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Conformal Symmetry in Two Dimensions: An Explicit Recurrence Formula for the Conformal Partial Wave Amplitude

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Abstract. An explicit recurrence relation for the conformal block functions is presented. This relation permits one to evaluate the X-expansion of these functions order-by-order and appropriate for numerical calculations.

The properties of infinite algebra of infinitesimal conformal transformations of two-dimensional space-time (its central extension is known as Virasoro algebra) and its consequences for field theory are now under extensive investigation [1, 3]. In this theory an important role belongs to the so-called conformal block functions, or "conformal partial amplitudes" [1], which are essentially sums over the "S-channel" contributions of all conformal fields of the same conformal class to the four-point function of a certain set of conformal fields (see ref. 1). For example, the associativity property of operator algebra in conformal field theory can be expressed in terms of these functions as a set of conformal bootstrap equations [1, 2]. So, the solution of the conformal bootstrap equations (e.g. numerical) requires an effective method to calculate conformal block functions.

In principle these functions could be evaluated straightforwardly as a series in powers of anharmonic ratio X of the correlation function, solving level-by-level in an appropriately chosen basis the following set of equations in the conformal module space, i.e., the space spanned by all operators of the same conformal class of dimension D:

$$L(k)|n+k)=(D+kd_1-d_2+n)|n), \tag{1}$$

where $L(k)$ are Virasoro generators of infinitesimal conformal transformations and $|n)$ is the n^{th} level contribution to the state

$$V_{d_1}(x)V_{d_2}(0)|0)=x^{D-d_1-d_2}\sum x^n|n), \tag{2}$$

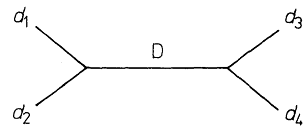
which is the intermediate state in the characteristic function:

$$F(D,d_i,C,x)=(0|V_{d_3}(\infty)V_{d_4}(1)V_{d_1}(x)V_{d_2}(0)|0). \tag{3}$$

Here we propose another approach which does not deal directly with the space of states and is based on the use of the explicit information about the degeneracy of conformal moduli given by the Kac formula [3].

Consider analytic properties of the conformal block function

$$F(D, d_i, C, x) = \qquad (4)$$



where d_i are anomalous dimensions of fields entering the correlation function; D, dimension of the conformal class contributing to it as an analytic function of the parameter; C, central charge in the Virasoro algebra:

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{C}{12}(m^3 - m)\delta(m+n). \qquad (5)$$

General arguments imply it has poles only in the finite C-plane, their positions being determined for every D by the Kac relation: there is one pole for each pair of positive integers m and n (n is greater than 1 in fact) for which

$$C = 13 - 6(T + 1/T)$$
$$T = ((2D + mn - 1) + \sqrt{(2D + mn - 1)^2 - (m^2 - 1)(n^2 - 1)})/(n^2 - 1). \qquad (6)$$

So one can write:

$$F(D, d_i, C, x) = f + \sum_{m,n} \frac{B(m,n)}{C - C(m,n)}, \qquad (7)$$

where f is the entire function of C and describes the behaviour of F near infinity, and $C(m,n)$ are given by Eq. (6). Quantities $B(m,n)$, being the residues of conformal block function in poles corresponding to its degeneration, themselves should be proportional to conformal blocks, corresponding to related submoduli of the degenerate module, hence one has:

$$F(D, d_i, C, x) = f + \sum \frac{R(m,n)}{C - C(m,n)} F(D + mn, d_i, C(m,n), x). \qquad (8)$$

Coefficients $R(m,n)$ in this formula depend on D and d_i, and due to the structure of Eq. (1) in fact, are polynomials in d_i. On the other hand, if C approaches Kac value $C(m,n)$, the module becomes degenerate, and in that case it is known the function F is well defined only if the definite relation between dimensions D and d_i is satisfied [1]. This relation can be most conveniently written if one defines $d_i = (C-1)/24 + l_i^2$, $D = (C-1)/24 + L^2$, and has the form:

$$L = l(m,n), \qquad l_1 \pm l_2 = l(p,q), \qquad l_3 \pm l_4 = l(p', q') \qquad (9)$$

for any pair of integers $p = 1-m, 3-m, \ldots, m-1$; and $q = 1-n, 3-n, \ldots, n-1$, where we have defined $l(p,q) = ((p+q)\sqrt{1-C} + (p-q)\sqrt{25-C})/2\sqrt{24}$. Thus

$R(m,n)$ should contain a set of corresponding factors, and in terms of d_i has the form:

$$R(m,n) \sim P(m,n),$$
$$P(m,n) = \prod_{p,q} \left(\frac{d_1+d_2}{2} - \frac{C-1}{24} + \frac{Y}{4} - \frac{(d_1-d_2)^2}{4Y} \right)$$
$$\cdot \left(\frac{d_3+d_4}{2} - \frac{C-1}{24} + Y/4 - \frac{(d_3-d_4)^2}{4Y} \right), \tag{10}$$

where $Y = l(p,q)^2$. Moreover, the d_i power counting in Eq. (1) shows that this product exhausts all the d_i dependence of these coefficients. Also, the structure of Eq. (5) means that if C tends to infinity only terms of the form $L(-1)L(-1)...L(-1)|0)$ survive in the state (2) which essentially are spatial derivatives of the invariant state $|D)$. This means that the free term in Eq. (8) is simply a hypergeometric function

$$f = {}_2F_1(D+d_1-d_2, D+d_3-d_4, 2D, x).$$

In what follows we shall consider for simplicity the symmetrical case $d = d_1 = d_2 = d_3 = d_4$ only; generalization is not difficult. In this case one writes

$$F(D,d,C,x) = {}_2F_1(D,D,2D,x) + \sum \frac{A(m,n)P(m,n)}{C-C(m,n)} F(D+mn,d,C,x), \tag{11}$$

where unknown coefficients $A(m,n)$ are functions of D only.

Note that the right-hand side of Eq. (11) contains F functions with internal dimensions which are by positive integer greater than dimension D in the left-hand side. Hence iterations of Eq. (11) may be used to produce the x expansion of F:

$$F(D,d,C,x) = x^{D-2d} \sum x^n F_n(D,d,C). \tag{12}$$

At each level coefficients F also may be represented as a sum of C-pole terms:

$$F_n = f_n + \sum_{m,n} \frac{B^{(n)}(m,n)}{C-C(m,n)}. \tag{13}$$

The following consideration may be proposed to obtain the quantities $A(m,n)$: simple speculations with Eqs. (1) and (5) lead to the conclusion that if one follows on each level the leading power of d and lets C tend to infinity, f_n behaves like $(2d/C)^{(n/2)}/(n/2)!$ (only even n are relevant in the symmetric case). This condition provides one with a set of equations which is sufficient to successively calculate all unknown $A(m,n)$. Analytical calculations soon become rather difficult, but this provides a suitable algorithm for numerical work.

Finally, it is worth noting that there is an explicit formula for coefficients $A(m,n)$ which is very likely to be exact. It was compared numerically with the results of the algorithm described above up to the 40th level with excellent agreement. For the special cases when $m=1$ or 2, $A(m,n)$ can be extracted from known expressions for a special choice of paramters when F satisfies differential equations of the second and third order [1] and also confirms the formula:

$$A(m,n)=\frac{12(T-1/T)}{mn((m^2-1)T-(n^2-1)/T)}\prod_{k=1}^{m-1}\prod_{l=1}^{n-1}\frac{1}{(k^2T-l^2/T)^2}\prod_{k=1}^{m-1}\frac{1}{k^2(k^2T^2-n^2)}$$
$$\cdot\prod_{l=1}^{n-1}\frac{1}{l^2(-m^2+l^2/T^2)}\prod_{p,q}(p^2T-q^2/T)\,, \tag{14}$$

where the last product is over the same set of p and q as in Eq. (10).

It is worth noting that one may consider analytical properties of F as a function of D instead of C to arrive at a recurrence relation analogous to Eq. (11):

$$F(D,d,C,x)=f+\sum\frac{A(m,n)P(m,n)}{D-D(m,n)}F(D(m,n)+mn,d,C,x)\,, \tag{15}$$

where $A(m,n)$ and $P(m,n)$ are essentially the same as in Eq. (11) but expressed now as functions of C instead of D.

This relation is in many respects much more favorable than Eq. (11). However in this case we can not write down any explicit expression for the free term f; the asymptotics of F as D tends to infinity is more complicated than in the previous case.

The numerical investigations of the conformal bootstrap equations which use the calculational algorithm based on the presented formula are now in progress.

References

1. Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Nucl. Phys. B**241**, 333 (1984)
2. Polyakov, A.M.: JETP Non-Hamiltonian approach to conformal quantum field theory. **39**, 10–18 (1974)
3. Kac, V.G.: Infinite dimensional Lie algebras. Progr. Math. Vol. **44**. Boston: Birkhäuser 1984

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