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Conformal slant submersions from cosymplectic manifolds

Yılmaz GÜNDÜZALP[1], **Mehmet Akif AKYOL**[2,*]
[1]Department of Mathematics, Faculty of Sciences, Dicle University, Diyarbakır, Turkey
[2]Department of Mathematics, Faculty of Science and Arts, Bingöl University, Bingöl, Turkey

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Abstract: Akyol [Conformal anti-invariant submersions from cosymplectic manifolds, Hacettepe Journal of Mathematics and Statistics 2017; 462: 177-192] defined and studied conformal antiinvariant submersions from cosymplectic manifolds. The aim of the present paper is to define and study the notion of conformal slant submersions (it means the Reeb vector field ξ is a vertical vector field) from cosymplectic manifolds onto Riemannian manifolds as a generalization of Riemannian submersions, horizontally conformal submersions, slant submersions, and conformal antiinvariant submersions. More precisely, we mention many examples and obtain the geometries of the leaves of vertical distribution and horizontal distribution, including the integrability of the distributions, the geometry of foliations, some conditions related to total geodesicness, and harmonicity of the submersions. Finally, we consider a decomposition theorem on the total space of the new submersion.

Key words: Second fundamental form of a map, almost contact metric manifold, conformal submersion, slant submersion, conformal slant submersion, horizontal distribution

1. Introduction

O'Neill [34] and Gray [22] independently studied the notion of Riemannian submersions between Riemannian manifolds in the 1960s. This notion is related to physics and has some applications in Yang–Mills theory [9, 49], supergravity and superstring theories [29, 32] and Kaluza–Klein theory [10, 28]. For Riemannian submersions, see also [27, 42]. After that, by using the notion of Riemannian submersion and the condition of almost complex mapping, Watson [48] introduced almost Hermitian submersions. In this case, the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total manifold of the submersion.

Şahin [40] defined the notion of antiinvariant Riemannian submersions from almost Hermitian manifolds. Afterwards, he also defined slant submersions from almost Hermitian manifolds in [41]. After that, many geometers studied this area and obtained lots of results on the new topic (see [3, 4, 6, 13, 21, 25, 26, 36, 38, 44–46]). Recent developments on the notion of Riemannian submersion can be found in [43].

In [14], Chinea introduced the notion of almost contact Riemannian submersions between almost contact metric manifolds. He obtained the differential geometric properties among total space, fibers, and base spaces.

A related topic of growing interest deals with the study of the so called *horizontally conformal submersions*: these maps, which provide a natural generalization of Riemannian submersion, were introduced

*Correspondence: mehmetakifakyol@bingol.edu.tr
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independently by Fuglede [20] and Ishihara [30]. As a generalization of holomorphic submersions, the notion of conformal holomorphic submersions was defined by Gudmundsson and Wood [23, 24] (see also [1, 2, 4, 7, 15–17, 23, 35, 37]). In 2017, Akyol and Şahin [5] defined a conformal slant submersion from almost Hermitian manifolds onto a Riemannian manifold. In this paper, we consider conformal slant submersions from a cosymplectic manifold onto a Riemannian manifold.

The paper is organized as follows. In Section 2, we recall several notions and formulas for other sections. In the third section, we introduce conformal slant submersions from cosymplectic manifolds onto Riemannian manifolds, mention a lot of examples, investigate the geometry of leaves of the vertical distribution and the horizontal distribution, and find necessary and sufficient conditions for a conformal slant submersion to be totally geodesic and harmonic, respectively. Finally, we consider a decomposition theorem on total space of the new submersion.

2. Cosymplectic manifolds

A $(2n+1)$-dimensional C^∞-manifold is said to have an almost contact structure if there exist on N a tensor field ϕ of type (1,1), a vector field ξ, and 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta o \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold N satisfying the following conditions:

$$g_1(\phi X_1, \phi X_2) = g_1(X_1, X_2) - \eta(X_1)\eta(X_2), \quad \eta(X_1) = g_1(X_1, \xi), \tag{2.2}$$

where $X_1, X_2 \in \Gamma(TN)$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J_1 on the product manifold $N \times \mathbb{R}$ is given by

$$J_1(X_1, f\frac{d}{dt}) = (\phi X_1 - f\xi, \eta(X_1)\frac{d}{dt}),$$

where f is a C^∞-function on $N \times \mathbb{R}$ having no torsion, i.e. J_1 is integrable. The condition for normality in terms of ϕ, ξ, and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on N, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally, the fundamental two-form Φ is defined $\Phi(X_1, X_2) = g_1(X_1, \phi X_2)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be cosymplectic if it is normal and both Φ and η are closed [8, 31], and the structure equation of a cosymplectic manifold is given by

$$(\nabla_{X_1}\phi)X_2 = 0 \tag{2.3}$$

for any X_1, X_2 tangent to N, where ∇ denotes the Riemannian connection of the metric g on N. Moreover, for a cosymplectic manifold, we have,

$$\nabla_{X_1}\xi = 0. \tag{2.4}$$

The canonical example of a cosymplectic manifold is given by the product $B^{2n} \times \mathbb{R}$ Kaehler manifold $B^{2n}(J, g)$ with the $\mathbb{R}$ real line. Now we will introduce a well-known cosymplectic manifold example of $\mathbb{R}^{2n+1}$.

Example 2.1 *([12, 33]) We consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates (u_i, v_i, t) $(i = 1, 2, ..., n)$ and its usual contact one-form $\eta = dt$. The Reeb vector field ξ is given by $\frac{\partial}{\partial t}$ and its Riemannian metric $g_{\mathbb{R}^{2n+1}}$ and tensor* *field ϕ are given by*

$$g_{\mathbb{R}^{2n+1}} = (dt)^2 + \sum_{i=1}^{n}((du_i)^2 + (dv_i)^2), \;\; \varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. The vector fields $e_i = \frac{\partial}{\partial v_i}$, $e_{n+i} = \frac{\partial}{\partial u_i}$, ξ form a ϕ-basis for the cosymplectic structure. On the other hand, it can be shown that $\mathbb{R}^{2n+1}(\phi, \xi, \eta, g)$ is a cosymplectic manifold.

3. Conformal submersions

Let $\psi : (N^m, g_N) \longrightarrow (B^n, g_B)$ be a smooth map between Riemannian manifolds, and let $q \in N$. Then ψ is said to be *horizontally weakly conformal or semiconformal* at q [7] if either (i) $d\psi_q = 0$, or (ii) $d\psi_q$ maps the horizontal space $\mathcal{H}_q = \{ker(d\psi_q)\}^\perp$ conformally onto $T_{\psi(q)}B$, i.e. $d\psi_q$ is surjective and there exists a number $\Lambda(q) \neq 0$ such that

$$g_B(d\psi_q X, d\psi_q Y) = \Lambda(q) g_N(X, Y), \;\; (X, Y \in \mathcal{H}_q).$$

We say that point q is of type (i) as a critical point if it satisfies the type (i), and we shall call the point q a regular point if it satisfies the type (ii). At a critical point, $d\psi_q$ has rank 0; at a regular point, $d\psi_q$ has rank n and ψ is submersion. Furthermore, the positive number $\Lambda(q)$ is called the *square dilation* of ψ at q. The map ψ is called *horizontally weakly conformal* or *semiconformal* on N if it is horizontally weakly conformal at every point of N and it has no critical point; then we call it a *horizontally conformal submersion*.

A vector field $X_1 \in \Gamma(TN)$ is called a basic vector field if $X_1 \in \Gamma((ker\psi_*)^\perp)$ and ψ-related with a vector field $\bar{X_1} \in \Gamma(TB)$, which means that $(\psi_{*q} X_{1q}) = \bar{X}_1(\psi(q)) \in \Gamma(TB)$ for any $q \in \Gamma(TN)$.

Define O'Neill's tensors T and A by

$$A_{X_1} X_2 = v\nabla_{hX_1} hX_2 + v\nabla_{hX_1} vX_2, \tag{3.1}$$

$$T_{X_1} X_2 = h\nabla_{vX_1} vX_2 + v\nabla_{vX_1} hX_2, \tag{3.2}$$

where for any $X_1, X_2 \in \Gamma(TN)$ and v, h are the vertical and horizontal projections (see [19]). Also, by using (3.1) and (3.2), for $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$ and $V_1, V_2 \in \Gamma(ker\psi_*)$, we have

$$\nabla_{V_1} V_2 = T_{V_1} V_2 + \hat{\nabla}_{V_1} V_2, \tag{3.3}$$

$$\nabla_{V_1} X_1 = h\nabla_{V_1} X_1 + T_{V_1} X_1, \tag{3.4}$$

$$\nabla_{X_1} V_1 = A_{X_1} V_1 + v\nabla_{X_1} V_1, \tag{3.5}$$

$$\nabla_{X_1} X_2 = h\nabla_{X_1} X_2 + A_{X_1} X_2, \tag{3.6}$$

where $\hat{\nabla}_{V_1} V_2 = v\nabla_{V_1} V_2$. If X_1 is basic, then $h\nabla_{V_1} X_1 = A_{X_1} V_1$. Then we easily obtain $-g_N(A_{X_1} E_1, E_2) = g_N(E_1, A_{X_1} E_2)$ and $-g_N(T_{V_1} E_1, E_2) = g_N(E_1, T_{V_1} E_2)$ for all $E_1, E_2 \in T_x N$. T is exactly the second fundamental form of the fibers of ψ. For the special case where ψ is horizontally conformal, we have the following:

Proposition 3.1 *([23]) Let $\psi : (N, g_N) \longrightarrow (B, g_B)$ be a horizontally conformal submersion with dilation λ and X_1, X_2 be horizontal vectors, then*

$$A_{X_1} X_2 = \frac{1}{2}\{v[X_1, X_2] - \lambda^2 g_N(X_1, X_2) grad_v(\frac{1}{\lambda^2})\}. \tag{3.7}$$

Let (N, g_N) and (B, g_B) be Riemannian manifolds and suppose that $\psi : N \longrightarrow B$ is a smooth map between them. The second fundamental form of ψ is given by

$$(\nabla \psi_*)(X_1, X_2) = \nabla^{\psi}_{X_1} \psi_*(X_2) - \psi_*(\nabla^N_{X_1} X_2) \tag{3.8}$$

for any $X_1, X_2 \in \Gamma(TN)$, where ∇^{ψ} is the pullback connection. It is obvious that the second fundamental form $(\nabla \psi_*)$ is symmetric.

Lemma 3.1 *[47] Let (N, g_N) and (B, g_B) be Riemannian manifolds and suppose that $\psi : N \longrightarrow B$ is a smooth map between them. Then we have*

$$\nabla^{\psi}_{X_1} \psi_*(X_2) - \nabla^{\psi}_{X_2} \psi_*(X_1) - \psi_*([X_1, X_2]) = 0 \tag{3.9}$$

for $X_1, X_2 \in \Gamma(TN)$.

From Lemma 2.1, for any X_1 a basic vector field and $V_1 \in \Gamma(ker\psi_*)$, we obtain $[X_1, V_1] \in \Gamma(ker\psi_*)$.

Remark 3.1 *In this paper, we assume that all horizontal vector fields are basic vector fields.*

Recall that ψ is called harmonic if the tension field $\tau(\psi) = trace(\nabla \psi_*) = 0$ (for details, see [7]).

Lemma 3.2 *[7] Let $\psi : N \longrightarrow B$ be a horizontally conformal submersion. Then we have:*

(a) $(\nabla \psi_*)(X_1, X_2) = X_1(\ln \lambda)\psi_* X_2 + X_2(\ln \lambda)\psi_* X_1 - g_N(X_1, X_2)\psi_*(\nabla \ln \lambda)$,

(b) $(\nabla \psi_*)(V_1, V_2) = -\psi_*(T_{V_1} V_2)$,

(c) $(\nabla \psi_*)(X_1, V_1) = -\psi_*(\nabla^N_{X_1} V_1) = -\psi_*(A_{X_1} V_1)$,

for any $V_1, V_2 \in \Gamma(ker\psi_*)$ and $X_1, X_2 \in \Gamma((ker\psi_*)^{\perp})$.

Finally, we will mention the following from [39].

Let g_N be a Riemannian metric tensor on the manifold $N = N_1 \times N_2$ and assume that the canonical foliations $\mathcal{D}_{N_1}$ and $\mathcal{D}_{N_2}$ intersect perpendicularly everywhere. Then g_N is the metric tensor of a usual product of Riemannian manifolds $\Longleftrightarrow$ $\mathcal{D}_{N_1}$ and $\mathcal{D}_{N_2}$ are totally geodesic foliations.

4. Conformal slant submersions

In this section, we introduce the notion of conformal slant submersions from cosymplectic manifolds onto Riemannian manifolds. We mention lots of examples and obtain the integrability of distributions, the geometry of foliations, some conditions related to totally geodesicness, and harmonicity of the map.

Definition 4.1 *Let $\psi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a horizontally conformal submersion, where $(N, \varphi, \xi, \eta, g_N)$ is a cosymplectic manifold and (B, g_B) is a Riemannian manifold. The map ψ is said to be slant if for any nonzero vector $V_1 \in \Gamma(ker\psi_*) - <\xi>$, the angle $\omega(V_1)$ between φV_1 and the space $ker\psi_*$ is a constant (which is independent of the choice of $p \in N$ and of $V_1 \in \Gamma(ker\psi_*) - <\xi>$). The angle ω is called the slant angle of the conformal slant submersion.*

Conformal holomorphic submersion and conformal antiinvariant submersions are conformal slant submersions with $\omega = 0$ and $\frac{\pi}{2}$, respectively. A conformal slant submersion that is not a conformal holomorphic submersion or conformal antiinvariant is called a proper conformal slant submersion.

Now we present some examples.

Example 4.1 $\mathbb{R}^5$ *has a cosymplectic structure as in Example 2.1. Let $\psi_1 : \mathbb{R}^5 \to \mathbb{R}^2$ be a map defined by $\psi_1(u_1, u_2, v_1, v_2, t) = e^7(u_1 \cos\alpha - v_1 \sin\alpha, u_2 \sin\beta - v_2 \cos\beta)$. Then, by direct calculations, we obtain the Jacobian matrix of ψ_1 as:*

$$e^7 \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha & 0 & 0 \\ 0 & \sin\beta & 0 & -\cos\beta & 0 \end{bmatrix}.$$

Since $rank(\psi_1*) = 2$, the map ψ_1 is a submersion. By a straightforward computation, we see that

$$ker\psi_1* = span\{V_1 = \sin\alpha\frac{\partial}{\partial u_1} + \cos\alpha\frac{\partial}{\partial v_1},\ \ V_2 = \cos\beta\frac{\partial}{\partial u_2} + \sin\beta\frac{\partial}{\partial v_2},\ \ V_3 = \xi = \frac{\partial}{\partial t}\}$$

and

$$(ker\psi_1*)^{\perp} = span\{X_1 = \cos\alpha\frac{\partial}{\partial u_1} - \sin\alpha\frac{\partial}{\partial v_1},\ \ X_2 = \sin\beta\frac{\partial}{\partial u_2} - \cos\beta\frac{\partial}{\partial v_2}\}.$$

Then the map ψ_1 is a conformal slant submersion with the slant angle ω and dilation $\lambda = e^7$ such that $\cos\omega = \mid \cos(\alpha + \beta) \mid$.

Example 4.2 $\mathbb{R}^5$ *has a cosymplectic structure as in Example 2.1. Let $\psi_2 : \mathbb{R}^5 \to \mathbb{R}^2$ be a map defined by $\psi_2(u_1, u_2, v_1, v_2, t) = \pi^5(\frac{u_1 - u_2}{\sqrt{2}}, v_2)$. Then, by direct calculations, we obtain the Jacobian matrix of ψ_2 as:*

$$\pi^5 \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since the rank of this matrix is equal to 2, the map ψ_2 is a submersion. After some computations, we obtain

$$ker\psi_2* = span\{H_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2},\ \ H_2 = \frac{\partial}{\partial v_1},\ \ H_3 = \xi = \frac{\partial}{\partial t}\}$$

and

$$(ker\psi_2*)^{\perp} = span\{Z_1 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}),\ \ Z_2 = \frac{\partial}{\partial v_2}\}.$$

Furthermore, $\psi_2(H_1) = -\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}$ and $\psi_2(H_2) = \frac{\partial}{\partial u_2}$ imply that $\mid g(\psi_2(H_1), H_2) \mid = \frac{1}{\sqrt{2}}$. Thus, the map ψ_2 is a conformal slant submersion with the slant angle $\omega = \frac{\pi}{4}$ and dilation $\lambda = \pi^5$.

Example 4.3 $\mathbb{R}^7$ *has a cosymplectic structure as in Example 2.1. Let* $\psi_3 : \mathbb{R}^7 \to \mathbb{R}^4$ *be a map defined by* $\psi_3(u_1, u_2, u_3, v_1, v_2, v_3, t) = e^{11}(u_1, \frac{v_1-v_2}{\sqrt{2}}, v_3, u_2)$*. Then, by direct calculations, we obtain the Jacobian matrix of* ψ_3 *as:*

$$e^{11}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We easily see that the map ψ_3 *is a submersion. After some computations, we derive*

$$ker\psi_{3}* = span\{\bar{H}_1 = \frac{\partial}{\partial u_3},\ \ \bar{H}_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}),\ \ \bar{H}_3 = \xi = \frac{\partial}{\partial t}\}$$

and

$$(ker\psi_{3}*)^{\perp} = span\{\bar{Z}_1 = \frac{\partial}{\partial u_1},\ \ \bar{Z}_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2}),\ \ \bar{Z}_3 = \frac{\partial}{\partial u_2}, \bar{Z}_4 = \frac{\partial}{\partial v_3}\}.$$

Moreover, $\psi_3(\bar{H}_1) = -\frac{\partial}{\partial v_1}$ *and* $\psi_3(\bar{H}_2) = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3})$ *imply that* $\mid g(\psi_3 \bar{H}_1, \bar{H}_2) \mid = \frac{1}{\sqrt{2}}$*. Thus, the map* ψ_3 *is a conformal slant submersion with the slant angle* $\omega = \frac{\pi}{4}$ *and dilation* $\lambda = e^{11}$*.*

Example 4.4 *Let* $(N, \varphi, \xi, \eta, g_N)$ *be an almost contact metric manifold. Suppose that* $\sigma : TN \to N$ *is the natural projection. Then the map* σ *is a conformal slant submersion with the slant angle* $\omega = 0$ *and dilation* $\lambda = 1$*.*

Example 4.5 *Let* $(N, \varphi, \xi, \eta, g_N)$ *be an almost contact metric manifold and* (B, g_B) *a Riemannian manifold. Suppose that* $\psi_3 : N \to B$ *is a slant submersion [18]. Then the map* ψ_3 *is a conformal slant submersion with dilation* $\lambda = 1$*.*

Example 4.6 *Let* $(N^{2n+1}, \varphi, \xi, \eta, g_N)$ *be an almost contact metric manifold and* (B^{2n}, g_B) *a Riemannian manifold. Suppose that* $\psi_4 : N \to B$ *is a horizontally conformal submersion with dilation* λ*. Then the map* ψ_4 *is a conformal slant submersion with the slant angle* $\omega = \frac{\pi}{2}$ *and dilation* $\lambda = 1$ *[1].*

Let $\psi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ to a Riemannian manifold (B, g_B). Then for any $V_1 \in \Gamma(ker\psi_*)$, we write

$$\varphi V_1 = DV_1 + EV_1, \tag{4.1}$$

where DV_1 and EV_1 are vertical and horizontal components of φV_1, respectively.

Given $X_1 \in \Gamma(ker\psi_*)^{\perp}$, we write

$$\varphi X_1 = dX_1 + eX_1, \tag{4.2}$$

where $dX_1 \in \Gamma(ker\psi_*)$ and $eX_1 \in \Gamma(ker\psi_*)^{\perp}$.

We denote the complementary orthogonal distribution to decomposition $E(ker\psi_*)$ in $(ker\psi_*)^{\perp}$ by μ. Then we get

$$(ker\psi_*)^{\perp} = E(ker\psi_*) \oplus \mu. \tag{4.3}$$

From (2.4), (3.3), and (3.5) we have

$$T_{V_1}\xi = 0, \quad A_{X_1}\xi = 0 \tag{4.4}$$

for $X_1 \in \Gamma((ker\psi_*)^{\perp})$ and $V_1 \in \Gamma(ker\psi_*)$.

By using (2.2), (4.1), and (4.2) we get the following result:

Lemma 4.1 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). Then we obtain*

$$DdX_1 + deX_1 = 0, \tag{4.5}$$

$$EdX_1 + e^2X_1 = -X_1, \tag{4.6}$$

$$D^2V_1 + dEV_1 = \varphi^2V_1, \tag{4.7}$$

$$EDV_1 + eEV_1 = 0, \tag{4.8}$$

for $X_1 \in \Gamma((ker\psi_*)^{\perp})$ and $V_1 \in \Gamma(ker\psi_*)$.

Using (3.3), (3.4), (4.1), and (4.2) we obtain

$$(\nabla_{V_1}D)V_2 = dT_{V_1}V_2 - T_{V_1}EV_2, \tag{4.9}$$

$$(\nabla_{V_1}E)V_2 = eT_{V_1}V_2 - T_{V_1}DV_2, \tag{4.10}$$

where

$$(\nabla_{V_1}D)V_2 = \hat{\nabla}_{V_1}DV_2 - D\hat{\nabla}_{V_1}V_2, \tag{4.11}$$

$$(\nabla_{V_1}E)V_2 = h\nabla_{V_1}EV_2 - E\hat{\nabla}_{V_1}V_2, \tag{4.12}$$

for $V_1, V_2 \in \Gamma((ker\psi_*)$. We call D and E parallel if $\nabla D = 0$ and $\nabla E = 0$, respectively.

Since the proof of the following theorem is quite similar to Theorem 2.2 of [11], we do not give the proof of it.

Theorem 4.1 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). Then we obtain*

$$D^2 = -\cos^2\omega(I - \eta \otimes \xi). \tag{4.13}$$

By (2.2), (4.1), and (4.13) we have the following result.

Corollary 4.1 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). Then we obtain*

$$g_N(DV_1, DV_2) = \cos^2\omega(g_N(V_1, V_2) - \eta(V_1)\eta(V_2)), \tag{4.14}$$

$$g_N(EV_1, EV_2) = \sin^2\omega(g_N(V_1, V_2) - \eta(V_1)\eta(V_2)), \tag{4.15}$$

for $V_1, V_2 \in \Gamma((ker\psi_*)$.

Proposition 4.1 *Let $\psi : (N, g_N, \varphi, \eta, \xi) \to (B, g_B)$ be a conformal slant submersion. If N is a cosymplectic manifold and E is parallel with respect to ∇ on $(ker\psi_*)$, then we have*

$$T_{DV_1}DV_1 = -\cos^2\omega T_{V_1}V_1 \tag{4.16}$$

for any $V_1 \in \Gamma(ker\psi_*)$.

Proof If E is parallel, then we derive $eT_{V_1}V_2 = T_{V_1}DV_2$ for any $V_1, V_2 \in \Gamma(ker\psi_*)$. Interchanging the role of V_1 and V_2, we get $eT_{V_2}V_1 = T_{V_2}DV_1$. Thus, we have

$$eT_{V_1}V_2 - eT_{V_2}V_1 = T_{V_1}DV_2 - T_{V_2}DV_1.$$

Since T is symmetric, we get $T_{V_1}DV_2 = T_{V_2}DX_1$. Then substituting V_2 by DV_1 we get $T_{V_1}D^2V_1 = T_{DV_1}DV_1$. By (4.4) and (4.13) we obtain (4.16).

□

Theorem 4.2 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N^{2n+1}, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B^s, g_B). Suppose that E is parallel with slant angle $\omega \in [0, \frac{\pi}{2})$. Then all the fibers of the map ψ are minimal.*

Proof Using (4.4) and Lemma 5 of [18], we have

$$\tau(\psi) = \sum_{i=1}^{n-\frac{s}{2}} (\nabla\psi_*)(E_i, E_i) = -\sum_{i=1}^{n-\frac{s}{2}} \psi_*(T_{E_i}E_i + T_{\sec\omega DE_i}\sec\omega DE_i) - \psi_*(T_\xi\xi).$$

Since $T_\xi\xi = 0$, we get

$$\tau = -\sum_{i=1}^{n-\frac{s}{2}} \psi_*(T_{E_i}E_i + \sec^2\omega T_{DE_i}DE_i).$$

By (4.16), we have

$$\tau = -\sum_{i=1}^{n-\frac{s}{2}} \psi_*(T_{E_i}E_i + \sec^2\omega(-\cos^2\omega T_{E_i}E_i) = -\sum_{i=1}^{n-\frac{s}{2}} \psi_*(T_{E_i}E_i - T_{E_i}E_i) = 0.$$

Thus, we prove that ψ is harmonic. □

Now we deal with the integrability of the distributions and the geometry of foliations.

Theorem 4.3 *Let* ψ *be a conformal slant submersion from a cosymplectic manifold* $(N, \varphi, \xi, \eta, g_N)$ *onto a Riemannian manifold* (B, g_B). *Then the following conditions are equivalent to each other:*

(i) The distribution $(ker\psi_*)^\perp$ *is integrable,*

(ii)
$$\begin{aligned}&\lambda^{-2} g_B(\nabla^\psi_{X_2}\psi_* eX_1 - \nabla^\psi_{X_1}\psi_* eX_2, \psi_* EV_1)\\
&= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)\\
&+ g_N(A_{X_1} dX_2 - A_{X_2} dX_1 - X_1(\ln\lambda) eX_2 + X_2(\ln\lambda) eX_1\\
&- eX_2(\ln\lambda) X_1 + eX_1(\ln\lambda) X_2\\
&+ 2g_N(X_1, eX_2)(\nabla \ln\lambda), EV_1)\end{aligned}$$

for any $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$ *and* $V_1 \in \Gamma(ker\psi_*)$.

Proof In view of (2.2), (2.3), and (4.4), we have

$$g_N([X_1, X_2], V_1) = g_N(\nabla_{X_1}\varphi X_2 - \nabla_{X_2}\varphi X_1, \varphi V_1) \tag{4.17}$$

for any $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$ and $V_1 \in \Gamma(ker\psi_*)$. Then, using (4.1), (4.2), and (4.17), we derive

$$\begin{aligned}g_M([X, Y], W) &= g_N(\nabla_{X_1} dX_2, DV_1) + g_N(\nabla_{X_1} dX_2, EV_1)\\
&+ g_N(\nabla_{X_1} eX_2, DV_1) + g_N(\nabla_{X_1} eX_2, EV_1)\\
&- g_N(\nabla_{X_2} dX_1, DV_1) - g_N(\nabla_{X_2} dX_1, EV_1)\\
&- g_N(\nabla_{X_2} eX_1, DV_1) - g_N(\nabla_{X_2} eX_1, EV_1).\end{aligned}$$

Using the property of ψ, (3.5), and (3.6) we get

$$\begin{aligned}g_N([X_1, X_2], V1) &= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)\\
&+ g_N(A_{X_1} dX_2 + h\nabla_{X_1} eX_2 - A_{X_2} dX_1 - h\nabla_{X_2} eX_1, EV_1)\\
&= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)\\
&+ g_N(A_{X_1} dX_2 - A_{X_2} dX_1, EV_1)\\
&+ \lambda^{-2} g_B(\psi_*(h\nabla_{X_1} eX_2 - \psi_*(h\nabla_{X_2} eX_1, \psi_* EV_1).\end{aligned}$$

Thus, by (3.8) and Lemma 3.2 we obtain

$$\begin{aligned}
g_N([X_1, X_2], V1) &= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)\\
&+ g_N(A_{X_1} dX_2 - A_{X_2} dX_1, EV_1)\\
&+ \lambda^{-2} g_B(-(\nabla\psi_*)(X_1, eX_2) + \nabla^{\psi}_{X_1} \psi_* eX_2 + (\nabla\psi_*)(X_2, eX_1) - \nabla^{\psi}_{X_2} \psi_* eX_1, \psi_* EV_1)\\
&= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)\\
&+ g_N(A_{X_1} dX_2 - A_{X_2} dX_1, EV_1) + \lambda^{-2} g_B(\nabla^{\psi}_{X_1} \psi_* eX_2 - \nabla^{\psi}_{X_2} \psi_* eX_1, \psi_* EV_1)\\
&+ \lambda^{-2} g_B(-X_1(\ln\lambda)\psi_* eX_2 - eX_2(\ln\lambda)\psi_* X_1 + g_N(X_1, eX_2)\psi_*(\nabla \ln\lambda)\\
&+ X_2(\ln\lambda)\psi_* eX_1 + eX_1(\ln\lambda)\psi_* X_2 - g_N(X_2, eX_1)\psi_*(\nabla \ln\lambda), \psi_* EV_1)\\
&= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)\\
&+ g_N(A_{X_1} dX_2 - A_{X_2} dX_1 - X_1(\ln\lambda) eX_2 + X_2(\ln\lambda) eX_1 - eX_2(\ln\lambda) X_1\\
&+ eX_1(\ln\lambda) X_2 + 2g_N(X_1, eX_2)(\nabla \ln\lambda), EV_1)\\
&+ \lambda^{-2} g_B(\nabla^{\psi}_{X_1} \psi_* eX_2 - \nabla^{\psi}_{X_2} \psi_* eX_1, \psi_* EV_1).
\end{aligned}$$

Hence, $(i) \Leftrightarrow (ii)$. □

We note that a horizontally conformal submersion $\psi : N \to B$ is said to be *horizontally homothetic* if the gradient of its dilation λ is vertical, i.e. $h(grad\lambda) = 0$ at $p \in N$, where h is the projection on the horizontal space $(ker\psi_*)^{\perp}$.

Theorem 4.4 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). Suppose that the distribution $(ker\psi_*)^{\perp}$ is integrable. Then the following conditions are equivalent to each other:*

(i) The map ψ is a horizontally homothetic submersion,

(ii) $\lambda^{-2} g_B(\nabla^{\psi}_{X_2} \psi_* eX_1 - \nabla^{\psi}_{X_1} \psi_* eX_2, \psi_* EV_1)$
$= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)$
$+ g_N(A_{X_1} dX_2 - A_{X_2} dX_1, EV_1)$

for $X_1, X_2 \in \Gamma((ker\psi_*)^{\perp})$ and $V_1 \in \Gamma(ker\psi_*)$.

Proof For any $X_1, X_2 \in \Gamma((ker\psi_*)^{\perp})$ and $V_1 \in \Gamma(ker\psi_*)$, by hypothesis, we get

$$\begin{aligned}
0 = g_N([X_1, X_2], V_1) &= g_N(v\nabla_{X_1} dX_2 + A_{X_1} eX_2 - v\nabla_{X_2} dX_1 - A_{X_2} eX_1, DV_1)\\
&+ g_N(A_{X_1} dX_2 - A_{X_2} dX_1 - X_1(\ln\lambda) eX_2 + X_2(\ln\lambda) eX_1 \qquad (4.18)\\
&- eX_2(\ln\lambda) X_1 + eX_1(\ln\lambda) X_2 + 2g_N(X_1, eX_2)(\nabla \ln\lambda), EV_1)\\
&+ \lambda^{-2} g_B(\nabla^{\psi}_{X_1} \psi_* eX_2 - \nabla^{\psi}_{X_2} \psi_* eX_1, \psi_* EV_1). \qquad (4.19)
\end{aligned}$$

By (4.19), we obtain $(i) \Leftrightarrow (ii)$. Conversely, using (4.19), we have

$$\begin{aligned} 0 &= g_N(-X_1(\ln\lambda)eX_2 + X_2(\ln\lambda)eX_1 - eX_2(\ln\lambda)X_1 + eX_1(\ln\lambda)X_2 \\ &+ 2g_N(X_1, eX_2)(\nabla \ln\lambda), EV_1). \end{aligned} \tag{4.20}$$

If $X_2 \in \Gamma(\mu)$, then by (4.3) and (4.20), we derive

$$0 = g_N(X_2(\ln\lambda)eX_1 - \varphi X_2(\ln\lambda)X_1 + 2g_N(X_1, \varphi X_2)(\nabla \ln\lambda), EV_1). \tag{4.21}$$

Now, taking $X_1 = \varphi X_2$ in (4.21), we obtain

$$\begin{aligned} 0 &= g_N(X_2(\ln\lambda)\varphi^2 X_2 - \varphi X_2(\ln\lambda)\varphi X_2 + 2g_N(\varphi X_2, \varphi X_2)(\nabla \ln\lambda), EV_1) \\ &= 2g_N(X_2, X_2)g_N(\nabla \ln\lambda, EV_1), \end{aligned} \tag{4.22}$$

which implies

$$g_N(\nabla\lambda, EV_1) = 0, \quad V_1 \in \Gamma(ker\psi_*). \tag{4.23}$$

Taking $X_1 = EV_1$ in (4.21), we get

$$\begin{aligned} 0 &= g_N(X_2(\ln\lambda)eEV_1 - \varphi X_2(\ln\lambda)EV_1, EV_1) \\ &= -\varphi X_2(\ln\lambda)g_N(EV_1, EV_1), \end{aligned}$$

which means

$$g_N(\nabla\lambda, X_3) = 0, \quad X_3 \in \Gamma(\mu). \tag{4.24}$$

Using (4.23) and (4.24), we obtain $(ii) \Leftrightarrow (i)$. □

Theorem 4.5 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). Then the following assertions are equivalent to each other:*

(i) $(ker\psi_*)^\perp$ *defines a totally geodesic foliation on the total space,*

(ii) $\lambda^{-2}g_B(\nabla^\psi_{X_1}\psi_* X_2, \psi_* EDV_1) - \lambda^{-2}g_B(\nabla^\psi_{X_1}\psi_* eX_2, \psi_* EV_1) = g_N(A_{X_1}dX_2, EV_1)$
$+ g_N(-X_1(\ln\lambda)eX_2 - eX_2(\ln\lambda)X_1 + g_N(X_1, eX_2)(\nabla \ln\lambda), EV_1)$
$- g_N(-X_1(\ln\lambda)X_2 - X_2(\ln\lambda)X_1 + g_N(X_1, X_2)(\nabla \ln\lambda), EDV_1)$

for $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$ and $V_1 \in \Gamma(ker\psi_*)$.

Proof Given $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$, $V_1 \in \Gamma(ker\psi_*)$ and by (2.2), (2.3), and (2.4), we get

$$g_N(\nabla_{X_1}X_2, V_1) = g_N(\nabla_{X_1}\varphi X_2, \varphi V_1).$$

From (2.3), (3.5), (3.6), (4.1), (4.2), and (4.13) we derive

$$\begin{aligned} g_N(\nabla_{X_1}X_2, V_1) &= g_N(\nabla_{X_1}\varphi X_2, DV_1 + EV_1) \\ &= \cos^2\omega g_N(\nabla_{X_1}X_2, V_1) - g_N(\nabla_{X_1}X_2, EDV_1) \\ &+ g_N(A_{X_1}dX_2 + h\nabla_{X_1}eX_2, EV_1) \\ \sin^2\omega g_N(\nabla_{X_1}X_2, V_1) &= -g_N(\nabla_{X_1}X_2, EDV_1) + g_N(A_{X_1}dX_2 + h\nabla_{X_1}eX_2, EV_1). \end{aligned}$$

Using the property of ψ, (3.8), and Lemma 3.2, we have

$$\begin{aligned}
\sin^2\omega g_N(\nabla_{X_1}X_2,V_1) &= -g_N(\nabla_{X_1}X_2,EDV_1)+g_N(A_{X_1}dX_2,EV_1)+\lambda^{-2}g_B(\psi_*h\nabla_{X_1}eX_2,\psi_*EV_1)\\
&= -g_N(\nabla_{X_1}X_2,EDV_1)+g_N(A_{X_1}dX_2,EV_1)\\
&+ \lambda^{-2}g_B(-(\nabla\psi_*)(X_1,eX_2)+\nabla^{\psi}_{X_1}\psi_*eX_2,\psi_*EV_1)\\
&= -g_N(\nabla_{X_1}X_2,EDV_1)+g_N(A_{X_1}dX_2,EV_1)+\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*eX_2,\psi_*EV_1)\\
&+ \lambda^{-2}g_B(-X_1(\ln\lambda)\psi_*eX_2-eX_2(\ln\lambda)\psi_*X_1+g_N(X_1,eX_2)\psi_*(\nabla\ln\lambda),\psi_*EV_1)\\
&= -g_N(\nabla_{X_1}X_2,EDV_1)+g_N(A_{X_1}dX_2,EV_1)+\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*eX_2,\psi_*EV_1)\\
&+ g_N(-X_1(\ln\lambda)eX_2-eX_2(\ln\lambda)X_1+g_N(X_1,eX_2)(\nabla\ln\lambda),EV_1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
g_N(\nabla_{X_1}X_2,EDV_1) &= \lambda^{-2}g_B(\psi_*\nabla_{X_1}X_2,\psi_*EDV_1)\\
&= \lambda^{-2}g_B(-(\nabla\psi_*)(X_1,X_2)+\nabla^{\psi}_{X_1}\psi_*X_2,\psi_*EDV_1)\\
&= \lambda^{-2}g_B(-X_1(\ln\lambda)\psi_*X_2-X_2(\ln\lambda)\psi_*X_1+g_N(X_1,X_2)\psi_*(\nabla\ln\lambda)\\
&+ \nabla^{\psi}_{X_1}\psi_*X_2,\psi_*EDV_1)\\
&= g_N(-X_1(\ln\lambda)X_2-X_2(\ln\lambda)X_1+g_N(X_1,X_2)(\nabla\ln\lambda),EDV_1)\\
&+ \lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*X_2,\psi_*EDV_1).
\end{aligned}$$

Thus, we obtain $(i)\Leftrightarrow(ii)$. □

Theorem 4.6 *Let ψ be a conformal slant submersion from a cosymplectic manifold (N,φ,ξ,η,g_N) onto a Riemannian manifold (B,g_B). Suppose that the distribution $(ker\psi_*)^{\perp}$ defines a totally geodesic foliation on the total space. Then the following assertions are equivalent to each other:*

(i) The map ψ is a horizontally homothetic submersion,

(ii) $\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*X_2,\psi_*EDV_1)-\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*eX_2,\psi_*EV_1)=g_N(A_{X_1}dX_2,EV_1)$

for any $X_2,X_1\in\Gamma((ker\psi_*)^{\perp})$ and $V_1\in\Gamma(ker\psi_*)$.

Proof Given $X_2,X_1\in\Gamma((ker\psi_*)^{\perp})$ and $V_1\in\Gamma(ker\psi_*)$, from Theorem 4.5, we get

$$\begin{aligned}
&\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*X_2,\psi_*EDV_1)-\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*eX_2,\psi_*EV_1)\\
&=g_N(A_{X_1}dX_2,EV_1)\\
&+g_N(-X_1(\ln\lambda)eX_2-eX_2(\ln\lambda)X_1+g_N(X_1,eX_2)(\nabla\ln\lambda),EV_1)\\
&-g_N(-X_1(\ln\lambda)X_2-X_2(\ln\lambda)X_1+g_N(X_1,X_2)(\nabla\ln\lambda),EDV_1),
\end{aligned}\tag{4.25}$$

which implies $(i)\Leftrightarrow(ii)$. Conversely, by (4.25), we obtain

$$\begin{aligned}
0 &= g_N(-X_1(\ln\lambda)eX_2-eX_2(\ln\lambda)X_1+g_N(X_1,eX_2)(\nabla\ln\lambda),EV_1)\\
&- g_N(-X_1(\ln\lambda)X_2-X_2(\ln\lambda)X_1+g_N(X_1,X_2)(\nabla\ln\lambda),EDV_1).
\end{aligned}\tag{4.26}$$

Now, taking $X_2 \in \Gamma(\mu)$, and using (4.26), we arrive at

$$\begin{aligned} 0 &= g_N(-\varphi X_2(\ln\lambda)X_1 + g_N(X_1,\varphi X_2)(\nabla \ln\lambda), EV_1) \\ &- g_N(-X_2(\ln\lambda)X_1 + g_N(X_1,X_2)(\nabla\ln\lambda), EDV_1). \end{aligned} \tag{4.27}$$

Taking $X_1 = \varphi X_2$ in (4.27) we find

$$0 = g_N(\varphi X_1, \varphi X_2)(\nabla \ln\lambda), EV_1), \tag{4.28}$$

which implies

$$0 = g_N(\nabla\lambda, EV_1), \quad V_1 \in \Gamma(ker\psi_*). \tag{4.29}$$

Taking $X_1 = EX_2$ in (4.27) and by (4.3) and (4.15), we have

$$\begin{aligned} 0 &= -g_N(EV_1, EV_1)g_N(\varphi X_2, \nabla\ln\lambda) + g_N(X_2, \nabla\ln\lambda)g_N(EV_1, EDV_1) \\ &= -g_N(EV_1, EV_1)g_N(\varphi X_2, \nabla\ln\lambda), \end{aligned} \tag{4.30}$$

which implies

$$0 = g_N(\nabla\lambda, X_3), \quad X_3 \in \Gamma(\mu). \tag{4.31}$$

Using (4.29) and (4.31), we obtain $(ii) \Leftrightarrow (i)$. □

Theorem 4.7 *Let ψ be a conformal slant submersion from a cosymplectic manifold (N,φ,ξ,η,g_N) onto a Riemannian manifold (B,g_B). Then for any $V_1,V_2 \in \Gamma(ker\psi_*)$ and $X_1 \in \Gamma((ker\psi_*)^\perp)$ the following assertions are equivalent to each other:*

(i) The distribution $ker\psi_*$ defines a totally geodesic foliation on the total space,

(ii) $g_N(\nabla_{V_1}EDV_2, X_1) = g_N(T_{V_1}EV_2, dX_1) + g_N(h\nabla_{V_1}EV_2, eX_1)$.

Proof For $V_1,V_2 \in \Gamma(ker\psi_*)$ and $X_1 \in \Gamma((ker\psi_*)^\perp)$, by (2.2) and (2.3) we obtain $g_N(\nabla_{V_1}V_2, X_1) = g_N(\nabla_{V_1}\varphi V_2, \varphi X_1)$. Using (2.2), (3.4),(4.1), (4.2), and (4.13) we arrive at

$$\begin{aligned} g_N(\nabla_{V_1}V_2, X_1) &= g_N(\nabla_{V_1}DV_2 + EV_2, \varphi X_1) \\ &= -g_N(\nabla_{V_1}D^2V_2 + \nabla_{V_1}EDV_2, X_1) + g_N(\nabla_{V_1}EV_2, dX_1 + eX_1) \\ &= \cos^2\omega g_N(\nabla_{V_1}V_2, X_1) - g_N(\nabla_{V_1}EDV_2, X_1) \\ &+ g_N(T_{V_1}EV_2, dX_1) + g_N(h\nabla_{V_1}EV_2, eX_1). \end{aligned}$$

Thus, we have

$$\begin{aligned} \sin^2\omega g_N(\nabla_{V_1}V_2, X_1) = &-g_N(\nabla_{V_1}EDV_2, X_1) \\ &+ g_N(T_{V_1}EV_2, dX_1) + g_N(h\nabla_{V_1}EV_2, eX_1) \end{aligned}$$

so that we obtain $(i) \Leftrightarrow (ii)$. □

Now we are going to investigate the harmonicity of ψ.

Let ψ be a horizontally conformal submersion from a Riemannian manifold (N, g_N) onto a Riemannian manifold (B, g_B) with dilation λ. Then the tension field $\tau(\psi)$ of ψ is given by

$$\tau(\psi) = -n\psi_* H + (2-s)\psi_*(\nabla \ln \lambda), \tag{4.32}$$

where H is the mean curvature vector field of the distribution $ker\psi_*$, $dimker\psi_* = n$, $dimB = s$ [7].

Using Theorem 4.2 and (4.32), we have:

Corollary 4.2 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B) with $dimB > 2$. Suppose that E is parallel with the slant angle $\omega \in [0, \frac{\pi}{2})$. Then the following assertions are equivalent to each other:*

(i) The map ψ is harmonic,

(ii) The map ψ is a horizontally homothetic submersion.

Corollary 4.3 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B) with $dimB = 2$. Suppose that E is parallel with the slant angle $\omega \in [0, \frac{\pi}{2})$. Then the map ψ is harmonic.*

Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). For $V_1 \in \Gamma(ker\psi_*)$ and $X_1 \in \Gamma(\mu)$, we call the map $\psi(Eker\psi_*, \mu)-$ totally geodesic if it satisfies $(\nabla\psi_*)(EV_1, X_1) = 0$.

Theorem 4.8 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). Then the following assertions are equivalent to each other:*

(i) The map ψ is a horizontally homothetic submersion,

(ii) The map ψ is $(Eker\psi_*, \mu)-$ totally geodesic.

Proof For $V_1 \in \Gamma(ker\psi_*)$, $X_1 \in \Gamma(\mu)$, from Lemma 3.2, we obtain

$$\begin{aligned}(\nabla\psi_*)(EV_1, X_1) &= EV_1(\ln\lambda)\psi_* X_1 + X_1(\ln\lambda)\psi_* EV_1 - g_N(EV_1, X_1)\psi_*(\nabla \ln\lambda) \\ &= EV_1(\ln\lambda)\psi_* X_1 + X_1(\ln\lambda)\psi_* EV_1.\end{aligned}$$

Since $g_B(\psi_* X_1, \psi_* EV_1) = \lambda^2 g_N(X_1, EV_1) = 0$, $\{\psi_* X_1, \psi_* EV_1\}$ is linearly independent for nonzero V_1, X_1. Thus, we get $(i) \Leftrightarrow (ii)$. □

Theorem 4.9 *Let ψ be a conformal slant submersion from a cosymplectic manifold $(N, \varphi, \xi, \eta, g_N)$ onto a Riemannian manifold (B, g_B). Then the following assertions are equivalent to each other:*

(i) The map ψ is a totally geodesic map,

(ii) (a) $e(T_{V_1}DV_2 + h\nabla_{V_1}EV_2) + E(T_{V_1}EV_2 + \hat{\nabla}_{V_1}DV_2) = 0$, (b) ψ is a horizontally homothetic map, (c) $e(A_{X_1}DV_1 + h\nabla_{X_1}EV_1) + E(A_{X_1}EV_1 + v\nabla_{X_1}DV_1) = 0$

for $X_1 \in \Gamma((ker\psi_*)^\perp)$ and $V_1, V_2 \in \Gamma(ker\psi_*)$.

Proof Given $V_1, V_2 \in \Gamma(ker\psi_*)$, using (2.2), (2.3), and (3.8) we obtain $(\nabla\psi_*)(V_1,V_2) = \psi_*(\varphi\nabla_{V_1}\varphi V_2)$. Using (3.3), (3.4), (4.1), and (4.2) we obtain

$$\begin{aligned}\nabla\psi_*)(V_1,V_2) &= \psi_*(\varphi(\nabla_{V_1}DV_2 + EV_2)))\\ &= \psi_*(\varphi(T_{V_1}DV_2 + \hat{\nabla}_{V_1}DV_2 + T_{V1}EV_2 + h\nabla_{V_1}EV_2))\\ &= \psi_*(dT_{V_1}DV_2 + eT_{V_1}DV_2 + D\hat{\nabla}_{V_1}DV_2 + E\hat{\nabla}_{V_1}DV_2\\ &+ DT_{V1}EV_2 + ET_{V1}EV_2 + d\nabla_{V_1}EV_2 + e\nabla_{V_1}EV_2)\\ &= \psi_*(eT_{V_1}DV_2 + E\hat{\nabla}_{V_1}DV_2 + ET_{V1}EV_2 + e\nabla_{V_1}EV_2).\end{aligned}$$

Thus, we have

$$(\nabla\psi_*)(V_1,V_2) = 0 \Leftrightarrow eT_{V_1}DV_2 + E\hat{\nabla}_{V_1}DV_2 + ET_{V1}EV_2 + e\nabla_{V_1}EV_2 = 0. \tag{4.33}$$

We claim that ψ is a horizontally homothetic map if and only if $(\nabla\psi_*)(X_1,X_2) = 0$ for $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$. From Lemma 3.2, we obtain

$$(\nabla\psi_*)(X_1,X_2) = X_1(\ln\lambda)\psi_*X_2 + X_2(\ln\lambda)\psi_*X_1 - g_N(X_1,X_2)\psi_*(\nabla\ln\lambda) \tag{4.34}$$

for $X_1, X_2 \in \Gamma((ker\psi_*)^\perp)$ so that the part from left to right is obtained. Conversely, using (4.34) we get

$$0 = X_1(\ln\lambda)\psi_*X_2 + X_2(\ln\lambda)\psi_*X_1 - g_N(X_1,X_2)\psi_*(\nabla\ln\lambda). \tag{4.35}$$

Applying $X_1 = X_2$ at (4.35), we have

$$0 = 2X_1(\ln\lambda)\psi_*X_1 - g_N(X_1,X_1)\psi_*(\nabla\ln\lambda). \tag{4.36}$$

Taking the inner product with ψ_*X_1 at (4.36), we derive

$$0 = \lambda^2 g_N(X_1,X_1)g_N(X_1,\nabla\ln\lambda),$$

which implies the result. For $X_1 \in \Gamma((ker\psi_*)^\perp)$ and $V_1 \in \Gamma(ker\psi_*)$, using (2.2), (2.3), and (3.8) we obtain $(\nabla\psi_*)(X_1,V_1) = \psi_*(\varphi\nabla_{X_1}\varphi V_1)$.

From (3.5), (3.6), (4.1), and (4.2) we get

$$\begin{aligned}\nabla\psi_*)(X_1,V_1) &= \psi_*(\varphi(\nabla_{X_1}DV_1 + EV_1)))\\ &= \psi_*(\varphi(A_{X_1}DV_1 + v\nabla_{X_1}DV_1 + A_{X_1}EV_1 + h\nabla_{X_1}EV_1))\\ &= \psi_*(eA_{X_1}DV_1 + Ev\nabla_{X_1}DV_1 + EA_{X_1}EV_1 + e\nabla_{X_1}EV_1).\end{aligned}$$

From here,

$$(\nabla\psi_*)(X_1,V_1) = 0 \Leftrightarrow eA_{X_1}DV_1 + Ev\nabla_{X_1}DV_1 + EA_{X_1}EV_1 + e\nabla_{X_1}EV_1 = 0. \tag{4.37}$$

Thus, we have $(i) \Leftrightarrow (ii)$. □

Finally, we consider a decomposition theorem. Denote by $N_{ker\psi_*}$ and $N_{(ker\psi_*)^\perp}$ the integral manifolds of $ker\psi_*$ and $(ker\psi_*)^\perp$, respectively. By Theorem 4.5 and Theorem 4.7, we have:

Theorem 4.10 *Let ψ be a conformal slant submersion from a cosymplectic manifold (N,φ,ξ,η,g_N) onto a Riemannian manifold (B,g_B). Then the following assertions are equivalent to each other:*

(i) (N,g_N) *is locally product manifold of the* $N_{(ker\psi_*)^\perp}\times N_{ker\psi_*}$,

(ii) $\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*X_2,\psi_*EDV_1)-\lambda^{-2}g_B(\nabla^{\psi}_{X_1}\psi_*eX_2,\psi_*EV_1)=g_N(A_{X_1}dX_2,EV_1)$

$$+\,g_N(-X_1(\ln\lambda)eX_2-eX_2(\ln\lambda)X_1+g_N(X_1,eX_2)(\nabla\ln\lambda),EV_1)$$

$$-\,g_N(-X_1(\ln\lambda)X_2-X_2(\ln\lambda)X_1+g_N(X_1,X_2)(\nabla\ln\lambda),EDV_1),$$

$$g_N(\nabla_{V_1}EDV_2,X_1)=g_N(T_{V_1}EV_2,dX_1)+g_N(h\nabla_{V_1}EV_2,eX_1)$$

for $X_1,X_2\in\Gamma((ker\pi_*)^\perp)$ *and* $V_1,V_2\in\Gamma(ker\pi_*)$.

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