}{g_{YM}^2}\mathrm{Tr}\Big[\frac{1}{2}[D_\mu, D_\nu]^2 + (D_\mu\Phi_i)^2 - \frac{1}{2}[\Phi_i, \Phi_j]^2 \\ + i\overline{\Psi}(\Gamma^\mu D_\mu\Psi + \Gamma^i\big[\Phi_i, \Psi\big]) + \partial_\mu\bar{c}D_\mu c + \zeta(\partial_\mu A_\mu)^2\Big] \tag{5.1}$$

with the covariant derivative defined by

$$D_\mu X = \partial_\mu X - i\big[A_\mu, X\big], \tag{5.2}$$

Γ^μ, Γ^i are ten real 16×16 Dirac matrices, c and $\bar{c}$ are the Faddeev-Popov ghosts and ζ is a gauge fixing term. The fields are $N \times N$ matrices,

$$\Phi_i = \Phi_i^a T^a, \quad A_\mu = A_\mu^a T^a,\ \Psi = \Psi^a T^a \tag{5.3}$$

where T^a are the generators of $U(N)$.

We shall study four point functions in this theory in the weak coupling regime, where $g_{YM}^2 N \ll 1$ and in the strong coupling limit where $g_{YM}^2 N \gg 1$. As was emphasized in the introduction, a four point function can be decomposed in the contribution of the operators that are exchanged in a given channel. This chapter is naturally divided into two parts, devoted to the weak and strong coupling expansions. In both cases we will focus on correlation functions of scalar primary operators.

5.1 Weak coupling

5.1.1 4-pt functions of protected operators

We will consider correlation functions of operators constructed out of the scalar fields of $\mathcal{N}=4$ SYM

$$\mathcal{O}_k(x,y)=y^{I_1}\dots y^{I_k}\mathrm{Tr}\big(\phi_{I_1}(x)\dots\phi_{I_k}(x)\big) \tag{5.4}$$

where the auxiliary variables y square to zero $y^2=0$ and encode the property that the indices I_i are symmetric and traceless and the $k=2$, 3 and 4. The I_i are the R-charge indices and run from 1 to 6. The four point function can be decomposed in irreducible representations of the R-charge group $SU(4)\simeq SO(6)$, *i.e.* the operators that are exchanged in the OPE transform in irreducible representations of the group $SU(4)$. The allowed irreps can be obtained by decomposing the tensor product of the external operators

$$\big[0,k,0\big]\times\big[0,k,0\big]=\sum_{p=0}^{k}\sum_{q=0}^{k-p}\big[q,2k-2q-2p,q\big]. \tag{5.5}$$

More explicitly we can write the four point function as

$$G_k=\langle\mathcal{O}_k(x_1,y_1)\mathcal{O}_k(x_2,y_2)\mathcal{O}_k(x_3,y_3)\mathcal{O}_k(x_4,y_4)\rangle=\frac{1}{(x_{12}^2x_{34}^2)^{2k}}\sum_r P_r(y_i,u,v)\mathcal{A}_r^k(u,v) \tag{5.6}$$

where the sum in r should run from over the irreps present in the tensor product. For example for $k=2$ there are 6 different representations. The explicit form of the projectors $P_r(y_i,u,v)$ are given in the appendix B.1.

There is a special class of operators which play an essential role in the Regge limit, these operators have the lowest dimension per spin and are called the leading twist operators. At three level they are given by a specific linear combination of

$$\mathrm{tr}\left(F_{\mu\nu_1}D_{\nu_2}\dots D_{\nu_{J-1}}F_{\nu_J}{}^{\mu}\right),\quad \mathrm{tr}\left(\phi_{AB}D_{\nu_1}\dots D_{\nu_J}\phi^{AB}\right),\quad \mathrm{tr}\left(\overline{\Psi}D_{\nu_1}\dots D_{\nu_{J-1}}\Gamma_{\mu_J}\Psi\right). \tag{5.7}$$

Notice that any of the terms has twist two (where twist is difference of the dimension and the spin), thus the correct linear combination can be found by diagonalizing the dilatation operator at one loop. Consider the following two state model where the Hamiltonian plays the role of the dilatation operator

$$\widehat{H}=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}\lambda&0\\0&-\lambda\end{pmatrix}+O\left(\lambda^2\right) \tag{5.8}$$

where λ should be understood as the coupling. At tree level any vector is an eigenstate of the

Hamiltonian. However the first correction leads to different eigenvalues, so there will be two unique, up to normalization, eigenvectors of the system. These eigenstates do not depend on the coupling λ. So, at finite 't Hooft coupling the degeneracy is lifted and there are three different Regge trajectories. [1] In appendix B.3, we give the precise linear combination to first order in perturbation theory. At strong coupling these operators are dual to massive string states, on the leading Regge trajectory of type IIB strings in AdS.

The structure of the four point function (5.6) is highly constrained by symmetry. This has been used to compute it perturbatively up to a very high order [39, 40, 41, 42, 43, 44, 45, 46, 47, 48]

$$G_2 = \left(\frac{y_{12}y_{34}}{x_{12}^2 x_{34}^2}\right)^2 \left[G_2^{(0)} + uR_2 F_2(u,v)\right]. \tag{5.9}$$

$$G_3 = \left(\frac{y_{12}y_{34}}{x_{12}^2 x_{34}^2}\right)^3 \left[G_3^{(0)} + 9R_3\left(F_3(u,v) + \frac{\sigma}{u^2}F_3\left(\frac{1}{u},\frac{v}{u}\right) + \frac{\tau}{v^2}F_3\left(\frac{u}{v},\frac{1}{v}\right)\right)\right] \tag{5.10}$$

$$G_4 = \left(\frac{y_{12}y_{34}}{x_{12}^2 x_{34}^2}\right)^4 \left[G_4^{(0)} + R_4\mathcal{H}(u,v;\sigma,\tau)\right] \tag{5.11}$$

where $G_k^{(0)}$ is the tree level contribution given by

$$\begin{aligned}
G_2^{(0)} &= \frac{(N^2-1)^2}{4(4\pi^2)^4 v^2}\left[v^2 + u^2v^2\sigma^2 + u^2\tau^2\right] + \frac{N^2-1}{(4\pi^2)^4 v}\left[u\tau + u^2\sigma\tau + uv\sigma\right] \\
G_3^{(0)} &= \frac{9u^2}{50(N^2-1)v^2}\left[(N^2-1)\left(\frac{v^2}{u^2} + uv^2\sigma^3 + \frac{u\tau^3}{v}\right) + \frac{v^2\sigma}{u} + uv\sigma^2\tau \right.\\
&\left. + \frac{v\tau}{u} + u\sigma\tau^2 + v^2\sigma^2 + \tau^2\right] \\
G_4^{(0)} &= 1 + u^4\left(\sigma^4 + \tfrac{\tau^4}{v^4}\right) + \frac{16u}{v^3}\left(\tau^3u^2(\sigma u+1) + \sigma v^3(\sigma u(\sigma u+1)+1)\right. \\
&\left. + \tau v^2(\sigma u+1)(\sigma u(\sigma u+1)+1) + uv(\tau+\sigma\tau u)^2\right)
\end{aligned} \tag{5.12}$$

with $\sigma = \frac{y_{13}y_{24}}{y_{12}y_{34}}$, $\tau = \frac{y_{14}y_{23}}{y_{12}y_{34}}$ and $y_{ij} = y_i \cdot y_j$. The prefactors R_2 and R_3 and R_4 are given by

$$R_2 = \frac{2(N^2-1)}{(4\pi^2)^4 v}\Big(\tau + \tau^2 u - \tau(\sigma u(1+v-u) + u + v) + v(\sigma((\sigma-1)u + v - 1) + 1)\Big) \tag{5.13}$$

$$R_3 = \frac{9u^2\sigma}{100v}\left(\tau\left(\sigma u^2 + u(\tau-\sigma-1) + 1\right) + v\left(\sigma^2 u - \tau - \sigma(\tau u + u + 1) + 1\right) + \sigma v^2\right) \tag{5.14}$$

$$R_4 = v + \sigma^2 uv + \tau^2 u + \sigma v(v-1-u) + \tau(1-u-v) + \sigma\tau u(u-1-v) \tag{5.15}$$

[1] These three trajectories are related by supersymmetry [38].
In fact, their anomalous dimensions are simply related by $\widetilde{\widetilde{\gamma}}(J) = \widetilde{\gamma}(J+2) = \gamma(J+4)$.

The function $F_2(u, v)$ is given up to three loops by

$$
\begin{aligned}
F_2(u,v) = \;& \frac{f(z,\bar{z})}{z-\bar{z}} \\
&+ \frac{1}{(z-\bar{z})^2}\Big(g(z,\bar{z}) + u g\big(\tfrac{1}{z},\tfrac{1}{\bar{z}}\big) + v g\big(\tfrac{1}{1-z},\tfrac{1}{1-\bar{z}}\big)\Big) \\
&+ \frac{1}{z-\bar{z}}\bigg(\frac{1+u}{1-u}h(z,\bar{z}) - \frac{1+v}{1-v}h(1-z,1-\bar{z}) + \frac{u+v}{u-v}h\big(1-\tfrac{1}{z},1-\tfrac{1}{\bar{z}}\big)\bigg)\,. \qquad (5.16)
\end{aligned}
$$

The functions f and h are antisymmetric in the exchange of z and $\bar{z}$ while g is symmetric. It may seem that we have taken a step backwards in expressing a single two-variable function F in terms of three two-variable functions f, g and h. However the functions f, g and h all have the property that their perturbation expansion is expressed in terms of pure transcendental functions of degree equal to twice the loop order. All three functions are single-valued in the Euclidean region where z and $\bar{z}$ are related by complex conjugation. We write the perturbative expansion of the functions f, g and h as follows[2],

$$
f(z,\bar{z}) = \sum_l g^{2l} f^{(l)}(z,\bar{z})\,, \qquad g(z,\bar{z}) = \sum_l g^{2l} g^{(l)}(z,\bar{z})\,, \qquad h(z,\bar{z}) = \sum_l g^{2l} h^{(l)}(z,\bar{z})\,. \quad (5.17)
$$

Up to two loops, the functions f and g can be expressed in terms of the following single-valued functions appearing in the ladder integrals,

$$
\phi^{(l)}(z,\bar{z}) = \sum_{r=0}^{l} \frac{(-1)^r(2l-r)!}{r!(l-r)!l!} \log^r(z\bar{z})\big(\mathrm{Li}_{2l-r}(z) - \mathrm{Li}_{2l-r}(\bar{z})\big)\,. \qquad (5.18)
$$

Explicitly we have at one loop [40, 49],

$$
f^{(1)}(z,\bar{z}) = -\phi^{(1)}(z,\bar{z})\,, \quad g^{(1)} = 0\,, \quad h^{(1)} = 0\,. \qquad (5.19)
$$

At two loops we have [42]

$$
\begin{aligned}
f^{(2)}(z,\bar{z}) &= 2\Big[\phi^{(2)}(z,\bar{z}) + \phi^{(2)}\big(1-\tfrac{1}{z},1-\tfrac{1}{\bar{z}}\big) + \phi^{(2)}\big(\tfrac{1}{1-z},\tfrac{1}{1-\bar{z}}\big)\Big]\,. \\
g^{(2)}(z,\bar{z}) &= \frac{1}{2}\phi^{(1)}(z,\bar{z})^2\,. \\
h^{(2)}(z,\bar{z}) &= 0\,. \qquad (5.20)
\end{aligned}
$$

At three loops, in addition to ladder integrals, there is the ‘tennis-court’ integral (which is in fact identical to the three-loop ladder [50]) as well as two genuinely new integrals, known as Easy and Hard [46]. These new three-loop integrals were evaluated in [48]. The result is that at three

[2] We hope that the context is sufficient to distinguish between the ’t Hooft coupling g^2 and the function $g(z,\bar{z})$.

loops we have

$$
\begin{aligned}
f^{(3)}(z,\bar{z}) =& -6\Big[\phi^{(3)}(z,\bar{z}) + \phi^{(3)}\big(1-\tfrac{1}{z},1-\tfrac{1}{\bar{z}}\big) + \phi^{(3)}\big(\tfrac{1}{1-z},\tfrac{1}{1-\bar{z}}\big)\Big] \\
& -2\Big[E(1-z,1-\bar{z}) - E\big(1-\tfrac{1}{z},1-\tfrac{1}{\bar{z}}\big) - E(z,\bar{z}) \\
& \quad + E\big(\tfrac{z}{z-1},\tfrac{\bar{z}}{\bar{z}-1}\big) + E\big(\tfrac{1}{z},\tfrac{1}{\bar{z}}\big) - E\big(\tfrac{1}{1-z},\tfrac{1}{1-\bar{z}}\big)\Big]\,, \\
g^{(3)}(z,\bar{z}) =& -2\phi^{(1)}(z,\bar{z})\phi^{(2)}\big(1-\tfrac{1}{z},1-\tfrac{1}{\bar{z}}\big) - H^{(a)}(z,\bar{z}) - H^{(a)}(1-z,1-\bar{z})\,, \\
h^{(3)}(z,\bar{z}) =& -2\Big[E(1-z,1-\bar{z}) + E(1-\tfrac{1}{z},1-\tfrac{1}{\bar{z}})\Big] - H^{(b)}(1-z,1-\bar{z})\,. \qquad (5.21)
\end{aligned}
$$

The function E arises in the Easy integral while the functions $H^{(a)}$ and $H^{(b)}$ arise in the Hard integral. Their explicit form can be found in [48]. The functions E and $H^{(b)}$ can be expressed in terms of single-variable harmonic polyogarithms. Only the function $H^{(a)}$ involves genuine multiple polylogarithms.

The four point function for $k=3,4$ is known up to two loops. The function F_3 is given by

$$
\begin{aligned}
F_3(z,\bar{z}) =& -4g^2\frac{\phi^{(1)}(z,\bar{z})}{z-\bar{z}} \\
&+\frac{8(g^2)^2}{z-\bar{z}}\left(\frac{1-\bar{z}+z(2\bar{z}-1)}{4(z-\bar{z})}\phi^{(1)}(z,\bar{z})^2 - \phi^{(2)}\big(z^{-1},\bar{z}^{-1}\big) - \phi^{(2)}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right)\right). \qquad (5.22)
\end{aligned}
$$

While the function $\mathcal{H}$ is given by

$$
\begin{aligned}
\mathcal{H}(u,v;\sigma,\tau) =& \frac{z\bar{z}\mathcal{F}(z,\bar{z})}{(1-z)(1-\bar{z})} + \frac{(z\bar{z})^2}{(1-z)(1-\bar{z})}\left(\sigma\widetilde{\mathcal{F}}(u,v) + \frac{\tau\widetilde{\mathcal{F}}(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1})}{((1-z)(1-\bar{z}))^2}\right) \\
&+\frac{(z\bar{z})^3}{((1-z)(1-\bar{z}))^2}\left(\sigma^2\mathcal{F}\left(\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}\right) + \frac{\tau^2\mathcal{F}(1-z,1-\bar{z})}{(1-z)(1-\bar{z})} + \sigma\tau\widetilde{\mathcal{F}}(z,\bar{z})\right) \qquad (5.23)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{F}(z,\bar{z}) =& -\frac{8\lambda\phi^{(1)}(z,\bar{z})}{N^2(z-\bar{z})} + \frac{4\lambda^2}{N^2(z-\bar{z})}\left[\frac{2+z+\bar{z}+z\bar{z}}{4(z-\bar{z})}\phi^{(1)}(z,\bar{z})^2 + \phi^{(2)}(z,\bar{z}) - \phi^{(2)}\big(\tfrac{z}{z-1},\tfrac{\bar{z}}{\bar{z}-1}\big)\right] \\
\widetilde{\mathcal{F}}(z,\bar{z}) =& -\frac{8\lambda\,\phi^{(1)}(z,\bar{z})}{N^2(z-\bar{z})} + \frac{4\lambda^2}{N^2(z-\bar{z})}\left[\frac{z\bar{z}\,\phi^{(1)}(z,\bar{z})^2}{4(z-\bar{z})} - \phi^{(2)}\big(z^{-1},\bar{z}^{-1}\big)\right]. \qquad (5.24)
\end{aligned}
$$

5.1.2 Lorentzian OPE limit

The four point function depends on two cross ratios since as it organizes the CFT data in a non-trivial way. We shall need to find an efficient procedure to extract this data since conformal Regge theory relates dimensions of operators and OPE coefficients to the pomeron pole and residue through (4.47) and (4.50). One of the most efficient ways to obtain the dimensions and the OPE coefficients is through the Lorentzian OPE limit in the four point function. In this limit the point x_2 is approaches the light-cone of the point x_1, so $x_{12}^2 \to 0$; in terms of cross ratios u and v (or z and $\bar{z}$) this corresponds to $u \to 0$ while v is kept fixed (or $\bar{z} \to 0$ while z is kept fixed).

5.1.2.1 Four point function in Lorentzian OPE

Lorentzian OPE limit for $k = 2$

It is instructive to consider the Lorentzian OPE limit of each perturbative contribution to the four-point function (5.9). The general structure is given by

$$F_2(u, v) = \sum_{l=1}^{\infty} g^{2l} \sum_{s=0}^{l} (\log u)^{l-s} \theta_s^{(l)}(v) + \mathcal{O}(u) \tag{5.25}$$

where $s = 0$ is called leading log, $s = 1$ next-to-leading log, etc. Recalling the form (1.16) of the cross-ratios in terms of z and $\bar{z}$, we see that the Lorentzian OPE limit can be implemented by taking $\bar{z} \to 0$, leaving z fixed. In this limit the variable v becomes simply $1 - z$, and it is convenient to express the results for $\theta_s^{(l)}(v)$ in terms of z.

Taking the limit on the form of $F_2(u, v)$ up to three loops, given in (5.16), we find that the rational prefactors simplify to be either $1/z$ or $1/z^2$. This means that up to three loops the functions $\theta_s^{(l)}(v)$ are always of the form

$$\theta_s^{(l)}(v) = \frac{1}{z}\, \kappa_s^{(l)}(z) + \frac{1}{z^2}\, \tilde{\kappa}_s^{(l)}(z) \tag{5.26}$$

for pure transcendental functions $\kappa_s^{(l)}$ and $\tilde{\kappa}_s^{(l)}$.

In order to present explicit results for the limits considered in this paper, it is useful to introduce the family of harmonic polylogarithms (or HPLs) [51]. We will need the harmonic polylogarithms whose weight vectors w are composed of the letters 0 or 1. For a string of n zeros we define $H_{0_n}(z) = \frac{1}{n!} \log^n z$. The remaining functions are then defined as follows,

$$H_{0,w}(z) = \int_0^z \frac{dt}{t} H_w(t)\,, \qquad H_{1,w}(z) = \int_0^z \frac{dt}{1-t} H_w(t)\,. \tag{5.27}$$

The classical polylogarithms functions are the harmonic polylogarithms whose weight vectors are are string of zeros followed by a single one, $\mathrm{Li}_n(z) = H_{0_{n-1},1}(z)$. As is common in the literature we will employ the shorthand notation whereby a string of $(k-1)$ zeros followed by a 1 is contracted to the label k, for example $\mathrm{Li}_4(z) = H_{0,0,0,1}(z) = H_4(z)$ or $H_{0,1,0,0,1}(z) = H_{2,3}(z)$. Expressing the OPE limit in terms of the harmonic polylogarithms we find at one loop,

$$\theta_0^{(1)}(v) = \frac{1}{z}\, H_1\,, \qquad \theta_1^{(1)}(v) = -\frac{2}{z} H_2\,. \tag{5.28}$$

where we have left the argument z of the HPLs implicit. At two loops we have

$$\theta_0^{(2)}(v) = \frac{2}{z}\,H_2 + \frac{2}{z^2}\,H_{1,1}\,,\quad \theta_1^{(2)}(v) = -\frac{2}{z}\big(6H_3 + H_{1,2} + H_{2,1}\big) - \frac{4}{z^2}\big(H_{1,2} + 2H_{2,1}\big) \tag{5.29}$$
$$\theta_2^{(2)}(v) = \frac{2}{z}\big(12H_4 + 3H_{1,3} + H_{2,2} - 4H_{3,1} + 2H_{1,1,2} - 2H_{1,2,1} + 6H_1\zeta_3\big) + \frac{8}{z^2}\big(H_{2,2} + 2H_{3,1}\big)$$

Moving to three loops we quote here the result for the coefficient of $\log^3 u$,

$$\theta_0^{(3)}(1-z) = -\frac{4}{3z}\big(H_{1,2} - 2H_3\big) + \frac{4}{3z^2}\big(4H_{1,2} + 2H_{2,1} + 3H_{1,1,1}\big) \tag{5.30}$$

and the coefficient of $\log^2 u$,

$$\begin{aligned}\theta_1^{(3)}(1-z) =& -\frac{4}{z}\big(8H_4 - H_{1,3} - H_{2,2} - 2H_{3,1} + H_{1,2,1} - H_{2,1,1}\big)\\ &- \frac{4}{z^2}\big(8H_{1,3} + 8H_{2,2} + 8H_{3,1} + 4H_{1,1,2} + 3H_{1,2,1} + 5H_{2,1,1}\big)\,.\end{aligned} \tag{5.31}$$

Lorentzian OPE limit for $k = 3$

For $k = 3$ and $k = 4$ we will analyze the Lorentzian for the singlet channel since this is the channel we will be interested in the Regge limit. The leading term in the Lorentzian OPE limit in the singlet channel up to two loops is given by

$$\begin{aligned}\mathcal{A}_{0,0,0}(u,v) =& 1 + \tfrac{3(z-2)}{2N^2(z-1)}u + \tfrac{3\lambda(6-6z+z^2)u}{z(z-1)}\Bigg[- \ln u H_1 + 2H_2\\ &+\lambda\Bigg(- \ln^2 u\tfrac{2zH_2+H_1^2}{z} + \tfrac{\ln u}{z}\big(z\big(6H_3 - 6\mathrm{Li}_3\big(\tfrac{z}{z-1}\big) - H_1^3\big) - 4(z-1)H_2H_1\big)\\ &\tfrac{4(z-2)H_2^2+24z\mathrm{Li}_4\left(\frac{z}{z-1}\right)-24zH_4-2zH_2H_1^2-12z\mathrm{Li}_3\left(\frac{z}{z-1}\right)H_1-zH_1^4}{2z}\Bigg)\Bigg]\end{aligned} \tag{5.32}$$

Lorentzian OPE limit for $k = 4$

In this section we present the leading term of the four point function in the Lorentzian OPE limit for the case with $k = 4$ in the singlet channel

$$\begin{aligned}\mathcal{A}_{0,0,0}(u,v) =& 1 + \tfrac{8u(2-z)}{3(N^2-1)(1-z)} + \tfrac{4\lambda u(6-6z+z^2)}{3(N^2-1)(z-1)z}\Bigg[- \ln u H_1 + 2H_2\\ &+\lambda\big(- \ln^2 u\tfrac{2zH_2+H_1^2}{4z} + \tfrac{\ln u}{4z}\big(z\big(6H_3 - 6\mathrm{Li}_3\big(\tfrac{z}{z-1}\big)\big) - H_1^3\big)\\ &-4(z-1)H_2H_1 + \tfrac{4(z-2)H_2^2+24z\mathrm{Li}_4\left(\frac{z}{z-1}\right)-24zH_4-2zH_2H_1^2-12z\mathrm{Li}_3\left(\frac{z}{z-1}\right)H_1-zH_1^4}{8z}\big)\big)\Bigg]\end{aligned} \tag{5.33}$$

5.1.2.2 Extracting CFT data: dimensions and OPE coefficient

As reviewed in section 2 the conformal block assumes a particularly simple form in this limit

$$G_{\Delta,J}(z,\bar{z}) = (z\bar{z})^{\frac{\Delta-J}{2}}\big(-\tfrac{z}{2}\big)^{J}\,{}_2F_1\big(\tfrac{\Delta+J}{2},\tfrac{\Delta+J}{2},\Delta+J,z\big) = (z\bar{z})^{\frac{\Delta-J}{2}}g_{\Delta,J}(z) \tag{5.34}$$

Extracting the CFT data of the leading twist operators amounts to decompose the four point function in terms of hypergeometric functions ${}_2F_1$. A systematic procedure to perform this decomposition was presented in [44] and uses the identity

$$z^n = \sum_{J=n}^{\infty} p_{n,J}\, g_{2+J,J}(z) \tag{5.35}$$

with $p_{n,J}$ given by

$$p_{n,J} = \frac{(-1)^n 2^J J!^2 (J+n)!}{(2J)!(J-n)!n!^2}. \tag{5.36}$$

Consider that the four point function can be written as

$$\mathcal{A}(z,\bar{z}) = z\bar{z} f(z) + O(\bar{z}^2), \tag{5.37}$$

then the function $f(z)$ can be decomposed in terms of $g_{2+J,J}(\bar{z})$

$$f(z) = \sum_{n=1}^{\infty} f_n z^{n+1} = \sum_{J=2}^{\infty} 2^{J+1}\frac{J!^2}{(2J)!}\widetilde{f}_J\, g_{2+J,J}(z) \tag{5.38}$$

where $\widetilde{f}_J$ is given by

$$\widetilde{f}_J = \sum_{n=1}^{J-1} f_n (-1)^{n+1}\frac{(J+n+1)!}{(n+1)!^2(J-n-1)!} \tag{5.39}$$

In practice, for the correlation function (5.6), this task is facilitated since the dimension and OPE coefficients can be expressed in terms of a finite basis of functions. These are the (nested) harmonic sums, which are recursively defined by

$$S_{a_1,a_2,\ldots,a_n}(x) = \sum_{y=1}^{x}\frac{(\mathrm{sign}(a_1))^y}{y^{|a_1|}}\, S_{a_2,\ldots,a_n}(y)\,, \tag{5.40}$$

starting from the trivial seed without indices, $S(y)=1$. Thus we can compute $\widetilde{f}_J$ for various values of J and fix the finite coefficients of an ansatz built using this basis.

In the following we will extract the CFT data, *i.e.* dimension of operators and OPE coefficients in the weak coupling limit of twist two operators

$$\Delta(J) = 2 + J + g^2\gamma_1(J) + g^4\gamma_2(J) + \dots \tag{5.41}$$

$$C^2(J) = C_0^2(J)\big(1 + g^2 c_J^{(1)} + g^4 c_J^{(2)} + \big)\dots \tag{5.42}$$

For simplicity let us assume that there exists just one operator with a given twist and spin J being exchanged in a four point function. Then conformal block decomposition in the Lorentzian OPE limit (2.9) predicts that the four point function should have the following structure at weak coupling

$$\mathcal{A}(u,v) = 1 + \sum_J u\, C_0^2(J)\big[g_0 + g^2\big(\ln u\, \frac{\gamma_1}{2} g_0 + \gamma_1 g_0' + c_J^{(1)} g_0\big) + g^4\bigg(\frac{\ln^2 u\gamma_1^2(J)}{8} g_0 \tag{5.43}$$

$$+ \frac{\ln u}{2}((\gamma_2 + c_J^{(1)})g_0 + \gamma_1^2 g_0') + (c_J^{(2)} g_0 + \big((c_J^{(1)}\gamma_1 + \gamma_2)g_0' + \frac{\gamma_1^2}{2} g_0''\big)\bigg) + O(g^6, u^2)$$

where the g_0 is the leading coefficient in (2.9) and derivatives are taken with respect to the Δ. As was shown in the previous section the four point function has precisely this type of structure. For example the four point function for $k = 2$ in the 20 channel[3] is given by

$$\mathcal{A}_{20}(u,v) = \frac{u(2-z)}{3(N^2-1)(1-z)} + g^2 u \frac{10z^2}{15(1-z)(N^2-1)}\bigg(\ln u \frac{1}{z} H_1 - \frac{2}{z} H_2\bigg) + O(u^2, g^4) \tag{5.44}$$

In this R-charge channel there is just one twist two operator flowing which allows to determine unambiguously. For example the series expansion of the tree level result is just

$$\frac{u(2-z)}{3(N^2-1)(1-z)} = \bar{z}\sum_{n=0}^{\infty} a_n\, z^n\,, \qquad a_n = \begin{cases} \frac{1}{3}, & n > 1 \\ \frac{2}{3}, & n = 0 \end{cases}. \tag{5.45}$$

Using (5.39) we conclude that the tree level OPE coefficient is given by $C_0^2 = \frac{2^{J+1} J!^2}{3(2J)!}$. Next we could look at the $\ln u$ coefficient in the expansion, perform the same analysis and read γ_1. This procedure can be repeated to extract $c_J^{(1)}$ by studying the regular in u term at one loop. Notice that this is possible because there is just one operator flowing in the OPE and so there is one equation for each undetermined coefficient. In the singlet channel, there are three operators that can flow and so this would not be a good method to determine these coefficients. However the dimension of operator and OPE coefficients of the operators that flow in the 20 channel and the singlet channel are related by supersymmetry. The dimension of the operators are related by a simple shift in the spin J

$$\gamma(J+2), \qquad\qquad \gamma(J), \qquad\qquad \gamma(J-2). \tag{5.46}$$

[3] Where 20 represents the irrep of SU(4) with dimension 20. Check appendix B.1 to see how to project into this channel.

The anomalous dimension for operators in the leading Regge trajectory, which contains the energy-momentum tensor, is the one with $\gamma=\gamma(J-2)$ and the $\gamma(J)$ is the anomalous dimension of the operator transforming in the 20 of $SU(4)$. For the case of $k=2$ the relation between the OPE coefficients can be obtained by analyzing the equation

$$A_1 - 1 + \frac{4u(1+v)}{(N^2-1)(1-v)^2} = 2\,\frac{1+4v+v^2}{(1-v)^2} A_{20} + O(u^2)\ . \tag{5.47}$$

which follows the fact that both channels are expressed in terms of the same function $F(u,v)$. On the other hand, the singlet channel four-point function can also be expanded in conformal blocks

$$\begin{aligned} A_1 = 1 &+ u \sum_{\substack{J=2\\ \text{even}}}^{\infty} b_J\, u^{\frac{\gamma(J-2)}{2}} \left(\frac{v-1}{2}\right)^J F\big(2+2J+\gamma(J-2), 1-v\big) \\ &+ u \sum_{\substack{J=2\\ \text{even}}}^{\infty} b'_J\, u^{\frac{\gamma(J)}{2}} \left(\frac{v-1}{2}\right)^J F\big(2+2J+\gamma(J), 1-v\big) \\ &+ u \sum_{\substack{J=0\\ \text{even}}}^{\infty} b''_J\, u^{\frac{\gamma(J+2)}{2}} \left(\frac{v-1}{2}\right)^J F\big(2+2J+\gamma(J+2), 1-v\big) + O(u^2)\,, \end{aligned} \tag{5.48}$$

where the 1 corresponds to the contribution of the identity and we denoted by b_J, b'_J and b''_J the product of OPE coefficients associated with the three trajectories of twist-two operators that exist in the singlet sector. Using the relation (5.47) between the two channels A_1 and A_{20}, and their conformal block expansions (5.48) and (5.43), we conclude that

$$b_2 \left(\frac{v-1}{2}\right)^2 F(6,1-v) + \frac{4(1+v)}{(N^2-1)(1-v)^2} = 2\,\frac{1+4v+v^2}{(1-v)^2}\, a_0\, F(2,1-v) \tag{5.49}$$

and

$$\begin{aligned} &12\left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{6}\right) \sum_{J=2}^{\infty} a_J\, u^{\gamma/2} \frac{z^J}{2^J} F\Big(1+J+\frac{\gamma}{2}, z\Big) = \\ &\sum_{J=2}^{\infty} \frac{z^J u^{\gamma/2}}{2^J} \Big(b_{J+2}\, \frac{z^2}{4} F\Big(3+J+\frac{\gamma}{2}, z\Big) + b'_J\, F\Big(1+J+\frac{\gamma}{2}, z\Big) + b''_{J-2}\, \frac{4}{z^2} F\Big(J-1+\frac{\gamma}{2}, z\Big)\Big)\,, \end{aligned} \tag{5.50}$$

where we used $v=1-z$ and shifted the summation variable J so that the function γ has argument always given by J. Equation (5.49) follows from the twist-two contributions to (B.7) with no anomalous dimensions (like the energy-momentum tensor), and leads to

$$a_0 = \frac{2}{3(N^2-1)}\ , \qquad b_2 = \frac{8}{45(N^2-1)}\ . \tag{5.51}$$

Equation (5.50) encodes the contributions of all other twist-two operators. It is valid for small, but still finite, coupling constant, as long as the anomalous dimensions of the twist-two operators are small enough that higher order terms in the expansion (5.48), from higher twist operators, are subleading. Thus, in this equation at finite coupling, terms with different J can not mix because they have different powers of u. This means that

$$
\begin{aligned}
&12\left(\frac{1}{z^2}-\frac{1}{z}+\frac{1}{6}\right)a_J\,F\Big(1+J+\frac{\gamma}{2},z\Big)= \\
&b_{J+2}\,\frac{z^2}{4}\,F\Big(3+J+\frac{\gamma}{2},z\Big)+b'_J\,F\Big(1+J+\frac{\gamma}{2},z\Big)+b''_{J-2}\,\frac{4}{z^2}\,F\Big(J-1+\frac{\gamma}{2},z\Big)\,,
\end{aligned}
\tag{5.52}
$$

for all $J=2,4,6,\ldots$. Analyzing the Taylor expansion in z on both sides, we see that this equation is satisfied if

$$
\begin{aligned}
&b''_{J-2}=3a_J\,, \qquad b'_J=\frac{\big(\gamma(J)+2J\big)\big(\gamma(J)+2J+2\big)}{2\big(\gamma(J)+2J-1\big)\big(\gamma(J)+2J+3\big)}\,a_J\,,\\
&b_{J+2}=\frac{3\big(\gamma(J)+2J+2\big)^2\big(\gamma(J)+2J+4\big)^2}{16\big(\gamma(J)+2J+1\big)\big(\gamma(J)+2J+3\big)^2\big(\gamma(J)+2J+5\big)}\,a_J\,.
\end{aligned}
\tag{5.53}
$$

These relations are non-perturbative. They follow from supersymmetry as explained in [52]. Notice that for $J=0$, we find $b_2=4a_0/15$ which is compatible with (5.51).

From the Lorentzian OPE analysis we can read off the anomalous dimensions for the leading twist operators

$$\gamma(J)=\Delta(J)-J-2=\sum_{n=1}^{\infty}g^{2n}\gamma_n(J)\,. \tag{5.54}$$

We use notation with coupling g related to the 't Hooft coupling $\lambda=g_{YM}^2N$ by

$$g^2=\frac{\lambda}{16\pi^2}\,. \tag{5.55}$$

The anomalous dimensions γ_n are known up to five loops [38, 53, 54, 55] and obey the principle of maximal transcendentality [38]. The first three terms in this expansion are

$$
\begin{aligned}
\gamma_1&=8S_1(x)\,,\\
\gamma_2&=-32\big(S_3(x)+S_{-3}(x)\big)+64S_{-2,1}(x)-64\big(S_1(x)S_2(x)+S_1(x)S_{-2}(x)\big)\,, &(5.56)\\
\gamma_3&=64\big(3S_{-5}+8S_{-4}S_1+S_{-2}^2S_1+6S_{-3}S_1^2+S_{-3}S_2+4S_{-2}S_1S_2+2S_1S_2^2+2S_{-2}S_3\\
&\quad+2S_1^2S_3+S_2S_3+3S_1S_4+S_5-6S_{-4,1}-12S_1S_{-3,1}-6S_{-3,2}-4S_1^2S_{-2,1}-2S_2S_{-2,1}\\
&\quad-10S_1S_{-2,2}-6S_{-2,3}+12S_{-3,1,1}+16S_1S_{-2,1,1}+12S_{-2,1,2}+12S_{-2,2,1}-24S_{-2,1,1,1}\big) &(5.57)
\end{aligned}
$$

where $x=J-2$ and the functions S are (nested) harmonic sums (5.40). The OPE coefficients can be obtained in the similar way up to three loops. In particular, [56] proposed a nice structure for the coefficients

$$a_J = \frac{2}{3(N^2-1)} \frac{2^J \left(1+\frac{\gamma(J)}{2}\right)_J^2}{\left(1+\gamma(J)\right)_{2J}} \sum_{n=0} g^{2n} a_n(J) \ , \tag{5.58}$$

using the results up to three loops. In appendix B.2 we present the results for the OPE coefficients. While this analysis was performed for $k=2$ it can be extended to $k=3,4$, relating the anomalous dimensions and the OPE coefficients. The dimension of the leading twist operators exchanged in the 20 channel is the same, the OPE coefficients are different and we present them in the appendix B.2.

5.1.3 Regge limit

The four point correlation function (5.9-5.11) is defined in the Euclidean region. The Regge limit is performed in the Lorentzian regime, thus we need to analytic continue the four point function to achieve this goal. The precise form of the analytic continuation is shown in 2.2. In the following we summarize the leading results in this limit after the analytic continuation is performed.

Regge limit for $k=2$

In the Regge limit the general structure of the function $F_2(u,v)$ is given by

$$uF_2(u,v) = \sum_{l=1}^{\infty} g^{2l} \sum_{k=0}^{l-2} (\log\sigma)^{l-2-k} \xi_k^{(l)}(\rho) + O(\sigma)\,, \tag{5.59}$$

where $k=0$ is called leading log, $k=1$ next-to-leading log, etc. The invariance of $F_2(u,v)$ under the interchange of z and $\bar{z}$ implies that $\xi_k^{(l)}(\rho) = \xi_k^{(l)}(-\rho)$. Note that only the two-loop and higher contributions to $F_2(u,v)$ enter in (5.59). In fact, if we take the Regge limit on the form of $F_2(u,v)$ up to three loops (5.16), we see that only the functions g and h contribute,

$$uF_2(u,v) = \frac{r}{(1-r)^2}\Big(g(z,\bar{z}) + g\big(\tfrac{1}{1-z}, \tfrac{1}{1-\bar{z}}\big)\Big) - \frac{r}{1-r^2}\, h(1-z, 1-\bar{z}) + O(\sigma)\,, \tag{5.60}$$

where $r = z/\bar{z} = e^{2\rho}$. Since g and h vanish at one loop, the four-point function at this order has a subdominant behavior in the Regge limit $\sigma \to 0$. The two-loop contribution was calculated in [37], where it was found that

$$uF^{(2)}(u,v) = \xi_0^{(2)}(\rho) + O(\sigma) = -\frac{8r}{(1-r)^2}\, \pi^2 \log^2 r + O(\sigma)\,. \tag{5.61}$$

As expected, the result is invariant under $r \to 1/r$. At three loops one finds from the explicit results of [48] that the coefficient of the log-divergent term is given by[4]

$$\xi_0^{(3)}(\rho) = -\frac{32r}{(1-r)^2}\,\pi^2\big(H_{0,0,0} + 2H_{1,0,0} - 2\zeta_3\big)\,, \tag{5.62}$$

where the argument of the harmonic polylogarithms is $r = e^{2\rho}$ and $\zeta_x = S_x(\infty)$ is the Riemann zeta function. The coefficient of the finite term is given by

$$\begin{aligned}\xi_1^{(3)}(\rho) = 64\pi^2\frac{r}{(1-r)^2}\bigg(&H_{0,0,0,0}(r) + 2H_{1,0,0,0}(r) + 4H_{1,1,0,0}(r) + 2H_{2,0,0}(r)\\ &- 2\zeta_3\left(H_0(r) + 2H_1(r)\right) + \frac{\zeta_2}{2}H_{0,0}(r) - 3\zeta_4\bigg) + \frac{i\pi}{2}\xi_0^{(3)}.\end{aligned} \tag{5.63}$$

Note that, although at three loops the function h is non-zero, in the Regge limit its analytic continuation is power suppressed as $\sigma \to 0$. Therefore at three loops we do not see any contribution to the four-point function with the form of the second term in (5.60), with its prefactor $r/(1-r^2)$.

5.1.4 Regge limit for $k = 3$ and $k = 4$

In this subsection we analyze the Regge limit for the correlation function (5.9-5.11). We have gather both these cases, $k = 3, 4$ since they give the same result up to a constant

$$\begin{aligned}\mathcal{A}_{0,0,0}(u,v) &= -\frac{72r\pi^2\ln^2 r}{(1-r)^2},\\ \mathcal{A}_{0,0,0}(u,v) &= -\frac{128r\pi^2\ln^2 r}{(1-r)^2}\end{aligned} \tag{5.64}$$

respectively.

5.1.4.1 Extracting CFT data: Pomeron spin and residue

The procedure to obtain the CFT data in the Regge limit is different from the Lorentzian OPE. According to (4.52) the leading term of the four point function in the Regge limit should be given by a simple power of σ times a non-trivial function of ρ. To extract the relevant CFT data in this regime we need to use the orthogonality of the $\Omega_\nu(\rho)$. For $d = 4$ this function reduces to

$$\Omega_{i\nu}(\rho) = \frac{\nu\sin(\rho\nu)}{4\pi^2\sinh\rho}\,. \tag{5.65}$$

which satisfies

$$16\pi^3\int_{-\infty}^{\infty} d\rho\,\sinh^2\rho\,\Omega_\nu(\rho)\Omega_{\overline{\nu}}(\rho) = \nu^2(\delta(\nu-\overline{\nu}) + \delta(\nu+\overline{\nu})). \tag{5.66}$$

[4]More details on the analytic continuation are given in Appendix B.4.

This completeness relation allows to extract, in principle, the pomeron spin and pomeron residue $j(\nu)$ and $\alpha(\nu)$.

The goal of this subsection is to extract the pomeron pole $j(\nu)$ and residue $\alpha(\nu)$ at weak coupling[5]. For this purpose we will just use the orthogonality property of the $\Omega_\nu(\rho)$ functions (5.66). We write the expansion of the pomeron spin $j(\nu)$ and the residue $\alpha(\nu)$ in the usual form

$$j(\nu) = 1 + \sum_{n=1} g^{2n} j_n(\nu)\,, \qquad \alpha(\nu) = \sum_{n=2} g^{2n} \alpha_n(\nu)\,. \tag{5.67}$$

k=2

Comparing equations (4.52), (5.59) and expression (B.7) for A_1, we conclude that

$$\frac{8}{N^2-1}\,\xi_0^{(2)}(\rho) = 2\pi i \int d\nu\, \alpha_2(\nu)\, \Omega_{i\nu}(\rho)\,, \tag{5.68}$$

$$\frac{8}{N^2-1}\,\xi_0^{(3)}(\rho) = -2\pi i \int d\nu\, \alpha_2(\nu)\, j_1(\nu)\, \Omega_{i\nu}(\rho)\,, \tag{5.69}$$

$$\frac{8}{N^2-1}\,\xi_1^{(3)}(\rho) = 2\pi i \int d\nu\, \alpha_3(\nu)\, \Omega_{i\nu}(\rho)\,. \tag{5.70}$$

Using the explicit expressions (5.61), (5.62) and (5.63) for the Regge limit of the four-point function up to three loops, we can determine the functions $\alpha_2(\nu)$, $\alpha_3(\nu)$ and $j_1(\nu)$ by inverting the integral transform above as explained in [16, 58]. We just need use the orthogonality relation (5.66).

In fact, the pomeron spin is known at leading order and next-to-leading order[6] for quite a long time [59, 38]

$$j_1(\nu) = 8\Psi(1) - 4\Psi\left(\frac{1+i\nu}{2}\right) - 4\Psi\left(\frac{1-i\nu}{2}\right), \tag{5.71}$$

$$j_2(\nu) = 4j_1''(\nu) + 24\zeta_3 - 2\zeta_2 j_1(\nu) \; - 8\Phi\left(\frac{1+i\nu}{2}\right) - 8\Phi\left(\frac{1-i\nu}{2}\right), \tag{5.72}$$

where $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the Euler Ψ-function and

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{2}{k+x} \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(k+1+r)^2}\,. \tag{5.73}$$

The function $\alpha(\nu)$ was computed at leading order in [37] and it is given by

$$\alpha_2(\nu) = i\,\frac{16\pi^5}{N^2-1}\,\frac{\tanh\left(\frac{\pi\nu}{2}\right)}{\nu\cosh^2\left(\frac{\pi\nu}{2}\right)}\,. \tag{5.74}$$

[5] In this analysis there is no issue about the contribution of other Regge trajectories, see for instance the discussion in section 4.2 of [57].

[6] Notice that the BFKL spin at NLO starts to contribute to the four-point function of the stress-energy tensor multiplet only at four loops.

Next we consider the computation of the next-to-leading order pomeron residue $\alpha_3(\nu)$. Up to next-to-leading order (at least), the functions $j(\nu)$ and $\alpha(\nu)$ obey the principle of maximal transcendentality. To see that more explicitly we write

$$j(\nu) = 1 + \sum_{n=1}^{\infty} g^{2n} \left[F_n\left(\frac{i\nu - 1}{2}\right) + F_n\left(\frac{-i\nu - 1}{2}\right)\right], \tag{5.75}$$

and

$$\alpha(\nu) = \frac{\pi^2}{N^2 - 1} \frac{1}{\nu} \sum_{n=2}^{\infty} g^{2n} \left[G_n\left(\frac{i\nu - 1}{2}\right) - G_n\left(\frac{-i\nu - 1}{2}\right)\right], \tag{5.76}$$

so that F_n and G_n have transcendentality $2n-1$ and can be written in terms of harmonic sums. In particular, for the above pomeron spin formulae (5.71) and (5.72), and for the pomeron residues determined from integrals (5.68) and (5.70), we have

$$\begin{aligned}
F_1(x) &= -4S_1\,, && (5.77)\\
F_2(x) &= 4\left(\pi^2 \ln 2 - \frac{3}{2}\zeta_3 + \frac{\pi^2}{3} S_1 + \pi^2 S_{-1} + 2S_3 - 4S_{-2,1}\right), && (5.78)\\
G_2(x) &= 16 S_3\,, && (5.79)\\
G_3(x) &= -128\left(2\zeta_3 S_2 - 4\zeta_3 S_{1,1} - 2S_{1,4} - 2S_{2,3} + 4S_{1,1,3} + 3\zeta_4\, S_1 + S_5 + \frac{\zeta_2}{2}\, S_3\right), && (5.80)\\
&\quad + i64\pi\left(S_4 + 2\zeta_3 S_1 - 2S_{1,3}\right),
\end{aligned}$$

where all harmonic sums have argument x. In [60] the next-to-leading order pomeron residue $\alpha_3(\nu)$ was computed using the techniques of operator expansion over color dipoles [61]. However, the proposed expression is different from our result obtained from taking directly the Regge limit of the three-loop contribution to the four-point function. We are confident that our result is correct, in part, because of the non-trivial consistency check imposed by the OPE coefficient relation (4.50). Nonetheless, we are not able to pinpoint any specific mistake in the calculations of [60]. It would be interesting to return to this question using the recent works [62, 63]. Finally, let us just make a technical remark, that will be useful when analyzing the residue of $\alpha_3(\nu)$ near $J = 1$. It turns out that, using expression (B.27) given in appendix B.2, we can express $\alpha_3(\nu)$ in terms of simpler functions

$$\alpha_3(\nu) = \frac{1}{4}\alpha_2\left(j_1^2 + 16\zeta_2\right) - 16 i\pi^7 \frac{\sinh\left(\frac{3\pi\nu}{2}\right) - 11\sinh\left(\frac{\pi\nu}{2}\right)}{3\nu\cosh^5\left(\frac{\pi\nu}{2}\right)} + \frac{i\pi}{2} j_1 \alpha_2\,. \tag{5.81}$$

Notice that the first two terms are imaginary and the last term is real because $\alpha_2(\nu)$ is imaginary for real ν.

The available data for $k = 3$ and $k = 4$ only allows to extract the leading order result for the pomeron residue $\alpha(\nu)$. Repeating the same procedure we obtain that these give the same result,

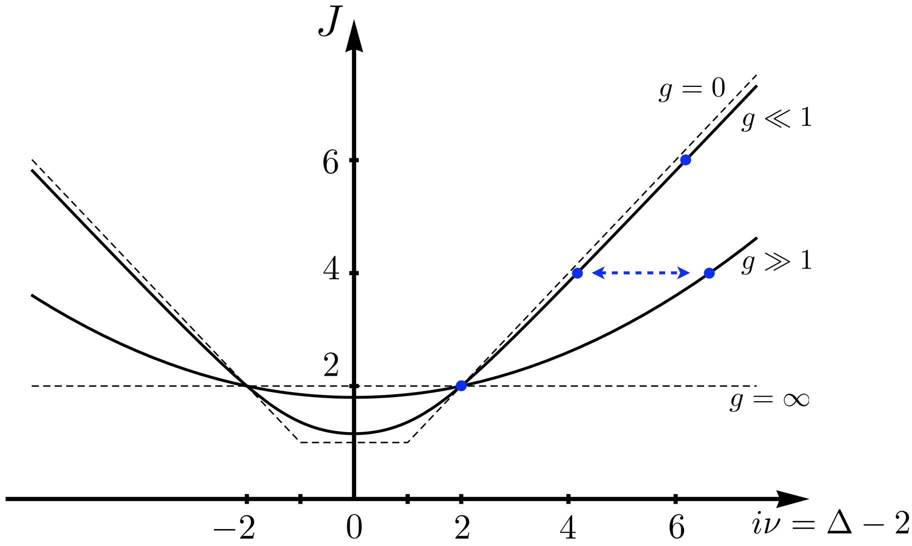


Figure 5.1: Leading Regge trajectory in the dimension–spin plane for various values of the 't Hooft coupling. The physical operators have even spin J and positive dimension, and are represented by blue dots along the curves. The horizontal dashed line $J = 2$ corresponds to the strong coupling limit $g \to \infty$. The weak coupling limit $g \to 0$, is described by the other dashed line, with 3 branches: $J = 1$, $\Delta - 2 = J$ and $2 - \Delta = J$. The intercept $j(0)$ moves from 1 to 2, as the coupling g goes from 0 to ∞.

up a multiplicative constant, as for the $k = 2$

$$G_2^{k=3}(x) = 36 S_3 \tag{5.82}$$

$$G_2^{k=4}(x) = 64 S_3 \tag{5.83}$$

where we used the notation (5.76). Notice that $G_2^k(x)$ can be written in a compact form as

$$G_2^k(x) = 4k^2 S_3(x) \tag{5.84}$$

which was computed previously in [37].

5.1.5 Regge theory relations

The goal of this section is to check the validity of the relations (4.47, 4.50) between dimensions, OPE coefficients, pomeron pole and residue for this specific correlation functon. We will start with the simplest relation involving the dimensions and pomeron pole (4.47). The weak coupling expansion (5.54) of the function $\Delta(J)$ is an expansion around the free theory line $\Delta(J) = J + 2$ (see figure 5.1). The function $\Delta(J)$ defines the Reggeon spin $j(\nu)$. By inverting (4.47) we have

$$\Delta\big(j(\nu)\big) = 2 \pm i\nu\,. \tag{5.85}$$

However, the Reggeon spin can also be computed directly from the Regge limit of the four point correlation function [37] as demonstrated in (5.69).

It is important to realize that, although the functions $\Delta(J)$ and $j(\nu)$ are basically the inverse of each other, their perturbative expansions, either at weak or at strong coupling, contain different information [64]. In other words, the process of inverting the functions does not commute with perturbation theory. Let us consider first the limit $j \to 1$ and $g^2 \to 0$, with $(j-1)/g^2$ fixed, of the BFKL spin (5.75). In this limit, only the leading order term in the expansion (5.75) survives, and we have

$$\frac{j(\nu)-1}{g^2} = -4S_1\left(\frac{i\nu-1}{2}\right) - 4S_1\left(\frac{-i\nu-1}{2}\right). \tag{5.86}$$

Now, the function in the RHS of this equation has simple poles at $i\nu = \pm 1$. If we expand this equation around one of this points, say around $i\nu = 1$, the fixed quantity in the LHS of this equation will be very large. This expansion has the following form

$$\frac{j(\nu)-1}{-4g^2} = \frac{2}{i\nu-1} - 2\sum_{k=1}^{\infty}\zeta(2k+1)\left(\frac{i\nu-1}{2}\right)^{2k}, \tag{5.87}$$

where the coefficients can be read from formulae presented in appendix B.2. We can now invert this equation, solving for $i\nu - 1 = \Delta(j) - 3$, to obtain the behavior of $\Delta(J)$ around $J = 1$, as a power expansion in the small quantity $g^2/(J-1)$. The result for the first order terms in this expansion is [64]

$$\Delta(J) - 3 = 2\left(\frac{-4g^2}{J-1}\right) + 0\left(\frac{-4g^2}{J-1}\right)^2 + 0\left(\frac{-4g^2}{J-1}\right)^3 - 4\zeta(3)\left(\frac{-4g^2}{J-1}\right)^4 + \dots. \tag{5.88}$$

The remarkable thing about this expression is that, after inversion of the leading order BFKL spin, one has a prediction for the leading singularities of the anomalous dimension function (5.54) around $J = 1$ at all orders in perturbation theory. This fact was explored in [64] and served as a guide for the computation of anomalous dimensions using integrability, most notably to check wrapping corrections that appear at four loops [65]. In the next section we shall follow a similar procedure to study the behavior of OPE coefficients.

Let us close these introductory remarks by explaining how higher order terms in the BFKL expansion can be taken into account in the above argument. In this case one considers the expansion (5.75) with $(j-1)/g^2$ fixed, keeping all terms in the g^2 expansion, instead of only the leading term as we did in (5.87). Then one expands around $i\nu = 1$ and inverts. The result is a prediction for the expansion of the anomalous dimension around $J = 1$ at all orders in perturbation theory. In particular, the next to leading BFKL spin allows one to predict the next to leading singularity around $J=1$ at all orders in perturbation theory [64]

$$\begin{aligned}\Delta-3=&\Big(0+(J-1)\Big)+\Big(2+0\,(J-1)\Big)\left(\frac{-4g^2}{J-1}\right)+\Big(0+0\,(J-1)\Big)\left(\frac{-4g^2}{J-1}\right)^2\\&-\Big(0+\zeta(3)(J-1)\Big)\left(\frac{-4g^2}{J-1}\right)^3-\left(4\zeta(3)+\frac{5\zeta(4)}{4}\,(J-1)\right)\left(\frac{-4g^2}{J-1}\right)^4+\dots.\end{aligned} \tag{5.89}$$

Finally let us also note that we can twist around these arguments, and use the knowledge of the anomalous dimension function (5.54) to some fixed order in perturbation theory, to study the behavior of the BFKL spin around $i\nu=1$. Considering the first two orders in perturbation theory for the anomalous dimension given in (5.56), one obtains the prediction

$$\begin{aligned}j(\nu)-1=&\Big(1+0(i\nu-1)\Big)\left(\frac{8g^2}{i\nu-1}\right)+\left(\frac{1}{i\nu-1}+0\,(i\nu-1)\right)\left(\frac{8g^2}{i\nu-1}\right)^2\\&+\left(\frac{2}{(i\nu-1)^2}+\frac{0}{i\nu-1}\right)\left(\frac{8g^2}{i\nu-1}\right)^3+\left(\frac{5}{(i\nu-1)^3}+\frac{0}{(i\nu-1)^2}\right)\left(\frac{8g^2}{i\nu-1}\right)^4+\dots.\end{aligned} \tag{5.90}$$

This result agrees with the known leading order and next to leading order BFKL spin.

Let us comment on the non-commutativity of the perturbation theory and the Regge limit. Assume that the dimension of the leading twist operators has the form

$$\Delta(J)=2+J+\sum_{l=1}^{\infty}g^{2l}\gamma_l(J) \tag{5.91}$$

then from (4.47) we have

$$j(\nu)+\sum_{l=1}g^{2l}\gamma_l(j(\nu))=i\nu. \tag{5.92}$$

Let us say that at tree level the pomeron pole $j(\nu)$ is given by $j(\nu)=a+\sum_{l=1}^{\infty}g^{2l}j_l(\nu)$ for constant a. Then (5.92) is consistent with our assumptions if $\gamma_l(J)$ develops a pole for $J=a$. Thus by inspecting the location of the poles in $\gamma_l(J)$ we can predict what should be the value of a. Besides this solution there is another one given by $j(\nu)=i|\nu|+O(g^2)$ that is the naive continuation from the physical particles that sit at integer $j(\nu)$. These two solutions meet at a point defined by $a=i\nu$ and this is the reason to expand (5.90) around $i\nu=1$.

Relation for OPE coefficients

The idea is to use the knowledge of the Regge residue $\beta(\nu)$ to derive non-trivial predictions for OPE coefficients. We remind the reader that we are considering OPE coefficients with operators normalized as in (2.3). The product of these normalized OPE coefficients is also defined by the ratio of correlators

$$b_J \equiv C_{11J}C_{33J} \sim \frac{\langle \mathcal{O}_1(x_1)\mathcal{O}_1^*(x_2)\mathcal{O}_J(x_5)\rangle \, \langle \mathcal{O}_J(x_6)\mathcal{O}_3(x_3)\mathcal{O}_3^*(x_4)\rangle}{\langle \mathcal{O}_1(x_1)\mathcal{O}_1^*(x_2)\rangle \, \langle \mathcal{O}_J(x_5)\mathcal{O}_J(x_6)\rangle \, \langle \mathcal{O}_3(x_3)\mathcal{O}_3^*(x_4)\rangle}\,. \tag{5.93}$$

The precise relation between the OPE coefficients and the ratio of correlators involves many kinematical factors that we give in appendix B.3. We omit these details to avoid dealing with all the indices in the main text. From direct computation of the the four point correlator in the Regge limit we can extract the Regge residue that appears in the correlation function (4.49). Then, using (4.50), this is related to the analytic continuation of the product of OPE coefficients $b_{j(\nu)}$.

We start with the weak coupling side of the story. We can compute in the OPE coefficient b_J of the spin J operator of the leading Regge trajectory in the OPE of two protected scalar operators of the form $\mathcal{O}_1 = \mathrm{tr}(\phi_{12}\phi^{12})$, where ϕ_{12} is a complex scalar field of SYM, in two ways. One can use (5.53) and the OPE coefficients extracted from the four point function (5.58) or we can compute the explicitly the OPE coefficients. Using the first approach we can access the three loop result which is very hard using using the second method. In the appendix B.3 we computed explicitly this OPE coefficient at tree level but let us emphasize we have this OPE coefficient up to three loops from the four point function data. This requires lifting the degeneracy of the twist two operators and some combinatorics in doing Wick contractions and the result is

$$b_J = \frac{1}{N^2}\frac{2^{1+J}J(J-1)\Gamma^2(J+1)}{(4J^2-1)\,\Gamma(2J+1)} + O(g^2)\,. \tag{5.94}$$

which agrees with the OPE extracted from the four point function. In particular, we can continue this result to the region around $J=1$, with an expansion of the form

$$b_J = \frac{J-1}{N^2}\left(\frac{2}{3} + O(J-1)\right) + O(g^2)\,. \tag{5.95}$$

Now let us look at the Regge residue $r(j(\nu))$ computed in perturbation theory from the four point correlation function. In [37] the Regge residue in position space $\alpha(\nu)$ was shown to be

$$\alpha(\nu) = i\,\frac{16\pi^5 g^4}{N^2}\frac{\tanh\left(\frac{\pi\nu}{2}\right)}{\nu\cosh^2\left(\frac{\pi\nu}{2}\right)} + O(g^6)\,. \tag{5.96}$$

Using (4.50) and (4.53) this translates into a residue $r(j(\nu))$ given by

$$r\big(j(\nu)\big) = -\frac{2^8\pi g^2}{N^2}\frac{\tanh\left(\frac{\pi\nu}{2}\right)}{\chi'(\nu)(1+\nu^2)^2} + O(g^4)\,, \tag{5.97}$$

where we used the leading term in the BFKL spin as written in (5.71). It is clear that (5.97) computes the behavior of the function $r(J)$ around $J=1$, which starts with a power of $J-1 \sim g^2$. The same thing happens to the square of the OPE coefficients (5.95) computed directly in free theory. This is not a coincidence because both $r(J)$ and b_J are related by (4.45). Thus, for $J = j(\nu)$, we can use (4.45) in the form

$$r\big(j(\nu)\big) = C^2\big(j(\nu)\big)\, K_{\Delta(j(\nu)),j(\nu)}\,, \tag{5.98}$$

to compute the OPE coefficients from the Regge residue in the region $J - 1 \sim g^2$, i.e. in the double limit $g^2 \to 0$ and $J \to 1$ with $g^2/(J-1)$ fixed. In particular, we will recover the above free field theory result and also make predictions to arbitrary high order in perturbation theory. The analysis is entirely analogous to that of the anomalous dimension reviewed above.

From (5.95) we conclude that the continuation of the OPE coefficients $C(J)$ to the region where $J - 1 \sim g^2$ admits the following general perturbative expansion

$$b_J = (J-1)\, f\!\left(\frac{g^2}{J-1}\right) + (J-1)^2\, h\!\left(\frac{g^2}{J-1}\right) + O\big((J-1)^3\big)\,, \tag{5.99}$$

with

$$f(x) = \sum_{n=0} f_n\, x^n\,, \qquad h(x) = \sum_{n=0} h_n\, x^n\,. \tag{5.100}$$

Then, expanding (4.53) near $\nu = -i$, we obtain a system of equations for the coefficients f_n and h_n. With the knowledge of $\alpha_2(\nu)$ and $\alpha_3(\nu)$, this can be solved for all f_n and h_n, with the first terms given by

$$\begin{aligned}
&f_0 = \frac{2}{3}\,, \qquad f_1 = \frac{64}{9}\,, \qquad f_2 = \frac{32}{27}\left(61 - 3\pi^2\right), \qquad f_3 = \frac{256}{81}\left(223 - 12\pi^2 - 27\zeta_3\right),\\
&h_0 = \frac{2}{9}(-8 + 3\ln 2)\,, \qquad\qquad h_1 = \frac{4}{27}(-244 + 9\pi^2 + 48\ln 2)\,,\\
&h_2 = \frac{16}{27}\left(153\zeta_3 - 892 + 122\ln 2 - 2\pi^2(3\ln 2 - 20)\right),\\
&h_3 = \frac{64}{1215}\Big(20\big(669\ln 2 - 27\zeta_3(3\ln 2 - 44) - 6320\big) + 171\pi^4 + \pi^2(6405 - 720\ln 2)\Big)\,.
\end{aligned} \tag{5.101}$$

Equations (5.53) and (5.58), together with the explicit results for the OPE coefficients up to three loops derived in [56] and reviewed in appendix B.2, can be used to check that all these coefficients are indeed correct. This check is extremely non-trivial, therefore confirming our NLO computation of the Regge residue $\alpha_3(\nu)$ in (5.80). Moreover, forthcoming computations of OPE coefficients at higher loops must pass this test of conformal Regge theory. We state here the prediction for four loops

$$\begin{aligned}
f_4 &= \frac{512(15800 - 915\pi^2 - 36\pi^4)}{1215}\,,\\
h_4 &= -\,\frac{256\big(9\pi^4(24\ln 2 - 221) + 90\pi^2(81\zeta_3 - 669 + 61\ln 2)\big)}{3645}\\
&\quad + \frac{1280(21870\zeta_5 - 218720 + 18960\ln 2 + 31293\zeta_3)}{3645}\,.
\end{aligned} \tag{5.102}$$

The same type of relations between the pomeron residue and OPE coefficients exist for the cases $k = 3, 4$. We have verified this relation for both cases up to two loops.

5.2 Strong coupling

The leading term in the strong coupling expansion of the four point functions (5.6) has been computed in [66, 67, 68, 69]. The dimensions of unprotected operators gain a large anomalous dimension in the strong coupling limit. Thus these operators do not show up in the Lorentzian OPE limit. In fact we will use this decoupling argument together with the flat space limit to obtain strong coupling correction to the supergravity result.

5.2.1 Regge limit of supergravity 4-pt function

In the following we will analyze the Regge limit of (5.9) in the strong coupling regime. From [66, 70] we know that $F_2(u, v)$ has the following form

$$F_2(u,v) = \tfrac{2025}{128}(1 + u\partial_u + v\partial_v)(uv\partial_u\partial_v)\frac{\phi^{(1)}(z,\bar{z})}{z-\bar{z}} + O(g^{-1}). \tag{5.103}$$

The analytic continuation of $\phi^{(1)}(u, v)$ is straightforward, thus the Regge limit of the four point function in the singlet channel is given by

$$\mathcal{A}(\sigma,\rho) = \frac{2025}{256}\frac{i\pi(36\rho + 24\rho\cosh(2\rho) - 28\sinh(2\rho) - \sinh(4\rho))}{N^2\sinh^7\rho}\,\sigma^{-1}. \tag{5.104}$$

The pomeron residue can be extracted using the orthogonality relation (5.66)

$$\alpha(\nu) = -\frac{\pi^3\nu^2\left(\nu^2+4\right)}{16N^2\sinh^2\left(\frac{\pi\nu}{2}\right)} \tag{5.105}$$

and the pomeron spin is just given by $j(\nu) = 2 + O(g^{-1})$.

5.2.2 Dimensions and BFKL spin

In the following we invert the logic, we will assume the Regge constraints (4.47,4.50) and check what are the predictions for the CFT data. Starting with the relation between spin and anomalous dimension of the operators in the leading Regge trajectory. [7] The anomalous dimensions of the leading twist operators can be computed at strong coupling from the energy of short strings in AdS, and admit an expansion of the type

$$\Delta(\Delta-4) = 4x\alpha_1(\lambda) + x^2(4\alpha_2(\lambda) + \alpha_1^2(\lambda)) + O(x^3)\,, \tag{5.106}$$

[7] Some of the results presented in this section were obtained after many discussions with Diego Bombardelli and Pedro Vieira, who also participated in some of these computations.

$$\Delta(\Delta - 4) = x\beta_1(\lambda) + x^2\beta_2(\lambda) + x^3\beta_3(\lambda) + O(x^4) \tag{5.107}$$

where we conveniently defined $x = J - 2$. The overall factor of x guarantees that the energy momentum tensor has protected dimension. This expansion was first considered by [71, 72] which also derived an exact expression for the first coefficient $\beta_1(\lambda)$. Recently, it was possible to derive an exact integral expression for the second coefficient $\beta_2(\lambda)$ using the $\mathbf{P}\mu$-system[73]. The strong coupling expansions for the slope function $\beta_1(\lambda)$ and curvature function $\beta_2(\lambda)$ are given by

$$\beta_1(\lambda) \approx 2\lambda^{1/2} - 1 + \frac{15}{4\lambda^{1/2}} + \frac{15}{4\lambda} + \frac{135}{64\lambda^{3/2}} - \frac{45}{16\lambda^2} + O\big(\tfrac{1}{\lambda^{5/2}}\big) \tag{5.108}$$

$$\beta_2(\lambda) \approx \frac{3}{2} - \frac{24\zeta_3 - 3}{8\lambda^{1/2}} - 9\frac{8\zeta_3 + 3}{16\lambda} + \frac{32\zeta_3 + 640\zeta_5 - 457)}{256\lambda^{3/2}} + O\big(\tfrac{1}{\lambda^2}\big). \tag{5.109}$$

The $\beta_3(\lambda)$ has not been analyzed in detail yet [74, 75, 73]

$$\beta_3(\lambda) \approx -\frac{3}{8\lambda^{1/2}} + \frac{60\zeta_5 + 60\zeta_3 - 17}{16\lambda} + O(\tfrac{1}{\lambda^{3/2}}). \tag{5.110}$$

Observing the coefficients $\beta_i(\lambda)$ we are led to conjecture that the leading order of $\beta_i(\lambda)$ should start as $\lambda^{\frac{2-i}{2}}$.

On the other hand, at strong coupling the Reggeon spin was computed using the dual string description [76, 16]

$$j(\nu) = 2 - \sum_{n=1}^{\infty} \frac{j_n(\nu^2)}{g^n} = 2 - \frac{4 + \nu^2}{2\sqrt{\lambda}} \left(1 + \sum_{n=2}^{\infty} \frac{\widetilde{j}_n(\nu^2)}{\lambda^{(n-1)/2}} \right), \tag{5.111}$$

where $\widetilde{j}_n(\nu^2)$, defined for $n \geq 2$, is a polynomial of degree $n-2$. The $n = 1$ term in this expansion was computed in [76] and gives the linear Regge trajectory of strings in the flat space limit. The general form that constrains the degree of the polynomial $\widetilde{j}_n(\nu^2)$ was derived in [16] by requiring that such limit is well defined. We will actually see that this polynomial can be further restricted.

Next we consider the limit $J \to 2$ and $\lambda \to \infty$, with $(J - 2)\sqrt{\lambda}$ fixed, of the expression for the anomalous dimension (5.106). Noting that $-\Delta(\Delta - 4) = 4 + \nu^2$, we can equate both expansion (5.106) and (5.111) to obtain new data for the polynomials $\widetilde{j}_n(\nu^2)$, with $n \geq 2$, that characterize the AdS graviton Regge trajectory. Writing

$$\widetilde{j}_n(\nu^2) = \sum_{k=0}^{n-2} c_{n,k}\nu^{2k}, \tag{5.112}$$

we can fix the coefficients $c_{n,n-2}$ and $c_{n,n-3}$. More precisely, we obtained that

$$c_{2,0} = \frac{1}{2}\,, \quad c_{3,0} = -\frac{1}{8}\,, \quad c_{3,1} = \frac{3}{8}\,, \quad c_{4,1} = -\frac{3}{32}\big(8\zeta_3 - 7\big)\,, \quad c_{5,2} = \frac{21}{64}\,,$$
$$c_{4,0} = -1 - \zeta_3\,, \quad c_{5,1} = -\frac{3(48\zeta_3 - 17)}{64}\,, \quad c_{5,0} = -\frac{361 + 1152\zeta_3}{128}\,, \quad c_{6,2} = \frac{137 - 204\zeta_3 - 60\zeta_5}{128}$$
$$c_{6,1} = -\frac{13 + 2880\zeta_3 + 480\zeta_5}{256} \quad c_{6,0} = -\frac{3(149 + 416\zeta_3)}{64} \quad c_{7,3} = \frac{45}{128} \tag{5.113}$$

and the remaining coefficients of this type vanish ($c_{n,n-2} = 0$ for $n \geq 4$, $c_{n,n-3} = 0$ for $n \geq 6$). In particular, we derived the next and the next to next leading order correction to the intercept.

$$j(0) = 2 - \frac{2}{\sqrt{\lambda}} - \frac{1}{\lambda} + \frac{1}{4\lambda^{3/2}} \tag{5.114}$$
$$+ \frac{6\zeta_3 + 2}{\lambda^2} + \frac{1152\zeta_3 + 361}{64\lambda^{5/2}} + \frac{1248\zeta_3 + 447}{32\lambda^3} + O\big(\tfrac{1}{\lambda^{7/2}}\big) \tag{5.115}$$

where the coefficients in the first line were computed in [27] while the terms in the second line were determined in [77, 73] From figure 5.2 we conclude that this strong coupling expansion works reasonably well for $g \gtrsim 0.3$. Such a strong coupling expansion has been recently used to construct phenomenological models of high energy processes in QCD that are dominated by Pomeron exchange, following the proposal of [76]. These models start from the conformal limit here studied, and then introduce a hard wall in AdS to cut off the IR scale. Data analysis of deep inelastic scattering (DIS) [78, 79, 80] and deeply virtual Compton scattering (DVCS) [81] gives an intercept in the region $j(0) = 1.2 - 1.3$. At a first glance it may seem surprising how the fits of data in a region reasonably close to $j(0) = 1$ are so successful, even better than those fits that use the weak coupling expansion (see [82] for the latest analysis on DIS). However, in SYM, figure 5.2 shows that indeed the strong coupling expansion seems to already work reasonably well around the region of $j(0) = 1.2 - 1.3$.

Finally, we remark that the coefficients $c_{n,k}$ can be further restricted if we assume that $H_l(x)$ is a polynomial of degree l. This assumption leads to the conclusion that for $n \geq 4$ the coefficients satisfy

$$c_{n,k} = 0 \quad \text{for} \quad \left[\frac{n}{2}\right] \leq k \leq n - 2\,. \tag{5.116}$$

5.2.3 OPE coefficients

5.2.3.1 Constraint from supergravity result

Let us start with the simple case of graviton exchange in AdS between external scalar fields dual to operators of protected dimension Δ_1 and Δ_3. In this strict $\lambda \to \infty$ limit the spin $j(\nu) = 2$ and the scattering is elastic. The Regge amplitude in position space (4.52) has real residue and

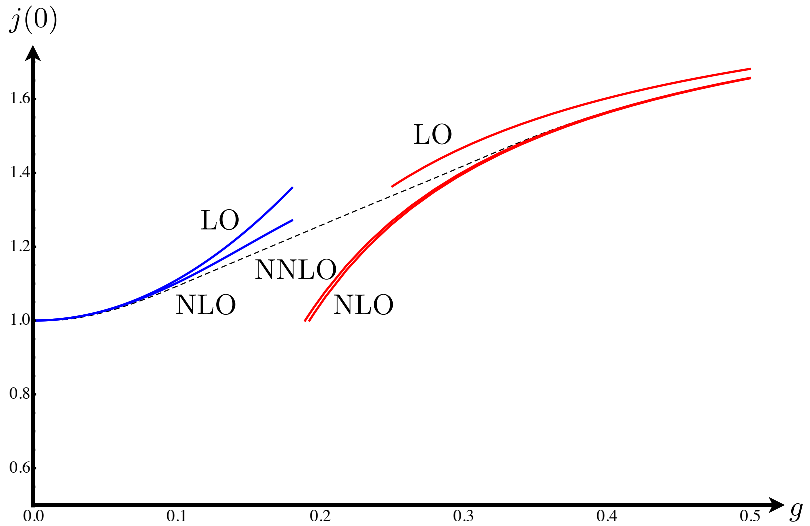


Figure 5.2: Weak (in blue) and strong (in red) coupling expansions of the BFKL intercept $j(0)$. The plot suggests a smooth interpolation, like the black dashed curve, from 1 at $g = 0$ to 2 at $g = \infty$.

is given by[8]

$$\alpha(\nu) = -\frac{4\pi}{N^2}\,\frac{1}{4+\nu^2}\,\frac{\Gamma\big(\Delta_1 + \frac{i\nu}{2}\big)\,\Gamma\big(\Delta_1 - \frac{i\nu}{2}\big)\,\Gamma\big(\Delta_3 + \frac{i\nu}{2}\big)\,\Gamma\big(\Delta_3 - \frac{i\nu}{2}\big)}{\Gamma(\Delta_1)\,\Gamma(\Delta_1 - 1)\,\Gamma(\Delta_3)\,\Gamma(\Delta_3 - 1)}\,. \tag{5.117}$$

which is the same result as obtained in (5.105) for $\Delta_1 = \Delta_3 = 2$. Using (4.53) and (4.50) we can relate the function $\alpha(\nu)$ to the residue $r(j(\nu))$, and therefore to the product of OPE coefficients. In the particular case of graviton exchange, we can use the first term in the expansion (5.111), $j(\nu) = 2 - (4+\nu^2)/(2\sqrt{\lambda}) + O(\lambda^{-1})$, to obtain the $\lambda \to \infty$ result

$$\alpha(\nu) = -r(2)\,\frac{16\pi^2}{4+\nu^2}\,\Gamma\bigg(\Delta_1 + \frac{i\nu}{2}\bigg)\,\Gamma\bigg(\Delta_1 - \frac{i\nu}{2}\bigg)\,\Gamma\bigg(\Delta_3 + \frac{i\nu}{2}\bigg)\,\Gamma\bigg(\Delta_3 - \frac{i\nu}{2}\bigg)\,. \tag{5.118}$$

Equating the previous two equations, we can determine $r(2)$ and therefore, using (4.45), the product of OPE coefficients

$$C^2(2) = \frac{2\Delta_1\Delta_3}{45N^2}\,. \tag{5.119}$$

[8] In [16, 37] we computed $\alpha(\nu)$ for external states of dimension $\Delta_i = 2$. From the results in those papers it is simple to see that, for arbitrary dimension of the external fields, graviton exchange in AdS gives

$$\alpha(\nu) = -\frac{\pi}{4N^2}V_1(\nu)\,\frac{1}{4+\nu^2}\,V_3(\nu)\,, \qquad V_k(\nu) = 4\,\frac{\Gamma\big(\Delta_k + \frac{i\nu}{2}\big)\,\Gamma\big(\Delta_k - \frac{i\nu}{2}\big)}{\Gamma(\Delta_k)\,\Gamma(\Delta_k - 1)}\,.$$

This is actually independent of λ because the three point function with the stress-energy tensor is determined by a Ward identity [83, 10]

$$C^2(2) = \frac{16\Delta_1\Delta_3}{9C_T}\,, \tag{5.120}$$

for $\Delta_2 = \Delta_1$ and $\Delta_3 = \Delta_4$. Note that the central charge C_T appears in the denominator because, as explained in (5.93), we are considering normalized operators. The central charge is known from gravity in AdS [84, 85]

$$C_T = 20\,\frac{\pi R^3}{G_N^{(5)}}\,. \tag{5.121}$$

Using $G_N^{(5)} R^{-3} = \pi/(2N^2)$, we obtain $C_T = 40N^2$, and reproduce exactly the result (5.119). In fact, we can do better. Using (5.118) gives

$$K_{h\pm i\nu,j(\nu)} C^2_{12j(\nu)} = \frac{2}{N^2\Gamma(\Delta_1-1)\Gamma(\Delta_1)\Gamma(\Delta_3-1)\Gamma(\Delta_3)}. \tag{5.122}$$

This relation constraint the form of the three point function at strong coupling. For example, imagine that the OPE coefficient has the following generic expansion around $J = 2$,

$$C^2_{12J} \approx C^2(2) + (J-2)F_1(\lambda) + (J-2)^2 F_2(\lambda) + \dots\,. \tag{5.123}$$

The consistency of (5.122), (5.123) and (5.111) imply that the functions F_i have the following strong coupling expansion,

$$F_i(\lambda) = c_i \lambda^{\frac{i}{2}}\left(1 + b_i\lambda^{-1/2} + \dots\right) \tag{5.124}$$

with all coefficients c_i determined by (5.122). For concreteness let us write the first coefficients for the particular case $\Delta_1 = \Delta_3 = 4$

$$c_0 = \frac{32}{45N^2}\,, \qquad c_1 = -\frac{188}{675N^2}\,, \qquad c_2 = \frac{9401}{81000N^2}\,, \qquad c_3 = \frac{6000\zeta_3 - 13441}{135000N^2} + \dots\,. \tag{5.125}$$

5.2.3.2 Constraining from flat space limit

We can actually compute the leading term in the strong coupling expansion of the function $C^2(J)$ for arbitrary integer J, therefore computing the OPE coefficients between the leading twist operators in the pomeron-graviton Regge trajectory and two external scalar operators. This can be done by considering the flat space limit of the CFT amplitude in the Regge limit, and then equating it to the flat space string theory S-matrix element with external scalar fields, also in the Regge limit, to read the function $C^2(J)$. As a specific example we shall consider the Virasoro-Shapiro amplitude reviewed in section 4.1 with external dilaton fields, which are dual to the Lagrangian operator of protected dimension $\Delta_i = 4$.

The string theory S-matrix for four external scalars in the Regge limit can be recovered from the flat space limit introduced in [23], [9]

$$\mathcal{T}(S,T) = \frac{1}{\mathcal{N}} \lim_{R\to\infty} V(S^5)\, R \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i}\, \alpha^{2-\frac{\sum_i \Delta_i}{2}} e^{\alpha}\, M\!\left(\frac{R^2 S}{2\alpha}, \frac{R^2 T}{2\alpha}\right), \tag{5.126}$$

with the Mellin amplitude given by the Regge theory form (4.49), the volume of the 5-sphere $V(S^5) = \pi^3 R^5$ and the constant $\mathcal{N}$ given by (A.2). The computation now is entirely similar to the one of appendix A.1.1 for the flat space limit of a single conformal partial wave, so we will not be so detailed here (see equation (A.6)). The integration over α produces a delta function in ν^2 with a characteristic width L, so in this case we have

$$\begin{aligned}\mathcal{T}(S,T) \approx \frac{1}{\mathcal{N}} \lim_{R\to\infty} V(S^5)\, R \int_{-\infty}^{\infty} d\nu\, \beta(\nu) \left(\frac{R^2 S}{2}\right)^{j(\nu)} \frac{e^{i\frac{\pi}{2}j(\nu)}}{\sin\!\left(\frac{\pi j(\nu)}{2}\right)} \\ \frac{1}{\sqrt{\nu^2}} \left(\frac{-\nu^2}{R^2 T}\right)^{\frac{1}{2}\sum \Delta_i + j(\nu) - 2} \big(-R^2 T\big)\, \delta_L\big(\nu^2 + R^2 T\big)\,.\end{aligned} \tag{5.127}$$

The function δ_L should be understood as a delta function when integrated against functions that vary in a scale $\delta\nu^2 \gg L$. On the other hand, for functions with characteristic scale $\delta\nu^2 \ll L$ one should take the average. The expression above is general but in practice it becomes simpler. The function $\beta(\nu)$ has is composed of two parts $\frac{\pi}{\nu} j'(\nu) K_{h\pm i\nu, j(\nu)}$ and $C_{12j(\nu)} C_{34j(\nu)}$. The first part has in it a rapidly varying function, however the three point function has poles, as demonstrated in appendix B.3.1 that smooth the dependence in ν. Thus the δ_L can be replaced by the usual delta function. It follows from the above

$$\mathcal{T}(S,T) \approx \frac{\pi R^6}{\mathcal{N}}\, \beta(T) \left(\frac{R^2 S}{2}\right)^{J(T)} \frac{e^{i\frac{\pi}{2}J(T)}}{\sin\!\left(\frac{\pi J(T)}{2}\right)}, \tag{5.128}$$

where for $R \to \infty$ the graviton Regge trajectory becomes the usual linear trajectory

$$J(T) = 2 + \frac{\alpha'}{2} T\,. \tag{5.129}$$

Before we analyze in more detail the implications of (5.128), let us check that in the simplest case of $T = 0$ (i.e. $J = 2$) we can derive again (5.119). We consider the case of scattering of four dilaton fields (dual to the Lagragian operator of dimension $\Delta_i = 4$), so that in this case $K_{\Delta(2),2} = 5/256$ is constant. Thus there is no issue with averaging. It is then a simple exercise to equate (5.128) near $T = 0$, to the S-matrix for graviton exchange between four dilatons, $T(S,T) = -8\pi G_N S^2/T$, checking again (5.119).

The S-matrix element (5.128) can be equated to a type IIB string theory S-matrix element in the

[9] From now on, we shall denote the usual flat space Mandelstam invariants with capital letters S and T, to distinguish them from the Mellin variables s and t.

Regge limit. Let us consider again the case of external fields given by the dilaton. Then we can equate (5.127) to the Virasoro Shapiro S-matrix element (4.22). Although the S-matrix element (5.128) was derived in the physical scattering region of $T < 0$, we can analytically continue this expression to positive T. In particular this means that we can consider J a positive even integer, and compute $C^2(J)$ at strong coupling. In this kinematical region the dimension of the exchanged leading twist operators is real. Since we work in the strong limit we have

$$\Delta(J) \approx i\nu \approx \lambda^{1/4}\sqrt{2(J-2)}\,. \tag{5.130}$$

Let us first look at the expansion of the function $K_{\Delta(J),J}$ at large Δ and for external operators of dimension four,

$$K_{\Delta(J),J} \approx \frac{2^{9+2J+2\Delta(J)}\big(\Delta(J)\big)^{-10-2J}}{\pi^3}\sin^2\!\left(\frac{\pi\Delta(J)}{2}\right). \tag{5.131}$$

The sin^2 piece is rapidly varying however, the poles in the OPE coefficients cancel this behavior. In section B.3.1 we show that there are poles in the OPE coefficients whenever there is almost level crossing with the double trace operators. So this justifies the assumption that $\beta(\nu)$ does not vary rapidly in ν.

Thus, after some straightforward algebra, we can write the flat space limit of this CFT amplitude in the following form

$$\begin{aligned}\mathcal{T}(S,T) \approx \frac{1}{\mathcal{N}}\frac{N^2 G_N}{2\pi\alpha'}\,&\big\langle K_{\Delta(j(\nu)),j(\nu)}\,C^2\big(j(\nu)\big)\big\rangle_T \\ &\left(\frac{R^4}{\alpha'^2}\right)^{\frac{\alpha' T}{4}} e^{\mp i\pi\frac{\alpha' T}{4}}\,\Gamma\!\left(1+\frac{\alpha' T}{4}\right)\Gamma\!\left(-\frac{\alpha' T}{4}\right)\left(\frac{\alpha' S}{2}\right)^{J(T)},\end{aligned} \tag{5.132}$$

where, for the external operators under consideration, $\mathcal{N} = 1/(1152\pi^2)$. Finally equating to the Virasoro-Shapiro amplitude in the Regge limit (4.28), we obtain the following strong coupling prediction for the OPE coefficient involving two Lagrangians and a spin J operator in the leading Regge trajectory,

$$C_{\mathcal{L}\mathcal{L}J} \approx \frac{\pi^{\frac{3}{2}}}{3N}\,\frac{(J-2)^{\frac{5+J}{2}}}{2^{\frac{5}{2}+J}\Gamma\big(\frac{J}{2}\big)}\,\lambda^{\frac{7}{4}}\,\frac{2^{-\lambda^{1/4}\sqrt{2(J-2)}}}{\sin^2\big(\frac{\pi\lambda^{1/4}\sqrt{2(J-2)}}{\sqrt{2}}\big)}\,. \tag{5.133}$$

The exponential dependence on the coupling comes precisely from the dimension of the spin J operator. This is expected since the AdS computation of the three point function should be dominated by the saddle point of the dual heavy short string. This result has been check recently [86].

5.3 Lagrangian four point function from flat space limit

The goal of this section is to show how it is possible to obtain/constraint the four point function of the Lagrangian density, at strong coupling, from the flat space limit[10]. Thus, it is not directly related with the Regge limit but it shows how it it possible to constraint the CFT data using another kinematical limit. It is convenient to perform the analysis using Mellin amplitudes

$$\langle \mathcal{L}(x_1)\dots\mathcal{L}(x_4)\rangle = \int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}\frac{u^{\frac{t}{2}}v^{\frac{8-s-t}{2}}M_{\mathcal{L}}(s,t)}{(x_{12}^2)^4(x_{34}^2)^4}\Gamma^2\left(\frac{8-t}{2}\right)\Gamma^2\left(\frac{8-s}{2}\right)\Gamma^2\left(\frac{s+t-8}{2}\right) \tag{5.134}$$

Notice that the Mellin amplitude $M_{\mathcal{L}}(s,t)$ should satisfy,

$$M_{\mathcal{L}}(s,t) = M_{\mathcal{L}}(t,s) = M_{\mathcal{L}}(s,16-s-t) \tag{5.135}$$

since the correlation function is invariant under permutation of the external points. In [88] the correlator of Lagrangian density was related to the correlation function of the primary operator $\mathcal{O}(x,y)$. More concretely we have,

$$\langle \mathcal{L}(x_1)\dots\mathcal{L}(x_4)\rangle = \frac{2}{x_{12}^8 x_{34}^8}\big(u^4 H(u,v) + H\big(1/u, v/u\big) + u^4/v^4 H\big(u/v, 1/v\big)\big), \tag{5.136}$$

where the function $H(u,v)$ is related to $F(u,v)$ in (5.9) by a eight-order differential operator[11]

$$H(u,v) = \frac{1}{72}D^2u^2v^2D^2\frac{F(u,v)}{uv}\,, \qquad D = u\partial_u^2 + v\partial_v^2 + (u+v-1)\partial_u\partial_v + 2\partial_u + 2\partial_v. \tag{5.137}$$

In the strong coupling limit it is natural to divide the $1/\lambda$ corrections to the Mellin amplitude from the supergravity approximation,

$$M_{\mathcal{L}}(s,t) = M_{\mathcal{L}}^{\text{SUGRA}}(s,t) + M_{\mathcal{L}}^{\lambda}(s,t) \tag{5.138}$$

The crucial point is that single trace unprotected operators that gain a large anomalous dimension, so in the strong coupling limit at each order in $1/\lambda$ there is no pole in $M_{\mathcal{L}}^{\lambda}(s,t)$ as can be

[10] A four point function related to this one by supersymmetry was studied recently [87] and they constructed all solutions consistent with crossing symmetry.

[11] Notice that [88] uses different notation, our $F(u,v)$ is their $\mathcal{F}(u,v)$. See Appendix A for more details. Moreover, we redefined our differential operator by a constant factor such that it maps the SUGRA result $M_F(s,t)$ to $M_{\mathcal{L}}(s,t)$.

seen from (3.37). Let us introduce the Mellin amplitude of $F(u,v)$[12]

$$F(u,v) = \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} u^{\frac{t-2}{2}} v^{\frac{2-s-t}{2}} M_F(s,t) \Gamma^2\left(\frac{4-t}{2}\right)\Gamma^2\left(\frac{4-s}{2}\right)\Gamma^2\left(\frac{s+t}{2}\right). \tag{5.139}$$

$$M_F(s,t) = M_F(t,s) = M_F(s, 4-s-t). \tag{5.140}$$

The action of the differential operator (5.137) on the function $F(u,v)$ is mapped to a difference equation in the Mellin representation,

$$M_{\mathcal{L}}(s,t) = \frac{1}{9216} \sum_{a,b=0}^{6} q_{a,b}(s,t) M_F(s-2a, t-2b) \tag{5.141}$$

and the functions $q_{a,b}(s,t)$ are given in appendix F.

The factor R is permutation symmetric and has weight one at each point, so it must satisfy,

$$F(u,v) = F(v,u) = F\big(1/u, v/u\big)/u. \tag{5.142}$$

This symmetry is translated in terms of Mellin amplitudes as (5.140). Each amplitude in (B.7) can be written in terms of the Mellin amplitude $M_F(s,t)$ given in (5.139). Imposing that absence of poles in the Mellin amplitudes corresponding to the amplitudes A_r would not lead to constraints on $M_F(s,t)$. Let us study how these constraints come about by analyzing the channel 105 of the four point function. Following the notation of [27] we write the Mellin amplitude of the channel 105 as

$$A_{105} = \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} u^{t/2} v^{-(s+t)/2} M_{105}(s,t) \Gamma^2\left(\frac{4-t}{2}\right)\Gamma^2\left(\frac{-s}{2}\right)\Gamma^2\left(\frac{s+t}{2}\right) \tag{5.143}$$

with $M_{105}(s,t)$ given by

$$M_{105}(s,t) = \frac{(t-4)^2(t-6)^2 M_F(4+s, t-4)}{40}. \tag{5.144}$$

The absence of poles in $M_{105}(s,t)$ allows $M_F(s,t)$ to have double poles at $t=0$ and $t=2$, *i.e.*

$$M_F(s,t) = \frac{h(s,t)}{t^2(t-2)^2} \tag{5.145}$$

with $h(s,t)$ a regular function in s and t. However we know that $M_F(s,t)$ satisfies,

$$M_F(s,t) = M_F(t,s) = M_F(s, 4-t-s). \tag{5.146}$$

[12]The symmetry $M_F(s,t)$ is inherited from $F(u,v)$. Recall that $F(u,v)$ satisfies $F(u,v) = F(v,u) = F(1/u, v/u)/u$ since R is symmetric regarding all points.

Notice that $M_F(s,t)$ cannot have poles, otherwise it is not possible to satisfy

$$\frac{M_F(s,t)}{M_F(t,s)}=\frac{s^2(s-2)^2h(s,t)}{t^2(t-2)^2h(t,s)}=1. \tag{5.147}$$

Thus we conclude that the absence of poles in the channel M_{105} implies that $M_F(s,t)$ is a meromorphic function of s and t. In particular this is useful to study the $1/\lambda$ corrections to the four point function. The simplest regular function is a constant, thus using (5.141) we get

$$\begin{aligned}M_F(s,t)=c \implies M_{\mathcal{L}}(s,t)=\frac{c}{48}\big[&504(t^2u^2+s^2t^2+s^2u^2)+4144(s^3+t^3+u^3)+\\&+17662(tu+st+su)-54001(s^2+t^2+u^2)\big].\end{aligned} \tag{5.148}$$

with $u=16-s-t$. The correlation function of four Lagrangians in $\mathcal{N}=4$ SYM is related to the scattering amplitude of four dilatons in superstring theory through the flat space limit. This is the limit of the four point function that focus the interaction region in AdS to be small, thus it only probes flat space physics. In [23, 30] it was shown that this leads to the relation,

$$\begin{aligned}&\lim_{\lambda\to\infty}\lambda^{-1/2}\int_{-i\infty}^{i\infty}\frac{d\alpha}{2\pi i}\frac{e^\alpha}{\alpha^6}M_{\mathcal{L}}\left(s=\sqrt{\lambda}\frac{S}{2\alpha},t=\sqrt{\lambda}\frac{T}{2\alpha}\right)\\&=-\frac{1}{N^2}\frac{1}{2^5 3^2}\frac{(S^2+ST+T^2)^2}{ST(S+T)}\frac{\Gamma(1-\frac{S}{4})\Gamma(1-\frac{T}{4})\Gamma(1+\frac{S+T}{4})}{\Gamma(1+\frac{S}{4})\Gamma(1+\frac{T}{4})\Gamma(1-\frac{S+T}{4})}.\end{aligned} \tag{5.149}$$

We should emphasize that this is a prediction/constraint for $M_{\mathcal{L}}(s,t)$ and gives information which is not easily accessible by other methods. From (5.138) we know that the corrections to the Mellin amplitude do not have poles. The flat space limit relation (5.149) constraints the $1/\lambda$ corrections to be polynomial functions in s and t at each fixed order in $1/\lambda$. The flat space limit relation allows us to write the Mellin amplitude $M_{\mathcal{L}}(s,t)$ as

$$M_{\mathcal{L}}(s,t)=M_{\mathcal{L}}^{\mathrm{SUGRA}}(s,t)+\sum_{n=0}^{\infty}\lambda^{-\frac{3+n}{2}}l_{n+4}(s,t) \tag{5.150}$$

$$M_F(s,t)=M_F^{\mathrm{SUGRA}}(s,t)+\sum_{n=0}^{\infty}\lambda^{-\frac{3+n}{2}}f_n(s,t). \tag{5.151}$$

with $l_n(s,t)$ and $f_n(s,t)$ polynomials of degree n. [13] Let us emphasize that $l_4(s,t)$ was completely determined, up to a constant, just from the relation between the four point function of $\mathcal{O}$ and $\mathcal{L}$, the regular behavior of the corrections to $M_F(s,t)$ and the existence of a flat space limit. In particular, the large s and t behavior of (5.141) tells us that $l_n(s,t)$ satisfies

$$\widetilde{l}_{n+4}(s,t)\equiv\lim_{b\to\infty}b^{-4-n}l_{n+4}(bs,bt)=\frac{(n+9)!}{288(n+5)!}(s^2+t^2+st)^2\widetilde{f}_n(s,t) \tag{5.152}$$

[13] It will be clear in the following why $M_F(s,t)$ has this form.

where $\widetilde{f}_n(s,t)$ is defined by $\widetilde{f}_n(s,t) = \lim_{b\to\infty} b^{-n} f_n(bs,bt)$. Notice that since the flat space limit is sensitive just to the highest power of s and t it is possible to extract all the coefficients of $\widetilde{f}_n(s,t)$. In fact, as we show in Appendix F, we can rewrite the flat space limit in terms for $M_F(s,t)$

$$\begin{aligned}
&\lim_{\lambda\to\infty} \lambda^{3/2} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \frac{e^{\alpha}}{\alpha^6} M_F\left(\frac{\sqrt{\lambda}S}{2\alpha}, \frac{\sqrt{\lambda}T}{2\alpha}\right) \\
&= -\frac{16}{N^2 ST(S+T)} \frac{\Gamma\left(1-\frac{S}{4}\right)\Gamma\left(1-\frac{T}{4}\right)\Gamma\left(1+\frac{S+T}{4}\right)}{\Gamma\left(1+\frac{S}{4}\right)\Gamma\left(1+\frac{T}{4}\right)\Gamma\left(1-\frac{S+T}{4}\right)},
\end{aligned} \tag{5.153}$$

From here we conclude that

$$\begin{aligned}
&M_F(s,t) = M_F^{\text{SUGRA}}(s,t) \\
&+ \frac{1}{\lambda^{\frac{3}{2}} N^2} \left(60\zeta_3 + \frac{b_1}{\lambda^{\frac{1}{2}}} + \frac{315\zeta_5\left(s^2+t^2+(4-s-t)^2\right)+b_2}{\lambda}\right) + O(\lambda^{-3}).
\end{aligned} \tag{5.154}$$

where b_1 and b_2 are undetermined coefficients that cannot be fixed from the flat space limit alone,

$$M_F^{\text{SUGRA}}(s,t) = \frac{4}{N^2(s-2)(t-2)(2-s-t)}, \tag{5.155}$$

is the supergravity result[66] and λ is the 't Hooft coupling[14].

[14]More specifically λ can be written in terms of the AdS radius and string length l_s as $\lambda = \frac{R^4}{l_s^4}$.

Chapter 6

Conclusions and open questions

Conformal field theories have proven to be an interesting research topic in theoretical physics. One of the most important properties of these theories, the OPE, has been used to constrain the structure and the allowed space of CFTs. The usual approach studies the consistency of the conformal block decomposition in the s and t channels using Euclidean kinematics. While this has been very important in deriving bounds on dimensions and OPE coefficients of operators, it misses the Lorentzian kinematical region which might be as relevant as the Euclidean.

The main goal of this thesis was to study the structure of a four point function in the Regge limit. Building on previous work on the subject we were able to relate the dimension and OPE coefficients to the pomeron pole and residue, under certain assumptions. This analysis was performed using Mellin amplitudes which makes more transparent the relation to scattering amplitudes. For example in this representation the Regge limit corresponds to the usual limit $s \to \infty$ with fixed t as in scattering amplitudes.

These relations between the relevant CFT data in the Euclidean region and the Regge limit have been checked perturbatively for specific correlation functions in $\mathcal{N} = 4$ SYM. A important consequence of these relations is that in this theory they do not commute with perturbation theory, giving infinite predictions for both dimensions of operators and OPE coefficients.

We have also studied another Lorentzian limit which probes the connection between a correlation function and the flat space scattering amplitude of the gravitational dual. Of course this only makes sense when there is a gravity dual. More precisely we have used it to constraint the stringy corrections to a particular four point function in $\mathcal{N} = 4$ SYM.

There are several open questions left unanswered with this work. One issue that remains to be studied is the assumptions made in the derivation of the Regge limit for correlation functions. While these assumptions seem to be true in the specific cases we have checked, we have no proof, that this should be the case in a generic conformal field theory. In this context, there are two main directions to pursue: one is to check the consequences of our assumptions in other correlations functions and other conformal field theories, preferably with less symmetry; another point that can be done is to phrase this assumptions in terms of concrete properties that four point correlation functions have to obey, or even to phrase these in terms of properties of the dimensions and OPE coefficients of the operators.

The conformal partial wave expansion (2.14) and the Sommerfeld Watson transform played a key role in analyzing the Regge limit. It would be nice to see if the same approach could be applied in the case of energy-energy correlation functions [89, 90, 91]. By this we mean to express the energy-energy correlation in terms of CFT data.

Appendix A

Mellin amplitudes in more detail

In this appendix, we collect several results that complement the description of Mellin amplitudes of section 3.

A.1 Flat space limit

The goal of this section is analyze the flat space limit of the conformal partial wave expansion (4.34). Let us just recall the expression for the flat space limit proposed in [23] and rederived in [30] using localized wave packets

$$\mathcal{T}(p_i) = \frac{1}{\mathcal{N}} \lim_{R\to\infty} R^{2h-3} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i}\, \alpha^{h-\frac{1}{2}\sum\Delta_i}\, e^{\alpha}\, M\left(\delta_{ij} = \frac{R^2}{2\alpha}\, p_i \cdot p_j\right), \tag{A.1}$$

where the integration contour runs to the right of all poles of the integrand and [1]

$$\mathcal{N} = \frac{1}{8\pi^h} \prod_{i=1}^{4} \frac{1}{\sqrt{\Gamma(\Delta_i)\Gamma(\Delta_i - h + 1)}}\,. \tag{A.2}$$

In formula (A.1), M is the Mellin amplitude of a CFT four-point function of single-trace operators $\mathcal{O}_i$ and $\mathcal{T}$ is the scattering amplitude of the dual bulk fields ϕ_i.

A relevant example for the present paper is the tree-level exchange of a spin J and mass m particle. In flat space, this gives rise to

$$\mathcal{T} = g^2\, T^{J-1} f\left(\frac{S}{T}, \frac{m^2}{T}\right), \tag{A.3}$$

where S and T are the Mandelstam invariants, g is a dimensionful coupling constant and f is a dimensionless function. Then, formula (A.1) tells us that the Mellin amplitude associated to the tree-level exchange of a spin J and dimension Δ field in AdS has the following asymptotic

[1] This normalization differs from [23] because here we are using operators normalized to have unit two-point function (2.3).

behavior

$$M(s,t) = g^2 R^{5-2h-2J}\, t^{J-1} \widetilde{f}\left(\frac{s}{t}, \frac{\Delta^2}{t}\right) + O(t^{J-2})\,, \tag{A.4}$$

with [2]

$$f\,(x,y) = \frac{2^{1-J}}{\mathcal{N}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i}\, \alpha^{h-\frac{1}{2}\sum\Delta_i - J+1}\, e^{\alpha}\, \widetilde{f}(x, 2y\alpha)\,. \tag{A.5}$$

A.1.1 Flat space limit of conformal partial wave expansion

The goal of this section is to study the flat space limit (A.1) of the conformal partial wave expansion (4.34),

$$\mathcal{T}(p_i) = \frac{1}{\mathcal{N}} \sum_{J=0}^{\infty} \lim_{R\to\infty} R^{2h-3} \int_{-\infty}^{\infty} d\nu\, b_J(\nu^2) \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i}\, \alpha^{h-\frac{1}{2}\sum\Delta_i}\, e^{\alpha}\, M_{\nu,J}\left(\frac{R^2 S}{2\alpha}, \frac{R^2 T}{2\alpha}\right). \tag{A.6}$$

In order to compute the large R limit of the integral it would be useful to know what is the integration region in ν and α that dominates the integral for large R. We shall start by assuming that the integral is dominated by $\nu^2 \gg 1$ and later check that this is indeed the case. Using the Stirling expansion of the Γ-function we find

$$\omega_{\nu,J}(t) \approx \frac{1}{\sqrt{\nu^2}} \left(\frac{-\nu^2}{2t}\right)^{\frac{1}{2}\sum\Delta_i + J - h} \exp\left\{\nu \arctan\left(\frac{\nu}{t}\right) - \frac{t}{2}\log\left(1+\frac{\nu^2}{t^2}\right)\right\}, \tag{A.7}$$

where we are assuming $-t = -\frac{R^2 T}{2\alpha} \gg |\nu| \gg 1$. In appendix A.3 we consider the limit $|t| \sim |s| \gg |\nu| \gg 1$ of the Mack polynomials, and obtain

$$P_{\nu,J}(s,t) \approx \left(\frac{t}{2}\right)^J P_J(z)\,, \tag{A.8}$$

where $P_J(z)$ are the partial waves in $(2h+1)$-dimensional flat spacetime and $z = 1 + \frac{2s}{t}$. Using these two approximations (A.6) becomes

$$\begin{aligned} \mathcal{T}(p_i) = \frac{1}{\mathcal{N}} \sum_{J=0}^{\infty} P_J(z) \lim_{R\to\infty} R^{2h-3} \left(\frac{R^2 T}{4}\right)^J \int_{-\infty}^{\infty} d\nu\, b_J(\nu^2)\, \frac{1}{\sqrt{\nu^2}} \left(\frac{-\nu^2}{R^2 T}\right)^{\frac{1}{2}\sum\Delta_i + J - h} \\ \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \exp\left\{\alpha + \nu \arctan\left(\frac{2\alpha\nu}{R^2 T}\right) - \frac{R^2 T}{4\alpha} \log\left(1 + \left(\frac{2\alpha\nu}{R^2 T}\right)^2\right)\right\}, \end{aligned} \tag{A.9}$$

[2] If the particle is massless in flat space ($m=0$) then the relation between f and $\widetilde{f}$ is very simple

$$f\,(x,0) = \frac{2^{1-J}}{\mathcal{N}\,\Gamma\left(\frac{1}{2}\sum\Delta_i - h + J - 1\right)}\, \widetilde{f}(x,0)\,.$$

where now $z = \cos\theta = 1 + \frac{2S}{T}$ encodes the flat space scattering angle. Let us discuss the integral (A.9). If we expand the exponent at large R, we obtain

$$\int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \exp\left\{\alpha\left(1+\frac{\nu^2}{R^2T}\right) - \frac{2\alpha^3\nu^4}{3R^6T^3} + O\left(\frac{1}{R^{10}}\right)\right\}. \tag{A.10}$$

Keeping only the first term in this exponential, the integral over α gives rise to a delta-function $\delta\left(1+\frac{\nu^2}{R^2T}\right)$. This justifies the initial assumption of large ν^2. However, one must be careful because the delta-function follows from taking the integrand to be a plane wave in α, for all values of α. This is clearly wrong since for $\alpha \sim R^2T/\nu^{4/3}$ the second term in the exponent becomes of order 1. In fact, we can perform the integral over α keeping only the first two terms in the exponent, obtaining

$$\frac{R^2T}{(2\nu^4)^{1/3}}\,\mathrm{Ai}\left(\frac{\nu^2+R^2T}{(2\nu^4)^{1/3}}\right), \tag{A.11}$$

where Ai is the Airy function. This expression means that the integral over ν is dominated by the region $\nu^2 \sim -R^2T \pm \nu^{4/3} \sim -R^2T \pm (-R^2T)^{2/3}$. At large R, both the mean value of ν^2 and the width of the region are large, but the mean is much larger than the width. Including higher order corrections in (A.10), leads to corrections to the function of ν^2 (A.11) in smaller scales than $(-R^2T)^{2/3}$ but still much larger than 1. Therefore, we conclude that the flat space limit of the conformal partial wave expansion gives the standard partial wave expansion,

$$\mathcal{T} = \sum_{J=0}^{\infty} P_J(z)\, a_J(T)\,, \tag{A.12}$$

with the flat space partial amplitudes given by the limit

$$a_J(T) = \frac{1}{\mathcal{N}} \lim_{R\to\infty} R^{2h-3}\left(\frac{R^2T}{4}\right)^J \langle b_J\rangle_T\,, \tag{A.13}$$

where

$$\langle b_J\rangle_T = \int dx\, \delta_L(x)\, b_J\big(-R^2T + x\big) \tag{A.14}$$

is an averaging of the conformal partial amplitudes $b_J(\nu^2)$ around $\nu^2 = -R^2T$ with a function $\delta_L(x)$ which is a regulated delta-function with characteristic width $L = (-R^2T)^{2/3}$.

The flat space limit of conformal blocks in Mellin space was first studied in [30]. The main novelty of our result is the averaging (A.14). In our example this averaging is not necessary since the functions vary slowly in ν. Thus, the δ_L can be replaced by a usual delta function.

A.2 Example: Witten diagrams

Consider the Witten diagram in figure A.1a associated with the exchange of a dimension Δ and spin J field in AdS. The OPE expansion of the corresponding four-point function in the (12)(34) channel contains double-trace operators and the single-trace operator dual to the exchanged

(a) (b)

Figure A.1: Witten diagrams of (a) exchange of a dimension Δ and spin J field in AdS and (b) contact interaction.

field in AdS [92, 93, 94]. The OPE expansion in the other channels only contains double-trace operators. This means that the only poles of the associated Mellin amplitude are given by equation (3.22). In addition, we know from the flat space limit analysis of the previous section that this Mellin amplitude is polynomially bounded at large values of s and t. Thus, we conclude that it can be written as a sum of poles plus an analytic piece, which is a polynomial $\mathcal{R}_{J-1}$ of degree $J-1$ in both variables s and t,

$$M(s,t) = C_{12k}C_{34k}\sum_{m=0}^{\infty}\frac{\mathcal{Q}_{J,m}(s)}{t-\Delta+J-2m} + \mathcal{R}_{J-1}(s,t)\,. \tag{A.15}$$

Let us see if this results agrees with the expectations from the bulk point of view. To compute the Witten diagram in figure A.1a we need to know what is the precise form of the cubic vertices. However, there is a unique cubic vertex between 2 scalar fields and a spin J field if we are allowed to use the equations of motion. This is directly related to the fact that there is a unique conformal three-point function between 2 scalar operators and a spin J operator (see [17] for a more complete discussion of this correspondence). On the other hand, the internal line of the diagram A.1a is not on-shell and, therefore, the equations of motion will not give zero, but will transform the internal propagator into a delta-function. This means that different cubic vertices will produce correlation functions that differ by contact diagrams like the one in figure A.1b. In fact, it is not hard to convince ourselves that this contact diagrams can have at most $2J-2$ derivatives. As explained in [23], this implies that the associated Mellin amplitude is a polynomial of degree $J-1$. Thus, the result (A.15) is exactly what one expects from the bulk point of view. The Mellin amplitude contains a polynomial $\mathcal{R}_{J-1}$ that encodes the precise choice of cubic couplings, and a sum of poles completely fixed by the OPE coefficients of the exchanged operator $\mathcal{O}_k$ in the OPEs of $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$. Similar arguments were recently given in [95].

Regge limit

Something nice happens in the Regge limit of large s and fixed t. Firstly, the non-universal part $\mathcal{R}_{J-1}$ of the Mellin amplitude (A.15) drops out. Secondly, the polynomials $Q_{J,m}(s)$, introduced in (3.23), can be replaced by their asymptotic. This gives

$$M(s,t) \approx C_{12k}C_{34k}\, f(t)\, s^J\,, \tag{A.16}$$

with

$$\begin{aligned} f(t) = -\, & \frac{2\Gamma(\Delta+J)(\Delta-1)_J}{4^J\Gamma\big(\frac{\Delta+J+\Delta_{12}}{2}\big)\,\Gamma\big(\frac{\Delta+J-\Delta_{12}}{2}\big)\,\Gamma\big(\frac{\Delta+J+\Delta_{34}}{2}\big)\,\Gamma\big(\frac{\Delta+J-\Delta_{34}}{2}\big)} \\ & \sum_{m=0}^{\infty} \frac{1}{m!(\Delta-h+1)_m\,\Gamma\big(\frac{\Delta_1+\Delta_2-\Delta+J-2m}{2}\big)\,\Gamma\big(\frac{\Delta_3+\Delta_4-\Delta+J-2m}{2}\big)\,(t-\Delta+J-2m)}\,. \end{aligned} \tag{A.17}$$

Fortunately, this sum has a nice integral representation [23]

$$f(t) = K_{\Delta,J}\int d\nu\, \frac{\omega_{\nu,J}(t)}{(\Delta-h)^2+\nu^2}\ , \tag{A.18}$$

where $\omega_{\nu,J}(t)$ is given in (4.36) and the normalization constant $K_{\Delta,J}$ is given in (2.18).

In the Regge limit, it is striking how similar is the behavior of the Mellin amplitude (A.16) for an exchange of a spin J field in AdS, and the corresponding flat space scattering amplitude. Both grow as s^J (or S^J) times a function of t (or T). In the integral representation (A.18) the infinite sequence of poles in t is generated by a single pole in ν^2. Indeed, this pole is the best analogue to the unique pole in T of the flat space scattering amplitude.

A.2.1 Double trace operators

Let us briefly remark how double trace operators, that also appear in the conformal block decomposition (2.5), are generated in the conformal partial wave expansion (4.34). The double traces are generated by poles of $\kappa_{\nu,J}$ at

$$i\nu = \Delta_1+\Delta_2+J+2m-h\,, \qquad m=0,1,2,\dots\,, \tag{A.19}$$

$$i\nu = \Delta_3+\Delta_4+J+2m-h\,, \qquad m=0,1,2,\dots\,. \tag{A.20}$$

The product of the OPE coefficients of a double-trace operator $\widetilde{\mathcal{O}}_k \sim \mathcal{O}_1\partial_{\mu_1}\dots\partial_{\mu_J}\partial^{2m}\mathcal{O}_2$, of dimension $\widetilde{\Delta}_k = \Delta_1+\Delta_2+\widetilde{J}+2m$ and spin $\widetilde{J}$, in the OPE $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$, is given by

$$\widetilde{C}_{12k}\widetilde{C}_{34k} = \frac{4^J\Gamma\left(\frac{\widetilde{J}+\Delta_3+\Delta_4-\widetilde{\Delta}_k}{2}\right)\Gamma\left(\frac{-2h+\widetilde{J}+\Delta_3+\Delta_4+\widetilde{\Delta}_k}{2}\right)\Gamma\left(\frac{\widetilde{J}-\Delta_{34}+\widetilde{\Delta}_k}{2}\right)\Gamma\left(\frac{\widetilde{J}+\Delta_{34}+\widetilde{\Delta}_k}{2}\right)}{m!\Gamma\left(\widetilde{J}+\widetilde{\Delta}_k\right)\left(h-\widetilde{\Delta}_k+1\right)_m\left(\widetilde{\Delta}_k-1\right)_{\widetilde{J}}}$$
$$\Gamma\left(\widetilde{J}+m+\Delta_1\right)\Gamma\left(\widetilde{J}+m+\Delta_2\right)b_{\widetilde{J}}\left(-\left(\widetilde{\Delta}_k-h\right)^2\right). \tag{A.21}$$

To illustrate the use of this formula consider the correlator associated to the Witten diagram of figure A.1a describing the exchange of a spin J and dimension Δ field in AdS. In the previous section, we concluded that the corresponding Mellin amplitude was a polynomial of degree J in the variable s. This implies that the conformal partial wave expansion (4.34) obeys $b_{J'}(\nu^2)=0$ for $J'>J$. Thus, the Regge limit of (4.34) is simply given by

$$M(s,t) \approx s^J \int_{-\infty}^{\infty} d\nu\, b_J(\nu^2)\,\omega_{\nu,J}(t)\,. \tag{A.22}$$

Comparing with the results (A.16) and (A.18) for the Regge limit of the Witten diagram of figure A.1a, we conclude that

$$b_J(\nu^2) = C_{12k}C_{34k}\,\frac{K_{\Delta,J}}{\nu^2+(\Delta-h)^2}\,, \tag{A.23}$$

where $C_{12k}C_{34k}$ is the product of the OPE coefficients of the operator dual to the field exchanged in AdS. Notice that in this case, the partial amplitude $b_J(\nu^2)$ is exactly given by the sum of the two simple poles predicted in (4.39). The partial amplitudes $b_{J'}(\nu^2)$ for $J'<J$ are more complicated and are not determined by the Regge limit. It is then trivial to use $b_J(\nu^2)$ given by (A.23) in (A.21), to immediately obtain the OPE coefficients of the double trace operators of maximal spin $\widetilde{J}=J$ produced by the Witten diagram of figure A.1a.

A.2.2 Analytic structure of partial amplitudes

The pole structure of the partial amplitudes $b_J(\nu)$ is directly related to the spectrum of single-trace operators that appear in both OPEs $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$. The mechanism is the following: the poles (3.22) of the Mellin amplitude arise from the integral over ν in (4.34) when the integration contour is pinched between two poles of the integrand as depicted in figure A.2. The partial wave $M_{\nu,J}(s,t)$ introduced in (4.35) has the following poles in the variable ν,

$$\pm i\nu = \Delta_1+\Delta_2+J-h+2m\,, \qquad m=0,1,2,\dots\,, \tag{A.24}$$
$$\pm i\nu = \Delta_3+\Delta_4+J-h+2m\,, \qquad m=0,1,2,\dots\,, \tag{A.25}$$
$$\pm i\nu = h-J-t+2m\,, \qquad m=0,1,2,\dots\,, \tag{A.26}$$
$$\pm i\nu = h-1+J-q\,, \qquad q=1,2,\dots,J\,, \tag{A.27}$$

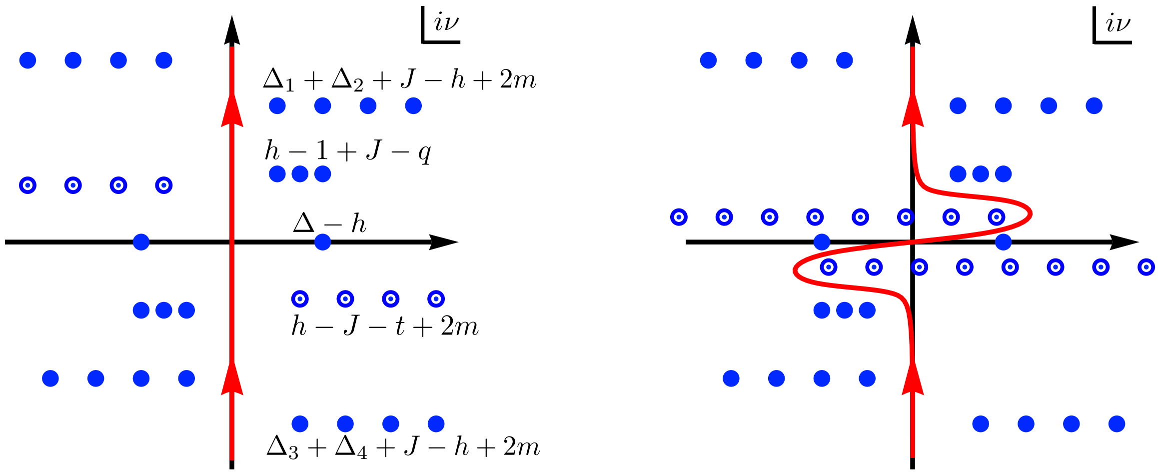


Figure A.2: Integration contour in the $i\nu$ complex plane used in the conformal partial wave expansion (4.34). The blue dots represent poles of the integrand given by (A.24-A.27) and (A.28). In order to make the figure readable, we have complexified several parameters to separate the poles better. The poles (A.26) that depend on t are marked with a small dot enclosed by a circle. As t varies these poles move and can collide with other poles pinching the integration contour, as shown on the right panel. This is the mechanism that generates the poles (3.22) of the Mellin amplitude.

where the first three sets of poles come from the function $\omega_{\nu,J}(t)$ defined in (4.36), and the last line are poles of the polynomial $P_{\nu,J}(s,t)$ defined in appendix A.3. In order to obtain poles in the variable t from the integral over ν in (4.34), a pole from (A.26) must collide with another ν pole. In fact, in order to reproduce the poles (3.22) with the correct residue one needs the partial amplitudes $b_J(\nu^2)$ to have the following pair of ν poles

$$b_J(\nu^2) \approx C_{12k}C_{34k}\,\frac{K_{\Delta,J}}{\nu^2+(\Delta-h)^2}\,, \tag{A.28}$$

where the normalization constant $K_{\Delta,J}$ is given in (2.18). When t approaches $\Delta-J+2m$ with $m=0,1,2,\dots$, two poles from (A.26) collide with the two poles (A.28), pinching the ν-contour in (4.34) and producing a divergent integral (see figure A.2). To check that the resulting poles in t of the Mellin amplitude have the correct residues it is sufficient to keep the contribution from the poles (A.28) to the integral (4.34) using the Cauchy theorem,

$$\begin{aligned} M(s,\Delta-J+2m+\delta t) &\approx C_{12k}C_{34k}\,K_{\Delta,J}\,\frac{2\pi}{\Delta-h}\,M_{i(\Delta-h),J}(s,\Delta-J+2m+\delta t) \\ &\approx C_{12k}C_{34k}\,\frac{\mathcal{Q}_{J,m}(s)}{\delta t}\,, \end{aligned} \tag{A.29}$$

in perfect agreement with (3.22). To obtain this result it was crucial to use the property (A.45) of the Mack polynomials. We conclude that for every single-trace operator that appears in both OPEs $\mathcal{O}_1\mathcal{O}_2$ and $\mathcal{O}_3\mathcal{O}_4$, the partial amplitudes $b_J(\nu^2)$ will have a pair of poles of the form (A.28).

Unfortunately, the story is slightly more complicated and cumbersome because $b_J(\nu^2)$ has other (spurious) poles that do not correspond to any operators appearing in the OPEs. To explain this let us systematically analyze all possible contour pinchings in (4.34) that can give rise to poles in t. Suppose a pole from (A.26) collides with a pole from (A.24). This would give rise to a pole at $t = \Delta_1 + \Delta_2 + 2m$ for $m = 0, 1, 2, \dots$. However, this pole is cancelled by a zero of $\omega_{\nu,J}(t)$ produced by the last Γ-functions in the denominator of (4.36). A similar statement applies to collision with the poles (A.25). Another possibility is the collision of 2 poles of the form (A.26) themselves. This happens when $J+t-h$ is a non-negative integer, which means that the colliding poles are located at integer values of $i\nu$. Thus, this collision also does not generate poles in t because the function $\omega_{\nu,J}(t)$ has zeros at integer values of $i\nu$. The final possibility is for the poles (A.26) to collide with the poles (A.27) of the Mack polynomials. Let us focus on the contribution of one the poles (A.27) for a fixed value of J and q. This gives rise to a series of poles in t of the form (3.22) with dimension Δ', spin J' and OPE coefficients $C'_{12k}C'_{34k}$ given by

$$\Delta' = 2h - 1 + J\,, \tag{A.30}$$

$$J' = J - q\,, \tag{A.31}$$

$$C'_{12k}C'_{34k} = \frac{\mathcal{Z}_{J,q}}{K_{\Delta',J'}}\, b_J\big(-(h-1+J-q)^2\big)\,, \tag{A.32}$$

where $K_{\Delta,J}$ is given in (2.18) and

$$\mathcal{Z}_{J,q} = \frac{J!}{(J-q)!q!}\frac{2(-2)^q\left(\frac{\Delta_1+\Delta_2+1-2h-q}{2}\right)_q\left(\frac{\Delta_3+\Delta_4+1-2h-q}{2}\right)_q\left(\frac{\Delta_{12}+1-q}{2}\right)_q\left(\frac{\Delta_{34}+1-q}{2}\right)_q}{\Gamma(q)(h+J-q)_{q-1}}\,. \tag{A.33}$$

In order to derive this result we used the property (A.48) of the Mack polynomials given in appendix A.3. This result looks strange because it says that the OPE will generically contain primary operators of dimension Δ' (which is an integer or half-integer). This can not be the case. In fact, what happens is that the partial amplitudes $b_{J'}(\nu^2)$, with $J' = J - q$ have other poles that cancel this effect. This requirement fixes the new residues to be

$$b_{J-q}(\nu^2) \approx -\frac{\mathcal{Z}_{J,q}\, b_J\big(-(h-1+J-q)^2\big)}{\nu^2+(h-1+J)^2}\,, \qquad q = 1, 2, 3, \dots J. \tag{A.34}$$

These poles were termed spurious poles in [16]. We believe that the relation (A.34) is the translation to our language of the identity (2.59b) of [36], which discusses a similar conformal partial wave expansion (although in position space).

A.3 Mack polynomials

With our normalizations, the polynomials introduced in [21] can be written as

$$P_{\nu,J}(s,t) = \sum_{r=0}^{[J/2]} a_{J,r} \frac{2^{J+2r} \left(\frac{h+i\nu-J-t}{2}\right)_r \left(\frac{h-i\nu-J-t}{2}\right)_r (J-2r)!}{(h+i\nu-1)_J\,(h-i\nu-1)_J}$$
$$\sum_{\sum k_{ij}=J-2r} (-1)^{k_{13}+k_{24}} \prod_{(ij)} \frac{(\delta_{ij})_{k_{ij}}}{k_{ij}!} \prod_{n=1}^{4} (\alpha_n)_{J-r-\sum_j k_{jn}} \,. \tag{A.35}$$

In this expression the labels (ij) run over the 4 possibilities (13), (14), (23) and (24). The variables δ_{ij} are as before,

$$\delta_{13} = \frac{\Delta_{34}-s}{2}\,, \quad \delta_{24} = -\frac{\Delta_{12}+s}{2}\,, \quad \delta_{23} = \frac{t+s}{2}\,, \quad \delta_{14} = \frac{t+s+\Delta_{12}-\Delta_{34}}{2}\,. \tag{A.36}$$

The variables α_n are given by

$$\begin{aligned} \alpha_1 &= 1 - \frac{h+i\nu-J+\Delta_{12}}{2}\,, \qquad & \alpha_2 &= 1 - \frac{h+i\nu-J-\Delta_{12}}{2}\,, \\ \alpha_3 &= 1 - \frac{h-i\nu-J+\Delta_{34}}{2}\,, \qquad & \alpha_4 &= 1 - \frac{h-i\nu-J-\Delta_{34}}{2}\,. \end{aligned} \tag{A.37}$$

The coefficients $a_{J,r}$ define the flat $(2h+1)$-dimensional spacetime partial waves

$$P_J(z) = \sum_{r=0}^{[J/2]} a_{J,r}\, z^{J-2r}\,, \qquad a_{J,r} = (-1)^r \frac{J!(h+J-1)_{-r}}{2^{2r} r!(J-2r)!}\,. \tag{A.38}$$

It is clear from the definition (A.35) that $P_{\nu,J}(s,t)$ is indeed a polynomial of degree J in both variables t and s. Let us check that the leading term is s^J, as stated in the main text. This must come from the $r=0$ term in the sum (A.35),

$$P_{\nu,J}(s,t) \approx \frac{s^J J!}{(h+i\nu-1)_J\,(h-i\nu-1)_J} \sum_{\sum k_{ij}=J} \prod_{(ij)} \frac{1}{k_{ij}!} \prod_{n=1}^{4} (\alpha_n)_{J-\sum_j k_{jn}}\,, \tag{A.39}$$

where we have kept only the leading term in s in the Pochhammer symbols $(\delta_{ij})_{k_{ij}}$. To perform the last sum we change to the variables $q_1 = J - k_{13} - k_{14}$ and $q_3 = J - k_{13} - k_{23}$. Then the sum over k_{ij} in (A.39) can be written as

$$\sum_{q_1=0}^{J} \sum_{q_3=0}^{J} \sum_{k_{13}=0}^{J-q_1} \frac{(\alpha_1)_{q_1}(\alpha_2)_{J-q_1}(\alpha_3)_{q_3}(\alpha_4)_{J-q_3}}{k_{13}!\,(J-q_1-k_{13})!\,(J-q_3-k_{13})!\,(q_1+q_3-J+k_{13})!}$$
$$= J! \sum_{q_1=0}^{J} \frac{(\alpha_1)_{q_1}(\alpha_2)_{J-q_1}}{q_1!\,(J-q_1)!} \sum_{q_3=0}^{J} \frac{(\alpha_3)_{q_3}(\alpha_4)_{J-q_3}}{(q_3)!\,(J-q_3)!} = \frac{(\alpha_1+\alpha_2)_J\,(\alpha_3+\alpha_4)_J}{J!}\,. \tag{A.40}$$

Using the definitions (A.37) it follows that $P_{\nu,J}(s,t) \approx s^J$.

There are several symmetry properties that follow from the formula (A.35) by relabelling the summation variables,

$$k_{13} \leftrightarrow k_{24} \quad \Rightarrow \quad P_{-\nu,J}\big(s,t,\Delta_{12},\Delta_{34}\big) = P_{\nu,J}\big(s,t,-\Delta_{34},-\Delta_{12}\big) \tag{A.41}$$

$$k_{14} \leftrightarrow k_{23} \quad \Rightarrow \quad P_{-\nu,J}\big(s,t,\Delta_{12},\Delta_{34}\big) = P_{\nu,J}\big(s+\Delta_{12}-\Delta_{34},t,\Delta_{34},\Delta_{12}\big) \tag{A.42}$$

and

$$\begin{cases} k_{13} \leftrightarrow k_{14} \\ k_{23} \leftrightarrow k_{24} \end{cases} \Rightarrow \quad P_{\nu,J}\big(s,t,\Delta_{12},-\Delta_{34}\big) = (-1)^J P_{\nu,J}\big(-s-t-\Delta_{12},t,\Delta_{12},\Delta_{34}\big) \tag{A.43}$$

$$\begin{cases} k_{13} \leftrightarrow k_{23} \\ k_{14} \leftrightarrow k_{24} \end{cases} \Rightarrow \quad P_{\nu,J}\big(s,t,-\Delta_{12},\Delta_{34}\big) = (-1)^J P_{\nu,J}\big(-s-t+\Delta_{34},t,\Delta_{12},\Delta_{34}\big) \tag{A.44}$$

In fact, there is a more basic invariance, $P_{-\nu,J}(s,t) = P_{\nu,J}(s,t)$, which is not obvious from the definition (A.35). These symmetries were first discussed in [12].

Another important property is that the Mack polynomials, at specific values of t, reduce to the polynomials $Q_{J,m}(s)$ that control the OPE as explained in chapter 3,

$$P_{i(\Delta-h),J}(s,\Delta-J+2m) = Q_{J,m}(s)\,. \tag{A.45}$$

Consider now the limit $t \sim s \gg 1$ and $\nu^2 \gg 1$. In equation (A.35), it is sufficient to replace the Pochhammer symbols $(x)_n$ of a large quantity x by the leading term x^n. This gives

$$\begin{aligned} P_{\nu,J}(s,t) &\approx \sum_{r=0}^{[J/2]} a_{J,r}\, 4^{r-J} \left(t^2+\nu^2\right)^r (J-2r)! \sum_{\sum k_{ij}=J-2r} \frac{s^{k_{13}+k_{24}}(t+s)^{k_{14}+k_{23}}}{k_{13}!\,k_{24}!\,k_{14}!\,k_{23}!} \\ &= \sum_{r=0}^{[J/2]} a_{J,r}\, 2^{-J} \left(t^2+\nu^2\right)^r \sum_{q=0}^{J-2r} \frac{(J-2r)!}{q!\,(J-2r-q)!}\, s^q (t+s)^{J-2r-q} \\ &= \left(\frac{t^2+\nu^2}{4}\right)^{\frac{J}{2}} \sum_{r=0}^{[J/2]} a_{J,r} \left(\frac{t+2s}{\sqrt{t^2+\nu^2}}\right)^{J-2r} = \left(\frac{t^2+\nu^2}{4}\right)^{\frac{J}{2}} P_J\!\left(\frac{t+2s}{\sqrt{t^2+\nu^2}}\right). \end{aligned} \tag{A.46}$$

If we further assume $|t| \gg |\nu|$, we obtain

$$P_{\nu,J}(s,t) \approx \left(\frac{t}{2}\right)^J P_J(z)\,, \tag{A.47}$$

where $z = 1 + 2s/t$. This limit was first studied in [30].

The definition (A.35) also makes it clear that $P_{\nu,J}(s,t)$ is polynomial in the parameters Δ_{12} and Δ_{34}. On the other hand, we see that $P_{\nu,J}(s,t)$ has poles at $\nu = \pm i(h+J-q-1)$ for

$q = 1, 2, \ldots, J$. We checked that the residues of these poles are described by the formula,

$$P_{\nu,J}(s,t) \approx \frac{2^q J! \left(\frac{\Delta_{12}-q+1}{2}\right)_q \left(\frac{\Delta_{34}-q+1}{2}\right)_q \left(\frac{2h-t-q+1}{2}\right)_q}{(h+J-q-1\pm i\nu)\, q!\, (J-q)!\, \Gamma(q)\, (h-1+J-q)_q} P_{i(h+J-1),J-q}(s,t)\,, \tag{A.48}$$

for all $J \leq 8$. We believe that this is an identity, but were unable to prove it.

A.4 Regge limit in position space

This appendix has two goals. The first one is to show that our definition of the Regge limit of the Mellin amplitude (large s and fixed t) corresponds to the Regge limit defined in position space in [16, 37]. This limit can be defined by $x_1^+ \to \lambda x_1^+$, $x_2^+ \to \lambda x_2^+$, $x_3^- \to \lambda x_3^-$, $x_4^- \to \lambda x_4^-$ and $\lambda \to \infty$, keeping the causal relations $x_{14}^2, x_{23}^2 < 0$ and all the other $x_{ij}^2 > 0$. This is depicted in figure 2.1. [3] We remark that by Fourier transforming to momentum space the position of the operators x_i, and defining the corresponding Mandelstam invariants, the Regge limit is just the usual Regge limit of large s and fixed t. The second goal of this appendix is to derive an expression for the position space correlator in the Regge limit corresponding to our main equation (4.49).

Let us then start by the definition of the Mellin representation for the time ordered Lorentzian correlation function [21]

$$A(x_i) = \int [d\delta]\, M(\delta_{ij}) \prod_{i<j} \Gamma(\delta_{ij}) \big(x_{ij}^2 + i\epsilon\big)^{-\delta_{ij}}\,. \tag{A.49}$$

Given the chosen causal relations for the Regge limit, we should rotate

$$v^{-(s+t)/2} \to v^{-(s+t)/2} e^{-i\pi(s+t)} \tag{A.50}$$

in the integral (3.21). With this phase, the convergence of the integral is not obvious when $s = ix$, $x \to +\infty$. To study this question, we approximate the second line of (3.21) using

$$\Gamma\Big(a + i\,\frac{x}{2}\Big)\, \Gamma\Big(b - i\,\frac{x}{2}\Big) \approx 2\pi\, e^{i\frac{\pi}{2}(a-b)} \left(\frac{x}{2}\right)^{a+b-1} e^{-\frac{\pi}{2}x}\,, \qquad x \to +\infty\,. \tag{A.51}$$

This gives

$$\mathcal{A}(u,v) \approx \int_{-i\infty}^{i\infty} \frac{dt}{4i}\, u^{t/2} v^{-t/2}\, \Gamma\Big(\frac{2\Delta_1 - t}{2}\Big)\, \Gamma\Big(\frac{2\Delta_3 - t}{2}\Big)\, e^{-i\pi\frac{t}{2}} \int^{\infty} dx\, M(ix,t) \left(\frac{x}{2}\right)^{t-2} v^{-\frac{i}{2}x}\,, \tag{A.52}$$

where we have restricted to the case $\Delta_{12} = \Delta_{34} = 0$. Following [16, 37], we introduce the

[3] Notice that in this paper we are labelling points differently from [16, 37]. The translation is simply the permutation $x_2 \leftrightarrow x_3$. The reason for a different notation is to follow the dominant convention in the OPE literature $\mathcal{O}_1\mathcal{O}_2 \sim \mathcal{O}_{\Delta,J}$.

variables σ and ρ via

$$u = \sigma^2\,, \qquad v = (1-\sigma e^{\rho})(1-\sigma e^{-\rho}) \approx 1 - 2\sigma\cosh\rho\,, \tag{A.53}$$

such that the Regge limit corresponds to $\sigma \to 0$ with fixed ρ. In this limit,

$$\mathcal{A} \approx \int_{-i\infty}^{i\infty} \frac{dt}{4i}\,\sigma^t\,\Gamma\!\left(\frac{2\Delta_1 - t}{2}\right)\Gamma\!\left(\frac{2\Delta_3 - t}{2}\right) e^{-i\pi\frac{t}{2}} \int^{\infty} dx\, M(ix,t)\left(\frac{x}{2}\right)^{t-2} e^{ix\sigma\cosh\rho}\,, \tag{A.54}$$

which shows that the small σ behavior of $\mathcal{A}$ is controlled by the large s behavior of the Mellin amplitude. We can now use our main result (4.49) to write

$$M(ix,t) \approx \int d\nu\,\beta(\nu)\,\frac{x^{j(\nu)}}{\sin\!\left(\frac{\pi j(\nu)}{2}\right)}\,\omega_{\nu,j(\nu)}(t)\,, \tag{A.55}$$

where we have chosen the appropriate phase for $s = ix$, $x \to +\infty$. After performing the integral over x, we find

$$\begin{aligned}\mathcal{A} \approx -\pi i \int d\nu\,\beta(\nu)\,\frac{e^{i\pi j(\nu)/2}}{\sin\!\left(\frac{\pi j(\nu)}{2}\right)}\,\sigma^{1-j(\nu)}2^{j(\nu)} \\ \int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\,\Gamma\!\left(\frac{2\Delta_1 - t}{2}\right)\Gamma\!\left(\frac{2\Delta_3 - t}{2}\right)\frac{\Gamma\big(j(\nu)+t-1\big)}{(2\cosh\rho)^{j(\nu)+t-1}}\,\omega_{\nu,j(\nu)}(t)\,.\end{aligned} \tag{A.56}$$

Finally, using the following integral representation for the harmonic functions $\Omega_{i\nu}(\rho)$ on $(2h-1)$-dimensional hyperbolic space

$$\begin{aligned}\Omega_{i\nu}(\rho) &= \int \frac{dz}{2\pi i}\,\frac{\Gamma(z)\,\Gamma\big(\frac{h+i\nu-z-1}{2}\big)\,\Gamma\big(\frac{h-i\nu-z-1}{2}\big)}{8\pi^h\Gamma(i\nu)\Gamma(-i\nu)}\,(2\cosh\rho)^{-z} \\ &= \frac{\nu\sinh(\pi\nu)\Gamma(h-1+i\nu)\Gamma(h-1-i\nu)\,{}_2F_1\big(h-1+i\nu,h-1-i\nu,h-\frac{1}{2},-\sinh^2\big(\frac{\rho}{2}\big)\big)}{2^{2h-1}\pi^{h+\frac{1}{2}}\Gamma\big(h-\frac{1}{2}\big)}\end{aligned} \tag{A.57}$$

we recover the general Regge behavior in position space written in (4.52), with residue given by (4.53).

Appendix B

Four point function in $\mathcal{N} = 4$ SYM

B.1 SO(6) projectors

In this section we collect the $SO(6)$ projectors for the cases of interest in the main text. The construction of these projectors is given in [70, 52].

k=2

The $SO(6)$ projectors used in the main text were constructed in [52]. In our notation, they read

$$
\begin{aligned}
P_1 &= \frac{1}{20} y_{12}^4 y_{34}^4 \,, && \text{(B.1)}\\
P_{15} &= \frac{1}{4} y_{12}^2 y_{34}^2 \left(y_{24}^2 y_{13}^2 - y_{23}^2 y_{14}^2\right) , && \text{(B.2)}\\
P_{20} &= \frac{1}{10} y_{12}^2 y_{34}^2 \left(3 y_{24}^2 y_{13}^2 + 3 y_{23}^2 y_{14}^2 - y_{12}^2 y_{34}^2\right) , && \text{(B.3)}\\
P_{84} &= \frac{1}{3}\left(y_{13}^4 y_{24}^4 + y_{23}^4 y_{14}^4\right) + \frac{1}{30} y_{12}^4 y_{34}^4 - \frac{2}{3} y_{13}^2 y_{32}^2 y_{24}^2 y_{41}^2 - \frac{1}{6} y_{12}^2 y_{34}^2 \left(y_{24}^2 y_{13}^2 + y_{23}^2 y_{14}^2\right) , && \text{(B.4)}\\
P_{105} &= \frac{1}{6}\left(y_{13}^4 y_{24}^4 + y_{23}^4 y_{14}^4\right) + \frac{1}{60} y_{12}^4 y_{34}^4 + \frac{2}{3} y_{13}^2 y_{32}^2 y_{24}^2 y_{41}^2 - \frac{2}{15} y_{12}^2 y_{34}^2 \left(y_{24}^2 y_{13}^2 + y_{23}^2 y_{14}^2\right) , && \text{(B.5)}\\
P_{175} &= \frac{1}{2}\left(y_{13}^4 y_{24}^4 - y_{23}^4 y_{24}^4\right) - \frac{1}{4} y_{12}^2 y_{34}^2 \left(y_{24}^2 y_{13}^2 - y_{23}^2 y_{14}^2\right) . && \text{(B.6)}
\end{aligned}
$$

Comparing with (5.9), we conclude that

$$
\begin{aligned}
A_1 &= 1 + \frac{u^2(1+v^2)}{20v^2} + \frac{u(u+10(v+1))}{15v(N^2-1)} + \frac{2u\left(u^2 - 8u(v+1) + 10(v(v+4)+1)\right)F(u,v)}{15v(N^2-1)} \,, \\
A_{15} &= \frac{u^2(v^2-1)}{20v^2} - \frac{2u(1-v)}{5v(N^2-1)} - \frac{2u(v-1)(u-2(v+1))F(u,v)}{5v(N^2-1)} \,, \\
A_{20} &= \frac{u^2(1+v^2)}{20v^2} + \frac{u(u+10(v+1))}{30v(N^2-1)} + \frac{u\left(u^2 - 5u(v+1) + 10(v-1)^2\right)F(u,v)}{15v(N^2-1)} \,, && \text{(B.7)}
\end{aligned}
$$

(B.8)

$$A_{84} = \frac{u^2(1+v^2)}{20v^2} - \frac{u^2}{10v(N^2-1)} - \frac{u^2(u-3(v+1))F(u,v)}{5v(N^2-1)},$$
$$A_{105} = \frac{u^2(1+v^2)}{20v^2} + \frac{u^2}{5v(N^2-1)} + \frac{2u^3F(u,v)}{5v(N^2-1)},$$
$$A_{175} = \frac{u^2(v^2-1)}{20v^2} + \frac{2u^2(v-1)F(u,v)}{5v(N^2-1)}.$$

k=3

The four point function for $k=3$ can be decomposed in several irreducible representations of $SO(6)$. In the main text we are just interested in the case of the singlet and 20 irreps that appear in this decomposition. They can be written as

$$\begin{aligned} A_1 = {} & \frac{3u\big(u^2(v+1)+15u\big(v^2+1\big)+50v(v+1)\big)}{100\mathrm{NN}^2v^2} + \frac{1}{50}\big(u^3\big(\frac{1}{v^3}+1\big)+50\big) \\ & + \frac{3u\big(u^3(v+1)-5u^2\big(2v^2+v+2\big)+5u\big(3v^3-v^2-v+3\big)+50v\big(v^2+4v+1\big)\big)\mathcal{F}(u,v)}{8v^2} \end{aligned} \tag{B.9}$$

$$\begin{aligned} A_{20} = {} & \frac{3u\big(u^2(v+1)+21u\big(v^2+1\big)+35v(v+1)\big)}{200(N^2-1)v^2} + \frac{u^3\big(v^3+1\big)}{50v^3} \\ & + \frac{3u\big(2u^3(v+1)-u^2\big(20v^2+7v+20\big)+7u\big(6v^3-v^2-v+6\big)+70(v-1)^2v\big)\mathcal{F}(u,v)}{32v^2} \end{aligned} \tag{B.10}$$

k=4

The four point function for $k=4$ can be decomposed in several irreducible representations of the $SO(6)$. As in the $k=3$ we need just the four point function in the singlet and the 20 channel. They can be written as

$$\begin{aligned} & A_1 = \tfrac{1}{105}\big(+105+u^4\big(\tfrac{1}{v^4}+1\big)\big) + \tfrac{4u\left(u^3\left(2v^2+v+2\right)+14u^2\left(3v^3+v^2+v+3\right)+35uv(v(3v+2)+3)+350v^2(v+1)\right)}{525(N^2-1)v^3} \\ & \frac{u}{20v^3}\Big(35v^2\big(u^2-8u(v+1)+10\big(v^2+4v+1\big)\big)\mathcal{F}(u,v)+u\big(uv\big(2u^2-8u(3v+1)+7\big(6v^2+8v+1\big)\big) \\ & \mathcal{F}\big(\tfrac{1}{v},\tfrac{u}{v}\big)+u\big(2u^2-8u(v+3)+7\big(v^2+8v+6\big)\big)\mathcal{F}(v,u)+u^3v\widetilde{\mathcal{F}}(u,v)-6u^2v^2\widetilde{\mathcal{F}}(u,v) \\ & +7u^2v^2\widetilde{\mathcal{F}}(v,u)-6u^2v\widetilde{\mathcal{F}}(u,v)+7u^2\widetilde{\mathcal{F}}\big(\tfrac{1}{v},\tfrac{u}{v}\big)+105v^4\widetilde{\mathcal{F}}(v,u)+7uv^3\widetilde{\mathcal{F}}(u,v)-70uv^3\widetilde{\mathcal{F}}(v,u) \\ & +210v^3\widetilde{\mathcal{F}}(v,u)+21uv^2\widetilde{\mathcal{F}}(u,v)+35v^2\widetilde{\mathcal{F}}\big(\tfrac{1}{v},\tfrac{u}{v}\big)-35uv^2\widetilde{\mathcal{F}}(v,u)+35v^2\widetilde{\mathcal{F}}(v,u)+7uv\widetilde{\mathcal{F}}(u,v) \\ & -35uv\widetilde{\mathcal{F}}\big(\tfrac{1}{v},\tfrac{u}{v}\big)+210v\widetilde{\mathcal{F}}\big(\tfrac{1}{v},\tfrac{u}{v}\big)-70u\widetilde{\mathcal{F}}\big(\tfrac{1}{v},\tfrac{u}{v}\big)+105\widetilde{\mathcal{F}}\big(\tfrac{1}{v},\tfrac{u}{v}\big)\big)\Big) \end{aligned} \tag{B.11}$$

$$A_{20} = \frac{4u^4\left(v^4+1\right)}{21v^4} + \frac{8u\left(u^3(v(11v+4)+11)+56u^2\left(6v^3+v^2+v+6\right)+196uv\left(3v^2+v+3\right)+980v^2(v+1)\right)}{525(N^2-1)v^3}$$
$$\frac{u}{10v^3}\Bigg(98v^2\big(u^2-5u(v+1)+10(v-1)^2\big)\mathcal{F}(u,v)+u^2v\big(11u^2-7u(21v+5)$$
$$+28(2v+1)(6v+1)\big)\mathcal{F}\big(\tfrac{1}{v},\tfrac{u}{v}\big)+u^2\big(11u^2-7u(5v+21)+28(v+2)(v+6)\big)\mathcal{F}(v,u)$$
$$+u^2v\big(4u^2-21u(v+1)+14(v(2v+3)+2)\big)\widetilde{\mathcal{F}}(u,v)+14u\big(2u^2-u(7v+20)+7(v(v+3)+6)\big)\widetilde{\mathcal{F}}\big(\tfrac{1}{v},\tfrac{u}{v}\big)$$
$$+14uv^2\big(-20uv+u(2u-7)+42v^2+21v+7\big)\widetilde{\mathcal{F}}(v,u)\Bigg) \qquad \text{(B.12)}$$

B.2 OPE coefficients

For completeness we present expressions for the OPE coefficients of leading twist operators derived in [56]. The perturbative expansion of the square of OPE coefficients in (5.58) is given by

$$a_0 = 1\,, \qquad a_1 = -4S_2\,, \qquad \text{(B.13)}$$
$$a_2 = 16\,(3\zeta_3 S_1 + c_{2,4})\,, \qquad a_3 = 64\,(\zeta_5 c_{3,1} + \zeta_3 c_{3,3} + c_{3,6})\,, \qquad \text{(B.14)}$$

where

$$c_{2,4} = \frac{5}{2}S_{-4} + S_{-2}^2 + 2S_{-3}S_1 + S_{-2}S_2 + S_2^2 + 2S_1S_3 + \frac{5}{2}S_4 - 2S_{-3,1} - S_{-2,2} - 2S_{1,3}\,, \qquad \text{(B.15)}$$
$$c_{3,1} = -\frac{25}{2}S_1\,, \qquad \text{(B.16)}$$
$$c_{3,3} = -3S_{-3} - 10S_{-2}S_1 + \frac{4}{3}S_1^3 - 6S_1S_2 - \frac{4}{3}S_3 + 6S_{-2,1}\,, \qquad \text{(B.17)}$$
$$\begin{aligned} c_{3,6} = &-11S_{-6} + \frac{5}{2}S_{-3}^2 - 5S_{-4}S_{-2} - \frac{41}{2}S_{-5}S_1 - S_{-3}S_{-2}S_1 - 5S_{-4}S_1^2 - 2S_{-2}^2S_1^2 \qquad \text{(B.18)}\\ &+ \frac{4}{3}S_{-3}S_1^3 - \frac{13}{2}S_{-4}S_2 - \frac{3}{2}S_{-2}^2S_2 - 10S_{-3}S_1S_2 - 2S_{-2}S_2^2 - S_2^3 - \frac{16}{3}S_{-3}S_3\\ &- 8S_{-2}S_1S_3 - 6S_1S_2S_3 - 3S_3^2 - 3S_{-2}S_4 + 9S_1^2S_4 - 4S_2S_4 + \frac{15}{2}S_1S_5 - \frac{13}{2}S_6\\ &+ 14S_{-5,1} + 11S_1S_{-4,1} + 9S_{-4,2} - 12S_1S_{-3,-2} + 10S_{-2}S_{-3,1} - 4S_1^2S_{-3,1}\\ &+ 8S_2S_{-3,1} + 4S_1S_{-3,2} + 9S_{-3,3} - 10S_{-3}S_{-2,1} + 14S_{-2}S_1S_{-2,1} - \frac{8}{3}S_1^3S_{-2,1}\\ &+ 4S_1S_2S_{-2,1} + \frac{20}{3}S_3S_{-2,1} + 10S_{-2,1}^2 + 10S_{-2}S_{-2,2} - 6S_1^2S_{-2,2} + 6S_2S_{-2,2}\\ &+ 6S_1S_{-2,3} + 11S_{-2,4} - 6S_2S_{1,3} - 4S_1S_{1,4} - 4S_{1,5} + 4S_1S_{2,3} + 4S_{2,4} - 12S_{-4,1,1}\\ &+ 8S_1S_{-3,1,1} - 2S_{-3,1,2} - 2S_{-3,2,1} - 24S_1S_{-2,-2,1} - 20S_{-2}S_{-2,1,1} + 16S_1^2S_{-2,1,1}\\ &- 8S_2S_{-2,1,1} + 16S_1S_{-2,1,2} - 6S_{-2,1,3} + 16S_1S_{-2,2,1} + 4S_{-2,2,2} - 6S_{-2,3,1} - 4S_1S_{1,1,3}\\ &- 8S_{1,1,4} + 8S_{1,3,2} - 8S_{-3,1,1,1} - 48S_1S_{-2,1,1,1} - 20S_{-2,1,1,2} - 20S_{-2,1,2,1} - 20S_{-2,2,1,1}\\ &+ 16S_{1,1,1,3} + 64S_{-2,1,1,1,1}\,. \end{aligned}$$

The definition of the harmonic sums is given in equation (5.40) of the next appendix, and we omitted their argument, which is J. For the case of $k = 3$ and $k = 4$ the expressions are quite similar.

k=3 OPE coefficients

The OPE coefficients exhcanged of the leading twist operator exchanged in the 20 channel for $k = 3$ are given by

$$a_J^{k=3} = \frac{21}{20(N^2-1)} \frac{2^J \left(1 + \frac{\gamma(J)}{2}\right)_J^2}{\big(1+\gamma(J)\big)_{2J}} \sum_{n=0} g^{2n} a_n^3(J) \ , \tag{B.19}$$

with

$$\begin{aligned} &a_0^3 = 1\,, \qquad a_1^3 = -4S_2 \\ &a_2^3 = 2S_4 - S_{1,3} - 2S_{-2,1,1} - 2S_{1,-2,1} - S_{-3,1} + \frac{5}{2}S_{-4} + \frac{S_{-2}^2}{2} \\ &+ 3S_{-3}S_1 + S_1^2 S_{-2} + S_2 S_{-2} + S_2^2 + 2S_1 S_3. \end{aligned} \tag{B.20}$$

k=4 OPE coefficients

The OPE coefficients exhcanged of the leading twist operator exchanged in the 20 channel for $k = 4$ are given by

$$a_J^{k=4} = \frac{2^6 7}{15(N^2-1)} \frac{2^J \left(1 + \frac{\gamma(J)}{2}\right)_J^2}{\big(1+\gamma(J)\big)_{2J}} \sum_{n=0} g^{2n} a_n^4(J) \ , \tag{B.21}$$

with

$$\begin{aligned} &a_0^4 = 1\,, \qquad a_1^4 = -S_2 \\ &a_2^4 = 2S_4 - S_{1,3} - 2S_{-2,1,1} - 2S_{1,-2,1} - S_{-3,1} + \frac{5}{2}S_{-4} + \frac{S_{-2}^2}{2} \\ &+ 3S_{-3}S_1 + S_1^2 S_{-2} + S_2 S_{-2} + S_2^2 + 2S_1 S_3. \end{aligned} \tag{B.22}$$

Harmonic sums and reflection symmetry

The simplest Harmonic sums are defined as,

$$\begin{aligned}
S_n(x) &= (-1)^{n-1}\,\frac{\Psi^{(n-1)}(x+1)-\Psi^{(n-1)}(1)}{\Gamma(n)} \overset{\text{integer x}}{=} \sum_{l=1}^{x}\frac{1}{l^n}\,, \qquad \text{(B.23)}\\
S_{-n}(x) &= (-1)^n\,\frac{\Psi^{(n-1)}\left(\frac{1}{2}+\frac{x}{2}\right)-\Psi^{(n-1)}\left(1+\frac{x}{2}\right)-\Psi^{(n-1)}\left(\frac{1}{2}\right)+\Psi^{(n-1)}(1)}{2^n\Gamma(n)} \overset{\text{even x}}{=} \sum_{m=1}^{x}\frac{(-1)^m}{m^n}\,,\\
S_{-a,b}(x) &= \zeta(-a,b)+\zeta(-(a+b))-\sum_{l=1}^{\infty}\frac{(-1)^l}{(l+x)^a}\,S_b(l+x)\,,
\end{aligned}$$

where $\zeta(-a)=\left(\frac{1}{2^{a-1}}-1\right)\zeta(a)$ and $\zeta(-a,b)$ are the Euler Zagier sums (or multivariate Zeta functions). The Euler Zagier $\zeta(-2,1)=\zeta(3)/8$. Notice that the analytically continued functions $S_{a_1,\ldots,a_n}(x)$ with one or more negative indices a_i, only match the definition (5.40) in terms of nested sums, for x an even integer.

The expansion of the Harmonic sums around the point $x=-1$ is

$$\begin{aligned}
S_n(-1+\omega) &= -\frac{1}{\omega^n}-\sum_{k=1}^{\infty}(-1)^k\begin{pmatrix} n+k-1\\ k\end{pmatrix}\zeta(n+k)\,\omega^k\,,\\
S_{-n}(-1+\omega) &= \frac{1}{\omega^n}+\zeta(-n)-\sum_{k=1}^{\infty}(-1)^k\begin{pmatrix} n+k-1\\ k\end{pmatrix}\zeta(-n-k)\,\omega^k\,, \qquad \text{(B.24)}\\
S_{-2,1}(-1+\omega) &= \frac{\zeta(2)}{\omega}-\frac{9\zeta(3)}{4}+\frac{33\zeta(4)}{16}\omega\,.
\end{aligned}$$

The Harmonic sum can be related to the sine function through the equality,

$$S_{-1}(x)+S_{-1}(-1-x)+\frac{\pi}{\sin(\pi x)}=-2\ln 2\,. \qquad \text{(B.25)}$$

There are certain quantities in the Regge kinematics that can be written as an antisymmetric or symmetric combination of harmonic sums at x and $-1-x$, where $x=(i\nu-1)/2$. For example, the BFKL spin is symmetric under the exchange of $x\to-1-x$ while the pre-factor α contains an antisymmetric factor under this symmetry. In the perturbative regime the coefficients of the $\ln\sigma$ terms in the four-point function are written as products like $j_k^n\alpha_q^m$ which have well defined symmetry under $x\to-1-x$, but involve products of harmonic sums with arguments x and $-x-1$. It is sometimes possible to express these functions in terms of linear combinations of harmonic sums without mixed arguments. As a simple example consider,

$$S_1(x)S_2(-1-x)+S_1(-1-x)S_2(x) \qquad \text{(B.26)}$$

which is equivalent to,

$$6\zeta_3 + \big(S_1(x)S_2(x) + S_3(x) - 2S_{2,1}(x) + (x \leftrightarrow -1-x)\big). \tag{B.27}$$

Identities like this one are sometimes useful to rewrite expressions in a simpler form as in equation (5.81). They can also be used in more technical aspects such as finding expansions near some point. We have discovered these identities performing numerical experiments but we are not aware of analytic derivations.

B.3 Explicit computation of three point function

Conformal symmetry imposes constrains on the form of two and three point functions between scalar and symmetric traceless operators [17]. In particular, the ratio of correlators like the one in (5.93) contains information about the OPE coefficients. More precisely, the structure of this ratio is fixed by conformal symmetry to be

$$\frac{\langle \mathcal{O}_1(x_1)\mathcal{O}_1^*(x_2)\mathcal{O}_J(x_5)\rangle\;\langle \mathcal{O}_J(x_6)\mathcal{O}_3(x_3)\mathcal{O}_3^*(x_4)\rangle}{\langle \mathcal{O}_1(x_1)\mathcal{O}_1^*(x_2)\rangle\,\langle \mathcal{O}_J(x_5)\mathcal{O}_J(x_6)\rangle\,\langle \mathcal{O}_3(x_3)\mathcal{O}_3^*(x_4)\rangle} = \tag{B.28}$$
$$C_{11J}C_{33J}\left(\frac{x_{13}^2x_{56}^4x_{24}^2}{x_{15}^2x_{35}^2x_{26}^2x_{46}^2}\right)^{\frac{\Delta+J}{2}}\frac{\big((w\cdot x_{15})\,x_{35}^2-(w\cdot x_{35})\,x_{15}^2\big)^J\big((w'\cdot x_{26})\,x_{46}^2-(w'\cdot x_{46})\,x_{26}^2\big)^J}{x_{13}^{2J}x_{24}^{2J}\big((w\cdot w')\,x_{56}^2-2(w\cdot x_{56})\,(w'\cdot x_{56})\big)^J},$$

where $x_{ij} = x_i - x_j$ and w and w' are *null polarization vectors* that allow us to write the symmetric traceless operator $\mathcal{O}_J$ as the polynomial $\mathcal{O}_J = w^{\mu_1}\dots w^{\mu_J}\mathcal{O}_{\mu_1\dots\mu_J}$ (see for example [17] for details).

In this appendix we compute the OPE coefficients C_{11J} to leading order in perturbation theory. The first step in the computation is to obtain the exact linear combination of operators that makes up the leading twist operator, as already mentioned in (5.7). This is achieved by diagonalizing the 1-loop dilatation operator and finding its eigenstates. The second step is to perform the perturbative (Wick contractions) computation of the three point functions.

Diagonalizing the 1-loop dilatation operator

The twist two operators are degenerate at tree level, however at finite t'Hooft coupling the degeneracy is lifted, making explicit which operator is in the leading Regge trajectory. This is done using the dilatation operator which can be written, at first order, using harmonic oscillators. By restricting its action to the subspace generated by states of the form (5.7) we find the eigenfunctions and eigenvalues of the dilatation operator.

In the following we review some definitions needed for the computation, following closely [96] and then apply it to our case.

Definitions

The elementary fields in SYM are $F_{\mu\nu}, \psi_{\alpha a}, \dot{\psi}^a_{\dot{\alpha}}$ and Φ_m, which can be written using harmonic oscillators as

$$\begin{aligned}
\mathcal{D}^k\mathcal{F} &\equiv \left(a^\dagger\right)^{k+2}\left(b^\dagger\right)^{k}\left(c^\dagger\right)^{0}|0\rangle\,, \\
\mathcal{D}^k\psi &\equiv \left(a^\dagger\right)^{k+1}\left(b^\dagger\right)^{k}\left(c^\dagger\right)^{1}|0\rangle\,, \\
\mathcal{D}^k\phi &\equiv \left(a^\dagger\right)^{k}\left(b^\dagger\right)^{k}\left(c^\dagger\right)^{2}|0\rangle\,, \\
\mathcal{D}^k\dot{\psi} &\equiv \left(a^\dagger\right)^{k}\left(b^\dagger\right)^{k+1}\left(c^\dagger\right)^{3}|0\rangle\,, \\
\mathcal{D}^k\dot{\mathcal{F}} &\equiv \left(a^\dagger\right)^{k}\left(b^\dagger\right)^{k+2}\left(c^\dagger\right)^{4}|0\rangle\,,
\end{aligned} \tag{B.29}$$

where $F_{\mu\nu} \sim \sigma_\mu^{\alpha\dot{\gamma}}\epsilon_{\dot{\gamma}\dot{\delta}}\sigma_\nu^{\dot{\delta}\beta}\mathcal{F}_{\alpha\beta} + \sigma_\mu^{\dot{\alpha}\gamma}\epsilon_{\gamma\delta}\sigma_\nu^{\delta\dot{\beta}}\dot{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}$, $\Phi_m \sim \sigma_m^{ba}\phi_{ab}$, the oscillators $a^\dagger_\alpha, b^\dagger_{\dot{\alpha}}$ have indices corresponding to the $\mathfrak{su}(2)\times\mathfrak{su}(2)$ Lorentz algebra and $c^\dagger_a$ has a $\mathfrak{su}(4)$ R-charge index. By definition $\mathcal{F}_{\alpha\beta}, \dot{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}$ are symmetric and ϕ_{ab} is antisymmetric in the indices. For example,

$$\mathcal{D}_{\dot{\alpha}\beta}\phi_{ab} \sim a^\dagger_\beta b^\dagger_{\dot{\alpha}} c^\dagger_a c^\dagger_b\,|0\rangle\,.$$

As expected, Bosonic oscillators commute and Fermionic oscillators anticommute

$$\left[a^\alpha, a^\dagger_\beta\right] = \delta^\alpha_\beta\,, \qquad \left[b^{\dot{\alpha}}, b^\dagger_{\dot{\beta}}\right] = \delta^{\dot{\alpha}}_{\dot{\beta}}\,, \qquad \left\{c^a, c^\dagger_b\right\} = \delta^a_b\,. \tag{B.30}$$

The state $|0\rangle$ is defined as the one annihilated by all oscillators $a^\alpha, b^{\dot{\alpha}}$ and c^a. Though this state is not physical state, as it gives a nonzero central charge

$$C = 1 - \frac{1}{2}\,a^\dagger_\alpha a^\alpha + \frac{1}{2}\,b^\dagger_{\dot{\alpha}} b^{\dot{\alpha}} - \frac{1}{2}\,c^\dagger_a c^a\,. \tag{B.31}$$

On the other hand, the elementary fields are obtained from the physical state $\phi_{34} \equiv c^\dagger_3 c^\dagger_4\,|0\rangle \equiv |\mathcal{Z}\rangle$ that is the highest weight. This leads to the redefinition of the oscillators

$$d^\dagger_1 = c^4\,, \qquad d^\dagger_2 = c^3\,, \qquad d^1 = c^\dagger_4\,, \qquad d^2 = c^\dagger_3\,, \tag{B.32}$$

which breaks the $\mathfrak{su}(4)$ into $\mathfrak{su}(2)\times\mathfrak{su}(2)$. This redefinition makes the state $|\mathcal{Z}\rangle$ the natural vacuum, since it is annihilated by a_α, $b_{\dot{\alpha}}$, c_1, c_2, d_1 and d_2.

Twist two operators

Twist two operators are defined by their classical dimension $\Delta_0 = 1 + (n_a + n_b)/2 = 2 + J$, where J is the spin. This implies that they are of the form

$$\mathrm{Tr}\left(\mathcal{W}_A\mathcal{W}_B\right),$$

where $\mathcal{W}_\mathcal{A} \in \left\{\mathcal{D}^k\mathcal{F}, \mathcal{D}^k\dot{\mathcal{F}}, \mathcal{D}^k\phi, \mathcal{D}^k\psi, \mathcal{D}^k\dot{\psi}\right\}$, or using oscillators

$$\left(a^\dagger\right)^{n_a}\left(b^\dagger\right)^{n_b}\left(c^\dagger\right)^{n_c}\left(d^\dagger\right)^{n_d}\left(a^\dagger\right)^{J-n_a}\left(b^\dagger\right)^{J-n_b}\left(c^\dagger\right)^{p-n_c}\left(d^\dagger\right)^{p-n_d}|\mathcal{Z}\mathcal{Z}\rangle\,. \tag{B.33}$$

This requires some explanation; the first four types of oscillators act on the first site and the others act on the second site; the number n_a of oscillators of type $a^\dagger$ on the first site is arbitrary in principle [1] but as we want spin J, the number of $a^\dagger$ on the second site has to be $J - n_a$; the same applies to oscillators of type $b^\dagger$; on the second site, there is no loss of generality when considering the number of oscillators of type $c^\dagger$ to be $p - n_c$, but the number of $d^\dagger$ on the same site follows because of the central charge condition, which now reads

$$C = \frac{1}{2}\sum_{\text{sites}}\left(b^\dagger_{\dot\alpha}b^{\dot\alpha} - a^\dagger_\alpha a^\alpha + d^\dagger_a d^a - c^\dagger_a c^a\right) = 0\ . \tag{B.34}$$

Requiring the state to be a $\mathfrak{su}(4)$ singlet, fixes $p = 2$. In the original operator basis, the $\mathfrak{su}(4)$ part of the state is $\epsilon^{abcd}c^\dagger_a c^\dagger_b c^\dagger_c c^\dagger_d\,|00\rangle$, which in the new basis becomes $c^\dagger_1 c^\dagger_2 d^1 d^2\,|00\rangle$ or $c^\dagger_1 c^\dagger_2 d^\dagger_1 d^\dagger_2\,|\mathcal{Z}\mathcal{Z}\rangle$. In the previous sentence, we did not specify in which site each operator acts because this is not relevant for the $\mathfrak{su}(4)$ singlet constraint. Note that, other $\mathfrak{su}(4)$ singlets, like $\mathcal{F}\mathcal{F}$ or $\dot{\mathcal{F}}\dot{\mathcal{F}}$, are excluded because they do not have the required Lorentz structure. Thus, the states can be labelled as

$$\Big|\underbrace{1\ldots1}_{n_a}\underbrace{2\ldots2}_{J-n_a}\underbrace{1\ldots1}_{n_b}\underbrace{2\ldots2}_{J-n_b}\,i_1\ldots i_4; a^\dagger\ldots a^\dagger b^\dagger\ldots b^\dagger c^\dagger_1 c^\dagger_2 d^\dagger_1 d^\dagger_2\Big\rangle\,, \tag{B.35}$$

where n_a is the number of the first set of 1's, $J - n_a$ is the number of the first set of 2's and, similarly, for n_b and $J - n_b$ in the following set of 1's and 2's. The i_j can be 1 or 2 encoding the site where $c^\dagger$ and $d^\dagger$ act.

In this representation the Hamiltonian can be written in the following way

$$\mathcal{H}\,|s_1\ldots s_n; A\rangle = \sum_{s'_1\ldots s'_n} c_{n,n_{12},n_{21}}\delta_{C_1,0}\delta_{C_2,0}\left|s'_1\ldots s'_n; A\right\rangle\,, \tag{B.36}$$

where A is a list of n operators, like the one in (B.35) and $s_1\ldots s_n$ is a list of 1's and 2's that specifies in which site each operator acts. The variable n_{12} counts the number of 1's in $s_1\ldots s_n$ that became 2's in $s'_1\ldots s'_n$. In other words, n_{12} and n_{21} is the number of oscillators hopping from site 1 to 2 and from 2 to 1, respectively. Finally,

$$c_{n,n_{12},n_{21}} = (-1)^{1+n_{12}n_{21}}\,\frac{\Gamma\left(\frac{n_{12}+n_{21}}{2}\right)\Gamma\left(1+\frac{n-n_{12}-n_{21}}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}\,. \tag{B.37}$$

It is clear that the subspace generated by states of the form (B.35) is closed under the action of the Hamiltonian. We have implemented a Mathematica program to find the eigenstates and

[1] Note that there is the restriction of $n_a < J$.

eigenvalues for values of $J = 2, \ldots, 8$.[2] The results allow us to confirm that, all the odd spin eigenstates are descendants, and that the eigenvalues for any J are $2S_1(J-2)$, $2S_1(J)$ and $2S_1(J+2)$, where S_1 is a harmonic sum. It also enabled us to confirm that the eigenvectors are a linear combination of the form

$$a \left|\phi\mathcal{D}^J\phi\right\rangle + b \left|\mathcal{F}\mathcal{D}^{J-2}\dot{\mathcal{F}}\right\rangle + c \left|\psi\mathcal{D}^{J-1}\dot{\psi}\right\rangle , \tag{B.38}$$

where we use the shorthand notation [97],[3]

$$\begin{aligned}
\left|\phi\mathcal{D}^J\phi\right\rangle &= \sum_{k=0}^{J}(-1)^k \binom{J}{k}^2 \mathrm{Tr}\Big(\mathcal{D}^k\phi\mathcal{D}^{J-k}\phi\Big) , \\
\left|\psi\mathcal{D}^{J-1}\dot{\psi}\right\rangle &= \sum_{k=0}^{J-1}(-1)^k \binom{J}{k}\binom{J}{k+1} \mathrm{Tr}\Big(\mathcal{D}^k\psi\mathcal{D}^{J-k-1}\dot{\psi}\Big) , \\
\left|\mathcal{F}\mathcal{D}^{J-2}\dot{\mathcal{F}}\right\rangle &= \sum_{k=0}^{J-2}(-1)^k \binom{J}{k}\binom{J}{k+2} \mathrm{Tr}\Big(\mathcal{D}^k F\mathcal{D}^{J-k-2}\dot{F}\Big) ,
\end{aligned} \tag{B.39}$$

with

$$\binom{J}{k} = \frac{J!}{(J-k)!k!} . \tag{B.40}$$

This was expected as it is possible to construct twist two primary operators at zero order restricting only to scalar, gauge or Fermionic fields [97], and so, at first order, the eigenvectors must be a linear combination of these zero order eigenstates.[4]

The data collected also allowed to infer the matrix form of the Hamiltonian for general J in the (non-normalized) basis $\{|1\rangle, |2\rangle, |3\rangle\} \equiv \Big\{\left|\phi\mathcal{D}^J\phi\right\rangle, \left|\psi\mathcal{D}^{J-1}\dot{\psi}\right\rangle, \left|\mathcal{F}\mathcal{D}^{J-2}\dot{\mathcal{F}}\right\rangle\Big\}$. We found

$$h_{ij} = \begin{pmatrix} 2S_1(J) & -\frac{2}{J+1} & \frac{1}{(J+1)(J+2)} \\ -\frac{6}{J} & 2S_1(J) - \frac{4}{J(J+1)} & \frac{J^2+J+2}{J(J+1)(J+2)} \\ \frac{12}{(J-1)J} & \frac{4\left(J^2+J+2\right)}{(J-1)J(J+1)} & 2S_1(J) - \frac{4\left(J^2+J+1\right)}{(J-1)J(J+1)(J+2)} \end{pmatrix} \tag{B.41}$$

where

$$\mathcal{H}\,|i\rangle = \sum_{j=1}^{3} h_{ji}\,|j\rangle \ . \tag{B.42}$$

It is then simple to determine the eigenvectors of $\mathcal{H}$. From highest to lowest eigenvalue, the three

[2] This was implemented in Mathematica by creating all possible states. Higher values of J were limited by this approach, as the number of states grows exponentially.

[3] On the right hand side of the equation we use (B.29).

[4] In perturbation theory in quantum mechanics the eigenvectors lag in relation to the eigenvalues.

eigenvectors obtained are

$$|V_1\rangle = \left|\phi\mathcal{D}^J\phi\right\rangle - 2\left|\psi\mathcal{D}^{J-1}\dot{\psi}\right\rangle - 2\left|\mathcal{F}\mathcal{D}^{J-2}\dot{\mathcal{F}}\right\rangle, \tag{B.43}$$

$$|V_2\rangle = \left|\phi\mathcal{D}^J\phi\right\rangle + \frac{3}{J}\left|\psi\mathcal{D}^{J-1}\dot{\psi}\right\rangle + \frac{6\,(2+J)}{J}\left|\mathcal{F}\mathcal{D}^{J-2}\dot{\mathcal{F}}\right\rangle, \tag{B.44}$$

$$|V_3\rangle = \left|\phi\mathcal{D}^J\phi\right\rangle + \frac{2\,(J+1)}{J}\left|\psi\mathcal{D}^{J-1}\dot{\psi}\right\rangle - \frac{2(J+2)(J+1)}{J(J-1)}\left|\mathcal{F}\mathcal{D}^{J-2}\dot{\mathcal{F}}\right\rangle. \tag{B.45}$$

Three-point function

In [97] two and three point between two scalars and the operator with highest eigenvalue (B.43) were computed. Their results can be easily adapted to the case of the leading twist operators, that corresponds to the state (B.45) with lowest eigenvalue. The only subtlety is that one needs to adapt field normalizations as follows $\phi \to \phi$, $F \to iF$ and $\psi \to \sqrt{2i}\psi$. Thus, in the conventions of [97], the leading twist operator is written as

$$\mathcal{O}_J = \left|\phi\mathcal{D}^J\phi\right\rangle + \frac{4i\,(J+1)}{J}\left|\psi\mathcal{D}^{J-1}\dot{\psi}\right\rangle + \frac{2(J+2)(J+1)}{J(J-1)}\left|\mathcal{F}\mathcal{D}^{J-2}\dot{\mathcal{F}}\right\rangle. \tag{B.46}$$

Its two and three point functions can then be obtained from [97],

$$\begin{aligned}
\langle\mathcal{O}_J(x_5)\mathcal{O}_J(x_6)\rangle &= \frac{2^{J+2}N^2}{(4\pi^2)^2}\left(12S_p + \left(\frac{J+1}{J}\right)^2 S_r + \left(\frac{(J+2)(J+1)}{J(J-1)}\right)^2 S_s\right) \\
&\qquad \frac{\left((w.w')x_{56}^2 - 2(w.x_{56})\,(w'.x_{56})\right)^J}{(x^2)^{2J+2}}\,, \\
\langle\mathcal{O}_1(x_1)\mathcal{O}_1(x_3)\rangle &= \frac{N^2}{8\pi^4}\,\frac{1}{\left(x_{13}^2\right)^2}\,, \\
\langle\mathcal{O}_1(x_1)\mathcal{O}_1(x_3)\mathcal{O}_J(x_5)\rangle &= \frac{2^{J+4}N^2\,\Gamma(J+1)}{(4\pi^2)^3}\,\frac{\left((w.x_{15})x_{35}^2 - (w.x_{35})x_{15}^2\right)^J}{\left(x_{12}^2\right)^3\left(x_{23}^2\right)^{1+J}\left(x_{13}^2\right)^{1+J}}\,,
\end{aligned} \tag{B.47}$$

where

$$S_p(J) = \Gamma(2J+1)\,, \qquad S_r(J) = 16J\,\frac{\Gamma(2J+1)}{J+1}\,, \qquad S_s(J) = 4\,\frac{J\,(J-1)\,\Gamma(2J+1)}{(J+1)(J+2)}\,. \tag{B.48}$$

Notice that the result for the two point function of $\mathcal{O}_J$ satisfies the constraint that the two point function of the stress energy momentum of the fermion is twice the value of the scalar and gauge part [83].

Thus, we finally conclude that $C^2(J) \equiv C_{11J}C_{33J}$ is given by

$$C^2(J) = \frac{2^{1+J}(J-1)J\,\Gamma^2(J+1)}{N^2\,(4J^2-1)\,\Gamma(2J+1)}\,, \tag{B.49}$$

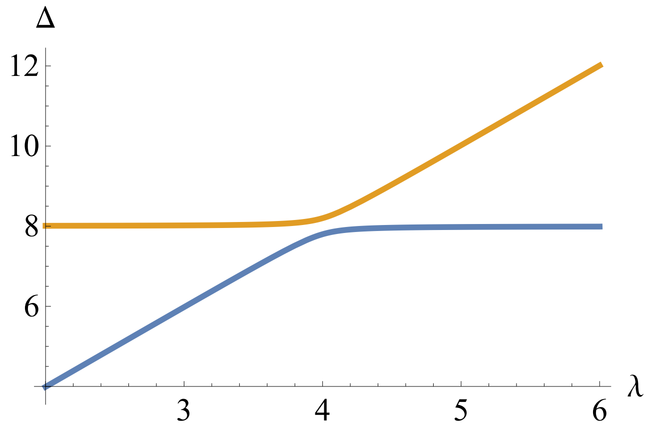


Figure B.1: Almost level crossing between single trace and double trace operator. In this schematic representation there is almost level crossing for $\lambda_* \approx 4$.

which satisfies the requirement $C^2(2) = 8/(45N^2)$ that must be satisfied independently of the t' Hooft coupling.

B.3.1 Poles in three point function

The goal of this section is to explain how the phenomenon of almost level crossing leads to poles in the three point function. This will necessarily happen in $\mathcal{N}=4$ SYM in the planar limit whenever an operator gains a large anomalous dimension. In the present case we will be interested in the level crossing between an unprotected single trace that gains a large anomalous dimension at strong coupling and the double trace operator associated with the Lagrangian density

$$\mathcal{O}^{dt}_{J,n} = \mathrm{Tr}(\mathcal{L})\partial_{\mu_1}\dots\partial_{\mu_J}(\partial^2)^n\mathrm{Tr}(\mathcal{L}) \tag{B.50}$$

whose dimension in the planar limit is fixed $8+J+2n$. As the 't Hooft coupling is varied the dimension of the unprotected operator will approach the dimension of the double trace operator. Let us assume a simplified model and consider just two operators whose dimension will come close. The Hamiltonian of these two operators close to the crossing can be represented as

$$\widehat{H} = (\Delta_{dt}+x)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}x & \epsilon\\ \epsilon & -x\end{pmatrix} + O\left(x^2\right) \tag{B.51}$$

where the variable x is defined by

$$x = \frac{1}{2}\left(\frac{\partial\Delta_{st}}{\partial\lambda}\right)_{\lambda_*}(\lambda-\lambda_*). \tag{B.52}$$

and λ_* is the point at which level cross would have occurred. Notice that dimension of the single trace can be expressed in a simple way $\Delta_{st} = \Delta_{dt} + 2x$. Let us be more concrete, for $\epsilon = 0$ there is level crossing

$$\widehat{H}|st\rangle = \widehat{H}\begin{pmatrix}1\\0\end{pmatrix} = \big(\Delta_{dt}+2x\big)|st\rangle = \Delta_{st}\big(\lambda\big)|st\rangle \tag{B.53}$$

$$\widehat{H}|dt\rangle = \widehat{H}\begin{pmatrix}0\\1\end{pmatrix} = \Delta_{dt}|dt\rangle. \tag{B.54}$$

In the case of $\epsilon \neq 0$ there is no level crossing

$$\widehat{H}|0\rangle = \widehat{H}\begin{pmatrix}\sqrt{x^2+\epsilon^2}-x\\-\epsilon\end{pmatrix} = \big(\Delta_{dt}+x-\sqrt{x^2+\epsilon^2}\big)|0\rangle \tag{B.55}$$

$$\widehat{H}|1\rangle = \widehat{H}\begin{pmatrix}\sqrt{x^2+\epsilon^2}+x\\\epsilon\end{pmatrix} = \big(\Delta_{dt}+x+\sqrt{x^2+\epsilon^2}\big)|1\rangle. \tag{B.56}$$

The minimal difference between the two energy levels is given by 2ϵ. Let us remark that the eigenstates of the Hamiltonian change their behavior once x passes through 0

$$\lim_{\epsilon\to0^+}\frac{|0\rangle}{\sqrt{\langle 0|0\rangle}} = \begin{cases}|st\rangle\,, & x<0\\ -|dt\rangle\,, & x>0\end{cases} \qquad \lim_{\epsilon\to0^+}\frac{|1\rangle}{\sqrt{\langle 1|1\rangle}} = \begin{cases}|dt\rangle\,, & x<0\\ |st\rangle\,, & x>0\end{cases}. \tag{B.57}$$

The three point function can be thought as the inner product of two states, one created by

$$|\psi\rangle = O_1(x_1)O_2(x_2)|vac\rangle \tag{B.58}$$

and the other

$$|st\rangle = O_3(x_3)|vac\rangle \tag{B.59}$$

Generically the state defined by $|\psi\rangle$ is not an eigenstate of the Hamiltonian,

$$|\psi\rangle = \frac{a}{N}|st\rangle + b|dt\rangle = \begin{pmatrix}a/N\\b\end{pmatrix} \tag{B.60}$$

where the coefficients a and b are of order 1. The correct way to compute the three point function is to take the inner product of $|\psi\rangle$ with an eigenstate of the Hamiltonian

$$\lim_{N\to\infty} N\frac{\langle\psi|0\rangle}{\sqrt{\langle 0|0\rangle}} = \begin{cases}a+\frac{bN\epsilon}{2x}, & x<0\\ -b, & x>0\end{cases} \qquad \lim_{N\to\infty} N\frac{\langle\psi|1\rangle}{\sqrt{\langle 1|1\rangle}} = \begin{cases}b, & x<0\\ a+\frac{bN\epsilon}{2x}, & x>0\end{cases} \tag{B.61}$$

So from this analysis we conclude that there is a pole in the three point function whenever there is almost level crossing, more specifically we have

$$C_{123}(\lambda) \approx a + \frac{bN\epsilon}{\Delta_{st}(\lambda) - \Delta_{dt}} \tag{B.62}$$

B.4 Analytic continuation of the four-point function

Here we give some details on the analytic continuation of the four-point function required to analyse the Regge limit described in Section 5.1.2. The two-loop calculation was performed in [37] and the only contribution came from the $[\phi^{(1)}(z,\bar{z})]^2$ contribution to g given in equation (5.20). At three loops we recall from (5.60) that we need to consider the contributions to the functions g and h. In fact all contributions to h at three loops (from the Easy function E and the Hard function $H^{(b)}$) are given in terms of single-valued combinations of harmonic polylogarithms [48] and calculating the analytic continuation is straightforward. The result is that at three loops the function h gives no contribution in the Regge limit (i.e. it is power suppressed in the limit $\sigma \to 0$).

The contributions to the function g are of two types. Firstly there are terms of the form $\phi^{(1)}\phi^{(2)}$ coming from the product of one-loop and two-loop ladder integrals. These ladder contributions are again given in terms of single-valued combinations of harmonic polylogarithms and again their analytic continuation is straightforwardly obtained. The resulting contribution to the analytically continued four-point function in the Regge limit is,

$$\text{ladders} \to 2\pi^2 \frac{r}{(1-r)^2} \log^2 r (2\pi^2 + \log^2 r - 4\log^2\sigma - 4i\pi\log\sigma)\,. \tag{B.63}$$

The second type of contribution to g comes from the Hard function $H^{(a)}$ which is given in terms of a single-valued combination of two-variable multiple polylogarithm functions [48]. In general these functions are specified by a weight vector $w = a_1 a_2 \ldots a_n$. If w is just a string of zeros, we define $G(0_n; x) = \frac{1}{n!}\log^n x$. Then, if we write a general weight vector w as $w = a_1 w'$ with $w' = a_2 \ldots a_n$, the remaining functions can be defined recursively via

$$G(w;x) = G(a_1, a_2, \ldots, a_n; x) = G(a_1, w'; x) = \int_0^x \frac{dt}{t-a_1} G(w';t)\,. \tag{B.64}$$

Such multiple polylogarithms obey a shuffle product relation,

$$G(w_1; z)G(w_2; z) = G(w_1 ⧢ w_2; z)\,, \tag{B.65}$$

and, if the word w does not have trailing zeros, a rescaling relation,

$$G(a_1, \ldots, a_n; x) = G(\lambda a_1, \ldots, \lambda a_n; \lambda x)\,, \qquad a_n \neq 0\,. \tag{B.66}$$

The harmonic polylogarithms used throughout are special cases where the weight vector consists only of zeros and ones. Due to unfortunate choices of conventions the precise relation involves a sign,

$$G(w;x) = (-1)^{d(w)} H(w;x)\,, \qquad a_i \in \{0,1\}\,, \tag{B.67}$$

where $d(w)$ is the number of "1" entries in w.

Obtaining the analytic continuation for the contributions to g coming from the Hard function $H^{(a)}$ is slightly more involved. Here we describe a method for obtaining the analytic continuation based on an integral formula for $H^{(a)}$ given in Appendix B of [48],

$$\begin{aligned} &H^{(a)}(1-z,1-\overline{z}) = \\ &\quad = \big(2H_{0,0}(u) + 4H_0(u)H_0(v) + 8H_{0,0}(v)\big)\big(\mathcal{L}_{0,0,1,1} + \mathcal{L}_{1,1,0,0} - \mathcal{L}_{0,1,1,0} - \mathcal{L}_{1,0,0,1}\big) \\ &\qquad - 8F_5\big(H_0(u) + 2H_0(v)\big) + F_6\,. \end{aligned} \tag{B.68}$$

In the above equation the $\mathcal{L}$ functions are single-valued combinations of harmonic polylogarithms, defined by Brown [98]. We refer the reader to [48] for all the conventions on single-valued polylogarithms. The functions F_5 and F_6 are given by integral formulae,

$$F_n(z,\overline{z}) = \int dt \left[\frac{X_{n-1}(t,\overline{z})}{t} - \frac{Y_{n-1}(t,\overline{z})}{1-t} + \frac{Z_{n-1}(t,\overline{z})}{t-\overline{z}}\right] \tag{B.69}$$

$$= F_n^X(z,\overline{z}) + F_n^Y(z,\overline{z}) + F_n^Z(z,\overline{z})\,. \tag{B.70}$$

For F_5 the integrand in (B.69) is given by the following three functions which are again single-valued combinations of harmonic polylogarithms,

$$\begin{aligned} X_4(x,\overline{x}) &= (\mathcal{L}_{0,0,1,1} - \mathcal{L}_{1,1,0,0} - \mathcal{L}_{0,1,1,1} + \mathcal{L}_{1,1,1,0})\,, \\ Y_4(x,\overline{x}) &= (\mathcal{L}_{0,0,0,1} - \mathcal{L}_{1,0,0,0} - \mathcal{L}_{0,0,1,1} + \mathcal{L}_{1,1,0,0})\,, \\ Z_4(x,\overline{x}) &= (\mathcal{L}_{0,0,1,1} + \mathcal{L}_{1,1,0,0} - \mathcal{L}_{0,1,1,0} - \mathcal{L}_{1,0,0,1})\,. \end{aligned} \tag{B.71}$$

For F_6 the integrand in (B.69) is given by the following three functions,

$$\begin{aligned} X_5 = {}& 20\mathcal{L}_{0,0,0,1,1} + 12\mathcal{L}_{0,0,1,1,0} - 32\mathcal{L}_{0,0,1,1,1} - 8\mathcal{L}_{0,1,0,1,1} - 12\mathcal{L}_{0,1,1,0,0} - 8\mathcal{L}_{0,1,1,0,1} \\ &+ 16\mathcal{L}_{0,1,1,1,1} - 8\mathcal{L}_{1,0,0,1,1} + 8\mathcal{L}_{1,0,1,1,0} - 20\mathcal{L}_{1,1,0,0,0} + 8\mathcal{L}_{1,1,0,0,1} + 8\mathcal{L}_{1,1,0,1,0} \\ &+ 32\mathcal{L}_{1,1,1,0,0} - 16\mathcal{L}_{1,1,1,1,0} - 16\mathcal{L}_{1,1}\zeta_3\,, \end{aligned} \tag{B.72}$$

$$\begin{aligned} Y_5 = {}& 20\mathcal{L}_{0,0,0,0,1} - 32\mathcal{L}_{0,0,0,1,1} - 8\mathcal{L}_{0,0,1,1,0} + 16\mathcal{L}_{0,0,1,1,1} - 8\mathcal{L}_{0,1,0,0,1} + \mathcal{L}_{0,1,1,0,0} \\ &- 20\mathcal{L}_{1,0,0,0,0} + 8\mathcal{L}_{1,0,0,1,0} + 16\mathcal{L}_{1,0,0,1,1} + 8\mathcal{L}_{1,0,1,0,0} + 32\mathcal{L}_{1,1,0,0,0} - 16\mathcal{L}_{1,1,0,0,1} \\ &- 16\mathcal{L}_{1,1,1,0,0} - 16\mathcal{L}_{1,0}\zeta_3 + 64\mathcal{L}_{1,1}\zeta_3\,. \end{aligned} \tag{B.73}$$

$$Z_5 = 32F_5\,. \tag{B.74}$$

In the case of both F_5 and F_6 the X and Y parts of the integrand can be integrated immediately to obtain F_n^X and F_n^Y in terms of harmonic polylogarithms. The analytic continuation of these terms can therefore be obtained easily. For the F_n^Z contributions we use the fact that the discontinuity around $z = 1$ can be moved through the integral sign,

$$\Delta_{1-z} F_n^Z(z, \bar{z}) = \int_1^z \frac{dt}{t - \bar{z}} \Delta_{1-t} Z_{n-1}(t, \bar{z}) \,. \tag{B.75}$$

Since Z_4 is a combination of harmonic polylogarithms its discontinuity can be easily calculated. Then to obtain $\Delta_{1-z} F_5$ it remains to perform the integral. This can be done using multiple (or Goncharov) polylogarithms (see equations (B.64) and (B.67)),

$$\int_0^z \frac{dt}{t - \bar{z}} H_w(t) = (-1)^{d(w)} G(\bar{z}, w; z) \,. \tag{B.76}$$

Having analytically continued around $z = 1$ (there is no contribution from $z = 0$) it remains to take the Regge limit $z, \bar{z} \to 0$ with fixed ratio $r = z/\bar{z}$. This can be done by first extracting any trailing zeros from the word w appearing in the G-functions above by using the shuffle relations (B.65). Having made any $\log z$ and $\log \bar{z}$ explicit, one may rescale the arguments of the G-functions using (B.66),

$$G(\bar{z}, w; z) = G(1, w/\bar{z}; r). \tag{B.77}$$

Then any G-functions exhibiting a letter $1/\bar{z}$ in the weight vector will be power suppressed in the Regge limit and may be dropped. Finally one obtains an expression in terms of powers of $H_0(\sigma) = \log \sigma$ and harmonic polylogarithms of argument r. The result for the Regge limit of F_5 is

$$F_5 \to 32\pi^2 \big(-2H_0(\sigma) H_{0,0}(r) + H_{0,0,0}(r) + 2H_{1,0,0}(r) - 2\zeta_3 + i\pi H_{0,0}(r) \big) \,. \tag{B.78}$$

For F_6 we can perform the same analysis as above with the only difference that $Z_5 = 32F_5$ so that we need to recycle our previous result for $\Delta_{1-z} F_5(z, \bar{z})$ when calculating $\Delta_{1-z} F_6^Z(z, \bar{z})$. This leads to some terms with two $\bar{z}$ appearing in the arguments of the G functions. Otherwise the analysis is very similar and we find that the Regge limit of F_6 is given by

$$\begin{aligned} F_6 \to 8\pi^2 \big(&- 8H_{0,0}(r) H_{0,0}(\sigma) + 4H_0(\sigma) H_{0,0,0}(r) + 8H_0(\sigma) H_{1,0,0}(r) - 4H_{2,0,0}(r) - H_{0,0,0,0}(r) \\ &- 4H_{1,0,0,0}(r) - 8H_{1,1,0,0}(r) + 4H_{0,0}(r)\zeta_2 + 6\zeta_4 + 4H_0(r)\zeta(3) - 8H_0(\sigma)\zeta_3 \\ &+ 8H_1(r)\zeta_3 + 4i\pi H_0(\sigma) H_{0,0}(r) - 2i\pi H_{0,0,0}(r) - 4i\pi H_{1,0,0}(r) + 4i\pi\zeta_3 \big) \,. \end{aligned} \tag{B.79}$$

Combining the above calculations we can now obtain the Regge limit of $H^{(a)}(1-z,1-\overline{z})$,

$$\begin{aligned} H^{(a)}(1-z,1-\overline{z}) \to 8\pi^2\big(& -2H_{0,0}(r)H_{0,0}(\sigma)+2H_0(\sigma)H_{0,0,0}(r)+4H_0(\sigma)H_{1,0,0}(r) \\ & -4H_{2,0,0}(r)-H_{0,0,0,0}(r)-4H_{1,0,0,0}(r)-8H_{1,1,0,0}(r)+4H_{0,0}(r)\zeta_2 \\ & +6\zeta_4+4H_0(r)\zeta_3-4H_0(\sigma)\zeta_3+8H_1(r)\zeta_3-2i\pi H_0(\sigma)H_{0,0}(r) \\ & +2i\pi H_{0,0,0}(r)+4i\pi H_{1,0,0}(r)-4i\pi\zeta_3\big) \end{aligned} \tag{B.80}$$

Similar calculations yield results for $H^{(a)}(z,\overline{z})$ and $H^{(a)}\big(\frac{1}{z},\frac{1}{\overline{z}}\big)$,

$$H^{(a)}(z,\overline{z}) \to 16\pi^2\big(-2H_{2,0,0}(r)-2H_{1,0,0,0}(r)-4H_{1,1,0,0}(r)+3\zeta_4+2H_0(r)\zeta_3+4H_1(r)\zeta_3\big)\,, \tag{B.81}$$

$$\begin{aligned} H^{(a)}\big(\tfrac{1}{z},\tfrac{1}{\overline{z}}\big) \to \ 8\pi^2\big(& -2H_{0,0}(r)H_{0,0}(s)+2H_0(s)H_{0,0,0}(r)+4H_0(s)H_{1,0,0}(r)-4H_{2,0,0}(r) \\ & -H_{0,0,0,0}(r)-4H_{1,0,0,0}(r)-8H_{1,1,0,0}(r)-2H_{0,0}(r)\zeta_2]+6\zeta_4+4H_0(r)\zeta_3 \\ & -4H_0(s)\zeta_3+8H_1(r)\zeta_3\big)\,. \end{aligned} \tag{B.82}$$

Finally, as dictated by equations (5.21) and (5.60), we need to take the combination

$$\frac{r}{(1-r)^2}\Big(-2H^{(a)}(z,\overline{z})-H^{(a)}(1-z,1-\overline{z})-H^{(a)}\big(\tfrac{1}{z},\tfrac{1}{\overline{z}}\big)\Big)\,, \tag{B.83}$$

and combine with the ladder contributions (B.63) to obtain the results quoted in equations (5.62) and (5.63) of the main text. Note that the $\log^2\sigma$ divergence from the ladder contribution (B.63) is cancelled by similar divergences in (B.80) and (B.81) (recall $2H_{0,0}(\sigma)=\log^2\sigma$). This is necessary since the leading divergence in the four-point function at three loops should only be a single power of $\log\sigma$. Likewise the imaginary divergent contribution from (B.63) is cancelled by a similar contribution from (B.80) which is necessary for the coefficient of the leading $\log\sigma$ divergence to be real.

Appendix C

Conformal Partial wave coefficient

The goal of this subsection is to write an expression for the the conformal partial wave coefficient, $b_J(\nu)$, in terms of the reduced amplitude $\mathcal{A}(u,v)$. To this end we shall use an orthogonality relation for the conformal partial wave $F_{\nu,J}(u,v)$. To avoid the cluttering of formulae and to make contact with [36] we introduce the following notation

$$V_J(x_1,x_2,x_3,z_3,c_1,c_2,c_3) = \frac{N_J(c_1,c_2,c_3)(x_{13}^2 x_{23}\cdot z_3 - x_{23}^2 x_{13}\cdot z_3)^J}{(2\pi)^h (x_{12}^2)^{\frac{h+c_1+c_2-c_3+J}{2}} (x_{23}^2)^{\frac{h+c_3+c_2-c_1+J}{2}} (x_{13}^2)^{\frac{h+c_1+c_3-c_2+J}{2}}} \tag{C.1}$$

$$G_c(x_1,x_2,z_1,z_2) = \frac{\Gamma(c+h+J)\Gamma(h-c-1)}{(2\pi)^h\Gamma(-c)\Gamma(h+J-c-1)} \frac{(x_{12}\cdot z_1 x_{12}\cdot z_2 - \frac{1}{2}z_1\cdot z_2 x_{12}^2)^J}{(x_{12}^2)^{h+c+J}} \tag{C.2}$$

$$N_J^2(c_1,c_2,c_3) = 2^{c_1+c_2+c_3+J}\frac{\Gamma(\frac{h+J+c_1+c_2+c_3}{2})\Gamma(\frac{h+J+c_1+c_2-c_3}{2})\Gamma(\frac{h+J+c_1+c_3-c_2}{2})\Gamma(\frac{h+J+c_2+c_3-c_1}{2})}{\Gamma(\frac{h+J-c_1-c_2-c_3}{2})\Gamma(\frac{h+J+c_3-c_1-c_2}{2})\Gamma(\frac{h+J+c_2-c_1-c_3}{2})\Gamma(\frac{h+J+c_1-c_2-c_3}{2})}. \tag{C.3}$$

The conformal partial wave $F_{\nu,J}(u,v)$ can be written as the integral over two three point functions, more specifically we have,

$$\begin{aligned}
&\frac{F_{\nu,J}(u,v)}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} \\
&= \frac{1}{\beta\, J!(\frac{d}{2}-1)_J}\int dx_5 V_J(x_1,x_2,x_5,D_5,c_1,c_2,i\nu)V_J(x_3,x_4,x_5,z_5,c_3,c_4,-i\nu),
\end{aligned} \tag{C.4}$$

where β is a normalization constant given by

$$\beta = \frac{2\pi^{1+h}K_{h-i\nu,J}(h-1+i\nu)_J\Gamma(1+i\nu)}{\nu^2\Gamma(h+J-i\nu)} \frac{\Gamma\left(\frac{h+J-\Delta_{12}-i\nu}{2}\right)\Gamma\left(\frac{h+J+\Delta_{12}-i\nu}{2}\right)}{\Gamma\left(\frac{h+J-\Delta_{12}+i\nu}{2}\right)\Gamma\left(\frac{h+J+\Delta_{12}+i\nu}{2}\right)}. \tag{C.5}$$

$$\times\frac{N_J(c_1,c_2,i\nu)N_J(c_3,c_4,-i\nu)}{(2\pi)^{2h}} \tag{C.6}$$

D_z is the differential operator that implements index contraction defined by

$$D_a = \left(h - 1 + z \cdot \frac{\partial}{\partial z}\right)\frac{\partial}{\partial z^a} - \frac{z_a}{2}\frac{\partial^2}{\partial z \cdot \partial z}. \tag{C.7}$$

and where $c_i = \Delta_i - h$. We can check that this function satisfies the differential equation defined in (2.6). The polarization vector z, satisfying $z^2 = 0$, was introduced for convenience since it encodes automatically the symmetric and traceless properties.

Now we use two identities involving the integral over two points of two three point functions and the integral of a two point function with a three point function[36]. The first identity is given by

$$\begin{aligned}
&\int dx_1 dx_2 V_J(x_1, x_2, x_3, z_3 - c_1, -c_2, -i\nu) V_J(x_1, x_2, x_4, z_4, c_1, c_2, i\nu') \qquad \text{(C.8)}\\
&= \tfrac{2\pi i}{\rho(\nu)}\delta_{J,J'}\big(\delta(\nu - \nu')(z_3 \cdot z_4)^J \delta(x_3, x_4) + \delta(\nu + \nu')G_{-i\nu}(x_3, x_4, z_3, z_4)\big)
\end{aligned}$$

with $\rho(c)$ given by

$$\rho(\nu) = \frac{((h + J - 1)^2 + \nu^2)\Gamma(J + h)\Gamma(h - 1 + i\nu)\Gamma(h - 1 - i\nu)}{2(2\pi)^h J!\Gamma(i\nu)\Gamma(-i\nu)} \tag{C.9}$$

While the second identity is

$$\int dx_0 G_{-i\nu}(x_3, x_0, z_3, D_{z_0}) V_J(x_1, x_2, x_0, z_0, c_1, c_2, i\nu) = V_J(x_1, x_2, x_0, z_0, c_1, c_2, -i\nu). \tag{C.10}$$

We can use (2.14), (C.4), (C.10) and (C.8) to derive an expression for the conformal partial wave coefficient $b_J(\nu)$ in terms of the reduced amplitude

$$\begin{aligned}
\frac{b_J(\nu)}{\beta} V_J(x_3, x_4, x_0, z_0, c_3, c_4, i\nu) &= \int \frac{dx_1 dx_2 \mathcal{A}(u, v)}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}}\\
\times \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} &\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} V_J(x_1, x_2, x_0, z_0, -c_1, -c_2, i\nu)
\end{aligned} \tag{C.11}$$

The dependence on the point x_0 can be removed by integrating both sides. The integral of a three point function is given by

$$\int dx_0 V_J(x_1, x_2, x_0, z_0, c_1, c_2, i\nu) = \frac{\pi^h \Gamma(i\nu)(h+i\nu-1)_J N_J(c_1,c_2,-i\nu)(x_{12}\cdot z_0)^J}{\Gamma(h-i\nu+J)(x_{12}^2)^{\frac{h+c_1+c_2+i\nu+J}{2}}} \tag{C.12}$$

where we have used

$$\int \frac{d^d x_0}{(x_{01}^2)^a (x_{02}^2)^b} = \frac{\pi^h \Gamma(a + b - h)}{(x_{12}^2)^{a+b-h}} \frac{\Gamma(h - a)\Gamma(h - b)}{\Gamma(a)\Gamma(b)\Gamma(2h - a - b)} \tag{C.13}$$

and

$$\frac{2^J(a)_J(b)_J(z_3\cdot x_{13}x_{23}^2 - z_3\cdot x_{23}x_{13}^2)^J}{(x_{13}^2)^{a+J}(x_{23}^2)^{b+J}} = D_J(a, z\cdot\nabla_1; b, z\cdot\nabla_2)\frac{1}{(x_{13}^2)^a(x_{23}^2)^b} \tag{C.14}$$

where $D_J(a,\alpha;b,\beta)$ is defined as

$$D_J(a,\alpha;b,\beta) = \sum_{k=0}^{J}\begin{pmatrix} J \\ k \end{pmatrix}(a+k)_{J-k}(b+J-k)_k(-\alpha)^k\beta^{J-k}. \tag{C.15}$$

Thus we obtain

$$\frac{b_J(\nu)(x_{34}\cdot z)^J}{(x_{34}^2)^{\frac{i\nu+J-h}{2}}} = \frac{\rho(\nu)\beta N_J(-c_1,-c_2,-\nu)}{4\pi i N_J(c_3,c_4,-\nu)}\int\frac{dx_1dx_2\mathcal{A}(u,v)}{(x_{12}^2)^{\frac{3h+i\nu+J}{2}}}(x_{12}\cdot z)^J\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}}$$

Now we just act with

$$(x_{34}\cdot D)^J = \big(x_{34}^\mu(h-1+z\cdot\partial_z)\partial_z^\mu - \frac{x_{34}\cdot z}{2}\partial_z\cdot\partial_z\big)^J \tag{C.16}$$

on both sides to obtain

$$\frac{b_J(\nu)}{(x_{34}^2)^{\frac{i\nu-h}{2}}} = \frac{J!^2\rho(\nu)N_J(-c_1,-c_2,-\nu)\beta}{4\pi i(2h-2)_J(h-1)_JN_J(c_3,c_4,-\nu)}\int\frac{dx_1dx_2\mathcal{A}(u,v)}{(x_{12}^2)^{\frac{3h+i\nu}{2}}}C_J^{h-1}\big(\frac{x_{12}\cdot x_{34}}{(x_{12}^2x_{34}^2)^{1/2}}\big)\big(\frac{x_{24}^2}{x_{14}^2}\big)^{\frac{\Delta_{12}}{2}}\big(\frac{x_{14}^2}{x_{13}^2}\big)^{\frac{\Delta_{34}}{2}} \tag{C.17}$$

We compute the conformal partial wave coefficient in appendix C.1 for a particular four point function.

C.1 Conformal partial wave coefficient

The goal of this section is to compute the partial wave coefficient for a particular correlation function. The partial wave coefficient defined in (C.17) involves the integration of two points x_1 and x_2. The connected tree level component of the correlator in the channel 15 has the simplest integrals

$$\begin{aligned} b_J(\nu) = &-\frac{2}{5(N^2-1)}\frac{J!q(\nu)\beta(x_{34}^2)^{\frac{i\nu-h+2}{2}}}{4\pi i\lambda(\nu)\psi_J(\nu)(2h-2)_J} \\ &\times\int\frac{dx_1dx_2}{(x_{12}^2)^{\frac{3h+i\nu-2}{2}}}\left[\frac{1}{x_{14}^2x_{23}^2}-\frac{1}{x_{13}^2x_{24}^2}\right]C_J^{h-1}\big(\frac{x_{12}\cdot x_{34}}{(x_{12}^2x_{34}^2)^{1/2}}\big). \end{aligned} \tag{C.18}$$

The amplitude is antisymmetric under the exchange of the points x_1 and x_2, so only odd spins give a non-zero contribution. The integrals to are of the type

$$\int\frac{dx_1(x_{12}\cdot x_{34})^q}{(x_{12}^2)^{\frac{3h+i\nu-2}{2}}x_{14}^2} = \sum_{m=0}^{q}\frac{(x_{34}^2)^{\frac{m}{2}}q!\,(x_{34}\cdot\partial_{x_2})^{q-m}}{2^q(q-m)!\frac{m}{2}!(\frac{3h+i\nu-2}{2}-q+\frac{m}{2})_{q-\frac{m}{2}}}\int\frac{dx_1}{(x_{12}^2)^{\frac{3h+i\nu-2+m-2q}{2}}x_{14}^2} \tag{C.19}$$

which can be evaluted in a systematic way. After doing this type of integrations we obtain that

$$b_J(\nu) = -\frac{4}{5(N^2-1)}\frac{J!q(\nu)\beta}{(2h-2)_J}\frac{\nu^2(\nu^2+(1+J)^2)(1+J)}{32\pi^3 i}$$
$$\frac{(-1)^{J+1}\pi^{2h}\Gamma^2(h-1)(2h-2)_J\Gamma\left(\frac{2+J-h-i\nu}{2}\right)\Gamma\left(\frac{2+J-h+i\nu}{2}\right)}{J!\Gamma\left(\frac{3h+J-i\nu-2}{2}\right)\Gamma\left(\frac{3h+J+i\nu-2}{2}\right)} \tag{C.20}$$

for odd J.

Appendix D

Factorization from the shadow operator formalism

In this appendix, we detail the calculations involved in the factorization method described in section 3.3.1. We consider exchanged operators of spin 0, 1 and 2. Auxiliary calculations are presented in the last three subsections: in D.3 we construct the projector for traceless symmetric tensors, in D.4 we evaluate some useful conformal integrals and in D.4.2 we derive an identity involving Mellin integrals.

D.1 Factorization on a scalar operator

In this subsection we fill in the gaps of the derivation presented in section 3.3.1. We shall use the notation

$$[x_{ab}]^{\gamma} = [x_a - x_b]^{\gamma} \equiv \frac{\Gamma(\gamma)}{(x_a - x_b)^{2\gamma}} \tag{D.1}$$

to shorten the expressions that follow.

Using (3.42) and (3.44), expression (3.40) can be written as follows

$$\frac{1}{\mathcal{N}_\Delta} \int [d\lambda][d\rho]\, I M_L M_R \prod_{1\leq a<b\leq k} [x_{ab}]^{\lambda_{ab}} \prod_{k<i<j\leq n} [x_{ij}]^{\rho_{ij}} \ , \tag{D.2}$$

where I is the scalar conformal integral

$$I = \int dydz\ [y-z]^{d-\Delta} \prod_{1\leq a\leq k} [x_a - y]^{\lambda_a} \prod_{k<i\leq n} [x_i - z]^{\rho_i} \tag{D.3}$$

which, in appendix (D.4), we show it can be written as

$$I = \pi^d \int [d\gamma]\, \frac{\Gamma(B)\Gamma(A-B)}{\Gamma(A)} \prod_{1\leq\mu\leq\nu\leq n} [x_{\mu\nu}]^{\gamma_{\mu\nu}} \tag{D.4}$$

with $B = \frac{2\Delta - d}{2}$ and $A = \sum_{1\le a<b\le k} \gamma_{ab} = \frac{\Delta - \gamma_{LR}}{2}$. Replacing this expression in (D.2) and shifting the integration variables $\gamma_{\mu\nu}$ to absorb the factors $[x_{ab}]^{\lambda_{ab}}$ and $[x_{ij}]^{\rho_{ij}}$ leads directly to (3.46).

We shall now determine the residue of F_L at $\gamma_{LR} = \Delta + 2m$ by deforming the integration contours in (3.47). Using the constraints (3.43) we can solve for

$$\lambda_{12} = -\frac{1}{2}\left[\Delta + 2\sum_{a=3}^{k} \lambda_{1a} + \sum_{a,b=2}^{k} \lambda_{ab}\right] \tag{D.5}$$

and use as independent integration variables $\lambda_{13}, \lambda_{14}, \ldots, \lambda_{1k}$ and λ_{ab} for $2 \le a < b \le k$. Then, the measure reads

$$\int [d\lambda] = \int_{-i\infty}^{i\infty} \prod_{3\le a\le k} \frac{d\lambda_{1a}}{2\pi i} \prod_{2\le a<b\le k} \frac{d\lambda_{ab}}{2\pi i} \,. \tag{D.6}$$

These $(k-2)(k+1)/2$ integrals can be done by deforming the contour to the right and picking up poles of the integrand in (3.47). There are explicit poles of the Γ-functions at $\lambda_{ab} = \gamma_{ab} + n_{ab}$ with $n_{ab} = 0, 1, 2, \ldots$, and possibly other poles of the Mellin amplitude M_L. This gives

$$\begin{aligned} F_L = \sum_{n_{ab}\ge 0} M_L(\gamma_{ab} + n_{ab}) \frac{\Gamma(\lambda_{12})\,\Gamma(\gamma_{12} - \lambda_{12})}{\Gamma(\gamma_{12})} \sideset{}{'}\prod_{1\le a<b\le k} \frac{(-1)^{n_{ab}} (\gamma_{ab})_{n_{ab}}}{n_{ab}!} \\ + \text{contributions from poles of } M_L \end{aligned} \tag{D.7}$$

where λ_{12} is given by (D.5) with $\lambda_{ab} = \gamma_{ab} + n_{ab}$,

$$\lambda_{12} = \gamma_{12} - \frac{1}{2}\left[\Delta - \gamma_{LR} + 2 \sideset{}{'}\sum_{1\le a<b\le k} n_{ab}\right] \tag{D.8}$$

and the prime denotes that $ab = 12$ is absent from the sum or product. From (D.7), it is clear that F_L will have poles when $\gamma_{12} - \lambda_{12} = -n_{12}$ with $n_{12} = 0, 1, 2, \ldots$. This corresponds to a pole at

$$\gamma_{LR} = \Delta + 2m \,, \qquad m = \sum_{1\le a<b\le k} n_{ab} \,, \tag{D.9}$$

with residue

$$F_L \approx \frac{-2(-1)^m}{\gamma_{LR} - \Delta - 2m} \sum_{\substack{n_{ab}\ge 0 \\ \sum n_{ab} = m}} M_L(\gamma_{ab} + n_{ab}) \prod_{1\le a<b\le k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \,. \tag{D.10}$$

D.2 Factorization on a vector operator

This section will be very similar to the scalar case. The main difference is that we will use the embedding formalism to simplify the calculations. The goal is to determine the poles and residues of the Mellin amplitude associated with

$$\langle \mathcal{O}_1(P_1)\dots\mathcal{O}_k(P_k)|\mathcal{O}|\mathcal{O}_{k+1}(P_{k+1})\dots\mathcal{O}_n(P_n)\rangle = \int dQ_1 dQ_2 \langle \mathcal{O}_1(P_1)\dots\mathcal{O}_k(P_k)\mathcal{O}(Q_1,Z_1)\rangle$$
$$\frac{\Gamma(d-\Delta+1)}{\mathcal{N}_{\Delta,1}} \frac{(\vec{Z}_1\cdot\vec{Z}_2)(Q_1\cdot Q_2)-(\vec{Z}_1\cdot Q_2)(Q_1\cdot\vec{Z}_2)}{(-2Q_1\cdot Q_2)^{d-\Delta+1}} \langle \mathcal{O}(Q_2,Z_2)\mathcal{O}_{k+1}(P_{k+1})\dots\mathcal{O}_n(P_n)\rangle \quad \text{(D.11)}$$

where we have used the projector for tensor operators described in appendix D.3. We shall use the notation

$$[P,Q]^a = \frac{\Gamma(a)}{(-2P\cdot Q)^a}\ , \qquad [P_{ij}]^a = [P_i,P_j]^a = \frac{\Gamma(a)}{(-2P_i\cdot P_j)^a} \quad \text{(D.12)}$$

to shorten the expressions that follow. We start by writing the correlation functions that appear in (D.11) in the Mellin representation,

$$\langle \mathcal{O}_1(P_1)\dots\mathcal{O}_k(P_k)\mathcal{O}(Q_1,Z_1)\rangle = \int [d\lambda] \sum_{l=1}^{k} (Z_1\cdot P_l)\, M_L^l \prod_{1\le a<b\le k} [P_{ab}]^{\lambda_{ab}} \prod_{1\le a\le k} [P_a,Q_1]^{\lambda_a+\delta_a^l}$$

where δ_a^l is the Kronecker-delta and

$$\lambda_a = -\sum_{b=1}^{k}\lambda_{ab}\ , \qquad \lambda_{aa}=-\Delta_a\ , \qquad \lambda_{ab}=\lambda_{ba}\ , \qquad \sum_{a,b=1}^{k}\lambda_{ab} = 1-\Delta\ . \quad \text{(D.13)}$$

Similarly,

$$\langle \mathcal{O}(Q_2,Z_2)\mathcal{O}_{k+1}(P_{k+1})\dots\mathcal{O}_n(P_n)\rangle = \int [d\rho] \sum_{r=k+1}^{n} (Z_2\cdot P_r) M_R^r \prod_{k<i<j\le n} [P_{ij}]^{\rho_{ij}} \prod_{k<i\le n} [P_i,Q_2]^{\rho_i+\delta_i^r}$$

where

$$\rho_i = -\sum_{j=k+1}^{n}\rho_{ij}\ , \qquad \rho_{ii}=-\Delta_i\ , \qquad \rho_{ij}=\rho_{ji}\ , \qquad \sum_{i,j=k+1}^{n}\rho_{ij} = 1-\Delta\ . \quad \text{(D.14)}$$

Expression (D.11) can then be written as follows

$$\begin{aligned} &\frac{1}{\mathcal{N}_{\Delta,1}} \int [d\lambda][d\rho] \sum_{l=1}^{k}\sum_{r=k+1}^{n} M_L^l M_R^r \prod_{1\le a<b\le k} [P_{ab}]^{\lambda_{ab}} \prod_{k<i<j\le n} [P_{ij}]^{\rho_{ij}} \\ &\times \int dQ_1 dQ_2 \prod_{1\le a\le k} [P_a,Q_1]^{\lambda_a+\delta_a^l} \prod_{k<i\le n} [P_i,Q_2]^{\rho_i+\delta_i^r} \\ &\times \left(\frac{\Delta-d}{2} (P_l\cdot P_r)[Q_{12}]^{d-\Delta} - (P_l\cdot Q_2)(P_r\cdot Q_1)[Q_{12}]^{d-\Delta+1} \right) \end{aligned} \quad \text{(D.15)}$$

where δ_a^l and δ_i^r are Kronecker-deltas. Expanding the last line, we obtain two integrals with different structure. The integral over Q_1 and Q_2 of the first term in (D.15) can be done using the conformal integral for scalars (D.57). The integral over Q_1 and Q_2 in the second term of (D.15) can be done using the vector conformal integral (D.66).

Putting all ingredients together we obtain that that the factorization for the vector case is given by

$$M_{\mathcal{O}}(\gamma_{\mu\nu}) = \frac{\pi^d}{\mathcal{N}_{\Delta,1}} \frac{\Gamma(B)\Gamma(A-B)}{4\Gamma(A+1)} \sum_{l=1}^{k} \sum_{r=k+1}^{n} \big(A(d-\Delta-1)\gamma_{lr} - B\gamma_l\gamma_r\big) F_L^l \times F_R^r \tag{D.16}$$

with $B = \frac{2\Delta-d}{2}$, $A = \sum_{1\leq a<b\leq k} \gamma_{ab} = \frac{\Delta-1-\gamma_{LR}}{2}$ and

$$F_L^l = \int [d\lambda] M_L^l(\lambda_{ab}) \prod_{1\leq a<b\leq k} \frac{\Gamma(\lambda_{ab})\Gamma(\gamma_{ab}-\lambda_{ab})}{\Gamma(\gamma_{ab})} \tag{D.17}$$

$$F_R^r = \int [d\rho] M_R^r(\rho_{ij}) \prod_{k<i<j\leq n} \frac{\Gamma(\rho_{ij})\Gamma(\gamma_{ij}-\rho_{ij})}{\Gamma(\gamma_{ij})} \tag{D.18}$$

Looking for the poles of γ_{LR} we obtain $\Gamma(A+1) \approx \frac{(-2)}{(m-1)!} \frac{(-1)^{m-1}}{\gamma_{LR}-\Delta-2m+1}$ and $F_L^l \approx \frac{(-2)(-1)^m}{\gamma_{LR}-\Delta-2m+1} L_m^l$ where

$$L_m^l = \sum_{\sum n_{ab}=m} M^l(\gamma_{ab}+n_{ab}) \prod_{1\leq a<b\leq k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \tag{D.19}$$

and similarly for R_m^r. The contribution of the physical operator $\mathcal{O}$ to the pole structure of M_O can therefore be written as

$$M \approx \frac{m!}{\left(1+\Delta-\frac{d}{2}\right)_m} \frac{\kappa_{\Delta,1}}{\gamma_{LR}-\Delta+1-2m} \sum_{l=1}^{k} \sum_{r=k+1}^{n} \left[\gamma_{lr} + \frac{d-2\Delta}{2m(\Delta-d+1)}\gamma_l\gamma_r\right] L_m^l R_m^r \ , \tag{D.20}$$

where we used $\Gamma(A-B) = (-1)^m \frac{\Gamma(d/2-\Delta)}{(1-d/2+\Delta)_m}$ and where we defined

$$\kappa_{\Delta,1} = \frac{\pi^d \Gamma\left(\frac{2\Delta-d}{2}\right)\Gamma\left(\frac{d-2\Delta}{2}\right)(\Delta-d-1)\Gamma(\Delta-d+1)}{2\mathcal{N}_{\Delta,1}} = \Delta\Gamma(\Delta-1) \ . \tag{D.21}$$

D.3 Projector for tensor operators

In the embedding formalism, the projector for tensor operators takes the form [13]

$$|\mathcal{O}| = \frac{\Gamma(d-\Delta+J)}{\mathcal{N}_{\Delta,J}} \int dP_1 dP_2 |\mathcal{O}(P_1,Z_1)\rangle \frac{\Big((\overleftarrow{Z}_1\cdot\vec{Z}_2)(P_1\cdot P_2) - (\overleftarrow{Z}_1\cdot P_2)(P_1\cdot\vec{Z}_2)\Big)^J}{(-2P_1\cdot P_2)^{d-\Delta+J}} \langle\mathcal{O}(P_2,Z_2)|$$

where the symbols $\overleftarrow{Z}_1$ and $\vec{Z}_2$ mean that we should expand and contract using

$$\vec{Z}^{B_1} \dots \vec{Z}^{B_J} \, Z^{A_1} \dots Z^{A_J} = \pi^{A_1 \dots A_J, B_1 \dots B_J} \ , \tag{D.22}$$

where $\pi^{A_1 \dots A_J, B_1 \dots B_J}$ is the projector onto traceless symmetric tensors with J indices. To determine the normalization constant $\mathcal{N}_{\Delta,J}$ we impose that

$$\langle \mathcal{O}(P,Z) \dots \rangle = \langle \mathcal{O}(P,Z) | \mathcal{O} | \dots \rangle \tag{D.23}$$

where the dots stand for any other operators. We normalize the operator $\mathcal{O}$ to have the following two point function

$$\langle \mathcal{O}(P,Z) \mathcal{O}(P_1,Z_1) \rangle = \frac{\left((Z \cdot Z_1)(-2P \cdot P_1) - 2(Z \cdot P_1)(P \cdot Z_1)\right)^J}{(-2P \cdot P_1)^{\Delta+J}} \ . \tag{D.24}$$

In general, the correlation function $\langle \mathcal{O}(P,Z) \dots \rangle$ of $\mathcal{O}$ with any other operators can be written as a linear combination (or integral) of

$$\frac{\left((Z \cdot Y_1)(P \cdot Y_2) - (Z \cdot Y_2)(P \cdot Y_1)\right)^J}{(-2P \cdot X)^{\Delta+J}} \tag{D.25}$$

with different X, Y_1 and Y_2. Therefore, equation (D.23) is equivalent to

$$\begin{aligned} &\frac{\left((Z \cdot Y_1)(P \cdot Y_2) - (Z \cdot Y_2)(P \cdot Y_1)\right)^J}{(-2P \cdot X)^{\Delta+J}} \\ =&\frac{2^J \Gamma(d-\Delta+J)}{\mathcal{N}_{\Delta,J}} \int \frac{dP_1 dP_2 \ \Omega(Z,P,P_1,P_2,Y_1,Y_2)}{(-2P \cdot P_1)^{\Delta+J} (-2P_1 \cdot P_2)^{d-\Delta+J} (-2P_2 \cdot X)^{\Delta+J}} \end{aligned} \tag{D.26}$$

where the numerator $\Omega(Z,P,P_1,P_2,Y_1,Y_2)$ is given by

$$\begin{aligned} &\Big((Z \cdot P_1)(P \cdot Z_1) - (Z \cdot Z_1)(P \cdot P_1)\Big)^J \Big((\overleftarrow{Z}_1 \cdot \vec{Z}_2)(P_1 \cdot P_2) - (\overleftarrow{Z}_1 \cdot P_2)(P_1 \cdot \vec{Z}_2)\Big)^J \\ &\Big((Z_2 \cdot Y_1)(P_2 \cdot Y_2) - (Z_2 \cdot Y_2)(P_2 \cdot Y_1)\Big)^J \end{aligned} \tag{D.27}$$

$$\begin{aligned} = \big[&(P_1 \cdot Z)(P \cdot Y_1)(P_2 \cdot P_1)(P_2 \cdot Y_2) - (P_1 \cdot P)(Z \cdot Y_1)(P_2 \cdot P_1)(P_2 \cdot Y_2) \\ &(P_1 \cdot Z)(P \cdot P_2)(Y_2 \cdot P_1)(P_2 \cdot Y_1) - (P_1 \cdot P)(Z \cdot P_2)(Y_2 \cdot P_1)(P_2 \cdot Y_1) \\ &(P_1 \cdot P)(Z \cdot P_2)(Y_1 \cdot P_1)(P_2 \cdot Y_2) - (P_1 \cdot Z)(P \cdot P_2)(Y_1 \cdot P_1)(P_2 \cdot Y_2) \\ &(P_1 \cdot P)(Z \cdot Y_2)(P_1 \cdot P_2)(P_2 \cdot Y_1) - (P_1 \cdot Z)(P \cdot Y_2)(P_1 \cdot P_2)(P_2 \cdot Y_1)\big]^J \end{aligned} \tag{D.28}$$

To perform the integrals we use the following trick

$$\int \frac{dP_2\ \Omega(Z,P,P_1,P_2,Y_1,Y_2)}{(-2P_1\cdot P_2)^{d-\Delta+J}(-2P_2\cdot X)^{\Delta+J}} \tag{D.29}$$

$$=\frac{\Gamma(\Delta-J)}{\Gamma(\Delta+J)}\Omega\left(P_2\to\frac{1}{2}\frac{\partial}{\partial X}\right)\int\frac{dP_2}{(-2P_1\cdot P_2)^{d-\Delta+J}(-2P_2\cdot X)^{\Delta-J}} \tag{D.30}$$

$$=\frac{\Gamma(\Delta-J)}{\Gamma(\Delta+J)}\Omega\left(P_2\to\frac{1}{2}\frac{\partial}{\partial X}\right)\frac{\pi^h\Gamma(\Delta-J-h)}{\Gamma(\Delta-J)}\frac{(-X^2)^{h-\Delta+J}}{(-2P_1\cdot X)^{d-\Delta+J}} \tag{D.31}$$

$$=\frac{\pi^h\Gamma(\Delta-J-h)}{\Gamma(\Delta+J)}(-X^2)^{h-\Delta+J}\Omega\left(P_2\to\frac{1}{2}\frac{\partial}{\partial X}+(h-\Delta+J)\frac{X}{X^2}\right)\frac{1}{(-2P_1\cdot X)^{d-\Delta+J}}$$

$$=\frac{\pi^h\Gamma(\Delta-J-h)}{(2J)!\Gamma(\Delta+J)}(-X^2)^{h-\Delta+J}\left(D_X\cdot\frac{\partial}{\partial P_2}\right)^{2J}\frac{\Omega(Z,P,P_1,P_2,Y_1,Y_2)}{(-2P_1\cdot X)^{d-\Delta+J}} \tag{D.32}$$

where $h=d/2$ and

$$D_X=\frac{1}{2}\frac{\partial}{\partial X}+(h-\Delta+J)\frac{X}{X^2}\ . \tag{D.33}$$

Doing the integral over P_1 using the same technique we obtain

$$\int \frac{dP_1 dP_2\ \Omega(Z,P,P_1,P_2,Y_1,Y_2)}{(-2P\cdot P_1)^{\Delta+J}(-2P_1\cdot P_2)^{d-\Delta+J}(-2P_2\cdot X)^{\Delta+J}} \tag{D.34}$$

$$=\frac{\pi^d\Gamma(\Delta-J-h)\Gamma(h-\Delta-J)}{(2J)!^2\Gamma(\Delta+J)\Gamma(d-\Delta+J)}(-X^2)^{h-\Delta+J}\left(D_X\cdot\frac{\partial}{\partial P_2}\right)^{2J} \tag{D.35}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial P_1}\right)^{2J}\frac{\Omega(Z,P,P_1,P_2,Y_1,Y_2)}{(-2P\cdot X)^{\Delta+J}}(-X^2)^{\Delta-h+J}$$

$$=\frac{\pi^d\Gamma(\Delta-J-h)\Gamma(h-\Delta-J)}{(2J)!^2\Gamma(\Delta+J)\Gamma(d-\Delta+J)} \tag{D.36}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial P_2}\right)^{2J}(-X^2)^{h-\Delta+J}\left(\frac{1}{2}\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial P_1}\right)^{2J}(-X^2)^{\Delta-h+J}\frac{\Omega(Z,P,P_1,P_2,Y_1,Y_2)}{(-2P\cdot X)^{\Delta+J}}$$

It is not hard to see that expanding the derivatives in the last expression leads to

$$\sum_{n=0}^{4J}\frac{Q_n(X,Z,P,Y_1,Y_2)}{(-2P\cdot X)^{\Delta+J+n}} \tag{D.37}$$

where Q_n are homogeneous polynomials of degree n in X, degree $(J+n)$ in P and degree J in Z, Y_1 and Y_2. Moreover, the polynomials Q_n inherit the following properties from the function Ω,

$$Q_n(X,Z,P,Y_1,Y_2)=Q_n(X,Z+\alpha P,P,Y_1,Y_2) \tag{D.38}$$

$$=Q_n(X,Z,P,Y_1+\alpha Y_2,Y_2) \tag{D.39}$$

$$=Q_n(X,Z,P,Y_1,Y_2+\alpha Y_1) \tag{D.40}$$

This means that Q_n can only depend on Z, Y_1 and Y_2 through the antisymmetric tensors $Z^{[A}P^{B]}$ and $Y_1^{[A}Y_2^{B]}$. All these properties together, imply that Q_n must be proportional to

$$(-2P\cdot X)^n \left((Z\cdot Y_1)(P\cdot Y_2)-(Z\cdot Y_2)(P\cdot Y_1)\right)^J \ . \tag{D.41}$$

Therefore, we conclude that

$$\begin{aligned}&\left(\frac{1}{2}\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial P_2}\right)^{2J}(-X^2)^{h-\Delta+J}\left(\frac{1}{2}\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial P_1}\right)^{2J}(-X^2)^{\Delta-h+J}\frac{\Omega(Z,P,P_1,P_2,Y_1,Y_2)}{(-2P\cdot X)^{\Delta+J}}\\ =&A_{\Delta,J}\frac{\left((Z\cdot Y_1)(P\cdot Y_2)-(Z\cdot Y_2)(P\cdot Y_1)\right)^J}{(-2P\cdot X)^{\Delta+J}}\end{aligned} \tag{D.42}$$

for some constant $A_{\Delta,J}$. Putting everything together, the normalization constant is given by

$$\mathcal{N}_{\Delta,J}=\frac{2^J\pi^d\Gamma\left(\Delta-\frac{d}{2}-J\right)\Gamma\left(\frac{d}{2}-\Delta-J\right)}{(2J)!^2\Gamma(\Delta+J)}A_{\Delta,J} \ . \tag{D.43}$$

Finally, we conjecture that

$$A_{\Delta,J}=(-1)^J2^{-2J}(2J)!^2(\Delta-1)_J\left(\Delta-\frac{d}{2}+1\right)_J\left(\Delta-\frac{d}{2}-J\right)_J(\Delta-d-J+2)_J \ . \tag{D.44}$$

Using *Mathematica* we verified this formula up to $J=3$. Unfortunately, higher values of J take too much time to compute all the derivatives in (D.42).

D.4 Conformal integrals

Integration over one point

The basic integral we need is given by Symanzik's formula

$$I=\int dQ\prod_{\mu=1}^{n}[P_\mu,Q]^{\Delta_\mu}=\pi^{\frac{d}{2}}\int[d\gamma_{\mu\nu}]\prod_{1\leq\mu<\nu\leq n}[P_{\mu\nu}]^{\gamma_{\mu\nu}} \tag{D.45}$$

where $\sum_{\mu=1}^{n} \Delta_\mu = d$ and the measure $[d\gamma_{\mu\nu}]$ is the usual measure over the constraint surface $\sum_{\nu:\nu\neq\mu}^{n} \gamma_{\mu\nu} = \Delta_\mu$. It will be useful to write this integral in alternative ways,

$$I = \int_0^\infty \prod_{\mu=1}^{n} dt_\mu t_\mu^{\Delta_\mu - 1} \int dQ e^{2Q\cdot(\sum t_\mu P_\mu)} \tag{D.46}$$

$$= \int_0^\infty \prod_{\mu=1}^{n} dt_\mu t_\mu^{\Delta_\mu - 1} \int dQ e^{2Q\cdot T} \int_0^\infty ds\, \delta\left(s - \sum_{\mu=1}^{n} t_\mu\right) \tag{D.47}$$

$$= \Gamma(d) \int_0^\infty \prod_{\mu=1}^{n} dt_\mu t_\mu^{\Delta_\mu - 1} \delta\left(1 - \sum_{\mu=1}^{n} t_\mu\right) \int \frac{dQ}{(-2Q\cdot T)^d} \tag{D.48}$$

$$= \pi^{d/2} \int_0^\infty \prod_{\mu=1}^{n} dt_\mu t_\mu^{\Delta_\mu - 1} \delta\left(1 - \sum_{\mu=1}^{n} t_\mu\right) \frac{\Gamma(d/2)}{(-T^2)^{d/2}} \tag{D.49}$$

where t_μ are real variables, $T^A = \sum_{\mu=1}^{n} t_\mu P_\mu^A$ are vectors in the embedding space $\mathbb{M}^{d+2}$ and we have used results of [13] (for example, equation (2.21)). Consider now the more general integral

$$I^{A_1\ldots A_l} = \int dQ\, Q^{A_1} \ldots Q^{A_l} \prod_{\mu=1}^{n} [P_\mu, Q]^{\Delta_\mu} \tag{D.50}$$

where $\sum_{\mu=1}^{n} \Delta_\mu = d + l$. We can write

$$I^{A_1\ldots A_l} = \Gamma(d+l) \int_0^\infty \prod_{\mu=1}^{n} dt_\mu t_\mu^{\Delta_\mu - 1} \delta\left(1 - \sum t_\mu\right) \int dQ \frac{Q^{A_1} \ldots Q^{A_l}}{(-2Q\cdot T)^{d+l}} \tag{D.51}$$

$$= \pi^{d/2}\Gamma(d/2 + l) \int_0^\infty \prod_{\mu=1}^{n} dt_\mu t_\mu^{\Delta_\mu - 1} \delta\left(1 - \sum t_\mu\right) \frac{T^{A_1} \ldots T^{A_l} - \text{traces}}{(-T^2)^{d/2+l}} \tag{D.52}$$

$$= \pi^{d/2} \sum_{\alpha_i=1}^{n} P_{\alpha_1}^{A_1} \ldots P_{\alpha_l}^{A_l} \int [d\gamma_{\mu\nu}^{(\alpha)}] \prod_{1\leq\mu<\nu\leq n} [P_{\mu\nu}]^{\gamma_{\mu\nu}^{(\alpha)}} - \text{traces} \tag{D.53}$$

where the integration variables have to satisfy a constraint that depends on the set of $\alpha_1, \ldots, \alpha_l$, namely $\sum_{\nu:\nu\neq\mu} \gamma_{\mu\nu}^{(\alpha)} = \Delta_\mu + \delta_\mu^{\alpha_1} + \cdots + \delta_\mu^{\alpha_l}$. In practice we will often need to compute only a piece of (D.53) because we will have a special point that we can now call P_1, and we will be interested only in the terms of $I^{A_1\ldots A_l}$ that are not proportional to $P_1^{A_i}$ for any $i = 1, \ldots l$. With this simplification we can rewrite the integral I^{A_1} in such a way to avoid the dependence on α of the constraint as follows

$$I^{A_1} = \pi^{d/2} \sum_{\alpha_i=2}^{n} P_{\alpha_1}^{A_1} \int [d\gamma_{\mu\nu}] \prod_{2\leq\mu<\nu\leq n} [P_{\mu\nu}]^{\gamma_{\mu\nu}} \prod_{2\leq\mu\leq n} [P_{1\mu}]^{\gamma_\mu + \delta_\mu^{\alpha_1}} + \ldots \tag{D.54}$$

where the dots stand by contributions proportional to $P_1^{A_1}$. We defined

$$\gamma_\mu = -\sum_{\substack{\nu=2 \\ \mu\neq\nu}}^{n} \gamma_{\mu\nu} + \Delta_\mu \tag{D.55}$$

in such a way that there is only one constraint to impose on the integration variables, namely

$$\sum_{\substack{\mu,\nu=2 \\ \mu\neq\nu}}^{n} \gamma_{\mu\nu} = \sum_{\mu=2}^{n} \Delta_\mu - \Delta_1 + 1 = d + 2 - 2\Delta_1 \ . \tag{D.56}$$

Conformal integral - integrating over two points

Scalar conformal integral

The goal of this section is to compute the integral,

$$I = \int dQ_1 dQ_2 \ [Q_{12}]^{d-\Delta} \prod_{1\leq a\leq k} [P_a, Q_1]^{\lambda_a} \prod_{k<i\leq n} [P_i, Q_2]^{\rho_i} \tag{D.57}$$

where the variables λ_a and ρ_i satisfy,

$$\sum_{a=1}^{k} \lambda_a = \Delta \, , \qquad \sum_{i=k+1}^{n} \rho_i = \Delta. \tag{D.58}$$

Let us compute first the Q_1 integral,

$$\int dQ_1 \, [Q_{12}]^{d-\Delta} \prod_{1\leq a\leq k} [P_a, Q_1]^{\lambda_a} = \pi^{\frac{d}{2}} \int [d\beta] \prod_{1\leq a\leq b\leq k} [P_{ab}]^{\beta_{ab}} \prod_{1\leq a\leq k} [P_a, Q_2]^{\beta_a} \tag{D.59}$$

where $\sum_{a=1}^{k} \beta_a = d - \Delta$ and $\sum_{b:a\neq b} \beta_{ab} = \lambda_a - \beta_a$, in particular we have $\sum_{a,b:a\neq b} \beta_{ab} = 2\Delta - d$. The integration over Q_2 can also be done using Symanzik's rule,

$$\int dQ_2 \prod_{k<i\leq n} [P_i, Q_2]^{\rho_i} \prod_{1\leq a\leq k} [P_a, Q_2]^{\beta_a} = \pi^{\frac{d}{2}} \int [d\tau] \prod_{1\leq \mu<\nu\leq n} [P_{\mu\nu}]^{\tau_{\mu\nu}} \tag{D.60}$$

where the variables $\tau_{\mu\nu}$ satisfy $\sum_{\mu=1}^{n} \tau_{\mu\nu} = 0$, $\tau_{aa} = -\beta_a$ $\tau_{ii} = -\rho_i$. The integral I can be written as

$$I = \pi^d \int [d\beta][d\tau] \prod_{1\leq a<b\leq k} [P_{ab}]^{\beta_{ab}} \prod_{1\leq \mu<\nu\leq n} [P_{\mu\nu}]^{\tau_{\mu\nu}}. \tag{D.61}$$

We can change variables,

$$\tau_{\mu\nu} = \begin{cases} \gamma_{ab} - \beta_{ab} & \text{if } \ \mu = a \leq k \ \text{ and } \ \nu = b \leq k \\ \gamma_{\mu\nu} & \text{otherwise} \end{cases} . \tag{D.62}$$

The function I can be rewritten as,

$$I = \pi^d \int [d\gamma][d\beta] \prod_{1\leq a<b\leq k} \frac{\Gamma(\beta_{ab})\Gamma(\gamma_{ab}-\beta_{ab})}{\Gamma(\gamma_{ab})} \prod_{1\leq\mu\leq\nu\leq n} [P_{\mu\nu}]^{\gamma_{\mu\nu}} , \tag{D.63}$$

where the integration variables $\gamma_{\mu\nu}$ satisfy the following constraints

$$\sum_{\mu=1}^{n} \gamma_{\mu\nu} = 0 , \qquad \gamma_{aa} = -\lambda_a , \qquad \gamma_{ii} = -\rho_i , \qquad \text{for } \begin{cases} a = 1, \dots k \\ i = k+1, \dots n \end{cases} . \tag{D.64}$$

The function I is then given by,

$$I = \pi^d \int [d\gamma] \frac{\Gamma(B)\Gamma(A-B)}{\Gamma(A)} \prod_{1\leq\mu\leq\nu\leq n} [P_{\mu\nu}]^{\gamma_{\mu\nu}} \tag{D.65}$$

with $B = \sum_{1\leq a<b\leq k} \beta_{ab} = \frac{2\Delta-d}{2}$ and $A = \sum_{1\leq a<b\leq k} \gamma_{ab} = \frac{\Delta-\gamma_{LR}}{2}$. In the derivation of this result we have used the identity (D.73).

D.4.1 Vector integral

The goal of this section is to compute the conformal integral,

$$I_{l,r} = \int dQ_1 dQ_2 \ (P_l \cdot Q_2)(P_r \cdot Q_1)[Q_{12}]^{d-\Delta+1} \prod_{1\leq a\leq k} [P_a, Q_1]^{\lambda_a+\delta_a^l} \prod_{k<i\leq n} [P_i, Q_2]^{\rho_i+\delta_i^r} \tag{D.66}$$

where the variables λ_a and ρ_i satisfy,

$$\sum_{a=1}^{k} \lambda_a = \Delta - 1 , \qquad \sum_{i=k+1}^{n} \rho_i = \Delta - 1. \tag{D.67}$$

This integral enters in the factorization of the vector and spin two, where we have the transversality condition (3.16). A moment of thought shows that it is sufficient to compute $I_{l,r}$ up to terms proportional to λ_l or ρ_r, so in the following we will drop these. Let us integrate first over

Q_1, using (D.54) we have

$$\begin{aligned}&\int dQ_1\,(Q_1\cdot P_r)\,[Q_{12}]^{d-\Delta+1}\prod_{1\le a\le k}[P_a,Q_1]^{\lambda_a+\delta_a^l}\\&=\pi^{\frac{d}{2}}\sum_{c=1}^{k}\frac{P_{cr}}{-2}\int[d\beta]\prod_{1\le a<b\le k}[P_{ab}]^{\beta_{ab}}\prod_{1\le a\le k}[P_a,Q_2]^{\beta_a+\delta_a^l+\delta_a^c}+\dots\end{aligned}\tag{D.68}$$

where the dots stand by terms proportional to $P_r\cdot Q_2$ that we can drop since they give rise to a contribution proportional to ρ_r and where the variables β_{ab} satisfy

$$\beta_a=-\sum_{b=1}^{k}\beta_{ab}\ ,\qquad \beta_{aa}=-\lambda_a\ ,\qquad \sum_{\substack{a,b=1\\a\neq b}}^{k}\beta_{ab}=2\Delta-d\ .\tag{D.69}$$

Now we compute the integral over Q_2 which is also of Symanzik type. Using (D.45) and shifting the integration variables in order to absorb the factors P_{cr} and $[P_{ab}]^{\beta_{ab}}$ in a single term, we obtain

$$I_{l,r}=\frac{\pi^d}{4}\sum_{c=1}^{k}\int[d\beta][d\gamma]\ \gamma_{cr}(\beta_l+\delta_l^c)\prod_{1\le\mu<\nu\le n}[P_{\mu\nu}]^{\gamma_{\mu\nu}}\prod_{1\le a<b\le k}\frac{\Gamma(\beta_{ab})\Gamma(\gamma_{ab}-\beta_{ab})}{\Gamma(\gamma_{ab})}\ ,$$

where the integration variables $\gamma_{\mu\nu}$ have to satisfy the following constraints

$$\sum_{\mu=1}^{n}\gamma_{\mu\nu}=0\ ,\qquad \gamma_{aa}=-\lambda_a\ ,\qquad \gamma_{ii}=-\rho_i\ ,\qquad \text{for}\ \begin{cases}a=1,\dots k\\ i=k+1,\dots n\end{cases}\ .\tag{D.70}$$

To integrate over β we use (D.73) and (D.74). The function $I_{l,r}$ can be simplified to

$$I_{l,r}=\frac{\pi^d}{4}\sum_{c=1}^{k}\int[d\gamma]\ \frac{\Gamma(B)\Gamma(A-B)}{\Gamma(A+1)}\ \gamma_{cr}\left(A(\lambda_l+\delta_l^c)+B(\gamma_l-\lambda_l)\right)\prod_{1\le\mu<\nu\le n}[P_{\mu\nu}]^{\gamma_{\mu\nu}}$$

where $A=\frac{\Delta-1-\gamma_{LR}}{2}$ and $B=\frac{2\Delta-d}{2}$ and where we defined as usual $\gamma_l=\sum_{i=k+1}^{n}\gamma_{li}$. Simplifying the sum over c and dropping terms proportional to λ_l we finally obtain [1]

$$I_{l,r}=\frac{\pi^d}{4}\int[d\gamma]\ \frac{\Gamma(B)\Gamma(A-B)}{\Gamma(A+1)}\ \big(A\gamma_{rl}+B\gamma_l\gamma_r\big)\prod_{1\le\mu<\nu\le n}[P_{\mu\nu}]^{\gamma_{\mu\nu}}\ .\tag{D.72}$$

[1] The full integral, without dropping terms proportional to λ_l or ρ_r, is obtained by adding

$$\frac{1}{B-1}\big[(A-B)(1+A-B)\lambda_l\rho_r+(A-B)(B-1)(\gamma_r\lambda_l+\gamma_l\rho_r)\big]\ .\tag{D.71}$$

to the bracket in the integrand of (D.72).

D.4.2 Constrained Mellin integral identity

The goal of this section is to analyze an integral over Mellin variables β_{ab} constrained by $\sum_{1\le a<b\le k}\beta_{ab}=B$.[2] Using recursively the first Barnes lemma we can prove the following identity

$$\int[d\beta]\prod_{1\le a<b\le k}\frac{\Gamma(\beta_{ab})\Gamma(\alpha_{ab}-\beta_{ab})}{\Gamma(\alpha_{ab})}=\frac{\Gamma(B)\Gamma(A-B)}{\Gamma(A)} \tag{D.73}$$

where $A=\sum_{1\le a<b\le k}\alpha_{ab}$. This type of integral can be easily generalized to the case where we have also a linear or quadratic dependence in β_{ab}. Let us consider first the linear case

$$\int[d\beta]\beta_{f_1p_1}\prod_{1\le a<b\le k}\frac{\Gamma(\beta_{ab})\Gamma(\alpha_{ab}-\beta_{ab})}{\Gamma(\alpha_{ab})}=\alpha_{f_1p_1}\frac{\Gamma(B+1)\Gamma(A-B)}{\Gamma(A+1)}. \tag{D.74}$$

where we have shifted the integration variables to reduce this case to the previous one. Given a function defined by

$$\mathcal{F}_{\{f_i,p_i\}}(\beta_{ab})\equiv\prod_{j=1}^{J}\left(\beta_{f_jp_j}+\sum_{\ell=1}^{j-1}\delta_{f_j}^{f_\ell}\delta_{p_j}^{p_\ell}+\delta_{f_j}^{p_\ell}\delta_{p_j}^{f_\ell}\right), \tag{D.75}$$

it easy to check that, shifting the integration variables, formula (D.73) can be generalized as follows

$$\int[d\beta]\mathcal{F}_{\{f_i,p_i\}}(\beta_{ab})\prod_{1\le a<b\le k}\frac{\Gamma(\beta_{ab})\Gamma(\alpha_{ab}-\beta_{ab})}{\Gamma(\alpha_{ab})}=\mathcal{F}_{\{f_i,p_i\}}(\alpha_{ab})\frac{\Gamma(B+J)\Gamma(A-B)}{\Gamma(A+J)}. \tag{D.76}$$

We can now generalize this type of integrals to the case of any polynomial dependence in β_{ab} just taking linear combination of (D.76), since $\mathcal{F}_{\{f_i,p_i\}}(\beta_{ab})$ can be used as a basis for the polynomials in β_{ab}. A useful example is given by a quadratic term in β_{ab}. In fact, using $\beta_{f_1p_1}\beta_{f_2p_2}=(\beta_{f_1p_1}+\delta_{f_1}^{f_2}\delta_{p_1}^{p_2}+\delta_{f_1}^{p_2}\delta_{p_1}^{f_2}-\delta_{f_1}^{f_2}\delta_{p_1}^{p_2}-\delta_{f_1}^{p_2}\delta_{p_1}^{f_2})\beta_{f_2p_2}$ and (D.76), we find

$$\begin{aligned}&\int[d\beta]\beta_{f_1p_1}\beta_{f_2p_2}\prod_{1\le a<b\le k}\frac{\Gamma(\beta_{ab})\Gamma(\alpha_{ab}-\beta_{ab})}{\Gamma(\alpha_{ab})}\\&=\alpha_{f_1p_1}\alpha_{f_2p_2}\frac{\Gamma(B+2)\Gamma(A-B)}{\Gamma(A+2)}-\alpha_{f_2p_2}(\delta_{f_1}^{f_2}\delta_{p_1}^{p_2}+\delta_{f_1}^{p_2}\delta_{p_1}^{f_2})\frac{\Gamma(B+1)\Gamma(A-B+1)}{\Gamma(A+2)}\ .\end{aligned} \tag{D.77}$$

[2] Imposing the constraint $\sum_{1\le a<b\le k}\beta_{ab}=B$, one can solve for β_{12} in terms of the other $(k+1)(k-2)/2$ variables β_{1a} for $a=3,\dots,k$ and β_{ab} for $2\le a<b\le k$. Then, the integrals over

$$\int[d\beta]=\int_{-i\infty}^{i\infty}\prod_{a=3}^{k}\frac{d\beta_{1a}}{2\pi i}\prod_{2\le a<b\le k}\frac{d\beta_{ab}}{2\pi i}$$

can be done by successive use of the first Barnes lemma.

Appendix E

Factorization from the conformal Casimir equation

In the main text we have derived a factorization formula using the shadow formalism. The goal of this appendix is to obtain the same result using the conformal Casimir equation.

Given a n-point function, we can perform a multiple OPE expansion of the first k operators as described in (3.35) to obtain a sum over the contributions of the exchanged primary operators $\mathcal{O}_p$ and their descendants,

$$\langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle = \sum_p G_p(P_1, \dots, P_n) \ . \tag{E.1}$$

Let us define the conformal Casimir for the firsts k operators as

$$\mathscr{C} = \frac{1}{2} \left[\sum_{i=1}^{k} \mathscr{J}_i \right]^2 , \tag{E.2}$$

where

$$\mathscr{J}_{AB} = P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A}, \qquad P \in \mathbb{M}^{d+2} \ , \tag{E.3}$$

are the generators of the Lorentz group acting on the embedding space $\mathbb{M}^{d+2}$. Then each $G_p(P_1, \dots, P_n)$ is an eigenfunction of the Casimir $\mathscr{C}$ with eigenvalue

$$c_{\Delta J} = \Delta(\Delta - d) + J(J + d - 2) \ ,$$

where J and Δ are the spin and the conformal dimension of the exchanged operator $\mathcal{O}_p$, *i.e.*

$$\mathscr{C} G_p(P_1, \dots, P_n) = -c_{\Delta J} G_p(P_1, \dots, P_n) \ . \tag{E.4}$$

This equation takes a simpler form in Mellin space. In fact, the Mellin transform $M_p(\gamma_{ij})$ of

$G_p(P_1, \dots, P_n)$ has to satisfy the following shifting relation [24]

$$\begin{aligned}&[(\gamma_{LR}-\Delta)(d-\Delta-\gamma_{LR})+J(J+d-2)]M_p+\\&\quad+\sum_{\substack{a,b=1\\a\neq b}}^{k}\sum_{\substack{i,j=k+1\\i\neq j}}^{n}\left[\gamma_{ai}\gamma_{bj}\left(M_p-[M_p]^{ai,bj}_{aj,bi}\right)+\gamma_{ab}\gamma_{ij}[M_p]^{ab,ij}_{ai,bj}\right]=0\ ,\end{aligned} \tag{E.5}$$

where we recall that $\gamma_{LR} = \sum_{a=1}^{k} \sum_{i=k+1}^{n} \gamma_{ai}$ and the square brackets are defined by

$$[f(\gamma_{ij})]^{ab} = f(\gamma_{ij} + \delta_i^a \delta_j^b + \delta_j^a \delta_i^b)\ , \qquad [f(\gamma_{ij})]_{ab} = f(\gamma_{ij} - \delta_i^a \delta_j^b - \delta_j^a \delta_i^b)\ . \tag{E.6}$$

with f being a generic function of the variables γ_{ij}.

The Mellin amplitude $M_p(\gamma_{\mu\nu})$ has the following pole structure

$$M_p \approx \frac{\mathcal{Q}_m}{\gamma_{LR} - (\Delta + 2m - J)}\ , \qquad m = 0, 1, 2, \dots\ . \tag{E.7}$$

where the residues $\mathcal{Q}_m$ are functions of the Mellin variables $\gamma_{\mu\nu}$ which satisfies the *on shell* condition $\gamma_{LR} = \Delta + 2m - J$. Therefore, the full Mellin amplitude $M = \sum_p M_p$ will also have these poles with the same residues. [1] Studying equation (E.5) close to the poles (E.7), we obtain an equation for the residues $\mathcal{Q}_m$, which can be written as

$$\widehat{C}\,(\mathcal{Q}_m) = 0\ , \qquad m = 0, 1, 2, \dots\ , \tag{E.8}$$

where $\widehat{C}$ is the operator

$$\widehat{C}(\mathcal{Q}_m) \equiv \eta \mathcal{Q}_m + \sum_{\substack{a,b=1\\a\neq b}}^{k}\sum_{\substack{i,j=k+1\\i\neq j}}^{n}\left[\gamma_{ai}\gamma_{bj}\left(\mathcal{Q}_m-[\mathcal{Q}_m]^{ai,bj}_{aj,bi}\right)+\gamma_{ab}\gamma_{ij}[\mathcal{Q}_{m-1}]^{ab,ij}_{ai,bj}\right] \tag{E.9}$$

and $\eta = (2m-J)(d-2\Delta-2m+J)+J(J+d-2)$. In particular, we notice that for $m > 0$ (E.8) is a recurrence equation for $\mathcal{Q}_m$ in terms of $\mathcal{Q}_{m-1}$, while for $m = 0$ (E.8) reduces to a constraint on $\mathcal{Q}_0$ (since $\mathcal{Q}_{-1} = 0$). In the rest of this section, we present a way to find $\mathcal{Q}_m$ using (E.8).

E.1 Factorization for scalar exchange

In the scalar case it is natural to guess a factorization formula of the kind

$$\mathcal{Q}_m = \kappa_{\Delta 0} \frac{m!}{(1 - \frac{d}{2} + \Delta)_m} L_m R_m\ , \tag{E.10}$$

where L_m and R_m are respectively functions of the Mellin variables on the left (γ_{ab} with $a, b = 1, \dots k$) and on the right (γ_{ij} with $i, j = k+1, \dots n$) such that $L_0 = M_L$ and $R_0 = M_R$. The

[1] In an interacting CFT, we do not expect that different primary operators give rise to coincident poles.

overall constant $\kappa_{\Delta\, 0}$ will be fixed later and $\frac{m!}{(1-\frac{d}{2}+\Delta)_m}$ is a function of m that we introduced for convenience and that could in principle be absorbed in the definition of L_m and R_m. Since $\mathcal{Q}_m$ does not depend on the mixed variables γ_{ai} (with $a = 1, \dots k$ and $i = k+1, \dots n$), then $[\mathcal{Q}_m]^{ai,bj}_{aj,bi} = \mathcal{Q}_m$ trivially. Therefore, equation (E.8) reduces to

$$2m(d-2\Delta-2m)\mathcal{Q}_m + \sum_{\substack{a,b=1\\a\neq b}}^{k} \sum_{\substack{i,j=k+1\\i\neq j}}^{n} \gamma_{ab}\gamma_{ij}[\mathcal{Q}_{m-1}]^{ab,ij} = 0 \ . \tag{E.11}$$

This equation is automatically satisfied for $m = 0$. Notice that the ansatz $\mathcal{Q}_m$ is consistent with equation (E.11). In fact, given a $\mathcal{Q}_{m-1}$ factorized in functions of left and right Mellin variables, (E.11) implies that $\mathcal{Q}_m$ is also factorized in the same way. Replacing $\mathcal{Q}_m$ in (E.11) we get a recurrence equation for L_m and R_m

$$L_m R_m = \Big(\frac{1}{2m} \sum_{\substack{a,b=1\\a\neq b}}^{k} \gamma_{ab}[L_{m-1}]^{ab}\Big)\Big(\frac{1}{2m} \sum_{\substack{i,j=k+1\\i\neq j}}^{n} \gamma_{ij}[R_{m-1}]^{ij}\Big) \ , \tag{E.12}$$

which can be solved separately for L_m and R_m in terms of $L_0 = M_L$ and $R_0 = M_R$. In section E.4.1, we show that

$$L_m = \frac{1}{2m} \sum_{\substack{a,b=1\\a\neq b}}^{k} \gamma_{ab}[L_{m-1}]^{ab} \qquad \Longleftrightarrow \qquad L_m = \sum_{\substack{n_{ab}\geq 0\\ \sum n_{ab}=m}} M_L(\gamma_{ab}+n_{ab}) \prod_{\substack{a,b=1\\a<b}}^{k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \tag{E.13}$$

and similarly for R_m. The final result exactly matches the one obtained using the shadow formalism up to an overall factor that cannot be fixed by the Casimir equation, since it is a homogeneous equation. This normalization can be fixed by comparing it to the factorization of the four point function or to the result obtained from the shadow formalism.

E.2 Factorization for vector exchange

For spin $J = 1$ the left and right Mellin amplitudes can be represented as functions M_L^a and M_R^i that satisfy the transersality condition $\sum_{a=1}^{k} \gamma_a M_L^a = 0$ as discussed in (3.8), where we recall that $\gamma_a = -\sum_{b=1}^{k} \gamma_{ab} = \sum_{i>k}^{n} \gamma_{ai}$.
The solution of (E.8) should depend on the left and the right Mellin amplitudes M_L^a and M_R^i in a form invariant under permutations of the left points P_a with $a = 1, \dots, k$ and of the right points P_i with $i = k+1, \dots, n$. Considering that the scalar solution takes the form (E.10), the first natural ansatz for the vector case is

$$\mathcal{Q}_m^{(1)} = \sum_{a=1}^{k} \sum_{i=k+1}^{n} \gamma_{ai} L_m^a R_m^i \ , \tag{E.14}$$

with L_m^a and R_m^i defined in (3.51). This ansatz is actually the complete solution in the case $m=0$, but it fails to solve the Casimir equation (E.8) for higher m. In fact, acting with the Casimir operator (E.9) on the ansatz $\mathcal{Q}_m^{(1)}$ times a function $f_m^{(1)}$ that does not depend on the Mellin variables, we find (see appendix E.3)

$$\widehat{C}(f_m^{(1)}\mathcal{Q}_m^{(1)}) = 2m\left((d-2\Delta-2m)f_m^{(1)}+2mf_{m-1}^{(1)}\right)\mathcal{Q}_m^{(1)}+2\left(f_m^{(1)}-f_{m-1}^{(1)}\right)\mathcal{Q}_m^{(2)} \tag{E.15}$$

where

$$\mathcal{Q}_m^{(2)} = \dot{L}_m\dot{R}_m\ , \qquad \text{with} \qquad \dot{L}_m = -\sum_{a=1}^{k}\gamma_a L_m^a \tag{E.16}$$

and similarly for $\dot{R}_m$. Notice that $L_0^a = M_L^a$ so that $\dot{L}_0 = -\sum_{a=1}^k \gamma_a M_L^a = 0$ due to the transversality condition (3.8). Therefore $\mathcal{Q}_0^{(2)} = 0$ and $\mathcal{Q}_0^{(1)}$ automatically solves (E.15) for $m=0$. Acting with the Casimir operator (E.9) on $f_m^{(2)}\mathcal{Q}_m^{(2)}$ we find (see section E.3)

$$\widehat{C}(f_m^{(2)}\mathcal{Q}_m^{(2)}) = \left(\eta f_m^{(2)}+4(m-1)^2 f_{m-1}^{(2)}\right)\mathcal{Q}_m^{(2)}. \tag{E.17}$$

Notice that the action of the Casimir operator $\widehat{C}$ on the structures $\mathcal{Q}_m^{(1)}$ and $\mathcal{Q}_m^{(2)}$ closes because it does not produce any new structure. Thus, we can find the solution of the problem fixing the functions $f_m^{(1)}$ and $f_m^{(2)}$ such that

$$\widehat{C}(f_m^{(1)}\mathcal{Q}_m^{(1)}+f_m^{(2)}\mathcal{Q}_m^{(2)}) = 0. \tag{E.18}$$

Since $\mathcal{Q}_m^{(1)}$ and $\mathcal{Q}_m^{(2)}$ are linearly independent, we need to set to zero the coefficients multiplying each structure in (E.18). Setting to zero the coefficient of $\mathcal{Q}_m^{(1)}$ we get a recurrence relation for $f_m^{(1)}$,

$$(d-2\Delta-2m)f_m^{(1)}+2mf_{m-1}^{(1)} = 0\ , \qquad m\geq 0\ , \tag{E.19}$$

that can be solved up to an overall constant $f_0^{(1)}$ that we will call $\kappa_{\Delta 1}$,

$$f_m^{(1)} = \kappa_{\Delta 1}\frac{m!}{(1-d/2+\Delta)_m}\ . \tag{E.20}$$

Setting to zero the terms multiplying $\mathcal{Q}_m^{(2)}$, we find a recurrence relation for $f_m^{(2)}$,

$$\eta f_m^{(2)}+4(m-1)^2 f_{m-1}^{(2)}+2\left(f_m^{(1)}-f_{m-1}^{(1)}\right) = 0\ , \qquad m\geq 1\ , \tag{E.21}$$

that, once we substitute (E.20), can be solved as

$$f_m^{(2)} = \kappa_{\Delta 1}\frac{m!}{(1-d/2+\Delta)_m}\frac{d-2\Delta}{2m(\Delta-d+1)}. \tag{E.22}$$

Therefore the final result is

$$\mathcal{Q}_m = \kappa_{\Delta 1} \frac{m!}{(1-d/2+\Delta)_m} \sum_{a=1}^{k} \sum_{i=k+1}^{n} L_m^a R_m^i \left(\gamma_{ai} + \frac{d-2\Delta}{2m(\Delta-d+1)} \gamma_a \gamma_i \right) \, , \tag{E.23}$$

which matches the result (3.50) that we found in the previous section using the shadow method. The same method could be applied to determine the factorization expressions for higher spin operators. However the number of structures that appear proliferate quite quickly. In [26] we have computed this formula for spin 2 using both shadow formalism and conformal Casimir equation.

E.3 Technical part of factorization for spin $J = 1$

In this subsection we prove formulas (E.15) and (E.17). First we consider that in the action of the casimir operator $\widehat{C}$ defined in (E.9) there is a term of the kind $[\,\cdot\,]^{ai,bj}_{aj,bi}$, which only shifts the mixed variable and it does not act on L_m and R_m. Moreover the action of $[\,\cdot\,]^{ai,bj}_{aj,bi}$ on a mixed variable γ_{cl} can be written in a simple way, namely

$$[\gamma_{cl}]^{ai,bj}_{aj,bi} = \gamma_{cl} + \delta^a_c \delta^i_l + \delta^b_c \delta^j_l - \delta^a_c \delta^j_l - \delta^b_c \delta^i_l \, . \tag{E.24}$$

So that we easily obtain

$$\sum_{\substack{a,b=1 \\ a\neq b}}^{k} \sum_{\substack{i,j=k+1 \\ i\neq j}}^{n} \gamma_{ai}\gamma_{bj} \left(\mathcal{Q}^{(1)}_m - [\mathcal{Q}^{(1)}_m]^{ai,bj}_{aj,bi} \right) = -2(\Delta - 1 + 2m)\mathcal{Q}^{(1)}_m + 2\mathcal{Q}^{(2)}_m \, , \tag{E.25}$$

$$\sum_{\substack{a,b=1 \\ a\neq b}}^{k} \sum_{\substack{i,j=k+1 \\ i\neq j}}^{n} \gamma_{ai}\gamma_{bj} \left(\mathcal{Q}^{(2)}_m - [\mathcal{Q}^{(2)}_m]^{ai,bj}_{aj,bi} \right) = 0 \, . \tag{E.26}$$

The second part of the computation is more subtle since $\widehat{C}$ also contains a term $[\mathcal{Q}_{m-1}]^{ab,ij}_{ai,bj}$ that has a shift both in the Mellin variables and in m. To simplify this term we find (see appendix E.4) the following recurrence relations to connect structures defined at $m-1$ to structures defined at m

$$L^c_m = \frac{1}{2m} \sum_{\substack{a,b=1 \\ a\neq b}}^{k} \gamma_{ab} \left[L^c_{m-1} \right]^{ab} \, , \tag{E.27}$$

$$\dot{L}_m = \sum_{\substack{a,b=1 \\ a\neq b}}^{k} \gamma_{ab} \left[L^a_{m-1} \right]^{ab} \, , \tag{E.28}$$

$$\dot{L}_m = \frac{1}{2(m-1)} \sum_{\substack{a,b=1 \\ a\neq b}}^{k} \gamma_{ab} \left[\dot{L}_{m-1} \right]^{ab} \, . \tag{E.29}$$

Using (E.27), (E.28), (E.29) we easily find

$$\sum_{\substack{a,b=1\\a\neq b}}^{k}\sum_{\substack{i,j=k+1\\i\neq j}}^{n}\gamma_{ab}\gamma_{ij}[\mathcal{Q}^{(1)}_{m-1}]^{ab,ij}_{ai,bj}=-2\mathcal{Q}^{(2)}_{m}+(2m)^2\mathcal{Q}^{(1)}_{m}\ , \tag{E.30}$$

$$\sum_{\substack{a,b=1\\a\neq b}}^{k}\sum_{\substack{i,j=k+1\\i\neq j}}^{n}\gamma_{ab}\gamma_{ij}[\mathcal{Q}^{(2)}_{m-1}]^{ab,ij}_{ai,bj}=4(m-1)^2\mathcal{Q}^{(2)}_{m}. \tag{E.31}$$

Formulas (E.15) and (E.17) descend respectively from (E.25) and (E.30) and from (E.26) and (E.31).

E.4 Recurrence relations

In this appendix we demonstrate formula (E.13) for the scalar case and (E.27), (E.28), (E.29) for the vector case. We also show some similar formulas useful in the spin two case.

First we write some formulas important to demonstrate the following results. We often deal with the set of integer compositions

$$\mathcal{A}^m=\left\{\{\lambda_1,\dots\lambda_n\}:\sum_{i=1}^{n}\lambda_i=m\,,\ \lambda_i\in\mathbb{N}_0\right\}\ ,$$

where from now on we will denote $\{\lambda_1,\dots\lambda_n\}\equiv\{\lambda_i\}$. We shall now show a simple property of the integer composition that will be very useful in the rest of this section. It is a trivial fact that for any $j\in\{1,\dots,n\}$ the set

$$\mathcal{A}^m_j\equiv\left\{\{\lambda_i+\delta^j_i\}:\sum_{i=1}^{n}\lambda_i=m\,,\ \lambda_i\in\mathbb{N}_0\right\}$$

is contained in the set $\mathcal{A}^{m+1}$. Moreover, it is clear that $\mathcal{A}^m_j$ contains all the elements of $\mathcal{A}^{m+1}$ except the ones that have $\lambda_j=0$, namely

$$\mathcal{A}^{m+1}\setminus\mathcal{A}^m_j=\left\{\{\lambda_i\}:\sum_{i=1}^{n}\lambda_i=m+1\,,\ \lambda_j=0\,,\ \lambda_i\in\mathbb{N}_0\right\}\ .$$

Using this fact it is easy to show the following formula. Given a function of n variables

$F(\{\lambda_1, \dots \lambda_n\}) \equiv F(\{\lambda_i\})$. If $F(\{\lambda_i\})\big|_{\lambda_j=0} = 0$ for a fixed $j \in \{1,\dots,n\}$ then [2]

$$\sum_{\sum_{i=1}^n \lambda_i = m} F(\{\lambda_i + \delta_i^j\}) = \sum_{\sum_{i=1}^n \lambda_i = m+1} F(\{\lambda_i\}) \ . \tag{E.32}$$

We can use (E.32) to find

$$\sum_{\sum_{i=1}^n \lambda_i = m} (\lambda_j + 1) G(\{\lambda_i + \delta_i^j\}) = \sum_{\sum_{i=1}^n \lambda_i = m+1} \lambda_j G(\{\lambda_i\}) \ , \tag{E.33}$$

$$\sum_{\sum_{i=1}^n \lambda_i = m} \sum_{j=1}^n (\lambda_j + 1) G(\{\lambda_i + \delta_i^j\}) = (m+1) \sum_{\sum_{i=1}^n \lambda_i = m+1} G(\{\lambda_i\}) \ , \tag{E.34}$$

for any function $G(\{\lambda_i\})$.

E.4.1 Demonstration of (E.13) and (E.27)

A proof of (E.13) can be given as follows

$$\begin{aligned}
\sum_{\substack{e,f=1 \\ e<f}}^{k} \gamma_{ef} [L_{m-1}]^{ef} &= \sum_{\substack{e,f=1 \\ e<f}}^{k} \gamma_{ef} \sum_{\sum n_{ab} = m-1} [M(\gamma_{ab} + n_{ab})]^{ef} \prod_{\substack{a,b=1 \\ a<b}}^{k} \frac{[(\gamma_{ab})_{n_{ab}}]^{ef}}{n_{ab}!} \\
&= \sum_{\substack{e,f=1 \\ e<f}}^{k} \sum_{\sum n_{ab} = m-1} (n_{ef} + 1) M(\gamma_{ab} + n_{ab} + \delta_a^e \delta_b^f) \prod_{\substack{a,b=1 \\ a<b}}^{k} \frac{(\gamma_{ab})_{n_{ab} + \delta_a^e \delta_b^f}}{(n_{ab} + \delta_a^e \delta_b^f)!} \\
&= m \sum_{\sum n_{ab} = m} M(\gamma_{ab} + n_{ab}) \prod_{\substack{a,b=1 \\ a<b}}^{k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \\
&= m L_m \ ,
\end{aligned}$$

where we used $\prod_{a<b} [(\gamma_{ab})_{n_{ab}}]^{ef} = \frac{1}{\gamma_{ef}} \prod_{a<b} (\gamma_{ab})_{n_{ab} + \delta_a^e \delta_b^f}$ and (E.34). Clearly the same demonstration holds for a Mellin amplitude with one or more indices that are not summed, namely

$$\sum_{\substack{e,f=1 \\ e<f}}^{k} \gamma_{ef} [L_{m-1}^{a_1 \dots a_J}]^{ef} = m L_m^{a_1 \dots a_J} \ , \qquad \text{with } L_m^{a_1 \dots a_J} = \sum_{\sum n_{ef} = m} M^{a_1 \dots a_J}(\gamma_{ef} + n_{ef}) \prod_{\substack{e,f=1 \\ e<f}}^{k} \frac{(\gamma_{ef})_{n_{ef}}}{n_{ef}!} \ .$$

In particular (E.27) holds.

---

[2] Property (E.32) holds because

$$\sum_{\{\lambda_i\} \in \mathcal{A}_j^m} F(\{\lambda_i\}) = \sum_{\{\lambda_i\} \in \mathcal{A}^{m+1}} F(\{\lambda_i\}) - \sum_{\{\lambda_i\} \in \mathcal{A}^{m+1} \setminus \mathcal{A}_j^m} F(\{\lambda_i\}) = \sum_{\{\lambda_i\} \in \mathcal{A}^{m+1}} F(\{\lambda_i\}) \ ,$$

where we used the fact that $F(\{\lambda_i\}) = 0$ for any $\{\lambda_i\} \in \mathcal{A}^{m+1} \setminus \mathcal{A}_j^m$.

E.4.2 Demonstration of (E.28)

First we note that $\dot{L}_m \equiv \sum_{e=1}^{k} \gamma_e L_m^e$ can be also defined as follows

$$\dot{L}_m = \sum_{\sum n_{ab}=m} \sum_{\substack{e,f=1 \\ e\neq f}}^{k} n_{ef} M^e(\gamma_{ab}+n_{ab}) \prod_{\substack{a,b=1 \\ a<b}}^{k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \ , \tag{E.35}$$

where we consider n_{ef} symmetric in its indeces, so that $n_{fe} \equiv n_{ef}$ when $f > e$. Formula (E.35) is true because of the transversality condition (3.8), in fact

$$\sum_{e=1}^{k} \gamma_e M^e(\gamma_{ab}+n_{ab}) = -\sum_{\substack{e,f=1 \\ e\neq f}}^{k} (\gamma_{ef}+n_{ef}-n_{ef}) M^e(\gamma_{ab}+n_{ab}) = \sum_{\substack{e,f=1 \\ e\neq f}}^{k} n_{ef} M^e(\gamma_{ab}+n_{ab}) \ .$$

Using (E.35) we can now prove (E.28)

$$\begin{aligned}
\sum_{\substack{e,f=1 \\ e\neq f}}^{k} \gamma_{ef} [L_{m-1}^e]^{ef} &= \sum_{\substack{e,f=1 \\ e<f}}^{k} \gamma_{ef} [L_{m-1}^e + L_{m-1}^f]^{ef} \\
&= \sum_{\substack{e,f=1 \\ e<f}}^{k} \gamma_{ef} \sum_{\sum n_{ab}=m-1} \Big[(M^e+M^f)(\gamma_{ab}+n_{ab})\Big]^{ef} \prod_{\substack{a,b=1 \\ a<b}}^{k} \frac{[(\gamma_{ab})_{n_{ab}}]^{ef}}{n_{ab}!} \\
&= \sum_{\sum n_{ab}=m} \sum_{\substack{e,f=1 \\ e<f}}^{k} n_{ef} (M^e+M^f)(\gamma_{ab}+n_{ab}) \prod_{\substack{a,b=1 \\ a<b}}^{k} \frac{(\gamma_{ab})_{n_{ab}}}{(n_{ab})!} \\
&= \dot{L}_m \ ,
\end{aligned}$$

where followed the same steps of demonstration in the previous subsection except that we used formula (E.34) instead of (E.33).

E.4.3 Demonstration of (E.29)

$$
\begin{aligned}
\sum_{\substack{a,b=1\\a\neq b}}^{k} \gamma_{ab}\left[\dot{L}_{m-1}\right]^{ab} &= \sum_{\substack{a,b=1\\a\neq b}}^{k} \gamma_{ab}\left[\sum_{c=1}^{k}\gamma_c L_{m-1}^{c}\right]^{ab} \\
&= \sum_{\substack{a,b=1\\a\neq b}}^{k} \gamma_{ab}\left(\sum_{c=1}^{k}\gamma_c \left[L_{m-1}^{c}\right]^{ab} - \left[L_{m-1}^{a}\right]^{ab} - \left[L_{m-1}^{b}\right]^{ab}\right) \\
&= \sum_{c=1}^{k}\gamma_c \sum_{\substack{a,b=1\\a\neq b}}^{k} \gamma_{ab}\left[L_{m-1}^{c}\right]^{ab} - 2\sum_{\substack{a,b=1\\a\neq b}}^{k} \gamma_{ab}\left[L_{m-1}^{a}\right]^{ab} \\
&= 2m\sum_{c=1}^{k}\gamma_c L_m^c - 2\dot{L}_m \\
&= 2(m-1)\dot{L}_m
\end{aligned}
$$

where in we have used the relations (E.27) and (E.28).

Appendix F

Symmetry and analyticity properties of the Mellin amplitude $M_F(s,t)$

The factor R is permutation symmetric and has weight one at each point, so it must satisfy,

$$F(u,v) = F(v,u) = F\big(1/u, v/u\big)/u. \tag{F.1}$$

This symmetry is translated in terms of Mellin amplitudes as (5.140). Each amplitude in (5.6) can be written in terms of the Mellin amplitude $M_F(s,t)$ given in (5.139). Imposing that absence of poles in the Mellin amplitudes corresponding to the amplitudes A_r would not lead to constraints on $M_F(s,t)$. Let us study how these constraints come about by analyzing the channel 105 of the four point function. Following the notation of [27] we write the Mellin amplitude of the channel 105 as

$$A_{105} = \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} u^{t/2} v^{-(s+t)/2} M_{105}(s,t) \Gamma^2\left(\frac{4-t}{2}\right) \Gamma^2\left(\frac{-s}{2}\right) \Gamma^2\left(\frac{s+t}{2}\right) \tag{F.2}$$

with $M_{105}(s,t)$ given by

$$M_{105}(s,t) = \frac{(t-4)^2(t-6)^2 M_F(4+s, t-4)}{40}. \tag{F.3}$$

The absence of poles in $M_{105}(s,t)$ allows $M_F(s,t)$ to have double poles at $t=0$ and $t=2$, *i.e.*

$$M_F(s,t) = \frac{h(s,t)}{t^2(t-2)^2} \tag{F.4}$$

with $h(s,t)$ a regular function in s and t. However we now that $M_F(s,t)$ satisfies,

$$M_F(s,t) = M_F(t,s) = M_F(s, 4-t-s). \tag{F.5}$$

Notice that $M_F(s,t)$ cannot have poles, otherwise it is not possible to satisfy

$$\frac{M_F(s,t)}{M_F(t,s)} = \frac{s^2(s-2)^2h(s,t)}{t^2(t-2)^2h(t,s)} = 1. \tag{F.6}$$

Thus we conclude that the absence of poles in the channel M_{105} implies that $M_F(s,t)$ is a meromorphic function of s and t. In particular this is useful to study the $1/\lambda$ corrections to the four point function.

Dilaton Four point function from $F(u,v)$

The goal of this section is derive the relation between $F(u,v)$ and the four point function of Lagrangians. We will use eqs (1.3), (2.23), (3.1) of [88]. The dilaton can be written in terms of fields L^+ and L^-,

$$L^+ + L^- = F_{\mu\nu}F^{\mu\nu} + \dots. \tag{F.7}$$

Then we just have to use,

$$\left\langle L_1^+ L_2^- L_3^+ L_4^- \right\rangle = \frac{1}{x_{13}^8 x_{24}^8} H(u,v) \tag{F.8}$$

together with the fact that a four point function with unequal number of fields L^+ and L^- gives zero. Using this we get (5.136).

Relation between Mellin amplitudes

In this appendix we show the precise form of the relation between $M_{\mathcal{L}}^{\lambda}(s,t)$ and $M_F^{\lambda}(s,t)$. The prescription to obtain (5.141) is simple, just act with the differential operator defined by (5.136-5.137) and then simplify using the symmetries of $M_F(s,t)$ to obtain

$$M_{\mathcal{L}}(s,t) = \frac{1}{9216} \sum_{a,b=0}^{6} q_{a,b}(s,t) M_F(s-2a, t-2b) \tag{F.9}$$

where the non-zero polynomials $q_{a,b}(s,t)$ are given by

$$q_{0,0}(s,t) = (s+t-2)^2(s+t-4)^2(s+t-6)^2(s+t-8)^2\,, \quad q_{0,1}(s,t) = \frac{4(t-6)(s+t-10)q_{0,0}(s,t)}{(s+t-2)^2}$$

$$q_{0,2}(s,t) = (s+t-6)^2(s+t-8)^2(t(t-2)(t-4)(t-6)+2(3(t-14)t+148)(s+t-12)(s+t-10))$$

$$\begin{aligned} q_{0,3}(s,t) = {} & 4(t-8)^2(s+t-8)^2(s^3(8-t)-3s^2(12-t)(8-t)-2((t-8)^2 \\ & (108-(16-t)t)-2s(868-t(262-(27-t)t)))) \end{aligned}$$

$$\begin{aligned} q_{0,4}(s,t) = {} & (t-8)^2(t-10)^2(38592-11120s+1148s^2-52s^3+s^4-16000t+3376st-216s^2t \\ & +4s^3t+2716t^2-384st^2+12s^2t^2-220t^3+16st^3+7t^4) \end{aligned}$$

$$q_{0,5}(s,t) = \frac{4(s+t-10)q_{0,0}(s,14-s-t)}{6-t}\,, \quad q_{0,6}(s,t) = q_{0,0}(-6,t)$$

$$q_{1,1}(s,t) = 4(104+3s(t-6)-18t)(12-s-t)(10-s-t)(8-s-t)^2(6-s-t)^2$$

$$\begin{aligned} q_{1,2}(s,t) = {} & -4(8-s-t)^2(1397760-604736s+93296s^2-6160s^3+148s^4-768512t \\ & +298912st-39960s^2t+2200s^3t-42s^4t+167200t^2-56776st^2+6176s^2t^2 \\ & -252s^3t^2+3s^4t^2-18208t^3+5184st^3-408s^2t^3+9s^3t^3+1016t^4-230st^4+10s^2t^4 \\ & -24t^5+4st^5)\,, \quad q_{1,5}(s,t) = \frac{4(6-s)q_{0,0}(s,14-s-t)}{6-t} \end{aligned}$$

$$\begin{aligned} q_{1,3}(s,t) = {} & 4(8-t)^2(1256448-617920s+115856s^2-10464s^3+460s^4-8s^5-620032t+264288st \\ & -40936s^2t+2872s^3t-90s^4t+s^5t+129248t^2-46488st^2+5584s^2t^2-264s^3t^2+4s^4t^2 \\ & -14624t^3+4320st^3-368s^2t^3+9s^3t^3+904t^4-210st^4+10s^2t^4-24t^5+4st^5) \end{aligned}$$

$$q_{1,4}(s,t) = 4(10-t)^2(8-t)^2(6-t)(t-4)(184-48s+3s^2-18t+3st)$$

$$
\begin{aligned}
\frac{q_{2,2}(s,t)}{4} &= 87736320 - 54491136s + 13977472s^2 - 1927680s^3 + 154960s^4 - 7104s^5 + 148s^6 \\
&- 54491136t + 31260800st - 7288464s^2t + 896544s^3t - 62972s^4t + 2460s^5t - 42s^6t \\
&+ 13977472t^2 - 7288464st^2 + 1504560s^2t^2 - 158200s^3t^2 + 9078s^4t^2 - 270s^5t^2 + 3s^6t^2 \\
&- 1927680t^3 + 896544st^3 - 158200s^2t^3 + 13288s^3t^3 - 546s^4t^3 + 9s^5t^3 + 154960t^4 \\
&- 62972st^4 + 9078s^2t^4 - 546s^3t^4 + 12s^4t^4 - 7104t^5 + 2460st^5 - 270s^2t^5 + 9s^3t^5 + 148t^6 \\
&- 42st^6 + 3s^2t^6 \\
q_{2,3}(s,t) &= -4(8-t)^2(73728 - 31680s + 7536s^2 - 1504s^3 + 180s^4 - 8s^5 - 74688t + 28224st \\
&- 4152s^2t + 392s^3t - 30s^4t + s^5t + 24944t^2 - 8808st^2 + 1040s^2t^2 - 48s^3t^2 + s^4t^2 \\
&- 3312t^3 + 1080st^3 - 108s^2t^3 + 3s^3t^3 + 148t^4 - 42st^4 + 3s^2t^4) \\
q_{2,4}(s,t) &= (10-t)^2(8-t)^2(7104 - 2064s + 188s^2 - 12s^3 + s^4 - 2960t + 840st - 60s^2t \\
&+ 296t^2 - 84st^2 + 6s^2t^2) \\
q_{3,3}(s,t) &= 4(8-s)^2(8-t)^2(768 - 400s + 96s^2 - 8s^3 - 400t + 88st - 12s^2t + s^3t \\
&+ 96t^2 - 12st^2 - 8t^3 + st^3)
\end{aligned}
$$

$$
\begin{aligned}
&q_{3,2}(s,t) = q_{2,3}(t,s)\,, \quad q_{4,2}(s,t) = q_{2,4}(t,s)\,, \quad q_{2,1}(s,t) = q_{1,2}(t,s)\,, \quad q_{3,1}(s,t) = q_{1,3}(t,s) \\
&q_{4,1}(s,t) = q_{1,4}(t,s)\,, \quad q_{5,1}(s,t) = q_{1,5}(t,s)\,, \quad q_{1,0}(s,t) = q_{0,1}(t,s)\,, \quad q_{2,0}(s,t) = q_{0,2}(t,s) \\
&q_{3,0}(s,t) = q_{0,3}(t,s)\,, \quad q_{4,0}(s,t) = q_{0,4}(t,s)\,, \quad q_{5,0}(s,t) = q_{0,5}(t,s)\,, \quad q_{6,0}(s,t) = q_{0,6}(t,s)
\end{aligned}
$$

Rewriting flat space limit in terms of M_F

The goal of this section is to express the flat space limit relation in terms of the Mellin amplitude $M_F(s,t)$. Notice that the Mellin amplitude $M_F(s,t)$ satisfies

$$
\begin{aligned}
&\lim_{\lambda\to\infty}(S^2+ST+T^2)^2\frac{d}{d\beta^4}\left[\lambda^{3/2}\beta^9 M_F(\beta\sqrt{\lambda}S,\beta\sqrt{\lambda}T)\right]_{\beta=1} \\
&\approx (S^2+ST+T^2)^2\sum_{n=0}^{\infty}\frac{\Gamma(10+n)}{\Gamma(6+n)}\widetilde{f}_n(S,T) = 4\sum_{n=0}^{\infty}\widetilde{l}_{n+4}(S,T)
\end{aligned}
\tag{F.10}
$$

where we have used (5.152). So we can rewrite $M_{\mathcal{L}}(s,t)$ in terms of $M_F(s,t)$ in this limit,

$$
\lim_{\lambda\to\infty}\frac{M_{\mathcal{L}}(\sqrt{\lambda}S,\sqrt{\lambda}T)}{\lambda} = \frac{(S^2+ST+T^2)^2}{4\lambda}\frac{d}{d\beta^4}\left[\beta^9 M_F(\beta\sqrt{\lambda}S,\beta\sqrt{\lambda}T)\right]_{\beta=1}
\tag{F.11}
$$

The flat space limit is then written as,

$$
\begin{aligned}
&\frac{(S^2+ST+T^2)^2}{2^6}\lim_{\lambda\to\infty}\lambda^{-3/2}\int_{-i\infty}^{i\infty}\frac{d\alpha}{2\pi i}\frac{e^\alpha}{\alpha}\frac{d}{d\beta^4}\left[\left(\frac{\beta}{\alpha}\right)^9 M_F\left(\frac{\beta\sqrt{\lambda}S}{2\alpha},\frac{\beta\sqrt{\lambda}T}{2\alpha}\right)\right]_{\beta=1} \\
&= -\frac{1}{N^2}\frac{\pi^2}{30}\left(\frac{T(S+T)}{S}+\frac{S(S+T)}{T}+\frac{ST}{S+T}\right)\frac{\Gamma(1-\frac{S}{4})\Gamma(1-\frac{T}{4})\Gamma(1+\frac{S+T}{4})}{\Gamma(1+\frac{S}{4})\Gamma(1+\frac{T}{4})\Gamma(1-\frac{S+T}{4})}\,,
\end{aligned}
\tag{F.12}
$$

Now we try to replace the derivative in β by a derivative in terms of α. This is accomplished by noticing that,

$$\frac{d}{d\beta^4} = x^8 \frac{d^4}{dx^4} + 12x^7 \frac{d^3}{dx^3} + 36x^6 \frac{d^2}{dx^2} + 24x^5 \frac{d}{dx} \tag{F.13}$$

with $x = \frac{1}{\beta}$. Schematically we have $\frac{d}{d\beta^4} g\big(\frac{1}{\alpha x}\big)$, thus we can trade derivatives in x by derivatives in α. Using the identity[1]

$$\int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \frac{e^\alpha}{\alpha} \left[\frac{d}{d\beta^4} g\left(\frac{\beta}{\alpha} \right) \right]_{\beta=1} = \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} e^\alpha \alpha^3 g\left(\frac{1}{\alpha} \right) \tag{F.14}$$

we obtain

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^{3/2} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \frac{e^\alpha}{\alpha^6} M_F\left(\frac{\sqrt{\lambda} S}{2\alpha}, \frac{\sqrt{\lambda} T}{2\alpha} \right) \\ &= -\frac{16}{N^2 ST(S+T)} \frac{\Gamma\big(1-\frac{S}{4}\big)\Gamma\big(1-\frac{T}{4}\big)\Gamma\big(1+\frac{S+T}{4}\big)}{\Gamma\big(1+\frac{S}{4}\big)\Gamma\big(1+\frac{T}{4}\big)\Gamma\big(1-\frac{S+T}{4}\big)} \,, \end{aligned} \tag{F.15}$$

[1] We assume that total derivatives give vanishing contributions in the integral.

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