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Conformality Lost: Broken Symmetries in the Early Universe

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Abstract

In this dissertation, we introduce and investigate a general framework to describe the dynamics of the early universe. This mechanism is based on spontaneously broken conformal symmetry; we find that spectator fields in the theory can acquire a scale invariant spectrum of perturbations under generic conditions. Before introducing the conformal mechanism, we first consider the landscape of cosmologies involving a single scalar field which can address the canonical early universe puzzles. We find that, generically, single field non-inflationary solutions become strongly-coupled. We are therefore led to consider theories with multiple fields. We introduce the conformal mechanism via specific examples before constructing the most general effective theory for the conformal mechanism by utilizing the coset construction familiar from particle physics to construct the lagrangian for the Goldstone field of the broken conformal symmetry. This theory may be observationally distinguished from inflation by considering the non-linearly realized conformal symmetries. We systematically derive the Ward identities associated to the non-linearly realized symmetries, which relate (N+1)-point correlation functions with a soft external Goldstone to N-point functions, and discuss observational implications, which cannot be mimicked by inflation. Finally, we consider violating the null energy condition (NEC) within the general framework considered. We show that the DBI conformal galileons, derived from the world-volume theory of a 3-brane moving in an Anti-de Sitter bulk, admit a background which violates the NEC. Unlike other known examples of NEC violation, such as ghost condensation and conformal galileons, this theory also admits a stable, Poincaré-invariant vacuum. However, perturbations around deformations of this solution propagate superluminally.

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CONFORMALITY LOST: BROKEN SYMMETRIES IN THE EARLY UNIVERSE

Kilgore Trout once wrote a short story which was a dialogue between two pieces of yeast. They were discussing the possible purposes of life as they ate sugar and suffocated in their own excrement. Because of their limited intelligence, they never came close to guessing that they were making champagne.

— Kurt Vonnegut, *Breakfast of Champions*

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Finally, I am immeasurably grateful to my family: my parents, Austin and Rita; and my siblings Kate, Bill and Mike, for their support and encouragement throughout my life and most especially during my graduate school years.

ABSTRACT

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In this dissertation, we introduce and investigate a general framework to describe the dynamics of the early universe. This mechanism is based on spontaneously broken conformal symmetry; we find that spectator fields in the theory can acquire a scale invariant spectrum of perturbations under generic conditions. Before introducing the conformal mechanism, we first consider the landscape of cosmologies involving a single scalar field which can address the canonical early universe puzzles. We find that, generically, single field non-inflationary solutions become strongly-coupled. We are therefore led to consider theories with multiple fields. We introduce the conformal mechanism via specific examples before constructing the most general effective theory for the conformal mechanism by utilizing the coset construction familiar from particle physics to construct the lagrangian for the Goldstone field of the broken conformal symmetry. This theory may be observationally distinguished from inflation by considering the non-linearly realized conformal symmetries. We systematically derive the Ward identities associated to the non-linearly realized symmetries, which relate $(N + 1)$-point correlation functions with a soft external Goldstone to N-point functions, and discuss observational implications, which cannot be mimicked by inflation. Finally, we consider violating the null energy condition (NEC) within the general framework considered. We show that the DBI conformal galileons, derived from the world-volume theory of a 3-brane moving in an Anti-de Sitter bulk, admit a background which violates the NEC. Unlike other known examples of NEC violation, such as ghost condensation and conformal galileons, this theory also admits a stable, Poincaré-invariant vacuum. However, perturbations around deformations of this solution propagate superluminally.

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Preface

This dissertation borrows liberally from the projects on which I worked while a student at Penn [2–9]. I have attempted to keep the discussion self-contained, but some details have necessarily been omitted and can be found in these references. I would like to again warmly thank my collaborators for their help and patience.

In Chapter 2, which is based on [2, 3], we consider the issue of perturbations in single-field cosmologies and argue that inflation is the unique mechanism which remains weakly-coupled—motivating us to consider multi-field cosmologies.

In Chapter 3, we introduce the conformal mechanism in its simplest incarnation: the negative quartic model of [10–13]. We show how conformal symmetry breaking naturally leads to a scale-invariant spectrum for spectator fields in the theory. This chapter is based upon work which appeared in [5, 7, 8].

In Chapter 4, we show that—as is often the case in particle physics—many of the details of the scenario are actually independent of the microphysical realization; instead, they follow solely from the symmetry breaking pattern. We use nonlinear realization techniques, in particular the coset construction of [14–16], to construct the most general low-energy effective action for the symmetry-breaking pattern of interest. Using this effective action, we verify that scale-invariance of spectator fields follows naturally in the theory. The discussion in this chapter first appeared in [5].

We make a slight digression in Chapter 5 to fill a gap in the construction of the previous chapter. We show that the low-energy lagrangian includes a Wess–Zumino term, which shifts by a total derivative under the symmetries of the theory. This term is not captured by the coset construction. We therefore introduce a cohomological construction of this term, which we demonstrate on a simpler example—the free point particle—before treating the case of interest. This chapter follows [5, 6].

A natural question to ask is: how does the (spontaneously broken) conformal symmetry act on correlation functions in the effective theory? In Chapter 6, we address this question by deriving the Ward identities corresponding to these symmetries. We verify these relations in a variety of cases and comment on model-independent observational consequences. This Chapter derives the results of [8] in a slightly different way by using field theoretic machinery as opposed to "background wave" type arguments.

In Chapter 7 we consider violating the null energy condition (NEC). This is a necessary requirement for any alternative to inflation and has proven to be extremely difficult in the context of quantum field theory. We attempt to construct a consistent field theory which violates the NEC. We are able to make significant progress, although the theory still has subtle problems—superluminality cannot be banished entirely. This Chapter is based on [9].

Chapter 1

Introduction

A central goal of modern cosmology is to understand the physics underlying the evolution of the early universe. At the simplest level, there are two distinct things to which cosmological observations are sensitive—the background cosmological evolution and small perturbations about this background. At the largest scales, the universe is very simple: observations indicate that the universe is very nearly homogeneous, isotropic and flat. Observations of relic Cosmic Microwave Background (CMB) radiation allow us measure small perturbations away from this background and allow us to infer the properties of the component driving the dynamics of the primordial universe. The CMB is homogeneous radiation left over from the hot big bang. Small temperature anisotropies in its spectrum were seeded by quantum fluctuations in the early universe, and measurements of these anisotropies [17–20] allow us to constrain the evolution of the early universe. The perturbations themselves turn out to be nearly the simplest imaginable—very nearly scale invariant and gaussian. Any scenario purporting to describe early universe evolution must address these observations.

Our observations of the background evolution of the universe carry with them some puzzles: we observe the universe to be very nearly homogeneous and very nearly flat, but this is a seemingly unnatural state in which to find the present-day universe. In the context of the standard big bang cosmology, one has to posit very *fine tuned* initial conditions in order to obtain a universe that looks like ours today. This apparent fine tuning cries out for an explanation.

The currently leading framework to explain these initial conditions is cosmic inflation [21–23]. By positing a phase where the universe expanded exponentially rapidly, inflation is able to explain the relative flatness and homogeneity of the observed universe—the visible universe all came from a tiny patch, so it is therefore not surprising that it is relatively homogeneous. Further, the exponential expansion of spatial slices drastically increases their radius of curvature, so locally they will appear very flat to an observer. Further, inflation makes predictions about the fluctuations about this background solution—it predicts that they will be very nearly scale-invariant and gaussian, which is in great accord with observations of the CMB [17–20].

However, it is important to ask to what extent the predictions of inflation are unique, and whether there are additional frameworks (observationally distinguishable from inflation) which can also solve the standard problems. In the past, this has led to various proposed alternatives to inflationary cosmology, for example, pre-big bang cosmology [24, 25], string gas cosmology [26–31] and the ekpyrotic scenario [2, 32–51].

In this dissertation, we will explore a general framework also capable of describing the physics of the early universe; we will investigate how spontaneously broken conformal symmetry can naturally solve the canonical puzzles of early universe cosmology. The conformal mechanism [5, 8, 12, 13, 52] is an alternative to inflation which postulates that instead of undergoing a phase of superluminal (de Sitter) expansion, the universe at very early times is cold, nearly static, and governed by an approximate conformal field theory (CFT) on approximately Minkowski space. The theory is invariant under the conformal algebra of 4-dimensional Minkowksi space, namely $\mathfrak{so}(4,2)$. The central ingredient of the scenario is that the dynamics allow for at least one scalar operator (of non-zero conformal weight) in the CFT to acquire a time-dependent expectation value which spontaneously breaks the $\mathfrak{so}(4,2)$ symmetries down to $\mathfrak{so}(4,1)$.

We will see that the conformal scenario naturally leads to a scale-invariant spectrum of perturbations, similar to inflation, under a broad range of conditions. Further, we will

see that the scenario can be distinguished from inflation through *sharp* observational tests. Finally, we will comment on a crucial hurdle facing any alternative to inflation: violation of the null energy condition (NEC). We will see that possibility of the universe transitioning from a contracting epoch to an expanding phase is intimately tied to the NEC. However, it has proven remarkably difficult to violate this condition within the context of quantum field theory. We present a marked improvement over previous attempts to violate the null energy condition. We will see a theory which possesses both a stable flat-space solution and a stable solution which violates the NEC, which up until now was impossible. There is still a subtle pathology in the theory—certain backgrounds allow superluminal propagation of signals. Aside from being of interest in order to construct non-singular bounces, whether or not it is possible to violate the NEC is a fundamental physics question. If it turns out to be impossible, it will tell us something profound about nature.

1.1 Background Friedmann–Robertson–Walker cosmology and puzzles

On the largest scales, we observe the universe to be homogeneous and isotropic. More precisely, we observe the universe to be isotropic relative to us observing from Earth. We imagine that we are unlikely to live at a distinguished point in the universe, therefore the universe must be isotropic about every point; which means that it is homogeneous. The most general metric describing a homogeneous and isotropic spatial geometry is of Friedmann–Robertson–Walker (FRW) type (see [53] for a nice, entirely geometric, proof of this fact):[1]

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) . \qquad (1.1.1)$$

Here κ can be one of $\{0, -1, 1\}$; when $\kappa = 1$, the spatial slices are 3-spheres, S^3. When $\kappa = -1$ the spatial slices are hyperbolic 3-spaces $\mathbb{H}_3$ and when $\kappa = 0$, the spatial slices are Euclidean 3-space, $\mathbb{R}^3$.

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[1]Of course this line element tells us only about the local geometry and not about the topology of our spatial slices, which can be quotiented by a subgroup of the isometries which acts freely. However, since the topology of spatial slices will not be important for our purposes, we will always work with the covering spaces.

The function $a(t)$ tells us about the time evolution of the geometry, which is governed by the Einstein equations, these follow from the Einstein–Hilbert action[2]

$$S = \frac{M_{\text{Pl}}^2}{2} \int \mathrm{d}^4 x \sqrt{-g} \Big(R - 2\Lambda \Big) + S_{\text{matter}} \; . \tag{1.1.2}$$

Varying this action with respect to $g^{\mu\nu}$ and minimizing, we obtain the *Einstein equations*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{M_{\text{Pl}}^2} T_{\mu\nu} \; , \tag{1.1.3}$$

where we have defined the energy-momentum tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \; . \tag{1.1.4}$$

These equations govern the dynamics of gravity: a given distribution of matter defines the stress tensor $T_{\mu\nu}$, which allows us to solve for the metric $g_{\mu\nu}$, which tells us about the geometry. Freely falling observers will move along geodesics defined by this metric. This is the origin of Wheeler's statement: *spacetime tells matter how to move, matter tells spacetime how to curve* [54].

For simplicity, we take the matter in the universe to be of the form of a perfect fluid

$$T_{\mu\nu}^{\text{fluid}} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} \; , \tag{1.1.5}$$

where ρ is the energy density, P is the pressure and u^μ is a time-like vector $(u_\mu u^\mu = -1)$. For a perfect fluid, we also have to specify a relation between the pressure and density through the *equation of state*

$$P = w\rho \; , \tag{1.1.6}$$

where w is typically constant. With this matter distribution and the metric ansatz (1.1.1),

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[2]Here and throughout, we will use the metric signature $(-, +, +, +)$ because I'm not a barbarian. Also, we have defined $M_{\text{Pl}}^2 \equiv (8\pi G_{\text{N}})^{-1}$.

the Einstein equations read[3] (for now we will take $\Lambda = 0$)

$$3M_{\text{Pl}}^2 H^2 = \rho - \frac{3M_{\text{Pl}}^2 \kappa}{a^2} \tag{1.1.7}$$

$$M_{\text{Pl}}^2 \dot{H} = -\frac{1}{2}(\rho + P) + \frac{M_{\text{Pl}}^2 \kappa}{a^2} \ , \tag{1.1.8}$$

where we have defined the Hubble parameter $H \equiv \dot{a}/a$.

We are now in position to understand two fundamental puzzles of early universe cosmology; these were most clearly pointed out by Guth in the original article on inflation [21]. They are essentially problems of initial conditions; for types of matter with which we are familiar, the initial state of FRW evolution appears rather fine-tuned, as we will see.

In these sections, we follow [55, 56] to elucidate the canonical problems with FRW models.

1.1.1 Horizon problem

The horizon problem is one of causal structure, so we define the *conformal time* variable by

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)} \ , \tag{1.1.9}$$

in terms of which the metric takes the form (for simplicity we restrict to the flat case, $\kappa = 0$)

$$\mathrm{d}s^2 = a(\eta)^2 \Big(- \mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2) \Big) \ . \tag{1.1.10}$$

The reason for defining this new time coordinate is that we are interested in the properties of photons, which are insensitive to the overall conformal factor.[4] We now consider a radial null geodesic

$$\mathrm{d}s^2 = 0 \implies \mathrm{d}\eta^2 = \mathrm{d}r^2 \ ; \tag{1.1.11}$$

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[3]There is also the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$; it is redundant with the equations written, but it is often easier to manipulate.

[4]This is most easily seen from the fact that electromagnetism is conformally invariant in $4d$.

we can integrate both sides to find the maximum coordinate distance traveled by a photon between the initial singularity $(t = 0)$ and some later time

$$\Delta r = \int_0^{t_f} \frac{\mathrm{d}t}{a(t)} = \int_0^{a_f} \frac{\mathrm{d}\log a}{aH} \ . \tag{1.1.12}$$

If we consider the universe to be filled with a perfect fluid, the qualitative behavior of this integral depends on the value of w. For a flat universe, we have [53, 55]

$$a(t) \propto t^{\frac{2}{3(1+w)}} \ , \tag{1.1.13}$$

from which we deduce that the *co-moving horizon* evolves as [55]

$$\frac{1}{aH} \propto a^{\frac{(1+3w)}{2}} \tag{1.1.14}$$

Using this, we can integrate (1.1.12) to find that the maximum casually connected distance also scales as

$$\Delta r \propto a^{\frac{(1+3w)}{2}} \ . \tag{1.1.15}$$

Now, conventional matter obeys the *strong energy condition*, which in terms of a perfect fluid means that $w \geq -1/3$. Therefore, in an *expanding* universe, Δr is monotonically increasing. This means that points that are just now entering the horizon must have been far from being in causal contact when the Cosmic Microwave Background was generated. This, however, is a problem: the temperature of the CMB is found to be uniform to a part in 10^5, but points on opposite sides of the sky have never been in causal contact![5] This is the *horizon problem.*

Notice that to solve the horizon problem, it suffices to have a sufficiently long period where the co-moving horizon shrinks. Like any good puzzle, the facts we have had to use in its derivation point us to the resolution. We had to assume two key things to arrive at our

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[5]In fact, the problem is much worse, one can verify that regions on the surface of last scattering separated by more than $\theta \sim 2°$ have never been in causal contact [55].

conclusion. The first is that matter obeys the strong energy condition—there is no apriori reason for this to be true. The second is that the universe is expanding, that is $a(t)$ is increasing. If we break either of these assumptions, we find the co-moving horizon now *decreases*, and there is no horizon problem.

Breaking the first condition—violating the strong energy condition—corresponds to accelerated expansion of the universe, or *inflation* [21–23]. Violating the second condition—considering a contracting universe—inspired scenarios such as the pre-big bang scenario [24], string gas cosmology [26, 27] and the Ekpyrotic universe [32].

A short digression: consider a scenario in which the universe is initially collapsing. We know that the universe is currently expanding, which implies that at some intermediate point, we have to transition from negative H to positive H; this necessarily implies that at some point we must have $\dot{H} > 0$. Recalling the Friedmann equations (1.1.8), we have, for a flat universe

$$M_{\mathrm{Pl}}^2 \dot{H} = -\frac{1}{2}(\rho + P) \ . \tag{1.1.16}$$

This implies that in order to transition from collapse to expansion, the component driving the evolution must satisfy $\rho + P < 0$. The condition $\rho + P \geq 0$ is the expression of the *null energy condition* for a perfect fluid. Therefore, any alternative to inflation must necessarily violate this condition at some point. Whether this is possible in a well-behaved theory is an open question, and it is one to which we will return.

1.1.2 Flatness problem

In this Section we discuss another cosmological puzzle, intimately related to the horizon problem. Consider the Friedmann equation for a universe with various perfect fluid components:

$$3H^2 M_{\mathrm{Pl}}^2 = \frac{3M_{\mathrm{Pl}}^2 \kappa}{a^2} + \frac{C_{\mathrm{matter}}}{a^3} + \frac{C_{\mathrm{radiation}}}{a^4} + \frac{C_{\mathrm{anisotropy}}}{a^6} + \ldots + \frac{C}{a^{3(1+w)}} \tag{1.1.17}$$

In an expanding universe, the curvature component is the most dangerous; we should expect all other sources of energy to redshift away and for the energy budget of the universe to be dominated by curvature. However, this is observationally not true; we measure the universe to be extremely close to flat, this is the *flatness problem*.

This problem can be addressed in roughly the same way as the horizon problem. In an expanding universe, if we have a component with $w < -1/3$, it will dilute away more slowly than curvature and drive the background to be flat. This is precisely the same condition that we found in the previous section in order to solve the horizon problem.

There is, however, another logical possibility: consider a contracting universe, now the most dangerous term is anisotropy, so we need a component with $w > 1$ in order to grow more quickly and smooth out the background. This corresponds precisely to *slow contraction* in a collapsing phase [39, 57].

1.2 Cosmological perturbations

Having considered background evolution of the universe and puzzles of its initial conditions we now turn to the question of perturbations about this background solution. One of the most exciting measurements in modern cosmology was the measurement of temperature anisotropies in the Cosmic Microwave Background.

The Cosmic Microwave Background (CMB) is relic radiation from the early universe. Immediately after the big bang, the universe was *radiation dominated*—filled by a hot, dense gas of photons in thermal equilibrium. During this epoch, photons scattered strongly off of electrons with a small mean free path, and the universe was optically opaque. Eventually, the universe cooled sufficiently for nuclei and electrons to form bound states and it became possible for photons to travel long distances without being scattered. It is the leftover radiation from this time of decoupling that we observe as the CMB.

The Cosmic Microwave Background is an exceptional black body, with a mean temperature
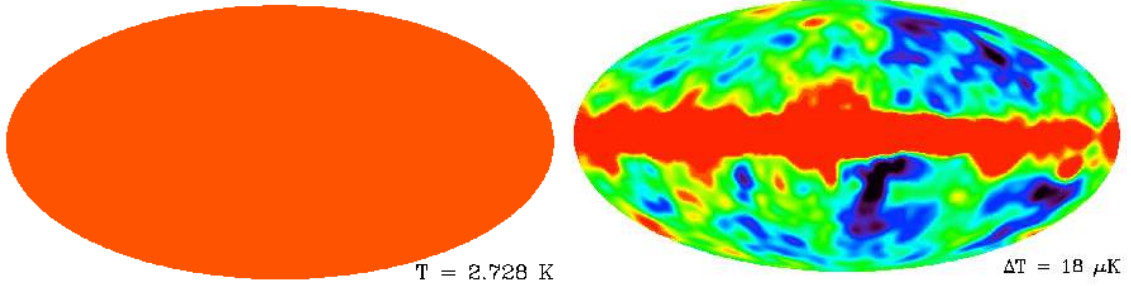
Figure 1: *Left*: Temperature map of the CMB showing a mean temperature of $T \sim 2.73$ K. This is a real map. *Right*: Temperature fluctuations as measured by the COBE experiment. Both maps taken from aether.lbl.gov/www/projects/cobe/COBE_Home/DMR_Images.html, compiled with COBE 4-year data

of approximately $T \sim 2.73$ K. Although the universe was very hot when the CMB was emitted, the expansion of the universe causes a redshift of photon frequencies, causing the mean temperature to actually be quite small. In Figure 1 we reproduce a temperature map from the Cosmic Background Explorer (COBE) satellite [58, 59], which makes manifest how amazingly uniform the observed spectrum really is.

If we subtract the mean temperature, there are small fluctuations in the observed spectrum, which are order $\delta T/T \sim 10^{-5}$. In Figure 1, we also reproduce the anisotropy map from the COBE experiment. Recently, the Planck satellite has greatly improved on the accuracy of this measurement, and we include their anisotropy map in Figure 2. The statistics of these temperature fluctuations can tell us a great deal about the physics of the early universe.

1.2.1 CMB temperature anisotropies

The temperature anisotropy is a scalar quantity, which depends on the direction we look in the sky, so we can decompose it in terms of eigenfunctions on the sphere[6] (spherical

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[6]The convention we are using for the spherical harmonics is

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \, P_\ell^m(\cos\theta) \, e^{im\varphi} \qquad (1.2.1)$$

where $P_\ell^m(\cos\theta)$ are *associated Legendre polynomials*.
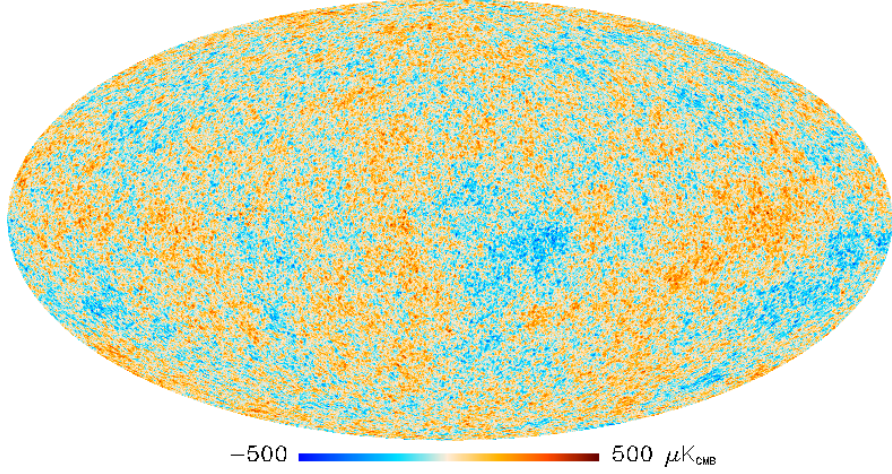
Figure 2: Map of temperature anisotropies measured by the Planck satellite [1]. Note the resolution difference between this and the COBE map.

harmonics) [55, 60–62]

$$\delta T(\hat{n}) = T(\hat{n}) - \bar{T} = \sum_{\ell,m} a_{\ell m} Y_\ell^m(\hat{n}) \; ; \qquad \bar{T} = \frac{1}{4\pi} \int \mathrm{d}\Omega \; T(\hat{n}) \; . \qquad (1.2.2)$$

where $\hat{n}$ is a unit vector and $\mathrm{d}\Omega$ is the standard measure on the 2-sphere. Given a particular real space function, $\delta T(\hat{n})$, the $a_{\ell m}$ can be calculated by inverting (1.2.2)

$$a_{\ell m} = \int \mathrm{d}\Omega \; \delta T(\hat{n}) Y_\ell^{*m}(\hat{n}) \; . \qquad (1.2.3)$$

We now use symmetries to place constraints on the $a_{\ell m}$. We assume that the universe is rotationally-invariant (there is no preferred direction), which implies [55, 60–62]

$$\langle a_{\ell m} \rangle = 0 \; ; \qquad \langle a_{\ell m} a_{\ell' m'}^* \rangle = \delta_{\ell \ell'} \delta_{mm'} C_\ell \; , \qquad (1.2.4)$$

where $\langle \cdots \rangle$ indicates either an average over all possible observer positions or an ensemble average over possible realizations of δT [60]. Using this, we can compute the two-point

correlation function for temperature fluctuations [55, 60, 62]

$$\langle \delta T(\hat{n}) \delta T(\hat{n}') \rangle = \sum_\ell C_\ell \left(\frac{2\ell+1}{4\pi} \right) P_\ell(\hat{n} \cdot \hat{n}') \ , \tag{1.2.5}$$

where P_ℓ are Legendre polynomials. What we measure is not precisely C_ℓ, which is an average over many realizations, but rather $\mathcal{C}_\ell$, which is an average over m for a single realization:

$$\mathcal{C}_\ell = \frac{1}{2\ell+1} \sum_m a_{\ell m} a_{\ell m}^* \ ; \tag{1.2.6}$$

the difference between C_ℓ and $\mathcal{C}_\ell$ is known as *cosmic variance*. Following Weinberg [60], we see that the mean-square fractional difference between the two is

$$\left\langle \left(\frac{C_\ell - \mathcal{C}_\ell}{C_\ell} \right)^2 \right\rangle = \frac{2}{2\ell+1} \ , \tag{1.2.7}$$

which decreases with increasing ℓ. The intuition is that at low ℓ, the physical size of a mode on the sky is rather large, so there are not many modes over which to average (equivalently, not many values of m over which to average). In Figure 3, the quantity $\mathcal{D}_\ell \equiv \ell(\ell+1)\mathcal{C}_\ell/2\pi$ is plotted versus ℓ, from the Planck data [1].

1.2.2 Primordial perturbations

We would like to connect the observed temperature fluctuations of the CMB to primordial physics. Roughly speaking, small fluctuations, ζ, are produced in the early universe during a phase where the co-moving horizon (1.1.14) is shrinking. These fluctuations then leave the horizon and stop evolving. Then, during the radiation-dominated era, these fluctuations re-enter the horizon and cause fluctuations in the plasma, these fluctuations evolve under gravity and lead to the temperature anisotropies we see.

The physics relating the primordial perturbations to the observed spectrum of fluctuations is beautiful, but quite complex and difficult to treat analytically. Therefore, we will merely
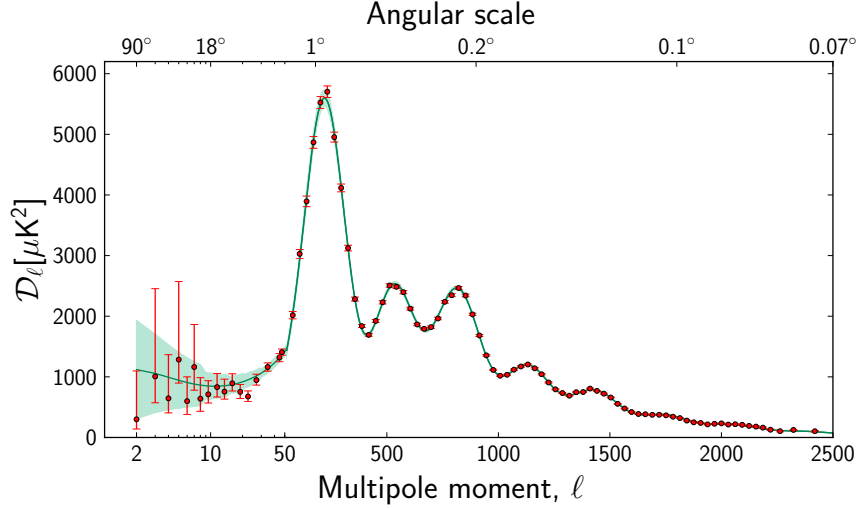
Figure 3: Planck angular power spectrum of temperature anisotropies. The green bands are cosmic variance error bars. The vertical axis plots the quantity $\mathcal{D}_\ell = \ell(\ell+1)\mathcal{C}_\ell/2\pi$. The horizontal scale is logarithmic until $\ell = 50$ and linear thereafter. Figure reproduced from [1].

say that the observed spectrum of $\mathcal{C}_\ell$s are related to the primordial fluctuations through

$$\mathcal{C}_\ell \sim \int \mathrm{d}k \; k^2 \mathcal{P}_\zeta(k) \Delta_{\ell T}^2(k) \;, \quad \text{where} \quad \mathcal{P}_\zeta(k) \equiv \frac{1}{2\pi^2} k^3 P_\zeta(k) \;, \quad (1.2.8)$$

where $P_\zeta(k) = \langle \zeta_k \zeta_{-k} \rangle$ and the funcion $\Delta_{\ell T}(k)$ is a transfer function which captures the evolution of perturbations through the radiation-dominated epoch. We now ask: *what do CMB observations tell us about the primordial perturbation, ζ?*

Nearly scale invariant

The first property satisfied by the primordial curvature perturbations is that they are nearly *scale invariant*. In order to understand what this means, we write the two-point function of the perturbations as

$$\langle \zeta(\vec{x}) \zeta(\vec{x} + \vec{a}) \rangle = \int \mathrm{d}\log k \, \frac{1}{2\pi^2} k^3 |\zeta_k|^2 e^{i\vec{k}\cdot\vec{a}} \;, \quad (1.2.9)$$

and note the appearance of the power spectrum $\mathcal{P}_\zeta(k)$. In order to fit the data, we posit a power-law dependence for the power spectrum $\mathcal{P}_\zeta$

$$\mathcal{P}_\zeta(k) \sim k^{n_s - 1} \ , \tag{1.2.10}$$

where n_s is referred to as the *spectral index* of the fluctuations. When $n_s = 1$, the power spectrum is independent of k; this is referred to as a *scale invariant* or *Harrison–Zel'dovich* spectrum. Planck data constrains the parameter n_s to be [18, 63]

$$n_s = 0.959 \pm 0.007 \ , \tag{1.2.11}$$

which establishes that $n_s \neq 1$ at 5σ. Since $\mathcal{P}_\zeta$ is slightly larger at smaller k, the spectrum is said to be red-tilted.

Gaussian

We can also infer that the statistics underlying the perturbations that seeded the CMB are very close to *gaussian*. By this, we mean that ζ is very close to being a gaussian random field. If ζ were exactly gaussian, all of the correlation functions in the theory would be completely determined by knowing $\langle \zeta^2 \rangle$. The first deviations from gaussianity would appear in the three-point function

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) f_{\mathrm{NL}} B(k_1, k_2, k_3) \ ; \tag{1.2.12}$$

here f_{NL} is a dimensionless amplitude and $B(k_1, k_2, k_3)$ is known as the *bispectrum*. Note that the delta function constrains the wavevectors to form a triangle. In general, the precise momentum dependence of the bispectrum is highly model-dependent. Different models produce non-gaussianities which peak in different configurations of the ks. In the following, we will focus on three fiducial shapes.

The first shape is the so-called *squeezed* or *local* shape; in momentum space, this corresponds

to a triangle with one side much shorter than the other two $k_1 \ll k_2 \sim k_3$. Large non-gaussianity in this configuration is a hallmark of multiple-field inflationary models, where conversion of entropic fluctuations to the adiabatic direction generates large local non-gaussianity. In single field inflation, local-type non-gaussianities must vanish [64–66].

Another commonly cited shape is the *equilateral* shape, which peaks when all of the momenta are roughly equal: $k_1 \sim k_2 \sim k_3$. Non-gaussianities of this shape are a signature of theories of inflation which involve higher derivatives, such as DBI inflation [67] or galileon inflation [68, 69].

The final shape we will discuss is the *orthogonal* shape. This shape peaks both in the equilateral and flattened triangle configurations [70]. It is not particularly intuitive, but it is considered—as its name suggests—because it is orthogonal (suitably understood) to the local and equilateral shapes.

Planck has constrained the values of f_{NL} for all three of these shapes, the constraints are [19, 20]

$$f_{\mathrm{NL}}^{\mathrm{local}} = 2.7 \pm 5.8 \; ; \qquad f_{\mathrm{NL}}^{\mathrm{equil.}} = -42 \pm 75 \; ; \qquad f_{\mathrm{NL}}^{\mathrm{ortho.}} = -25 \pm 39 \; . \qquad (1.2.13)$$

All of these values are consistent with zero at 1σ, indicating that the primordial fluctuations are extremely close to being gaussian.

Adiabatic

The last thing we learn from the CMB about primordial perturbations is that they were *adiabatic*. What this means is that the overall density varies from place to place, but the *relative* densities of various particle species do not vary appreciably. What this means is that the perturbations of the various components of the early universe all have the same origin. Another way of saying this in terms of the ratio of the densities of non-relativistic

species to that of relativistic species is that

$$\delta \left(\frac{\rho_{\text{non-rel.}}}{\rho_{\text{rel.}}} \right) = 0 \tag{1.2.14}$$

is satisfied in the early universe [55].

1.3 Inflation in brief

Here we briefly review inflation, how it solves the horizon and flatness problems, and the quantum production of perturbations during inflation. Inflation posits that prior to the conventional FRW hot big bang, the universe underwent a phase of quasi-de Sitter expansion, driven by a component with equation of state w very close to -1. There are, of course, many ways to source such a solution, but here we focus on the simplest example: *slow-roll inflation*. In this model, inflation is driven by a scalar field coupled to gravity

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right) , \tag{1.3.1}$$

which rolls down a nearly flat potential $V(\phi)$, schematically of the form in Figure 4. The equation of motion obeyed by the scalar field is (assuming a homogeneous profile $\phi = \phi(t)$ and an FRW ansatz for the metric)

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 . \tag{1.3.2}$$

In order for inflation to occur, two *slow-roll* conditions must be satisfied

$$\ddot{\phi} \ll 3H\dot{\phi} ; \qquad \epsilon \equiv \frac{\dot{\phi}^2}{2M_{\text{Pl}}^2 H^2} \ll 1 , \tag{1.3.3}$$

which are essentially constraints on the flatness of the potential $V(\phi)$. If these conditions are met, the background solution for the metric is approximately that of *de Sitter space*

$$\mathrm{d}s^2 \simeq -\mathrm{d}t^2 + e^{2Ht} \mathrm{d}\vec{x}^2 . \tag{1.3.4}$$

15
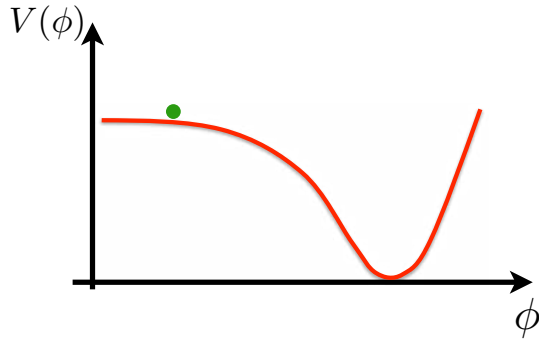
Figure 4: Schematic slow-roll potential for the inflaton.

Recalling that the equation of state for a homogeneous scalar field is given by

$$w_\phi = \frac{\frac{1}{2}\dot\phi^2 - V(\phi)}{\frac{1}{2}\dot\phi^2 + V(\phi)} \simeq -1 \; , \tag{1.3.5}$$

where in the last equality we have assumed the kinetic energy is negligible compared to the potential energy, which is an excellent approximation on the slow-roll solution. Recalling Sections 1.1.1 and 1.1.2, we know that a component with $w < -1/3$ will solve the horizon and flatness problems.

1.3.1 Quantum fluctuations seed the CMB

Now we want to study perturbations around the inflationary solution. Schematically, we want to expand

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \; ; \qquad\qquad \phi = \bar\phi + \delta\phi \; , \tag{1.3.6}$$

about the de Sitter solution and study the properties of scalar fluctuations. In practice—because of the gauge freedom of Einstein gravity—this turns out to be a somewhat intricate task, which we will undertake in Chapter 2. For now we just note that, in a particular limit, the scalar fluctuations are well-described at quadratic order by a free scalar field on de Sitter

space[7]

$$S = \frac{1}{2} \int d^3x d\eta \frac{1}{\eta^2} \left(\zeta'^2 - (\vec{\nabla}\zeta)^2 \right) , \qquad (1.3.7)$$

Where we have gone to conformal time (1.1.9) and $' \equiv d/d\eta$. We now proceed to canonically quantize this scalar field, following Maldacena [64].[8] The equation of motion following from (1.3.7) is (in Fourier space)

$$\zeta_k'' - \frac{2}{\eta}\zeta_k' + k^2\zeta_k = 0 . \qquad (1.3.8)$$

This equation has two solutions[9]

$$\zeta_k(\eta) = \frac{1}{\sqrt{2k^3}}(1 + ik\eta)e^{ik\eta} ; \qquad \zeta_k^*(\eta) = \frac{1}{\sqrt{2k^3}}(1 - ik\eta)e^{-ik\eta} . \qquad (1.3.9)$$

We now expand the field ζ in these modes and promote the coefficients to operators in the usual way[10]

$$\hat{\zeta}(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} \left(\hat{a}_k^\dagger \zeta_k(\eta) + \hat{a}_{-k}\zeta_k^*(\eta) \right) e^{i\vec{k}\cdot\vec{x}} . \qquad (1.3.12)$$

Now, we choose a vacuum such that $\hat{a}_k|\Omega\rangle = 0$. This is known as the *adiabatic* or *Bunch–Davies* vacuum. It is chosen to coincide with the Minkowski vacuum as $k \to \infty$ (equivalently $\eta \to -\infty$). With this vacuum choice, we can compute the two-point function for ζ:

$$\langle \zeta_k(\eta)\zeta_{k'}(\eta) \rangle = \delta^{(3)}(\vec{k} + \vec{k}')\frac{H^2}{2M_{\rm Pl}^2}\frac{1}{k^3}(1 + k^2\eta^2) , \qquad (1.3.13)$$

---

[7]Technically, here we are working in the *decoupling limit*, where we take $\epsilon \to 0$ and $M_{\rm Pl} \to \infty$, but keep the product, $M_{\rm Pl}^2\epsilon$ fixed. We have absorbed the factors of $M_{\rm Pl}$ and H into the definition of ζ, and will restore them when appropriate.

[8]Actually, Maldacena told us that this computation is so important that we should be prepared to do it if awoken suddenly in the middle of the night.

[9]Note that we have chosen the normalization so that the canonical commutation relations for the ladder operators are satisfied: $[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta^{(3)}(\vec{k} - \vec{k}')$

[10]We can also define the canonical momentum

$$\hat{\Pi}_{\zeta_k}(\eta) = \frac{\partial\mathcal{L}}{\partial\zeta_k'} = \frac{1}{\eta^2}\left(\hat{a}_k^\dagger\zeta_k' + \hat{a}_{-k}\zeta_k^{*'} \right) , \qquad (1.3.10)$$

so that the field and its momentum satisfy the canonical commutation relations

$$[\hat{\zeta}_k(\eta), \hat{\Pi}_{\zeta_{k'}}(\eta)] = i\delta^{(3)}(\vec{k} - \vec{k}') . \qquad (1.3.11)$$

where we have restored the factors of H and $M_{\rm Pl}$. Note that at late times ($k\eta \to 0$), this is precisely of the form (1.2.10) with $n_s = 1$. So we see that *quantum fluctuations during inflation naturally lead to a nearly scale invariant spectrum of perturbations.* Even better, generically models of slow-roll inflation lead to a slight red tilt, in agreement with the data [63].

1.3.2 Non-gaussianity and the consistency relation

While more complicated models of inflation can produce appreciable amounts of non-gaussianity, slow-roll inflation predicts negligible non-gaussianity for all correlation functions. However, all *single-field* inflationary models are subject to a powerful theorem which constrains their non-gaussian signature in the squeezed limit. This is the momentum configuration where one of the momenta is very small while the other two are of comparable size to each other. As was first noted by Maldacena [64], in this limit, the three-point function can be related to a product of two-point functions

$$f_{\rm NL}^{\rm local} \sim \lim_{k_1 \to 0} \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \sim (n_s - 1) \langle \zeta_{k_2} \zeta_{-k_2} \rangle \langle \zeta_{k_3} \zeta_{-k_3} \rangle \ . \tag{1.3.14}$$

This result can be made precise and is known as the *consistency relation*; it is a powerful observational test, any measured violation of it would rule out all single-field models of inflation.[11]

1.4 Alternatives to inflation?

We have seen that cosmic inflation can both solve the horizon and flatness problems and also give rise to primordial perturbations consistent with the observed statistics of the CMB. Why then, are we interested in exploring alternatives to the inflationary paradigm? One theoretical motivation is that it is important to know to what extent the predictions of inflation are unique—if inflation turns out to be the unique theory in agreement with the

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[11]Of note is that the consistency relation can only really be used as a null test of inflation, the equality is phrased in terms of *comoving* coordinates; the right hand size of the equality is not measurable if one transforms to physical coordinates, where observations are performed. See [71, 72] for details.

data, it will only bolster our confidence in the theory.

There are also fundamental questions about inflation that we do not know how to answer: for example there is the famous *measure problem* which asks how likely it is that our universe inflated for a sufficiently long time to account for present observations. It is not currently understood how to answer this question, with various proposals for answers finding that inflation is either extremely likely [73] or exponentially unlikely [74]. A second foundatonal problem for cosmic inflation is that inflation starting the first place requires extremely low entropy initial conditions; the universe must be smooth on a patch larger than its Hubble radius [75], but these are rather unnatural initial conditions—recall that inflation was introduced to alleviate initial condition problems itself! Finally, there is a more pragmatic problem; it has proven very difficult to embed inflation in a larger framework, such as string theory. Of course, much progress has been made—see for example [76–78]—but there is still much to be done. Generically, it is very difficult to construct a flat enough potential $V(\phi)$ for inflation to occur or a long enough time. It is therefore worthwhile to see if there are alternative mechanisms which can also account for observations which can be more easily embedded in a ultraviolet theory.

In Chapter 2 we will ask a relatively simple question: *what single field cosmologies are capable of producing a spectrum of perturbations consistent with observations*? We find that, while there are non-inflationary solutions which produce a scale invariant spectrum of curvature perturbations (1.3.13), they all become strongly-coupled and thus non-predictive after producing a finite range of modes. This is undesirable, because we would then have to explain why the modes we see today happen to be the scale invariant ones. We are therefore led to the conclusion that inflation is essentially the *unique* viable single field cosmology.

This "no-go" result points us toward multiple field models, if we want to consider alternative mechanisms. In particular, we will introduce and investigate a *conformal mechanism* for the generation of density perturbations. This mechanism is deeply rooted in symmetries, which makes it plausible that it could be free of some of the initial conditions worries that plague

inflation. Aside from this, we will find a rich theoretical structure, and many techniques familiar from particle physics will make an appearance in the analysis of these theories. The conformal mechanism, it should be emphasized, is similar to inflation in that it is a broad mechanism for producing a scale invariant spectrum of density perturbations, rather than being one particular microphysical mode. This is both an advantage and a disadvantage— it is advantageous in that there are many ways to realize such a mechanism, giving more opportunities for agreement with the data. However, it is also a disadvantage in that there is some inherent model dependence in predictions—in order to compute things we must choose a particular lagrangian.

We first introduce the conformal mechanism in particular incarnations, and show how these fiducial models work. then, we abstract these results and construct an effective theory for the symmetry breaking pattern of interest which accurately captures the low energy dynamics of any realization of the mechanism. The approach is similar to the effective field theory of inflation approach [79]; inflation may be thought of as a theory of spontaneously broken time diffeomorphism invariance, and the curvature perturbation is the goldstone of this symmetry breaking pattern. In our case, the relevant symmetry breaking will be of the global conformal algebra down to its de Sitter subgroup, but the ideas are the same. This emphasis on symmetry allows us to actually make some model-independent predictions; they follow from similar considerations to those that lead to the consistency relations for inflationary correlators.

Finally, we will discuss an open theoretical problem: *is it possible to violate the null energy condition with a sensible theory?* For now, we will not be precise about what we mean by a sensible theory, but note that it has been extremely difficult within the context of quantum field theory or string theory to violate the NEC in a well-behaved way. It is something any alternative to inflation must do, and is obviously of interest from this perspective, but it is also of more theoretical interest. If there is a fundamental tension with a cherished pillar of physics, it will be interesting to see in what way this conflict manifests.

Chapter 2

Prelude: Cosmology of a single scalar

2.1 A scalar coupled to gravity

Whatever physics describes the early universe and solves the horizon and flatness problem should also naturally give rise to the observed nearly Harrison–Zel'dovich spectrum of curvature perturbations. In this Section, we consider a single scalar field coupled to gravity and derive the action governing ζ. In the following Section, we will ask what background cosmologies are capable of giving rise to the observed spectrum.

We begin by considering a single scalar field coupled to Einstein gravity

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{M_{\mathrm{Pl}}^2 R}{2} + P(X, \phi) \right) , \qquad (2.1.1)$$

where $X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ and $P(X, \phi)$ is an arbitrary function of the field and this Lorentz-invariant combination of its first derivatives. This action is diffeomorphsim invariant, under which the variables transform as

$$\delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\nu\rho} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho \; ; \qquad \delta_\xi \phi = \xi^\mu \partial_\mu \phi \qquad (2.1.2)$$

The motivation for considering this type of scalar sector is that its energy-momentum tensor is given by

$$T_{\mu\nu} = P_{,X} \partial_\mu \phi \partial_\nu \phi + P g_{\mu\nu} . \qquad (2.1.3)$$

By defining $u_\mu = \partial_\mu \phi / \sqrt{2X}$, we can put this in the perfect fluid form

$$T_{\mu\nu} = 2X P_{,X} u_\mu u_\nu + P g_{\mu\nu} \, , \tag{2.1.4}$$

from which we can read off the pressure, energy density and equation of state parameter

$$P = P \, ; \qquad \rho = 2X P_{,X} - P \, ; \qquad w = \frac{P}{2X P_{,X} - P} \, . \tag{2.1.5}$$

It is also often useful to define the sound speed

$$c_s^2 = \frac{P_{,X}}{P_{,X} + 2X P_{,XX}} \, . \tag{2.1.6}$$

In this way we see that non-canonical scalar fields can mimic perfect fluids of arbitrary equation of state, for suitable choices of the function $P(X, \phi)$.

We are interested in studying the scalar fluctuations about FRW solutions of the combined system of gravity and a scalar. In order to facilitate this, we follow Maldacena [64] and use the Arnowitt–Deser–Misner (ADM) decomposition of the metric [80, 81]

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + h_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t) \, , \tag{2.1.7}$$

here h_{ij} is the metric on spatial slices; the quantities N and N^i are known as the *lapse* function and *shift* vector, respectively. In the ADM formalism, these two quantities play the role of Lagrange multipliers and their equations of motion are the constraints of the theory. In terms of these variables, the action (2.1.1) takes the form [64, 82–84]

$$S = \int \mathrm{d}^4 x \sqrt{h} N \left[\frac{M_{\mathrm{Pl}}^2}{2} \left(R^{(3)} + K_{ij} K^{ij} - K^2 \right) + P(\tilde{X}, \phi) \right] \, , \tag{2.1.8}$$

where $R^{(3)}$ is the Ricci curvature on spatial slices, K_{ij} is the extrinsic curvature and $\tilde{X}$ is

X in terms of this metric:

$$K_{ij} = \frac{1}{2N} \left(\dot{h}_{ij} - D_i N_j - D_j N_i \right) \; ; \quad \tilde{X} = \frac{1}{2N^2} \left(\dot{\phi} - N^i \partial_i \phi \right)^2 - \frac{1}{2} (\partial_i \phi)^2 \; , \qquad (2.1.9)$$

where D_i is the covariant derivative of the spatial metric. If we split the diffeomorphism parameter $\xi^\mu = (\xi, \xi^i)$, the ADM variables transform under (2.1.2) as [83]

$$\delta_\xi h_{ij} = \xi^k \partial_k h_{ij} + h_{jk} \partial_i \xi^k + h_{ik} \partial_j \xi^k + \xi \dot{h}_{ij} + N_i \partial_j \xi + N_j \partial_i \xi \qquad (2.1.10)$$

$$\delta_\xi N^i = \xi^j \partial_j N^i - N^j \partial_j \xi^i + \frac{\mathrm{d}}{\mathrm{d}t} \left(\xi N^i \right) + \dot{\xi}^i - \left(N^2 h^{ij} + N^i N^j \right) \partial_j \xi \qquad (2.1.11)$$

$$\delta_\xi N = \xi^i \partial_i N + \frac{\mathrm{d}}{\mathrm{d}t} \left(\xi N \right) - N N^i \partial_i \xi \qquad (2.1.12)$$

$$\delta_\xi \phi = \xi \dot{\phi} + \xi^i \partial_i \phi \; . \qquad (2.1.13)$$

2.1.1 Background equations of motion

We first want to derive the Friedmann equations governing the background evolution. Working with a flat slicing for simplicity, we have $h_{ij} = a^2(t) \delta_{ij}$. Plugging this into the action (2.1.8), and specializing to a homogeneous profile for the scalar, $\phi = \phi(t)$, we obtain

$$S = \int \mathrm{d}^4 x \, a^3 N \left[-3 M_{\mathrm{Pl}}^2 N^{-2} \left(\frac{\dot{a}}{a} \right)^2 + P \left(\frac{1}{2} N^{-2} \dot{\phi}^2, \phi \right) \right] \; . \qquad (2.1.14)$$

Varying with respect to the lapse, N, we obtain the Friedmann equation

$$3 M_{\mathrm{Pl}}^2 H^2 = 2 X P_{,X} - P \; , \qquad (2.1.15)$$

while varying with respect to the scale factor, a, yields the equation

$$2 M_{\mathrm{Pl}}^2 \dot{H} + 3 M_{\mathrm{Pl}}^2 H^2 = -P \; . \qquad (2.1.16)$$

2.1.2 Perturbations

Having derived the equations governing the background, we want to consider fluctuations about solutions to these equations. Schematically, we expand the metric and scalar field around a solution to the Friedmann equation

$$\phi = \bar{\phi} + \delta\phi \; ; \qquad\qquad h_{ij} = \bar{h}_{ij} + \delta h_{ij} \; , \qquad\qquad (2.1.17)$$

and want to analyze the perturbations $\delta\phi$ and δh_{ij}. General Relativity is a gauge theory, so in order to isolate the physical degrees of freedom we choose a gauge. In the gauge-fixed lagrangian there will be three degrees of freedom, the two transverse-traceless polarizations of the graviton and a single scalar fluctuation, coming from ϕ.

A particularly convenient and popular choice is ζ-gauge (also called co-moving gauge), which was used by Maldacena in his seminal paper [64].[12] This gauge choice is defined by choosing the spatial slices to be level sets of the scalar $\phi(t)$. Then, the scalar fluctuation is shuffled into the metric[13]

$$\delta\phi = 0 \; ; \qquad h_{ij} = a^2(t)e^{2\zeta(\vec{x},t)}\left(e^{\gamma}\right)_{ij} \; ; \qquad \gamma^i{}_i = \partial_i\gamma^i{}_j = 0 \; . \qquad (2.1.18)$$

In the remainder, we will not be concerned with tensor perturbations, and will therefore take $h_{ij} = a^2(t)e^{2\zeta(\vec{x},t)}\delta_{ij}$. Our goal is to derive an action for the variable ζ, which is related to the Ricci curvature of spatial slices through

$$R^{(3)} = -\frac{4}{a^2}\nabla^2\zeta \; , \qquad\qquad (2.1.19)$$

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[12]Note that, contrary to the conventional wisdom, this gauge choice does not completely fix the gauge; there are residual *large gauge transformations*, corresponding to diffeomorphisms which do not die off at infinity. Recently, these residual symmetries have been used to derive relationships between different correlation functions in inflation. See [83] for an excellent discussion.

[13]For this reason it is also sometimes called unitary gauge because of its similarity to unitary gauge in the standard model where the Goldstone degrees of freedom from electroweak symmetry breaking are 'eaten' to become the longitudinal polarization of the vector bosons.

and for this reason it is often called the curvature perturbation. The procedure is the following: we solve the constraint equations for N and N^i order by order in ζ and then substitute the result back into the action to obtain an action for the field ζ.

Varying with respect to N and N^i, we obtain the Hamiltonian and momentum constraints

$$\frac{1}{2}\left[R^{(3)} - N^{-2}\left(E_{ij}E^{ij} - E^2\right)\right] + P - 2XP_{,X} = 0 \;,$$

$$\nabla_i N^{-1}\left(E^i_j - \delta^i_j E\right) = 0 \;, \tag{2.1.20}$$

where we have defined $E_{ij} \equiv NK_{ij}$. We will solve these constraint equations order by order; first write [64, 82, 84]

$$N = 1 + \alpha \;; \qquad N_i = \partial_i\psi + \widetilde{N}_i \;; \qquad \partial_i\widetilde{N}^i = 0 \;; \tag{2.1.21}$$

then, we can expand α, ψ and $\widetilde{N}_i$ in powers of ζ

$$\begin{aligned}
\alpha &= \alpha_1 + \alpha_2 \ldots \\
\psi &= \psi_1 + \psi_2 + \ldots \\
\widetilde{N}_i &= \widetilde{N}_i^{(1)} + \widetilde{N}_i^{(2)} + \ldots
\end{aligned} \tag{2.1.22}$$

In what follows, we will only need to work to first order, so we will just take $\psi_1 \equiv \psi$, $\alpha_1 \equiv \alpha$ and $\widetilde{N}_i^{(1)} \equiv \widetilde{N}_i$ to simplify notation. At first order in the perturbations, the equations (2.1.20) have the solution [64, 82, 84]

$$\alpha = \frac{\dot{\zeta}}{H} \;; \qquad \widetilde{N}_i = 0 \;; \qquad \psi = -\frac{\zeta}{H} + \chi \;; \quad \text{where} \quad \partial^2\chi = \frac{a^2\epsilon}{c_s^2} \;. \tag{2.1.23}$$

We take these solutions and plug them back into the action (2.1.8) and expand up to cubic order[14] in ζ. This is a straightforward, but extremely laborious, process; at quadratic order

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[14]It is somewhat surprising—but true—that it is only necessary to solve the constraints to first order to obtain the cubic action. This is because the contributions at higher order in the lapse and shift multiply the lower order constraint equations [64, 84].

we obtain the following action for the curvature perturbation [82, 84, 85]

$$S_2 = \int \mathrm{d}^3 x \mathrm{d}\tau \; z^2 \left[\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} \right)^2 - c_s^2 (\vec{\nabla}\zeta)^2 \right] \; , \tag{2.1.24}$$

where we have defined $z^2 \equiv a^2 \epsilon / c_s^2$. Continuing to expand up to cubic order yields (after a truly impressive amount of integration by parts) [82, 84]

$$S_3 = \int \mathrm{d}^3 x \mathrm{d}t \left[-a^3 \left[\Sigma \left(1 - \frac{1}{c_s^2} \right) + 2\lambda \right] \frac{\dot{\zeta}^3}{H^3} + \frac{a^3 \epsilon}{c_s^4} \left(\epsilon - 3 + 3c_s^2 \right) \zeta \dot{\zeta}^2 + \right.$$
$$\frac{a\epsilon}{c_s^2} \left(\epsilon - 2\epsilon_s + 1 - c_s^2 \right) \zeta (\partial\zeta)^2 - \frac{2a\epsilon}{c_s^2} \dot{\zeta} \partial\zeta \partial\chi + \frac{a^3 \epsilon}{2c_s^2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\eta}{c_s^2} \right) \zeta^2 \dot{\zeta} + \tag{2.1.25}$$
$$\left. \frac{\epsilon}{2a} \partial\zeta \partial\chi \partial^2\chi + \frac{\epsilon}{4a} \partial\zeta (\partial\chi)^2 + 2f(\zeta) \frac{\delta\mathcal{L}}{\delta\zeta} \Big|_1 \right] \; ,$$

where $\eta = H^{-1} \mathrm{d} \ln\epsilon / \mathrm{d}t$, and

$$\lambda = X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX} \; ; \qquad \Sigma = X P_{,X} + 2X P_{,XX} = \frac{H^2 \epsilon}{c_s^2} \; . \tag{2.1.26}$$

Although we focus on a particular class of microphysical models, namely $P(X, \phi)$ theories, the model–dependence of the action is encoded only in λ. The final term in the action is given by the complicated expression

$$f(\zeta) = \frac{\eta}{4c_s^2} \zeta^2 + \frac{1}{Hc_s^2} \zeta \dot{\zeta}^2 + \frac{1}{4a^2 H^2} \left(-(\partial\zeta)^2 + \frac{\partial^i \partial^j}{\nabla^2} (\partial_i \zeta \partial_j \zeta) \right) + \frac{1}{2a^2 H} \left(\partial\zeta \partial\chi - \frac{\partial^i \partial^j}{\nabla^2} (\partial_i \zeta \partial_j \chi) \right)$$

and where $\frac{\delta\mathcal{L}}{\delta\zeta} \Big|_1$ is the equation of motion of the quadratic action

$$\frac{\delta\mathcal{L}}{\delta\zeta} \Big|_1 = a \left(\dot{\Lambda} + H\Lambda - \epsilon \partial^2 \zeta \right) \; , \qquad \Lambda = \partial^2 \chi = \frac{a^2 \epsilon}{c_s^2} \dot{\zeta}^2 \; . \tag{2.1.27}$$

Notice that if we were computing scattering amplitudes, this last term would be a redundant coupling, which we could eliminate via a field redefinition, and would not affect any observables. However, in cosmology, we are interested in equal-time correlation functions, which are sensitive to the field variables. Nonetheless, the terms coming from $f(\zeta)$ will not

play an important role in our arguments. We now turn to the analysis of these quadratic and cubic actions.

2.2 Scale invariant cosmologies and strong coupling

Now that we have set up the formalism, in this section we ask a simple question: *what cosmologies are capable of giving rise to a scale invariant spectrum of perturbations?* By scale invariant, we mean that the two-point function for the curvature perturbation should be of the form (1.3.13). In fact, production of a scale invariant spectrum of fluctuations by itself is not enough—it is also highly desirable for the background solution to be an *attractor.*[15] Technically, this is achieved by demanding that the curvature perturbation on uniform-density hypersurfaces, ζ, goes to a constant in the long wavelength limit $k \to 0$. In this limit, $\zeta \approx \delta a/a$ is interpreted as a constant perturbation of the scale factor, which may therefore be absorbed locally by a spatial diffeomorphism [89].

For models involving a single, canonical scalar field (*i.e.,* with unit sound speed, $c_s = 1$) minimally coupled to Einstein gravity, it is known that there are only *three* independent cosmological solutions which produce a scale invariant spectrum of curvature perturbations on an attractor background [88, 90], assuming adiabatic vacuum initial conditions. We will review this classification: the most well–known of these solutions is of course inflation [21–23], which relies on exponential expansion of the background with $\epsilon \equiv -\dot{H}/H^2 \simeq 0$. More recently, the adiabatic ekpyrotic [50, 51] scenario has been proposed, in which a scale invariant spectrum is produced by a rapidly evolving equation of state $\epsilon \sim 1/t^2$ on a slowly contracting background. The third solution can be viewed as a variant of adiabatic ekpyrosis, where curvature perturbations are again sourced by a rapidly changing equation of state, but this time on a slowly expanding background [2]. At the level of the two-point function, these three scenarios yield indistinguishable power spectra.

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[15]The requirement that the background be an attractor may not be essential. Indeed, there are scenarios where instabilities play a crucial role and have important consequences, for example the matter bounce scenario [86] in the single–field case, as well as the curvaton mechanism and the phoenix universe [87] in the multi–field case. See [88] for a detailed discussion of single–field, non–attractor scenarios.

However, the degeneracy is broken at the three-point level. The non-inflationary solutions have strongly scale dependent non-gaussianities [51, 90], which can be traced to the rapid growth of the equation of state parameter. In these models, $f_{\rm NL} \sim k$ grows rapidly at small scales and perturbative control is lost when $f_{\rm NL}\zeta \sim 1$. This difficulty can be avoided by suitably modifying the potential so that ζ becomes much smaller on small scales. But this in turn restricts the range of scale invariant and gaussian modes to about 5 decades ($\sim 10^5$) or $\simeq 12$ e-folds in k-space [51, 90]. As a result, with $c_s = 1$ and attractor background, inflation is the *unique* single field mechanism capable of producing many decades of scale invariant and gaussian perturbations.[16]

Here we generalize the analysis to the case of time-varying sound speed, $c_s(t)$, as obtained, for instance, with the non-canonical scalar fields considered in the previous section. With Einstein gravity plus a single degree of freedom, the sound speed is the only remaining knob at our disposal.[17] As shown in [57], allowing for $c_s(t)$ greatly broadens the realm of allowed cosmologies that yield a scale invariant power spectrum. In particular, any cosmology with constant equation of state can be made scale invariant by suitably choosing the evolution of the sound speed. In this work we show that non-gaussianities impose stringent constraints on the allowed cosmologies. Our analysis is very general and applies to *arbitrary* time-dependent $\epsilon(t)$ and $c_s(t)$, with the only restriction that the null energy condition be satisfied: $\epsilon \geq 0$.

We begin by reviewing how the time–dependence of the sound speed results in an effective cosmological background for the curvature perturbation, as was first shown in [57]. In this effective background, which depends both on the evolution of the scale factor and the sound speed, ζ propagates at the speed of light. We derive a consistency equation that the scale

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[16]Of course, as a theory of the early universe inflation must still surmount some foundational issues, such as the measure problem and low-entropy initial conditions [74]. Here we leave aside these critical questions and note that inflation, viewed as a mechanism for generating density perturbations, remains weakly–coupled over a large range of modes.

[17]One could also consider alternative theories of gravity. This analysis applies to any theory of gravity which admits an Einstein frame description in terms of some field variables, such as generic scalar tensor theories.

factor and the sound speed must satisfy in order to have scale invariance at the two-point level. In the spirit of [57], given an evolution for the scale factor, solving this equation gives a suitable evolution for the sound speed for which ζ has a scale invariant two–point function on an attractor background. This shows that a time-dependent sound speed vastly increases the degeneracy at the two-point level.

As in the canonical case, this degeneracy is generically broken by the three-point function. In particular, if the three-point function is strongly scale–dependent, we generically expect the theory to become strongly coupled either in the infrared (IR) or in the ultraviolet (UV). To avoid such perturbative breakdown, we demand that certain contributions to the three-point function be scale invariant. This turns out to be extremely restrictive: we show that slow-roll inflation is the *unique* cosmology with this property. Conversely, if the three-point function is not scale invariant, then non–gaussianities will increase rapidly with scale, resulting in a finite range ($\lesssim 10^5$ modes) of perturbations consistent with observations, as in the canonical case. This is a remarkable fact; it is extremely surprising, in light of the vast degeneracy afforded by a variable sound speed, that slow-roll inflation should be the unique possibility.

2.2.1 Scale invariance with variable sound speed

We begin by considering what types of cosmological evolution allow for a scale invariant two-point function for ζ. We make no assumptions about the underlying dynamics, only that they may be well modeled by a perfect fluid. Perturbing around a Friedmann–Robertson–Walker (FRW) background in ζ-gauge, the quadratic action for ζ is given by (2.1.24)

$$S_2 = M_{\rm Pl}^2 \int \mathrm{d}^3x \mathrm{d}\tau \ z^2 \left[\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} \right)^2 - c_s^2 \left(\partial \zeta \right)^2 \right] \ , \qquad (2.2.1)$$

where $z \equiv a\sqrt{\epsilon}/c_s$, and τ denotes conformal time, $a\mathrm{d}\tau = \mathrm{d}t$. This action is familiar from canonical single field models, except for the sound speed factor multiplying the spatial gradient term and appearing in the measure factor. In order to eliminate this complication,

following [57] we define the sound horizon time coordinate by $\mathrm{d}y = c_s \mathrm{d}\tau$.[18] Additionally, we define

$$q \equiv \sqrt{c_s} z = \frac{a\sqrt{\epsilon}}{\sqrt{c_s}} \, . \tag{2.2.2}$$

In terms of these new variables, the quadratic ζ action takes the familiar form

$$S_2 = M_{\mathrm{Pl}}^2 \int \mathrm{d}^3 x \mathrm{d}y \; q^2 \left[\zeta'^2 - (\partial \zeta)^2 \right] \, , \tag{2.2.3}$$

where $' \equiv \mathrm{d}/\mathrm{d}y$. The virtue of this change of variables is manifest—ζ now propagates luminally, but in *effective* cosmological background defined both by the scale factor and the sound speed.

The mode functions of the canonically normalized field, $v \equiv \sqrt{2} M_{\mathrm{Pl}} \; q \cdot \zeta$, obey the Mukhanov–Sasaki equation

$$v_k'' + \left(k^2 - \frac{q''}{q} \right) v_k = 0 \, . \tag{2.2.4}$$

Assuming the usual adiabatic (Bunch–Davies) vacuum, it is well known that (2.2.4) will yield a scale invariant spectrum of perturbations provided that

$$\frac{q''}{q} = \frac{2}{y^2} \, . \tag{2.2.5}$$

Note that modes freeze out when $k|y| \sim 1$, which corresponds to sound-horizon crossing in the constant c_s case, hence we take $-\infty < y < 0$. The solution for the mode functions is then

$$v_k(y) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{ky} \right) e^{iky} \, , \tag{2.2.6}$$

which describes a scale invariant spectrum, $v_k \sim k^{-3/2}$, in the limit $y \to 0$.

Equation (2.2.5) has two solutions, $q \sim 1/(-y)$ and $q \sim y^2$, but only the former describes a background which is a dynamical attractor. To see this, note that in the long wavelength

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[18]Note that when $c_s = $ constant, the variable y measures the size of the sound horizon.

$(k \to 0)$ limit we have the following expression for the power spectrum of the solution (2.2.6)

$$P_\zeta = \frac{1}{2\pi^2} k^3 |\zeta_k|^2 \sim \frac{1}{q^2 y^2} \ , \tag{2.2.7}$$

which is indeed independent of k. When $q \sim 1/(-y)$, $\zeta \to$ constant outside the horizon, indicating perturbative stability [89]. The other solution $q \sim y^2$, however, implies that ζ grows outside the horizon, $\zeta \sim y^{-3}$, signaling that the background is unstable. Since we are interested in attractor backgrounds, we henceforth ignore the $q \sim y^2$ solution.

We digress slightly to make an important point: notice that when $q \sim 1/t$, the action (2.2.3) takes the form

$$S_2 \sim \int \mathrm{d}^3 x \mathrm{d}y \frac{1}{y^2} \left[\zeta'^2 - (\partial \zeta)^2 \right] \ ; \tag{2.2.8}$$

if we define the *effective metric*

$$g_{\mu\nu}^{\mathrm{eff}} \sim \frac{1}{y^2} \eta_{\mu\nu} \ , \tag{2.2.9}$$

this action can be rewritten as

$$S_2 \sim \int \mathrm{d}^4 x \sqrt{-g_{\mathrm{eff}}} \left(g_{\mathrm{eff}}^{\mu\nu} \partial_\mu \zeta \partial_\nu \zeta \right) \ . \tag{2.2.10}$$

This is precisely the action for a massless scalar field on *de Sitter space*, where y plays the role of conformal time; the problem of classifying all cosmologies which produce a scale invariant spectrum on an attractor background is therefore identitical to the problem of finding all cosmologies on which the scalar ζ propagates on an *effective* de Sitter space!

Recalling the definition of q, the condition for scale invariance in an attractor background may therefore be succinctly expressed as

$$q^2 = \frac{a^2 \epsilon}{c_s} = \frac{\beta}{y^2} \ , \tag{2.2.11}$$

where β is an arbitrary (positive) constant. It is important to note that a and ϵ are not

independent degrees of freedom, but are related by $\epsilon = -\dot{H}/H^2$. Changing time variables to y, this relation becomes

$$\epsilon = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{H} = \frac{c_s}{a}\left(\frac{a^2}{c_s a'}\right)' \ . \tag{2.2.12}$$

Using the condition (2.2.11) for scale invariance, we can rewrite this to obtain the master equation

$$a\left(\frac{a^2}{c_s a'}\right)' = \frac{\beta}{y^2} \ . \tag{2.2.13}$$

This equation ensures a scale invariant spectrum on an attractor background. As noted in [88], in the case where $c_s = $ constant, this equation may be recast as a particular instance of the generalized Emden–Fowler equation.

For completeness, we review the results of [88, 90]. In the case of constant sound speed (without loss of generality, we may take $c_s = 1$), there are three distinct scale invariant solutions:

- *Inflation* is a solution where the scale factor grows as $a_{\mathrm{inf}} \sim 1/(-\tau)$ and the equation of state parameter is constant $\epsilon_{\mathrm{inf}} \ll 1$ [21–23]. To check that this is in fact scale invariant, we note that $q_{\mathrm{inf}}^2 \sim 1/\tau^2$, where $y \sim \tau$ because c_s is constant.

- *Adiabatic ekpyrosis* is a solution where the equation of state parameter varies rapidly, $\epsilon_{\mathrm{ek}} \sim 1/\tau^2$, while the background remains nearly static, $a_{\mathrm{ek}} \sim 1$. Again, we can check that this gives a scale invariant spectrum $q_{\mathrm{ek}}^2 \sim 1/\tau^2$. In fact, this corresponds to two distinct solutions, one where the background is slowly *contracting* [50, 51] and one where the background is slowly *expanding* [2]. It is important to note that in these scenarios modes freeze out on sub–Hubble scales and are subsequently pushed outside the horizon during a contracting ekpyrotic phase with constant $\epsilon \gg 1$.

Returning to the general case, given any evolution for a we can find an evolution of c_s that will make the spectrum of perturbations scale invariant by solving (2.2.13). Alternatively, specifying a relation between the evolution of c_s and a is sufficient to determine the evolu-

tion. As a result, we see that there is an enormous amount of degeneracy at the two–point level.

An excellent illustration of this degeneracy is the case of cosmologies with constant ϵ. With constant c_s, as reviewed above inflation is the only solution that has constant ϵ. But for more general sound speed, there is a power-law evolution for c_s that yields a scale invariant spectrum for arbitrary positive values of ϵ. Indeed, constancy of ϵ and $\epsilon_s \equiv \dot{c}_s/Hc_s$ is sufficient to deduce the scaling solutions

$$a \sim (-y)^{\frac{1}{\epsilon+\epsilon_s-1}} , \qquad c_s \sim (-y)^{\frac{\epsilon_s}{\epsilon+\epsilon_s-1}} . \qquad (2.2.14)$$

Inserting these expressions into (2.2.13), we find that the solution is scale invariant for $\epsilon_s = -2\epsilon$, in agreement with [57].

2.2.2 The cubic action and strong coupling

Non-gaussianities offer a powerful tool for differentiating between the different cosmologies with degenerate power spectra. Since the precise form of the cubic action depends the underlying physics, we must choose to parameterize the microphysics in some way. A convenient and quite general choice is to consider a non–canonical scalar field ϕ, described by a $P(X,\phi)$ lagrangian of the type considered in the previous section (2.1.1). Making the transformation to the sound horizon time variable $dy = c_s d\tau$, in the action (2.1.25) and ignoring the piece that may be field-redefined away, the action takes the form

$$
\begin{aligned}
S_3 = \int d^3x dy \bigg[&- ac_s^2 \left[\Sigma \left(1 - \frac{1}{c_s^2} \right) + 2\lambda \right] \frac{\zeta'^3}{H^3} + \frac{a^2\epsilon}{c_s^3} \left(\epsilon - 3 + 3c_s^2 \right) \zeta \zeta'^2 \\
&+ \frac{a^2\epsilon}{c_s^3} \left(\epsilon - 2\epsilon_s + 1 - c_s^2 \right) \zeta \left(\partial \zeta \right)^2 - \frac{2a^2\epsilon^2}{c_s^3} \zeta' \partial \zeta \frac{\partial \zeta'}{\nabla^2} \\
&+ \frac{a^2\epsilon}{2c_s} \left(\frac{\eta}{c_s^2} \right)' \zeta^2 \zeta' + \frac{a^2\epsilon^3}{2c_s^3} \partial \zeta \frac{\partial \zeta'}{\nabla^2} \zeta' + \frac{a^2\epsilon^3}{4c_s^3} \nabla^2 \zeta \left(\frac{\partial \zeta'}{\nabla^2} \right)^2 \bigg] ,
\end{aligned}
\qquad (2.2.15)
$$

where $\eta = H^{-1} \mathrm{d} \ln \epsilon / \mathrm{d}t$, and

$$\lambda = X^2 P_{,XX} + \frac{2}{3} X^3 P_{,XXX} \; ; \qquad \Sigma = X P_{,X} + 2 X P_{,XX} = \frac{H^2 \epsilon}{c_s^2} \; . \qquad (2.2.16)$$

Although we focus on a particular class of microphysical models, namely $P(X, \phi)$ theories, the model–dependence of the action is encoded only in λ. All other vertices in the cubic action are functions of the scale factor and the sound speed. For example, for a DBI action, the ζ'^3 term in (2.2.15) vanishes identically [57, 84]. Since the form of this first term will not be material to our arguments, our analysis even at the cubic level is rather general, but there may be some potential model–dependent effects from the ζ'^3 vertex which we have not considered.

To estimate non–gaussianities, a useful approximation is the *horizon–crossing approximation*, whereby f_{NL} is estimated by

$$f_{\mathrm{NL}} \sim \left. \frac{\mathcal{L}_3}{\zeta \cdot \mathcal{L}_2} \right|_{k|y|=1} . \qquad (2.2.17)$$

Here $\mathcal{L}_2$ and $\mathcal{L}_3$ are terms in the quadratic and cubic lagrangians, respectively. Since temporal and spatial gradients are comparable at horizon crossing ($\partial_y \sim \partial_i \sim k$), we may trade them freely in (2.2.17). The horizon-crossing approximation generally offers a good estimate of f_{NL} since modes are in their ground state at early times—when they are far inside the horizon—and become constant outside the horizon. We therefore expect non-gaussianities to peak around horizon crossing.[19]

At a classical level, perturbations are highly non-gaussian for $f_{\mathrm{NL}}\zeta \gtrsim 1$, corresponding to $\mathcal{L}_3/\mathcal{L}_2 \gtrsim 1$, and classical perturbation theory breaks down. At a quantum level, the right hand side of (2.2.17) also offers an estimate for the magnitude of loop corrections to the two-point function [91]. Thus, classical and quantum perturbation theory break down, and

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[19]An important exception is the adiabatic ekpyrotic solution, where the ϵ^3 contributions peak at late times, well after horizon crossing [51]. Although ζ goes to a constant outside the horizon, the rapid growth of the vertex, $\epsilon^3 \sim 1/t^6$, overwhelms the suppression from ζ derivatives becoming small. Thus, the horizon-crossing approximation is a conservative estimate of non-gaussianities.

the theory becomes strongly coupled, whenever

$$\frac{\mathcal{L}_3}{\mathcal{L}_2} \sim 1 \ , \tag{2.2.18}$$

or $f_{\mathrm{NL}}\zeta \sim 1$. This is the same strong coupling criterion used in [88].

In particular, if f_{NL} is strongly scale dependent, then the growth of non-gaussianities will generically lead to a breakdown of perturbation theory either in the IR or in the UV. This expectation is borne out by the analysis of the canonical case [2, 51, 90]. Even if the two-point function is scale invariant, strong coupling indicates that perturbation theory will only be valid for a finite range of modes; this reintroduces a cosmological puzzle, we would then have to explain why it is these weakly-coupled scale invariant modes which we observe in the CMB. We want to avoid strong coupling, thus we demand that f_{NL} be approximately scale invariant.

Amongst the terms in the cubic action is the vertex[20]

$$S_3 \supset \int \mathrm{d}^3x \mathrm{d}y \ \frac{a^2\epsilon^3}{2c_s^3} \partial\zeta \frac{\partial\zeta'}{\nabla^2}\zeta' \ . \tag{2.2.19}$$

Evaluating this vertex at horizon–crossing, we find that its non-gaussian contribution is

$$f_{\mathrm{NL}}^{\epsilon^3} \sim \left. \frac{\frac{a^2\epsilon^3}{2c_s^3}\partial\zeta\frac{\partial\zeta'}{\nabla^2}\zeta'}{\frac{a^2\epsilon}{c_s}\zeta \cdot \zeta'^2}\right|_{k|y|=1} \sim \left(\frac{\epsilon}{c_s}\right)^2 . \tag{2.2.20}$$

Substituting the condition (2.2.11) for scale invariance at the two–point level, $a^2\epsilon/c_s \sim 1/y^2$, this reduces to

$$f_{\mathrm{NL}}^{\epsilon^3} \sim \left. \frac{1}{a^4y^4}\right|_{k|y|=1} . \tag{2.2.21}$$

Now, in order for the full three–point function to be scale invariant, a necessary condition is that the contribution from this vertex be scale invariant, barring miraculous cancellations.

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[20]This vertex is the leading contribution to non-gaussianity in the adiabatic ekpyrotic scenarios [2, 50, 51, 90].

This implies that the scale factor must be growing as

$$a \sim \frac{1}{(-y)} \ , \tag{2.2.22}$$

which corresponds to an effective de Sitter geometry. Remarkably, simply demanding scale invariance of the two- and three-point correlation functions, without any consideration of the independent dynamics of a and c_s, has led us to focus on backgrounds that are effectively de Sitter, albeit in terms of the y variable.

Remarkably, simply demanding scale invariance of the two- and three-point correlation functions, without any consideration of the independent dynamics of a and c_s, has led us to focus on backgrounds that are effectively de Sitter, albeit in terms of the y variable. Thus the question becomes—is it possible to have inflation without inflation? By this, we mean, is there an evolution where the modes see an effective de Sitter space in terms of the y variable but for which the true geometry is far from de Sitter? Unfortunately, the answer appears to be no, as we now argue.

With $a(y) \sim 1/(-y)$, (2.2.11) immediately implies that $\epsilon/c_s = \gamma$, where γ is an arbitrary (positive) constant. This is all we need to solve (2.2.13), with the result

$$c_s(y) = \frac{-1}{\gamma \log (y/\bar{y})} \ ; \qquad\qquad \epsilon(y) = \frac{-1}{\log (y/\bar{y})} \ ; \tag{2.2.23}$$

where $0 \leq |y| \leq |\bar{y}|$. Both ϵ and c_s start out infinite and decrease rapidly to zero. By construction, this solution is scale invariant at the two–point level and the aforementioned three–point vertex is also scale invariant. At first sight, we might expect that this solution is far from de Sitter because $\epsilon \gg 1$ initially. However, because ϵ is decreasing so rapidly, by the time $|y| < e^{-1}|\bar{y}|$; ϵ is already less than unity, indicating an inflationary spacetime. As such, this solution is only a small deformation away from the de Sitter geometry, specifically only about one e–fold of evolution is non–inflationary.

This is a rather interesting result; it seems that even in the presence of an arbitrarily evolving

speed of sound, inflation remains the unique single-field mechanism which is capable of remaining weakly coupled for an extended period of cosmological evolution. One might think that allowing for superluminal values of the sound speed might alleviate these problems, this case is studied in detail in [3]. Not only do these solutions also become strongly-coupled, but there are questions about whether such theories with superluminal propagation can descend from a local theory in the ultraviolet [92].

Chapter 3

Introducing the conformal mechanism

Having surveyed the landscape of single-field cosmologies, we have come to the conclusion that (attractor) non-inflationary solutions suffer from strong coupling problems. Therefore, in order to explore alternatives to inflation, we are led to consider scenarios which either rely on an instability (as in the matter-dominated scenario of [86, 93, 94]) or involve multiple fields (as in the New Ekpyrotic scenario [45, 46, 48] or the pre-big bang scenario [24, 25]). It is this latter tack that we will take.

Here we introduce a novel cosmological scenario, the *conformal mechanism*, which is able to both address the canonical puzzles of FRW cosmology and produce a scale-invariant spectrum of cosmological perturbations. It evades our no-go theorem of Section 2.2 by involving multiple fields; it is a spectator field that acquires a scale invariant spectrum of perturbations, which must be later converted to the adiabatic direction.

Roughly, the logic is similar to that employed in Section 2.2.1; the goal is to construct an *effective* de Sitter space while keeping the true geometry far from de Sitter. While this was impossible in the single-field case, it turns out to be possible with multiple fields. Conformal symmetry dictates particular couplings between scalar fields in the theory; by causing one of the scalar fields to get a time-dependent background value, the other fields in the theory will feel as though they are living on de Sitter space.

This mechanism first appeared in an explicit model of Rubakov [12], and has been investigated in [5, 7, 8, 11–13, 52, 95–99]. A point which we want to stress, which was pointed

out in [5, 13], is that much of the relevant physics depends only on the symmetry breaking pattern in the theory—namely $\mathfrak{so}(4,2) \rightarrow \mathfrak{so}(4,1)$ and is independent of the microscopic realization. Nevertheless, in this Chapter we introduce the mechanism through concrete models, preferring to leave abstraction to the general case to Chapter 4.

3.1 A crash course in conformal symmetry

The mechanism relies on spontaneous breaking of global conformal symmetry, so we begin by quickly summarizing the basics of conformal symmetry in field theory, mostly following [100, 101]. We work in an arbitrary number of dimensions, $d \geq 3$. Recall that the *conformal group* is the group of diffeomorphisms that rescale the metric by an overall function

$$g_{ab} \longmapsto \Omega^2(x)g_{ab} \ . \tag{3.1.1}$$

Also recall that an infinitesimal diffeomorphism acts as

$$\delta_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \ ; \tag{3.1.2}$$

in what follows, we will restrict to flat space $g_{ab} = \eta_{ab}$. In order for this transformation to be conformal, we must have

$$\partial_a \xi_b + \partial_b \xi_a = \Omega^2(x)\eta_{ab} \ , \tag{3.1.3}$$

tracing over both sides yields $\Omega^2(x) = 2\partial_c \xi^c/d$ [100, 101]. Therefore, we find that conformal transformations satisfy the differential equation

$$\partial_a \xi_b + \partial_b \xi_a = \frac{2}{d} \left(\partial_c \xi^c \right) \eta_{ab} \ . \tag{3.1.4}$$

This equation can be solved, the corresponding conformal Killing vectors are (for constant vectors a^c, b^c) [101]

$$\xi^c = a^c \qquad\qquad\qquad\qquad\qquad \text{translations} \qquad (3.1.5)$$

$$\xi^c = \epsilon^{ca} x_a \qquad \text{where} \quad \epsilon_{ca} = -\epsilon_{ac} \qquad \text{rotations \& boosts} \qquad (3.1.6)$$

$$\xi^c = \lambda x^c \qquad\qquad\qquad\qquad\qquad \text{dilation} \qquad (3.1.7)$$

$$\xi^c = 2(x \cdot b)x^c - b^c x^2 \qquad\qquad \text{special conformal transformations} \qquad (3.1.8)$$

The first of these (translations, rotations and boosts) are familiar, they form the Poincaré group of flat space-time. The latter transformations (dilations and special conformal transformations) are less familiar. These transformations are generated by [100]

$$P_a = -\partial_a \qquad\qquad\qquad\qquad\qquad \text{translations} \qquad (3.1.9)$$

$$J_{ab} = x_a \partial_b - x_b \partial_a \qquad\qquad\qquad \text{rotations \& boosts} \qquad (3.1.10)$$

$$D = -x^a \partial_a \qquad\qquad\qquad\qquad\qquad \text{dilation} \qquad (3.1.11)$$

$$K_a = -2x_a x^b \partial_b + x^2 \partial_a \qquad\qquad \text{special conformal transformations} \qquad (3.1.12)$$

Taken together, these generators obey the *conformal algebra*

$$\begin{aligned}
&[D, P_a] = -P_a \,, && [D, K_a] = K_a \,, \\
&[J_{ab}, K_c] = \eta_{ac} K_b - \eta_{bc} K_a \,, && [J_{ab}, P_c] = \eta_{ac} P_b - \eta_{bc} P_a \,, \qquad\qquad (3.1.13)\\
&[K_a, P_b] = 2J_{ab} - 2\eta_{ab} D \,, && [J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} \,.
\end{aligned}$$

In fact, this algebra—in d dimensions—is isomorphic to $\mathfrak{so}(d, 2)$; this can be seen by defining the linear combinations

$$\begin{aligned}
J_{ab} &= J_{ab} \,, && J_{(d+1)a} = \tfrac{1}{2}(P_a + K_a) \,, \\
J_{(d+1)(d+2)} &= D \,, && J_{(d+2)a} = \tfrac{1}{2}(P_a - K_a) \,,
\end{aligned} \qquad (3.1.14)$$

which then satisfy the $\mathfrak{so}(d,2)$ algebra

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} - \eta_{BC}J_{AD} + \eta_{BD}J_{AC} - \eta_{AD}J_{BC} \ , \tag{3.1.15}$$

where $\eta_{AB} = \mathrm{diag}(\eta_{ab}, 1, -1)$.

3.1.1 Field transformations

Most relevant for our purposes is the action of these transformations on fields. To figure out the irreducible representations of the conformal group, we follow [100]. We begin by considering the subset of generators which leave the point $x = 0$ invariant (which is everything but translations), we define the generators $S_{ab}, \tilde{\Delta}, \kappa_a$, which are J_{ab}, D and K_a which act at $x = 0$. For example

$$J_{ab}\Phi(0) = S_{ab}\Phi(0) \ , \tag{3.1.16}$$

where Φ is an arbitrary irreducible representation of the Lorentz group. Then, we can use the Hausdorff formula[21] to deduce the action at finite x. For example, J_{ab} at finite x is given by [100]

$$e^{x^c P_c} S_{ab} e^{-x^c P_c} = S_{ab} - x_a P_b + x_b P_a \ , \tag{3.1.18}$$

so acting on a field we have $J_{ab}\Phi(x) = (x_a\partial_b - x_b\partial_a)\Phi(x) + S_{ab}\Phi(x)$. This is precisely how Lorentz transformations normally act; the gradient part is universal and the S_{ab} part is the extra piece that takes care of spin-ful fields. We can then play the same game with κ_a and

---

[21] The Hausdorff formula is enormously useful for various algebraic manipulations involving objects that do not commute, it states

$$e^X Y e^{-X} = e^{\mathrm{ad}_X} Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \dots \ , \tag{3.1.17}$$

where $\mathrm{ad}_X Y = [X, Y]$.

$\tilde{\Delta}$ to obtain [100]

$$D\Phi(x) = (\tilde{\Delta} - x^a \partial_a)\Phi(x)$$

$$K_a \Phi(x) = \left(\kappa_a + 2x_a \tilde{\Delta} - x^\nu S_{ab} - 2x_a x^b \partial_b + x^2 \partial_a \right) \Phi(x) \tag{3.1.19}$$

Now, κ_a, S_{ab} and $\tilde{\Delta}$ obey the algebra (3.1.13) (omitting the commutators involving P_a); since $\tilde{\Delta}$ commutes with S_{ab}, Shur's lemma implies that $\tilde{\Delta}$ must be proportional to a constant [100]. In fact, we have $\tilde{\Delta} = -\Delta$, which is the *conformal weight* of the field. Then, the commutator $[\tilde{\Delta}, \kappa_a] = \kappa_a$ implies that $\kappa_a = 0$. Inserting these back into (3.1.19), we can deduce the action of the conformal generators on a field of arbitrary spin. For concreteness, we focus on a Lorentz scalar, $\Phi \equiv \phi$, on which the generators act as

$$\delta_{P_a}\phi = -\partial_a \phi \ , \qquad\qquad \delta_{J_{ab}}\phi = (x_a \partial_b - x_b \partial_a)\phi \ ,$$
$$\delta_D \phi = -(\Delta + x^a \partial_a)\phi \ , \qquad\qquad \delta_{K_a}\phi = \left(-2\Delta x_a - 2x_a x^b \partial_b + x^2 \partial_a\right)\phi \ . \tag{3.1.20}$$

This is roughly all of the information we will need about conformal symmetry for the time being, we now turn to the conformal mechanism.

3.2 Cosmology of coupling a CFT to gravity

We now show how a (classical) conformal field theory (CFT) can address the background cosmological puzzles discussed in Chapter 2. We now specialize to $4d$: imagine that we have a conformal theory which is set up such that a scalar operator, $\mathcal{O}$, of conformal weight[22] Δ acquires a time-dependent expectation value

$$\bar{\mathcal{O}}(t) \sim \frac{1}{t^\Delta} \ , \tag{3.2.2}$$

---

[22]Recall that an operator of conformal weight Δ transforms under a dilation as

$$\mathcal{O}(\lambda x) = \lambda^{-\Delta} \mathcal{O}(x) \ . \tag{3.2.1}$$

where $-\infty < t < 0$. This expectation value breaks some of the conformal symmetries—in particular it breaks P_0, J_{0i} and K_0. The residual symmetries which annihilate this background can be repackaged into the generators

$$J_{ij} = J_{ij} \ , \qquad J_{56} = D \ , \qquad J_{5i} = \frac{1}{2}\left(P_i + K_i\right) \ , \qquad J_{6i} = \frac{1}{2}\left(P_i - K_i\right), \qquad (3.2.3)$$

which have the commutation relations of the $\mathfrak{so}(4,1)$ algebra,

$$[\delta_{J_{ab}}, \delta_{J_{cd}}] = \eta_{ac}\delta_{J_{bd}} - \eta_{bc}\delta_{J_{ad}} + \eta_{bd}\delta_{J_{ac}} - \eta_{ad}\delta_{J_{bc}} \ , \qquad (3.2.4)$$

where $\eta_{ab} = \mathrm{diag}\left(\delta_{ij}, 1, -1\right)$.

3.2.1 Einstein frame cosmology

Now, consider coupling this theory minimally to Einstein gravity

$$S = \int \mathrm{d}^4 x \sqrt{-g}\left(\frac{M_{\mathrm{Pl}}^2}{2}R + \mathcal{L}_{\mathrm{CFT}}\left[g_{\mu\nu}\right]\right) . \qquad (3.2.5)$$

Of course, this breaks conformal symmetry at the $1/M_{\mathrm{Pl}}$ level, but this is a mild breaking; at sufficiently early times (to be made precise shortly), gravity is negligible, hence the solution (3.2.2) is approximately valid. Since the background in the broken phase only depends on time and is invariant under dilation, the pressure and energy density must both scale as $1/t^4$. But energy conservation implies $\rho \simeq$ const. at zeroth order in $1/M_{\mathrm{Pl}}$, hence $\rho \simeq 0$. Thus, the assumed symmetries completely fix the form of the energy density and pressure of the CFT,

$$\rho_{\mathrm{CFT}} \simeq 0 \ ; \qquad\qquad P_{\mathrm{CFT}} \simeq \frac{\beta}{t^4} \ , \qquad (3.2.6)$$

up to a constant parameter β. For instance, for the quartic potential model we will discuss in Section 3.3, $\beta = 2/\lambda > 0$ corresponding to positive pressure. In the Galilean Genesis scenario [52]—which we will discuss in Section 3.4—on the other hand, $\beta < 0$, and the CFT violates the null energy condition.

Integrating $M_{\rm Pl}^2 \dot{H} = -(\rho_{\rm CFT} + P_{\rm CFT})/2$ gives the Hubble parameter

$$H(t) \simeq \frac{\beta}{6t^3 M_{\rm Pl}^2}\,, \qquad (3.2.7)$$

which corresponds to a contracting or expanding universe depending on the sign of β. In particular, the universe is contracting in the quartic potential case ($\beta = 2/\lambda$), and expanding in the Galilean Genesis scenario ($\beta < 0$). We can integrate once more to obtain the scale factor

$$a(t) \simeq 1 - \frac{\beta}{12 t^2 M_{\rm Pl}^2}\,. \qquad (3.2.8)$$

This self-consistently shows that the universe is indeed nearly static at early times. Specifically, neglecting gravity is valid for $t \ll t_{\rm end}$, with

$$t_{\rm end} \equiv -\frac{\sqrt{\beta}}{M_{\rm Pl}}\,. \qquad (3.2.9)$$

Note that in the ϕ^4 example, for instance, this corresponds to $\phi(t_{\rm end}) \sim M_{\rm Pl}$, where one in any case expects $M_{\rm Pl}$ suppressed operators to regulate the potential.

Finally, note that the evolution (3.2.7) implies the CFT equation of state

$$w_{\rm CFT} \simeq \frac{P_{\rm CFT}}{\rho_{\rm CFT}} = \frac{12}{\beta} t^2 M_{\rm Pl}^2\,. \qquad (3.2.10)$$

Over the range $-\infty < t < t_{\rm end}$, the equation of state decreases from $+\infty$ to a value of $O(1)$. A contracting phase with $w \gg 1$ is characteristic of ekpyrotic cosmologies. The key difference here compared to earlier ekpyrotic scenarios is that w is rapidly decreasing in time, as opposed to being nearly constant [32] or growing rapidly [2, 50, 51]. A phase of contraction/expansion with $|w| \gg 1$ is well known to drive the universe to be increasingly flat, homogeneous and isotropic [39]. Hence the background of interest is a dynamical attractor, even in the presence of gravity.

This is all that we will say in full generality for now; we now turn to some specific examples

of the conformal scenario: the negative quartic model, Galilean Genesis, and the world-volume theory of a brane probing an AdS_5 geometry. We will see that each of these examples share the same symmetry breaking pattern, and all can naturally leads to a scale invariant spectrum of fluctuations for spectator fields in the theory. In Chapter 4 we will generalize these examples and consider the most general low energy effective theory describing these dynamics.

3.3 Negative quartic example

As a first explicit realization of the conformal mechanism, we consider a conformal scalar field ϕ with *negative* ϕ^4 potential. The negative ϕ^4 example was considered in the context of a holographic dual to an AdS_5 bouncing cosmology by [11], discussed in the present context in a series of papers by Rubakov [12, 95–99], and further developed in [13].

Consider the action

$$S_\phi = \int \mathrm{d}^4 x \left(-\frac{1}{2} (\partial \phi)^2 + \frac{\lambda}{4} \phi^4 \right) , \tag{3.3.1}$$

with "wrong-sign" potential, $\lambda > 0$. The potential is unbounded from below, so we must imagine that higher-dimensional (*e.g.*, Planck-suppressed) operators stabilize the field at large ϕ [13]. At the classical level, this theory is invariant under the 15 conformal transformations (3.1.20), under which ϕ is a field of weight $\Delta = 1$,

$$
\begin{aligned}
\delta_{P_\mu} \phi &= -\partial_\mu \phi , & \delta_{J_{\mu\nu}} \phi &= (x_\mu \partial_\nu - x_\nu \partial_\mu), \phi \\
\delta_D \phi &= -(1 + x^\mu \partial_\mu) \phi , & \delta_{K_\mu} \phi &= \left(-2x_\mu - 2x_\mu x^\nu \partial_\nu + x^2 \partial_\mu \right) \phi .
\end{aligned}
\tag{3.3.2}
$$

The equation of motion for the action (3.3.1), assuming a homogeneous field profile, is

$$\ddot{\phi} - \lambda \phi^3 = 0 . \tag{3.3.3}$$

This equation admits a first integral of motion

$$\frac{1}{2} \dot{\phi}^2 - \frac{\lambda}{4} \phi^4 = E , \tag{3.3.4}$$

which has the zero-energy solution

$$\bar{\phi}(t) = \sqrt{\frac{2}{\lambda}} \frac{1}{(-t)} \equiv \frac{M_\pi}{H(-t)} \ . \tag{3.3.5}$$

This solution is a dynamical attractor [13], essentially because the growing mode solution for small perturbations $\delta\phi$ can be absorbed at late times into a time shift of the background. To see this, we expand the action (3.3.1) about the solution $\phi = \bar{\phi} + \delta\phi$ to obtain

$$S = \int d^4x \left(-\frac{1}{2}(\partial\delta\phi)^2 + \frac{3}{t^2}\delta\phi^2 \right) \ . \tag{3.3.6}$$

The Fourier space equation of motion for perturbations reads

$$\ddot{\delta\phi}_k + k^2\delta\phi_k - \frac{6}{t^2}\delta\phi = 0 \ ; \tag{3.3.7}$$

in the long-wavelength ($k \to 0$) limit, this equation has the two solutions

$$\delta\phi_k \sim \frac{1}{t^2} \ ; \qquad\qquad \delta\phi_k \sim t^3 \ . \tag{3.3.8}$$

At first glance, the growing mode solution $\delta\phi_k \sim 1/t^2$ appears dangerous; however, noting [13]

$$\bar{\phi}(t + \epsilon) \simeq \bar{\phi}(t) + \epsilon\dot{\bar{\phi}}(t) \sim \frac{1}{(-t)} + \frac{\epsilon}{t^2} \ , \tag{3.3.9}$$

it becomes clear that this growing mode is nothing more than a harmless time translation of the background solution. Therefore the solution (3.3.5) is an attractor. Finally, we note in passing at this point that through a field redefinition, $\phi = \bar{\phi} + \delta\phi = \bar{\phi}e^\pi = \frac{M_\pi}{H(-t)}e^\pi$, and introducing an effective de Sitter metric

$$g_{\mu\nu}^{\text{eff}} \equiv \frac{1}{H^2t^2}\eta_{\mu\nu} \ , \tag{3.3.10}$$

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the quadratic action (3.3.6) can be put in the form

$$S = M_\pi^2 \int \mathrm{d}^4x \sqrt{-g_{\mathrm{eff}}} \left(-\frac{1}{2} g_{\mu\nu}^{\mathrm{eff}} \partial^\mu \pi \partial^\nu \pi + 2H^2 \pi^2 \right) \ . \tag{3.3.11}$$

The profile (3.3.5) spontaneously breaks the symmetry algebra of the action (3.3.1) to its $\mathfrak{so}(4,1)$ de Sitter subalgebra. Indeed, the subalgebra of conformal generators (3.1.20) that annihilate the background (3.3.5) is spanned by

$$\left\{ \delta_{P_i}, \ \delta_D, \ \delta_{J_{ij}}, \ \delta_{K_i} \right\} \ . \tag{3.3.12}$$

Now, let us consider coupling a weight-0 spectator, *i.e.*, a field χ which transforms under (3.1.20) with $\Delta = 0$, to the rolling field ϕ.[23] In order for the action to be dilation invariant, the action for χ up to quadratic order (and second order in derivatives) must be of the form

$$S_\chi = \int \mathrm{d}^4x \left(-\frac{1}{2} \phi^2 (\partial \chi)^2 - \frac{m_\chi^2}{2} \phi^4 \chi^2 + \kappa \phi \Box \phi \chi^2 \right) \ . \tag{3.3.13}$$

In fact, this action is invariant under the full conformal group where χ transforms as a weight-0 field. When ϕ gets the profile (3.3.5), we may think of the χ field as coupling via the effective metric

$$g_{\mu\nu}^{\mathrm{eff}} = \bar{\phi}^2 \eta_{\mu\nu} = \frac{2}{\lambda t^2} \eta_{\mu\nu} \ , \tag{3.3.14}$$

which is the metric of de Sitter space in a flat slicing. Thus, the χ field feels as though it lives on de Sitter space. It is emphasized that this is *not* the physical metric—everything takes place in flat Minkowski space. It should not be surprising in light of the fact that χ lives in an effective de Sitter space that it can acquire a scale-invariant spectrum of perturbations. Indeed, if m_χ and κ are sufficiently small, in the long wavelength limit the power spectrum

---

[23]It is well-known that weight-zero fields are forbidden by unitary bounds [102], but these assume a stable conformally invariant vacuum. The conformal mechanism does not assume such a vacuum, only a time-dependent symmetry-breaking background. The conformal vacuum can be unstable or not exist.

is [13]

$$P_\chi = \frac{1}{2\pi^2} k^3 |\chi_k| \simeq \frac{\lambda}{2(2\pi)^2} \ , \tag{3.3.15}$$

which is scale invariant. The key insight of [13] is that weight-0 fields acquiring a scale-invariant spectrum is a feature generic to the symmetry breaking pattern $\mathfrak{so}(4,2) \to \mathfrak{so}(4,1)$.

3.3.1 Rubakov's U(1) model

As a special example of the above discussion, we consider the negative quartic U(1) model of Rubakov [12], which was further discussed in [95–99]. The action takes the form

$$S = \int d^4x \left(-\frac{1}{2} \partial \Phi^* \partial \Phi + \frac{\lambda}{4} (\Phi^* \Phi)^2 \right) \ , \tag{3.3.16}$$

where now Φ is a complex field and the action is invariant under a global U(1) symmetry, where $\phi \mapsto e^{i\theta} \phi$. If we write this action in terms of angular variables $\Phi = \phi e^{i\chi}$, we have

$$S = \int d^4x \left(-\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} \phi^2 (\partial \chi)^2 + \frac{\lambda}{4} \phi^4 \right) \ , \tag{3.3.17}$$

which is precisely the model considered above, with $m_\chi = \kappa = 0$.

3.4 Galilean genesis

Another example of the general conformal mechanism is the Galilean genesis scenario of [52]. This theory is based on the conformal galileons [6, 103–105], which can violate the null energy condition in a stable way.

In its simplest guise, Galilean Genesis [52] is achieved with a (wrong-sign) kinetic term plus a cubic conformal galileon term:

$$S = \int d^4x \left(f^2 e^{2\Pi} (\partial \Pi)^2 + \frac{f^3}{\Lambda^3} \Box \Pi (\partial \Pi)^2 + \frac{f^3}{2\Lambda^3} (\partial \Pi)^4 \right) \ , \tag{3.4.1}$$

where the scales f, Λ have dimensions of mass, and the scalar field Π is dimensionless. This action is also invariant under the conformal group SO(4,2), but in this case dilations and

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special conformal transformations act non-linearly to start with. The equation of motion following from this action admits a background solution of the form [52]

$$e^{\bar{\Pi}} = \frac{1}{H(-t)} \;, \qquad \text{where} \quad H^2 \equiv \frac{2\Lambda^3}{3f} \;.$$

(3.4.2)

This solution preserves the de Sitter subgroup of the conformal group. Perturbing about this solution $\Pi = \bar{\Pi} + \pi$, and introducing the effective de Sitter metric (3.3.10), the quadratic action takes exactly the form (3.3.11).

In this theory, additional fields must couple as

$$\mathcal{L} = M_\chi^2 e^{2\Pi}(\partial\chi)^2 \;,$$

(3.4.3)

in order to preserve the non-linearly realized conformal symmetries. When Π has the background solution (3.4.2), the field χ couples to an effective de Sitter metric, exactly in the same way as in Section 3.3.

It should be noted that perturbations about the solution (3.4.2) propagate exactly luminally, because of the $SO(4,1)$ symmetry of the solution; however, around slight deformations of this solution, perturbations generically propagate super-luminally [52]. This pathology may be avoided in two ways: the first is to (softly) explicitly break the conformal symmetry of the original action [106], alternatively, we can consider a different non-linear realization of the conformal group, which we will do presently [7, 9].

3.5 A nonlinear example: a brane probing AdS_5

A 'relativistic' extension of the conformal mechanism can be obtained by exploiting the isomorphism between the conformal group and the group of isometries of Anti-de Sitter space by considering the conformal Dirac–Born–Infeld (DBI) action [7, 13]

$$S_{\text{DBI}} = \int \mathrm{d}^4x\, \phi^4 \left(1 + \frac{\lambda}{4} - \gamma^{-1}\right) \;,$$

(3.5.1)

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where we have introduced the Lorentz factor, $\gamma \equiv 1/\sqrt{1 + (\partial\phi)^2/\phi^4}$. This action is the lowest order world-volume theory of a brane probing a bulk AdS_5 spacetime. The details of the construction can be found in [105, 107], which we summarize in Appendix A for convenience. Note that in the limit of small field gradients, $|(\partial\phi)^2| \ll \phi^4$, this action reduces to the negative quartic model (3.3.1).

This action is invariant under the 15 symmetries,

$$\delta_{P_\mu}\phi = -\partial_\mu\phi \,,$$

$$\delta_{J_{\mu\nu}}\phi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\phi \,,$$

$$\delta_D\phi = -\left(\Delta_\phi + x^\nu\partial_\nu\right)\phi \,,$$

$$\delta_{K_\mu}\phi = -2x_\mu\left(\Delta_\phi + x^\nu\partial_\nu\right)\phi + x^2\partial_\mu\phi + \frac{1}{\phi^2}\partial_\mu\phi \,, \tag{3.5.2}$$

where $\Delta_\phi = 1$. Although the special conformal transformations now act in a non-linear way, these transformations nonetheless satisfy the algebra (3.1.13).

Looking for purely time-dependent solutions, $\phi = \bar{\phi}(t)$, the equation of motion derived from (A.1.9) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\gamma}\,\dot{\bar{\phi}}\right) = \bar{\phi}^3\left(4 + \lambda - 2\bar{\gamma}^{-1} - 2\bar{\gamma}\right) \,, \tag{3.5.3}$$

where $\bar{\gamma} = 1/\sqrt{1 - \dot{\bar{\phi}}^2/\bar{\phi}^4} \geq 1$. We look for solutions of the form

$$\bar{\phi}(t) = \frac{\alpha}{(-t)} \,, \qquad -\infty < t < 0 \,, \tag{3.5.4}$$

where α can be assumed positive without loss of generality since the theory is $\mathbb{Z}_2$ symmetric. On the background (3.5.4), the relativistic factor γ is a constant,

$$\bar{\gamma}(\alpha) = \frac{1}{\sqrt{1 - 1/\alpha^2}} > 1 \,, \tag{3.5.5}$$

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and the equation of motion (3.5.3) becomes

$$\bar{\gamma}(\alpha) = 1 + \frac{\lambda}{4} \, . \tag{3.5.6}$$

In the "non-relativistic" limit, $\alpha \gg 1$, we recover the solution (3.3.5). More generally, since $\gamma \geq 1$ the existence of a non-trivial solution requires $\lambda > 0$.

The solution (3.5.4) breaks some of the symmetries; is annihilated by the 10 generators D, P_i, K_i, and J_{ij}, but not by the 5 generators P_0, K_0, or J_{0i}, which act as

$$\delta_{P_0}\bar{\phi} = \frac{\bar{\phi}}{t} \, ; \qquad \delta_{J_{0i}}\bar{\phi} = \frac{x^i \bar{\phi}}{t} \, ; \qquad \delta_{K_0}\bar{\phi} = -\left(x^2 + \frac{1}{\bar{\phi}^2}\right)\frac{\bar{\phi}}{t} \, . \tag{3.5.7}$$

Our background therefore spontaneously breaks the $\mathfrak{so}(4,2)$ symmetry of the DBI action down to its $\mathfrak{so}(4,1)$ subalgebra, realizing pseudo-conformal symmetry breaking in the same manner as the background (3.3.5). In a similar way, it is also possible to realize Galilean genesis in this DBI context [7].

3.5.1 Massless spectators

Coupling additional scalars to ϕ in this context has an elegant geometric interpretation; instead of considering a brane probing a pure AdS geometry, we consider the product space Ads$_5 \times S^1$ [7]. Performing the same steps as in the pure AdS case (summarized in Appendix A.1.1), we obtain the action

$$S_{\phi\theta} = \int \mathrm{d}^4 x \, \phi^4 \left(1 + \frac{\lambda}{4} - \sqrt{1 + \frac{(\partial\phi)^2}{\phi^4} + \frac{(\partial\theta)^2}{\phi^2} + \frac{(\partial\phi)^2(\partial\theta)^2 - (\partial\phi \cdot \partial\theta)^2}{\phi^6}}\right) \tag{3.5.8}$$

This action admits a solution where $\theta = \bar{\theta} = \text{const.}$, and ϕ satisfies (3.5.4). Perturbing about the background solution $\phi = \bar{\phi} + \varphi$ and $\theta = \bar{\theta} + \vartheta$, we find that the two-point function for ϑ is scale invariant [7]

$$\langle \vartheta_k \vartheta_{k'} \rangle' = \frac{\bar{\gamma}^2 - 1}{2} \frac{1}{k^3} \, . \tag{3.5.9}$$

3.6 Jordan-frame de Sitter description

Although the cosmological evolution in Einstein frame is non-inflationary—the scale factor is either slowly contracting or expanding—we have already mentioned that weight-0 spectator fields experience an effective de Sitter metric—see *e.g.,* (3.3.14). One may wonder whether the scenario is secretly inflation when cast in terms of this other metric. To shed light on this issue, consider for concreteness a single time-evolving scalar field ϕ of weight 1, as in the example of Section 3.3. As in (3.3.13), weight-0 fields are assumed to couple to an effective, "Jordan-frame" metric[24]

$$g_{\mu\nu}^{\text{eff}} = \phi^2 g_{\mu\nu} \,. \tag{3.6.1}$$

Let us see how the de Sitter background arises in Jordan frame. Upon the conformal transformation (3.6.1), the action (3.2.5) becomes

$$S = \int \mathrm{d}^4 x \sqrt{-g_{\text{eff}}} \left(\frac{M_{\text{Pl}}^2}{2\phi^2} R_{\text{eff}} + \frac{3M_{\text{Pl}}^2}{\phi^4} g_{\text{eff}}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{\phi^4} \mathcal{L}_{\text{CFT}} \left[\phi^{-2} g_{\mu\nu}^{\text{eff}} \right] \right) \,. \tag{3.6.2}$$

The Friedmann and scalar field equations that derive from (3.6.2) take the simple form

$$3H_{\text{eff}}^2 \simeq 6H_{\text{eff}} \frac{\dot{\phi}}{\phi^2} - 3\frac{\dot{\phi}^2}{\phi^4} \,,$$

$$\frac{\ddot{\phi}}{\phi^3} + 3H_{\text{eff}} \frac{\dot{\phi}}{\phi^2} - 3\frac{\dot{\phi}^2}{\phi^4} - \frac{R_{\text{eff}}}{6} = -\frac{\beta}{4\phi^2 M_{\text{Pl}}^2 t^4} \,, \tag{3.6.3}$$

where $H_{\text{eff}} = \phi^{-1} \mathrm{d} \ln a_{\text{eff}} / \mathrm{d}t$ is the Jordan-frame Hubble parameter, and dots are time derivatives with respect to the time coordinate t (we have not changed coordinates, only conformal frames). We have used (3.2.6) to substitute for the energy density and pressure of the CFT. The β term on the right hand side of the second equation of (3.6.3) is suppressed by $1/M_{\text{Pl}}$ and hence is negligible at sufficiently early times (specifically when $t \ll t_{\text{end}}$ from (3.2.9)). In this regime, the equations allow for a solution $\phi \sim 1/t$ and $H_{\text{eff}} =$ constant, consistent with the Einstein-frame analysis. Thus the effective geometry is indeed

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[24]The effective metric $g_{\mu\nu}^{\text{eff}}$ thus defined carries units, but this is inconsequential to our arguments; alternatively, one could write $g_{\mu\nu}^{\text{eff}} = (\phi^2/M^2)g_{\mu\nu}$ and carry the mass scale M throughout the calculation.

approximately de Sitter. But this is emphatically *not* inflation in any usual sense. The de Sitter expansion results from the non-minimal coupling of ϕ to gravity in this Jordan frame. In particular, the effective Planck scale $M_{\text{Pl}}^{\text{eff}} \sim 1/\phi$ varies by order unity in a Hubble time.

Chapter 4

Construction of an effective action

We have seen a number of concrete realizations of the symmetry breaking pattern $\mathfrak{so}(4,2) \to \mathfrak{so}(4,1)$ in Chapter 3. All of these realizations shared some properties: as we argued in Section 3.2, when coupled to gravity this symmetry breaking pattern naturally leads to a background that solves the canonical cosmological puzzles. Further, we found in each of the explicit examples that additional fields coupled in the theory acquired a scale-invariant spectrum of perturbations, which is required by observations. In [13] it was argued—at least at the quadratic level—that this is a generic feature of the symmetry breaking pattern.

In this Chapter, we systematically construct the low energy effective action corresponding to the symmetry breaking pattern

$$\mathfrak{so}(4,2) \longrightarrow \mathfrak{so}(4,1) \ , \tag{4.0.1}$$

and verify that it generically leads to a scale invariant spectrum for $\Delta = 0$ spectator fields. This effective lagrangian allows us to go beyond quadratic order as well, and make generic predictions about higher-order correlation functions, which we do in Chapter 6.

4.1 Nonlinear realizations and the coset construction

In order to construct the effective action, we employ machinery well-known to particle physicists, but which is slightly more obscure to cosmologists. We are interested in constructing the action for the Goldstone mode of the symmetry breaking pattern (4.0.1) and coupling

matter fields to it—to do this, we use the so-called coset construction.

Motivated by the successes of phenomenological Lagrangians in describing low energy pion scattering [108], Callan, Coleman, Wess and Zumino [14, 15], as well as Volkov [16], developed a powerful formalism for constructing the most general effective action for a given symmetry breaking pattern. This is the now well-known technique of non-linear realizations, or coset construction, which we review briefly here. More comprehensive reviews are given in [109, 110].

4.1.1 Spontaneously broken internal symmetries

We begin by reviewing the problem of constructing a Lagrangian for Goldstone fields corresponding to the breaking of an internal (*i.e.*, commuting with the Poincaré group) symmetry group G down to a subgroup H; that is, we seek the most general Lagrangian which is invariant under G transformations, where the H transformations act linearly on the fields and those not in H act non-linearly. As is well known [14, 15], there will be $\dim(G/H)$ Goldstone bosons, which parametrize the space of (left) cosets G/H.

However, to start with, we use fields $V(x)$ that take values in the group G; $V(x) \in G$, so that there are $\dim(G)$ fields. We then count as equivalent fields that differ by an element of the the subgroup, so $V(x) \sim V(x)h(x)$, where $h(x) \in H$. To implement this equivalence, we demand that the theory be gauge invariant under local $h(x)$ transformations $V(x) \mapsto V(x)h(x)$. There are $\dim(H)$ gauge transformations, so the number of physical Goldstone bosons will be $\dim(G) - \dim(H) = \dim(G/H)$, the expected number.

The global G transformations act on the left as $V(x) \mapsto gV(x)$, where $g \in G$. The theory should therefore be invariant under the symmetries

$$V(x) \longmapsto gV(x)h^{-1}(x), \tag{4.1.1}$$

where g is a global G transformation, and $h^{-1}(x)$ (written as an inverse for later convenience)

is a local H transformation.

A Lie group, G, possesses a distinguished left-invariant Lie algebra-valued 1-form, the so-called Maurer–Cartan form, given by $\omega = V^{-1}\mathrm{d}V$. Since this is Lie algebra-valued we may expand over a basis $\{V_I, Z_a\}$ where $\{V_I\}$, $I = 1, \ldots, \dim(H)$ is a basis of the Lie algebra $\mathfrak{h}$ of H, and $\{Z_a\}$, $a = 1, \ldots, \dim(G/H)$ is any completion to a basis of $\mathfrak{g}$. We expand the Maurer–Cartan form over this basis,

$$\omega = V^{-1}\mathrm{d}V = \omega_V^I V_I + \omega_Z^a Z_a \ , \tag{4.1.2}$$

where ω_V^I and ω_Z^a are the coefficients, which depend on the fields and their derivatives. The Maurer–Cartan form (4.1.2), and hence the coefficients in the expansion on the right hand side, are invariant under global G transformations.

Under the local $h(x)$ transformation, the pieces $\omega_V \equiv \omega_V^I V_I$ and $\omega_Z \equiv \omega_Z^I Z_I$ transform as

$$\omega_Z \longmapsto h\,\omega_Z h^{-1},$$
$$\omega_V \longmapsto h\,\omega_V h^{-1} + h\,\mathrm{d}h^{-1} \ . \tag{4.1.3}$$

We see that ω_Z transforms covariantly in the adjoint representation of the subgroup, and we use it as the basic ingredient to construct invariant Lagrangians [14–16, 109]. On the other had, ω_V transforms as a gauge connection.[25] If we have additional matter fields $\psi(x)$, which transform under some linear representation D of the local group H (and do not change under global G transformations),

$$\psi \longmapsto D\left(h\right)\psi \ , \tag{4.1.4}$$

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[25]This is a reflection of the well-known fact that the pullback of the Maurer–Cartan form defines a natural H-connection on G/H [111, 112].

we may construct a covariant derivative using ω_V via

$$\mathcal{D}\psi \equiv \mathrm{d}\psi + D(\omega_V)\psi, \qquad \mathcal{D}\psi \mapsto D(h)\,\mathcal{D}\psi\;. \tag{4.1.5}$$

Thus, the most general Lagrangian is any Lorentz and globally H-invariant scalar constructed from the components of ω_Z, ψ, and the covariant derivative,

$$\mathcal{L}\left(\omega_{Z\,\mu}^{I}, \psi, \mathcal{D}_\mu\right)\;. \tag{4.1.6}$$

To obtain a theory with global G symmetry, we fix the $h(x)$ gauge symmetry by imposing some canonical choice for $V(x)$, which we call $\tilde{V}(x)$. This canonical choice should smoothly pick out one representative element from each coset, so $\tilde{V}(x)$ contains $\dim(G/H)$ fields. In general, a global g transformation will not preserve this choice, so a compensating h transformation—depending on g and $\tilde{V}$—will have to be made at the same time to restore the gauge choice. The gauge fixed theory will then have the global symmetry

$$\tilde{V}(x) \longmapsto g\tilde{V}(x)h^{-1}(g, \tilde{V}(x)). \tag{4.1.7}$$

If we can choose the parametrization such that the transformation (4.1.7) is linear in the fields $\tilde{V}$ only when $g \in H$, then we will have realized the symmetry breaking pattern $G \to H$. When the commutation relations of the algebra are such that the commutator of a broken generator with a subgroup generator is again a subgroup generator $[V_I, Z] \sim Z$, (which is true if G is a compact group), one way to accomplish this is to choose the parametrization

$$\tilde{V}(x) = e^{\xi(x) \cdot Z}\;. \tag{4.1.8}$$

Here the real scalar fields $\xi^a(x)$ are the $\dim(G/H) = \dim G - \dim H$ different Goldstone fields associated with the symmetry breaking pattern. Under left action by some $g \in G$,

(4.1.7) gives the transformation law for the $\xi^a(x)$ as,

$$e^{\xi \cdot Z} \mapsto e^{\xi' \cdot Z} = g e^{\xi \cdot Z} h^{-1}(g, \xi) \ , \tag{4.1.9}$$

As can be seen using the Baker–Campbell–Hausdorff formula and the commutation condition $[V_I, Z] \sim Z$, the action on ξ is linear when $g \in H$.

4.1.2 Spontaneously broken space-time symmetries

In the preceding subsection we reviewed the case of spontaneously broken internal symmetries. However, the symmetry breaking pattern of interest, (4.0.1), corresponds to a breaking of *space-time*symmetries. Consequently, we must extend the coset procedure to account for subtleties involved in non-linear realizations of symmetries which do not commute with the Poincaré group. This was worked out comprehensively by Volkov [16] and is reviewed nicely in [109]. While the construction is generally similar to the internal symmetry case, the main subtlety is that now we must explicitly keep track of the generators of space-time symmetries in the coset construction.

Following [109], we assume that our full symmetry group G contains the unbroken generators of space-time translations P_α, unbroken Lorentz rotations $J_{\alpha\beta}$, an unbroken symmetry subgroup H generated by V_I (which all together form a subgroup), and finally the broken generators denoted by Z_a. The broken generators may in general be a mix of internal and space-time symmetry generators. As before, we want to parameterize the coset G/H, but the parameterization now takes the form [16, 109, 113]

$$\tilde{V} = e^{x \cdot P} e^{\xi(x) \cdot Z} \ . \tag{4.1.10}$$

Note that we treat the unbroken translation generators on the same footing as the broken generators, with the coefficients simply the space-time coordinates.[26] As in the case of the

---

[26]This is little more than bookkeeping, as the coordinates formally transform non-linearly under a translation $x^\mu \mapsto x^\mu + \epsilon^\mu$. One intuitive way to understand this is to think of Minkowski space as the coset Poincaré/Lorentz, as is pointed out in [113].

internal symmetries, under left action by some $g \in G$, (4.1.10) transforms non-linearly

$$e^{x \cdot P} e^{\xi(x) \cdot Z} \longmapsto e^{x' \cdot P} e^{\xi'(x') \cdot Z} = g \, e^{x \cdot P} e^{\xi(x) \cdot Z} h^{-1}(g, \xi(x)) \, , \qquad (4.1.11)$$

where $h(g, \xi(x))$ belongs to the unbroken group spanned by V_I and $J_{\mu\nu}$, but has dependence on ξ.

As in the internal symmetry case, the object in which we are interested is the Maurer–Cartan form

$$\omega = \tilde{V}^{-1} \mathrm{d} \tilde{V} = \omega_P^\alpha P_\alpha + \omega_Z^a Z_a + \omega_V^I V_I + \frac{1}{2} \omega_J^{\alpha\beta} J_{\alpha\beta} \, , \qquad (4.1.12)$$

where we have again expanded in the basis of the Lie algebra $\mathfrak{g}$. We may act with the transformation (4.1.11) to determine that the components, $\omega_P \equiv \omega_P^\alpha P_\alpha$, $\omega_Z \equiv \omega_Z^a Z_a$, $\omega_V \equiv \omega_V^I V_I$, $\omega_J \equiv \frac{1}{2} \omega_J^{\alpha\beta} J_{\alpha\beta}$ of the Maurer–Cartan 1-form transform as [109]

$$\omega_P \longmapsto h \, \omega_P \, h^{-1},$$

$$\omega_Z \longmapsto h \, \omega_Z \, h^{-1},$$

$$\omega_V + \omega_J \longmapsto h \, (\omega_V + \omega_J) \, h^{-1} + h \, \mathrm{d} h^{-1} \, . \qquad (4.1.13)$$

The covariant transformation rule for ω_P and ω_Z tells us that these are the ingredients to use in constructing invariant Lagrangians [16, 109, 113]. The form ω_P, expanded in components is

$$\omega_P = \mathrm{d} x^\nu (\omega_P)_\nu^\alpha P_\alpha, \qquad (4.1.14)$$

Here the components $(\omega_P)_\nu^\alpha$ should be thought of as an invariant vielbein, with α a Lorentz index, from which we can construct an invariant metric

$$g_{\mu\nu} = (\omega_P)_\mu^\alpha (\omega_P)_\nu^\beta \eta_{\alpha\beta}, \qquad (4.1.15)$$

and an invariant measure

$$-\frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\omega_P^\alpha \wedge \omega_P^\beta \wedge \omega_P^\gamma \wedge \omega_P^\delta = \mathrm{d}^4 x \sqrt{-g} \ . \tag{4.1.16}$$

The form ω_Z, expanded in components

$$\omega_Z = \mathrm{d}x^\mu (\omega_Z)_\mu{}^a Z_a, \tag{4.1.17}$$

yields the basic ingredient $\mathcal{D}_\alpha \xi^a$, the covariant derivative of the Goldstones, through

$$(\omega_Z)_\mu{}^a = (\omega_P)_\mu{}^\alpha \mathcal{D}_\alpha \xi^a. \tag{4.1.18}$$

We can construct covariant derivatives $\mathcal{D}$ for matter fields ψ, transforming as some combined Lorentz and H representation, which we call D, by using $\omega_V + \omega_J$ as a connection,

$$\omega_P^\alpha \bar{\mathcal{D}}_\alpha \psi = \mathrm{d}\psi + D(\omega_V)\psi + D(\omega_J)\psi \ . \tag{4.1.19}$$

This can also be used to take higher covariant derivatives of the Goldstones. From these pieces, $e_\mu{}^\alpha$, $\mathcal{D}_\alpha \xi^a$, ψ and $\bar{\mathcal{D}}_\alpha$, we can build the most general invariant Lagrangian by combing the pieces in a Lorentz and H invariant way, and integrating against the invariant measure (4.1.16).

4.1.3 Inverse Higgs constraint

There is another subtlety that arises in extending the coset construction to the case of space-time symmetries—there can be non-trivial relations between different Goldstone modes leading to fewer degrees of freedom than naïve counting would suggest. This is the well-known statement that the counting of massless degrees of freedom in Goldstone's theorem fails in the case of broken space-time symmetries [16, 113–119]; that is, the number of Goldstone modes will not in general be equal to $\dim(G/H)$. This phenomenon is sometimes

referred to as the *inverse Higgs effect* [114].

Accounting for this is simple—if the commutator of an unbroken translation generator with a broken symmetry generator, say Z_1, contains a component along some linearly independent broken generator, say Z_2,

$$[P, Z_1] \sim Z_2 + \dots \, , \qquad (4.1.20)$$

(where the dots represent a component along the broken directions), it is possible to eliminate the Goldstone field corresponding to the generator Z_1 [113, 114, 117]. The relation between the Goldstone modes is obtained by setting the coefficient of Z_2 in the Maurer–Cartan form to zero.

This is a covariant constraint; *i.e.*, it is invariant under G because the Maurer–Cartan form itself is invariant. However, there is no reason that we are forced to impose it; in almost all cases, it is equivalent to integrating out the redundant Goldstone field via its equation of motion [117]. There do exists cases where this is not true, though; the viewpoint that we will take is that since we are free to impose the inverse Higgs constraint and obtain a lagrangian with the desired symmetry properties, we will.

4.2 Breaking conformal to de Sitter

We now turn to the case of principal interest—spontaneously breaking the conformal algebra to its de Sitter subalgebra

$$\mathfrak{so}(4,2) \longrightarrow \mathfrak{so}(4,1) \, . \qquad (4.2.1)$$

To our knowledge, the coset construction for this symmetry breaking pattern has not appeared previously in the literature. (The case of breaking conformal to the *Anti*-de Sitter algebra $\mathfrak{so}(3,2)$ was considered in [120].) To this end, it is convenient to parameterize the conformal algebra by the generators $J_{\mu\nu}$, K_μ, D and

$$\hat{P}_\mu \equiv P_\mu + \frac{1}{4} H^2 K_\mu \, , \qquad (4.2.2)$$

where the dimensionful parameter H will turn out to be the Hubble constant for the effective de Sitter metric. In this basis, the algebra takes the form

$$
\begin{aligned}
&\left[\hat{P}_\mu, \hat{P}_\nu\right] = H^2 J_{\mu\nu}, && \left[D, \hat{P}_\mu\right] = -\hat{P}_\mu + \tfrac{1}{2} H^2 K_\mu, \\
&[D, K_\mu] = K_\mu, && \left[\hat{P}_\mu, K_\nu\right] = 2\eta_{\mu\nu} D + 2 J_{\mu\nu}, \\
&[J_{\mu\nu}, K_\rho] = \eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu, && \left[J_{\mu\nu}, \hat{P}_\rho\right] = \eta_{\mu\rho} \hat{P}_\nu - \eta_{\nu\rho} \hat{P}_\mu, \\
&[J_{\mu\nu}, J_{\sigma\rho}] = \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma}.
\end{aligned}
\tag{4.2.3}
$$

This parameterization of the conformal algebra appears also in [116] in the context of breaking the conformal algebra to Poincaré. The advantage of working with $\hat{P}_\mu$ rather than the P_μ is that the set $\{\hat{P}_\mu, J_{\nu\rho}\}$ generates an $\mathfrak{so}(4,1)$ subalgebra.[27] This can be made manifest by adding a fifth index and writing $J_{5\mu} \equiv \hat{P}_\mu$, in terms of which the commutation relations of $\{\hat{P}_\mu, J_{\nu\rho}\}$ take the $\mathfrak{so}(4,1)$ form,

$$
[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} ,
\tag{4.2.4}
$$

where $\eta_{ab} = \mathrm{diag}\,(-1, 1, 1, 1, 1)$ is the metric of 4+1 dimensional Minkowski space.

4.2.1 Constructing the Effective Action

Since the broken symmetries correspond to D and K_μ in this basis, we parameterize the group coset by

$$
\tilde{V} = e^{y \cdot \hat{P}} e^{\pi D} e^{\xi \cdot K} ,
\tag{4.2.5}
$$

where the inner product is taken with respect to the vielbein metric η_{mn}. As we will see shortly, the space-time coordinates y^μ corresponding to $\hat{P}_\mu$ parametrize a particular coordinate system on de Sitter space. At the end of the day, however, it will be possible to express all of our results in a coordinate-independent way.

---

[27]Although this is not our main focus, one might also be interested in breaking the conformal algebra to its Anti-de Sitter subalgebra $\mathfrak{so}(3,2)$. This breaking pattern follows straightfowardly by defining $\bar{P}_\mu \equiv P_\mu - \tfrac{1}{4} H^2 K_\mu$. Then, the set of generators $\{\bar{P}_\mu, J_{\nu\rho}\}$ generates an $\mathfrak{so}(3,2)$ subalgebra of $\mathfrak{so}(4,2)$. This symmetry breaking pattern was considered in [120], using a different parameterization of the algebra. In order to obtain actions equivalent to theirs (but algebraically simpler), one can analytically continue $H^2 \to -H^2$ in the following sections.

We can pull back the Maurer–Cartan form on the conformal group by this local section and expand it in components,

$$\omega_{\hat{P}}^m = e^\pi \bar{e}_\mu^m \mathrm{d}y^\mu \ ,$$

$$\omega_D = \mathrm{d}\pi + 2e^\pi \xi_m \bar{e}_\mu^m \mathrm{d}y^\mu \ ,$$

$$\omega_K^m = \mathrm{d}\xi^m - \omega_{\mathrm{spin}}^{mn} \xi_n + 2e^\pi \xi_n \xi^m \bar{e}_\mu^n \mathrm{d}y^\mu - e^\pi \xi^2 \bar{e}_\mu^m \mathrm{d}y^\mu - \frac{H^2}{2} \sinh \pi \bar{e}_\mu^m \mathrm{d}y^\mu + \xi^m \mathrm{d}\pi \ ,$$

$$\frac{1}{2}\omega_J^{mn} = e^\pi \mathrm{d}y^\mu \left(\xi^n \bar{e}_\mu^m - \xi^m \bar{e}_\mu^n \right) + \omega_{\mathrm{spin}}^{mn} \ . \tag{4.2.6}$$

Here, the vielbein is given by $e_\mu^m = e^\pi \bar{e}_\mu^m$ where,

$$\bar{e}_\mu^m(y) = \left(\delta_\mu^m - \frac{y_\mu y^m}{y^2} \right) \frac{\sin \sqrt{H^2 y^2}}{\sqrt{H^2 y^2}} + \frac{y_\mu y^m}{y^2} \ , \tag{4.2.7}$$

and the spin connection on de Sitter is given by

$$\omega_{\mathrm{spin}}^{mn}(y) = \mathrm{d}y^\mu \omega_\mu^{mn} = \left(\cos \sqrt{H^2 y^2} - 1 \right) \left[\frac{y^n \mathrm{d}y^m - y^m \mathrm{d}y^n}{y^2} \right] \ . \tag{4.2.8}$$

Although this is by no means obvious, these represent a vielbein and spin connection for de Sitter space.[28] Keeping this in mind, we leave the coordinates arbitrary and consider a general de Sitter metric

$$\bar{g}_{\mu\nu}^{\mathrm{eff}} = \bar{e}_\mu^m \bar{e}_\nu^n \eta_{mn} \ , \tag{4.2.12}$$

allowing us to write everything in terms of space-time indices.

---

[28]To see this explicitly, consider the coordinate transformation [120]

$$y^\mu = x^\mu \sqrt{\frac{4}{H^2 x^2}} \arctan \sqrt{\frac{H^2 x^2}{4}} \ . \tag{4.2.9}$$

This brings the vielbein into diagonal form

$$\bar{e}_\mu^m(x) = \left(\frac{1}{1 + \frac{1}{4}H^2 x^2} \right) \delta_\mu^m \ , \tag{4.2.10}$$

corresponding to the better-known coordinitization of de Sitter with metric

$$\bar{g}_{\mu\nu}^{\mathrm{eff}} = \left(\frac{1}{1 + \frac{1}{4}H^2 x^2} \right)^2 \eta_{\mu\nu} \ . \tag{4.2.11}$$

This makes it clear that the y^μ coordinates are in fact coordinates on de Sitter space, as claimed earlier.

There is an inverse Higgs constraint to be implemented which will give a relation between Goldstone fields. The commutator

$$[\hat{P}_\mu, K_\nu] = 2\eta_{\mu\nu}D + 2J_{\mu\nu} \tag{4.2.13}$$

implies that the Goldstone fields ξ^μ associated to the K_μ's can be removed in favor of π. This is implemented by setting $\omega_D = 0$, which gives the relation[29]

$$\xi_\mu = -\frac{1}{2}e^{-\pi}\partial_\mu\pi . \tag{4.2.14}$$

The expression (4.2.6) for the Maurer–Cartan form thus simplifies,

$$\omega_{\hat{P}}^\mu = e^\pi \mathrm{d}y^\mu ,$$

$$\omega_D = \mathrm{d}\pi + 2e^\pi \xi_\mu \mathrm{d}y^\mu ,$$

$$\omega_K^\mu = \mathrm{d}y^\nu \bar{\nabla}_\nu \xi^\mu - e^\pi \xi^2 \mathrm{d}y^\mu - \frac{H^2}{2}\sinh\pi \mathrm{d}y^\mu ,$$

$$\frac{1}{2}\omega_\mu^{ab}{}_J = e^\pi \left(\xi^b e_\mu^a - \xi^a e_\mu^b \right) + \omega_\mu^{ab}{}_{\text{spin}} , \tag{4.2.15}$$

where the contraction $\xi^2 = \bar{g}_{\text{eff}}^{\mu\nu}\xi_\mu\xi_\nu$ is everywhere understood as taken with respect to the de Sitter metric $\bar{g}_{\mu\nu}^{\text{eff}}$, and $\bar{\nabla}_\nu$ is the covariant derivative associated to this metric. As before, we define the covariant derivative of the Goldstone field ξ^μ by

$$\omega_K^\mu = \omega_{\hat{P}}^\nu \mathcal{D}_\nu \xi^\mu , \tag{4.2.16}$$

which implies

$$\mathcal{D}_\nu \xi_\mu = e^\pi \left[\bar{\nabla}_\nu \xi_\mu - \left(e^\pi \xi^2 + \frac{H^2}{2}\sinh\pi \right) \bar{g}_{\mu\nu} \right] . \tag{4.2.17}$$

---

[29]Although the form of the relation is the same as in the case where the conformal group is broken to Poincaré, here the space-time indices should be understood as being raised and lowered with a de Sitter metric instead of the flat metric.

The covariant derivative can be written explicitly in terms of π using (4.2.14) as

$$\mathcal{D}_\nu \xi_\mu = \frac{1}{2}\partial_\nu \pi \partial_\mu \pi - \frac{1}{2}\bar{\nabla}_\nu \bar{\nabla}_\mu \pi - \frac{1}{4}\bar{g}^{\alpha\beta}\partial_\alpha \pi \partial_\beta \pi \bar{g}_{\mu\nu} - \frac{H^2}{4}e^{2\pi}\bar{g}_{\mu\nu} + \frac{H^2}{4}\bar{g}_{\mu\nu} \ . \qquad (4.2.18)$$

The other key ingredient for writing down invariant actions is the metric. Noting that the appropriate vielbein is $e_\mu^m = e^\pi \bar{e}_\mu^m$, we see that the appropriate metric with which to contract indices is

$$g_{\mu\nu}^{\mathrm{eff}} = e^{2\pi}\bar{g}_{\mu\nu}^{\mathrm{eff}} \ . \qquad (4.2.19)$$

Finally, the invariant volume element is given by

$$\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_{\hat{P}}^\sigma = \mathrm{d}^4 y\sqrt{-\bar{g}_{\mathrm{eff}}}\ e^{4\pi} = \mathrm{d}^4 y\sqrt{-g_{\mathrm{eff}}} \ . \qquad (4.2.20)$$

Although expressed in terms of y^μ coordinates, the answer is manifestly diffeomorphism invariant and hence holds in any coordinate system.

The Goldstone action is then formed by building scalars from these ingredients. (As before we are allowed to use the matter covariant derivative, $\bar{\mathcal{D}}_\mu = \nabla_\mu$—the covariant derivative associated to $g_{\mu\nu}^{\mathrm{eff}}$—but for the lowest order actions we will not need it.) The simplest action is just the conformally invariant volume,

$$S_0 = M_{\mathrm{v}}^4 \int \mathrm{d}^4 y\sqrt{-\bar{g}_{\mathrm{eff}}}e^{4\pi} \ . \qquad (4.2.21)$$

Meanwhile, the kinetic term for the Goldstone field arises from[30]

$$S_1 = M_\pi^2 \int \mathrm{d}^4 y\sqrt{-g_{\mathrm{eff}}}\mathcal{D}_\mu \xi^\mu = M_\pi^2 \int \mathrm{d}^4 y\sqrt{-\bar{g}_{\mathrm{eff}}}\left[\frac{1}{2}e^{2\pi}(\partial\pi)^2 + H^2 e^{2\pi} - H^2 e^{4\pi}\right] \ , \qquad (4.2.22)$$

where all contractions are performed with the de Sitter metric $\bar{g}_{\mu\nu}^{\mathrm{eff}}$ and where we have

---

[30]Incidentally, S_1 can be realized as a wedge product as follows

$$S_1 = -\frac{M_\pi^2}{3!}\int \epsilon_{\mu\nu\rho\sigma}\omega_K^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_{\hat{P}}^\sigma \ .$$

integrated by parts. Note that this expression has a tadpole contribution which may be canceled by adding a suitable multiple of the invariant measure, thereby setting the relative coefficient between S_1 and S_0.

At the four-derivative level, we have[31]

$$
\begin{aligned}
S_2 &= \int d^4 y \sqrt{-g_{\text{eff}}} \, (\mathcal{D}_\mu \xi^\mu)^2 \\
&= \frac{1}{4} \int d^4 y \sqrt{-\bar{g}_{\text{eff}}} \left[(\bar{\Box}\pi)^2 + 2\bar{\Box}\pi(\partial\pi)^2 + (\partial\pi)^4 - 4H^2(\partial\pi)^2 \right] - \frac{8H^2}{M_0^2} S_1 - \frac{4H^4}{M_{\text{v}}^4} S_0 ,
\end{aligned}
\tag{4.2.23}
$$

where we have dropped a total derivative and a constant (π-independent) term. The last two terms can of course be absorbed into the coefficients of the lower-order action S_0 and S_1.

There is of course another four-derivative term, obtained from $(\mathcal{D}_\mu \xi_\nu)^2$, but the corresponding action turns out to be a linear combination of S_2, S_1 and S_0:

$$
S_2' = \int d^4 y \sqrt{-g_{\text{eff}}} \, (\mathcal{D}_\mu \xi_\nu)^2 = - \int \bar{g}_{\mu\nu}^{\text{eff}} \omega_K^\mu \wedge \star_4 \omega_K^\nu = S_2 + \frac{6H^2}{M_0^2} S_1 + \frac{3H^4}{M_{\text{v}}^4} S_0 . \tag{4.2.24}
$$

However, this degeneracy is an accident of $d = 4$ dimensions. In fact one can construct an orthogonal linear combination as a Wess–Zumino term form for the conformal group; the procedure, detailed in Chapter 5, leads to[32]

$$
S_{\text{wz}} = \int d^4 y \sqrt{-\bar{g}_{\text{eff}}} \left[(\partial\pi)^4 + 2\bar{\Box}\pi(\partial\pi)^2 + 6H^2(\partial\pi)^2 \right] . \tag{4.2.26}
$$

---

[31] As before, this term may also be constructed directly as a wedge product of Maurer–Cartan coefficients:

$$
S_2 = - \int \left(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} \omega_K^\mu \wedge \omega_K^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_{\hat{P}}^\sigma + \bar{g}_{\mu\nu}^{\text{eff}} \omega_K^\mu \wedge \star_4 \omega_K^\nu \right) .
$$

where $\star_4$ is the Hodge dual with respect to the conformal metric, $\star_4 \omega_K^\alpha = \frac{1}{3!}\epsilon_{\mu_0\mu_1\mu_2\mu_3} \mathcal{D}^{\mu_0} \xi^\alpha \omega_{\hat{P}}^{\mu_1} \wedge \omega_{\hat{P}}^{\mu_2} \wedge \omega_{\hat{P}}^{\mu_3}$.

[32] For now, we just note that WZ term can be constructed in five dimensions as

$$
S_{\text{wz}} = \int_M \epsilon_{\mu\nu\rho\sigma} \omega_D \wedge \omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma , \tag{4.2.25}
$$

and then pulled back to the physical four-dimensional space-time using Stokes' theorem [6].

The construction of the effective action can be extended in this way to arbitrary derivative order. To summarize, the most general Goldstone lagrangian consistent with the symmetry breaking pattern (4.2.1), up to fourth order in derivatives, is[33]

$$
\begin{aligned}
S_\pi = \int \mathrm{d}^4 x \sqrt{-\bar{g}_{\text{eff}}} \Bigg[& M_\pi^2 \left(-\frac{1}{2} e^{2\pi} (\partial\pi)^2 - H^2 e^{2\pi} + \frac{H^2}{2} e^{4\pi} \right) \\
& + M_1 \left((\bar{\Box}\pi)^2 + 2\bar{\Box}\pi(\partial\pi)^2 + (\partial\pi)^4 - 4H^2(\partial\pi)^2 \right) \\
& + M_2 \left((\partial\pi)^4 + 2\bar{\Box}\pi(\partial\pi)^2 + 6H^2(\partial\pi)^2 \right) + \dots \Bigg] ,
\end{aligned}
\tag{4.2.27}
$$

where the relative coefficient between the $e^{2\pi}$ and $e^{4\pi}$ terms has been fixed to cancel the π tadpole. For later use, we write the most general action for π with two derivatives and up to third order in fields

$$
S_\pi = M_\pi^2 \int \mathrm{d}^4 x \sqrt{-\bar{g}_{\text{eff}}} \left(-\frac{1}{2}(\partial\pi)^2 + 2H^2\pi^2 - \pi(\partial\pi)^2 + 4H^2\pi^3 \right) .
\tag{4.2.28}
$$

4.2.2 Transformation of π

Up to this point, we have not specified how π transforms under the non-linearly realized conformal symmetries, though it is implicit in the construction. A straightforward way to determine this transformation rule explicitly is to act on the left of (4.2.5) by a group element, $\bar{g} \in G$, and determine how π transforms. Note that this will be tied to a particular coordinitization of de Sitter space.

There is, in fact, a simpler method to derive the transformation rule for π in a coordinate-independent way. This method is closely tied to a technique we will use in Sec. 4.4 as an alternative to the coset construction. Consider the metric $g_{\mu\nu}^{\text{eff}} = e^{2\pi} \bar{g}_{\mu\nu}^{\text{eff}}$, where $\bar{g}_{\mu\nu}^{\text{eff}}$ is the Sitter metric in an arbitrary coordinate system. Clearly $g_{\mu\nu}^{\text{eff}}$ non-linearly realizes the

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[33]Note that the M_1 and M_2 higher-derivative terms include $H^2(\partial\pi)^2$ corrections to the kinetic term, which were not included in the two-derivative analysis of [13]. However, in order for the effective field theory paradigm to be useful, we are assuming that there is a hierarchy of scales such that the higher-order terms are sub-dominant, *i.e.*, $M_{1,2} \ll H^2$. The benefit of this approach is that it allows us to systematically include the effects of such corrections, but for the time being we ignore them.

conformal group through the dilaton field π. We can extract the transformation properties for the scalar mode π from the general transformation properties of the metric under an infinitesimal diffeomorphism, under which the metric changes by the Lie derivative

$$\delta g_{\mu\nu} = -\pounds_\xi\, g_{\mu\nu} = -g_{\rho\nu}\nabla_\mu\xi^\rho - g_{\mu\rho}\nabla_\nu\xi^\rho \ . \tag{4.2.29}$$

We assume that the background metric $\bar{g}^{\text{eff}}_{\mu\nu}$ remains fixed (this restricts us to isometries of de Sitter plus conformal transformations), so we have

$$2\delta\pi g_{\mu\nu} = -g_{\rho\nu}\nabla_\mu\xi^\rho - g_{\mu\rho}\nabla_\nu\xi^\rho \ , \tag{4.2.30}$$

tracing over both sides gives $\delta\pi = -\frac{1}{4}\nabla_\rho\xi^\rho$. This is the divergence of a vector, so we may write

$$\delta\pi = -\frac{1}{4\sqrt{-g}}\partial_\rho\left(\sqrt{-g}\xi^\rho\right) = -\xi^\rho\partial_\rho\pi - \frac{1}{4}\bar{\nabla}_\rho\xi^\rho \ . \tag{4.2.31}$$

So we have the transformation rule for π,

$$\delta\pi = -\xi^\rho\partial_\rho\pi - \frac{1}{4}\bar{\nabla}_\rho\xi^\rho \ . \tag{4.2.32}$$

From this transformation rule, it is clear that that π will transform linearly under isometries of the dS metric ($\bar{\nabla}_\rho\xi^\rho = 0$) and will transform in a nonlinear fashion under broken transformations. To make this explicit, we must make a choice of de Sitter slicing. Choosing the planar inflationary slicing:

$$\bar{g}^{\text{eff}}_{\mu\nu} = \frac{1}{H^2 t^2}\eta_{\mu\nu} \ , \tag{4.2.33}$$

we find

$$\delta\pi = -\xi^\rho\partial_\rho\pi - \frac{1}{4}\partial_\mu\xi^\mu + \frac{1}{t}\xi^0 \ . \tag{4.2.34}$$

Then, plugging in the Killing vectors (3.1.5)–(3.1.8) we obtain the transformation rules

$$\delta_{P_\mu}\pi = -\partial_\mu\pi + \delta_\mu^0\frac{1}{t},$$

$$\delta_{J_{\mu\nu}}\pi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\pi + \left(\delta_\mu^0\frac{x_\nu}{t} - \delta_\nu^0\frac{x_\mu}{t}\right),$$

$$\delta_D\pi = -x^\mu\partial_\mu\pi,$$

$$\delta_{K_\mu}\pi = -(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\pi - \delta_\mu^0\frac{x^2}{t} . \tag{4.2.35}$$

Consistent with the discussion of Chapter 3, the symmetries associated to P_0, K_0 and J_{0i} are non-linearly realized, while the others are linearly realized. Furthermore, π transforms as a weight 0 field under dilations.

4.2.3 Matter Fields

In the pseudo-conformal scenario, the progenitor of density perturbations is not the Goldstone field π associated with the time-evolving field, but rather a weight-0 spectator field, χ. As a result, we need to couple matter fields to the Goldstone in a way that non-linearly realizes the conformal group. Of course, the coset machinery is also capable of this task.

Recall that the covariant derivative of an arbitrary matter field, ψ, is given by

$$\omega_{\hat{P}}^\mu\bar{\mathcal{D}}_\mu\psi = \mathrm{d}\chi + \omega_V^i D(V_i)\psi + \frac{1}{2}\omega_J^{\mu\nu}D(J_{\mu\nu})\psi . \tag{4.2.36}$$

For this symmetry-breaking pattern, there are no elements, ω_V, of the Maurer–Cartan form that play the role of a gauge connection, so we only need to concern ourselves with the spin connection piece ω_J. Note under the Weyl transformation

$$\tilde{e}_\mu^m = e^\pi\bar{e}_\mu^m , \tag{4.2.37}$$

the spin connection transforms as

$$\widetilde{\omega}_\mu^{mn} = \omega_\mu^{mn} + e_\mu^n \partial^m \pi - e_\mu^m \partial^n \pi \ . \tag{4.2.38}$$

Thus the spin connection (4.2.15) is in fact the spin connection associated to the metric $g_{\mu\nu}^{\text{eff}} = e^{2\pi}\bar{g}_{\mu\nu}^{\text{eff}}$, where $\bar{g}_{\mu\nu}^{\text{eff}}$ is a metric on de Sitter space. In other words, the covariant derivative for ψ is just the geometric covariant derivative associated to this metric

$$\bar{\mathcal{D}}_\mu \psi = \nabla_\mu \psi \ . \tag{4.2.39}$$

An action for ψ can be obtained by contracting indices with the conformal metric, $g_{\mu\nu}^{\text{eff}}$, which will introduce a natural coupling between ψ and π. There is the additional freedom to promote any of the mass scales in the Goldstone lagrangian (4.2.27) to a function of ψ, being careful about integration by parts. (An important exception is the Wess–Zumino term (4.2.26). This term shifts by a total derivative under conformal transformations, hence its coefficient must remain independent of ψ.)

With these caveats in mind, we are free to write down any Lorentz-invariant action using χ, the effective metric $g_{\mu\nu}^{\text{eff}}$ and its covariant derivative ∇_μ. At the end of the day, the result can be expressed in terms of the effective de Sitter metric $\bar{g}_{\mu\nu}^{\text{eff}}$. Here we write the two-derivative effective lagrangian for χ (written in terms of the effective de Sitter metric $\bar{g}_{\mu\nu}^{\text{eff}}$):

$$\begin{aligned}
\mathcal{L}_\psi &\sim \sqrt{-g_{\text{eff}}}\left(-\frac{1}{2}(\partial\psi)^2 - V(\psi) + f(\psi)\mathcal{L}_1^\pi\right) \\
&= \sqrt{-\bar{g}_{\text{eff}}}\left(-\frac{1}{2}e^{2\pi}(\partial\psi)^2 - e^{4\pi}V(\psi) + e^{4\pi}f(\psi)\mathcal{L}_1^\pi\right) \ ,
\end{aligned} \tag{4.2.40}$$

where here

$$\mathcal{L}_1^\pi = \frac{1}{2}e^{2\pi}(\partial\pi)^2 + \frac{1}{2}e^{2\pi}\Box\pi - H^2 e^{2\pi} + \frac{H^2}{2}e^{4\pi} \ . \tag{4.2.41}$$

Since our aim will only be to verify symmetry statements in a variety of examples, we consider the case where $V(\psi) = \frac{m_\psi^2}{2}\psi^2 + \lambda\psi^3$ and $f(\psi) = 0$. The lagrangian then takes the

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form

$$\mathcal{L}_\psi \sim \sqrt{-\bar{g}_{\text{eff}}} \left(-\frac{1}{2} e^{2\pi} (\partial\psi)^2 - \frac{m_\psi^2}{2} e^{4\pi} \psi^2 - \lambda e^{4\pi} \psi^3 \right) . \tag{4.2.42}$$

Expanding about $\psi = \pi = 0$ to quartic order yields the action

$$S_\psi = M_\psi^2 \int \mathrm{d}^4 x \sqrt{-\bar{g}_{\text{eff}}} \left(-\frac{1}{2} (\partial\psi)^2 - \frac{m_\psi^2}{2} \psi^2 - 2m_\psi^2 \pi\psi^2 - \pi(\partial\psi)^2 - \lambda\psi^3 - 4\lambda\pi\psi^3 \right) . \tag{4.2.43}$$

4.3 Analysis of the quadratic action

In this section, we consider the two point function for a weight zero field, χ, coupled to the Goldstone π. The most general quadratic action for the combined π, χ system is [5]

$$S_{\pi\chi} = \int \mathrm{d}^4 x \sqrt{-\bar{g}_{\text{eff}}} \left[M_\pi^2 \left(-\frac{1}{2} (\partial\pi)^2 + 2H^2\pi^2 \right) - \frac{M_\chi^2}{2} (\partial\chi)^2 - \frac{m_\chi^2 + \bar{M}_\pi^2 H^2}{2} \chi^2 \right] , \tag{4.3.1}$$

4.3.1 Two-point function for the Goldstone

First we consider the two-point function for the Goldstone mode π. The quadratic action for π is

$$S_\pi = M_\pi^2 \int \mathrm{d}^4 x \sqrt{-\bar{g}_{\text{eff}}} \left[-\frac{1}{2} (\partial\pi)^2 + 2H^2\pi^2 \right] . \tag{4.3.2}$$

To proceed, we must choose a coordinatization of de Sitter. A convenient choice is the flat slicing

$$\mathrm{d}s^2 = \frac{1}{H^2 t^2} \left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 \right) . \tag{4.3.3}$$

Here we have written the conformal time coordinate as t because it is really the physical Minkowski space-time coordinate, it merely acts as a conformal time coordinate on the effective de Sitter space that spectator fields feel. In terms of this metric, the action takes the form

$$S_\pi = M_\pi^2 \int \mathrm{d}^4 x \left[\frac{1}{2H^2 t^2} \dot{\pi}^2 - \frac{1}{2H^2 t^2} (\vec{\nabla}\pi)^2 + \frac{2}{H^2 t^4} \pi^2 \right] . \tag{4.3.4}$$

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The equation of motion for the π field is given in Fourier space by[34]

$$\ddot{\pi}_k + k^2\pi_k - \frac{2}{t}\dot{\pi}_k - \frac{4}{t^2}\pi_k = 0 \tag{4.3.5}$$

After performing a field redefinition to the canonically-normalized variable, $v = \frac{M_\pi}{(-Ht)}\pi$, the mode function equation becomes

$$\ddot{v}_k + \left(k^2 - \frac{6}{t^2}\right)v_k = 0 \ . \tag{4.3.6}$$

Assuming adiabatic vacuum initial conditions, it is well-known that this equation admits a solution in terms of a Hankel function of the first kind

$$v_k(t) = \sqrt{\frac{\pi(-t)}{4}}H_{5/2}^{(1)}(-kt) \ . \tag{4.3.7}$$

Inverting our field redefinition to get an expression for π we find

$$\pi_k(t) = -i\frac{H(-t)^{3/2}}{M_\pi}\sqrt{\frac{\pi}{4}}H_{5/2}^{(1)}(-kt) = \frac{-3H}{\sqrt{2k^5}(-t)M_\pi}\left(1 + ikt - \frac{k^2t^2}{3}\right)e^{-ikt} \tag{4.3.8}$$

Using the asymptotic expansion for the Hankel function, $H_{5/2}^{(1)}(x) \sim -3i\sqrt{2/\pi}x^{-5/2}$ for $x \ll 1$, the long-wavelength ($|kt| \ll 1$) power spectrum for π is

$$\mathcal{P}_\pi = \frac{1}{2\pi^2}k^3|\pi_k|^2 \sim \frac{9H^2}{(2\pi)^2 M_\pi^2}\frac{1}{(-kt)^2} \ . \tag{4.3.9}$$

Note that this spectrum peaks at long wavelengths and is thus strongly red-tilted.

4.3.2 Two-Point Function for Massless Spectator Fields

Now let us compute the power spectrum for the weight-0 spectator field χ. Recall that this is the field that we envision will lead to a scale-invariant spectrum of curvature perturbations

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[34]Notice that at long wavelengths ($k \to 0$), this equation has a solution where $\pi \sim 1/t$. Though this is naïvely unstable, it may be re-absorbed by a time translation $\delta_{P_0}\pi = 1/t$, precisely as in the negative quartic case.

once these entropic perturbations have been converted to the adiabatic direction. A detailed analysis of the conversion of perturbations is beyond the scope of this paper, but is the subject of current work.

At quadratic order in χ, the action of χ is

$$S_\chi = \int \mathrm{d}^4 x \sqrt{-\bar{g}_{\mathrm{eff}}} \left[-\frac{M_\chi^2}{2} (\partial \chi)^2 - \frac{m_\chi^2 + \bar{M}_\pi^2 H^2}{2} \chi^2 \right] \ , \tag{4.3.10}$$

which just describes a massive scalar field on de Sitter space. It is well-known that the field will acquire a scale-invariant spectrum of fluctuations provided that its mass is sufficiently small: $m_\chi^2/(M_\chi^2 H^2)$ and $\bar{M}_\pi^2/M_\chi^2 \ll 1$. Indeed, ignoring the mass term, the solution for the canonically normalized variable $\hat{\chi} = \frac{M_\chi}{(-Ht)} \chi$ is

$$\hat{\chi}_k = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{kt} \right) e^{-ikt} \ , \tag{4.3.11}$$

where the usual adiabatic vacuum has been assumed. This implies that the long-wavelength power spectrum for χ_k is scale invariant

$$\mathcal{P}_\chi = \frac{1}{2\pi^2} k^3 |\chi_k|^2 \sim \frac{H^2}{(2\pi)^2 M_\chi^2} \ . \tag{4.3.12}$$

4.4 Curvature invariant construction

The coset construction machinery of the previous sections, while extremely powerful, is technically involved, hence it is pedagogically helpful to present an alternative way of deriving our effective lagrangians. The technique is an extension of the method used in [103] to obtain the conformal galileon combinations, which we foreshadowed in deriving the transformation rule for π in the last section.

The basic idea is the following. To linearly realize the de Sitter group, $SO(4, 1)$, our theory should be cast in terms of a (fictitious) de Sitter metric, $\bar{g}_{\mu\nu}^{\mathrm{eff}}$, and its covariant derivative. In

addition, we also want to non-linearly realize the conformal group $SO(4, 2)$. This is achieved by introducing the conformal mode:

$$g_{\mu\nu}^{\text{eff}} = e^{2\pi} \bar{g}_{\mu\nu}^{\text{eff}} \ . \tag{4.4.1}$$

This metric is clearly conformally invariant, with π transforming in some non-linear fashion under a general conformal transformation. To simplify the notation, we will omit the subscript "eff", with the implicit understanding that all metrics in the effective theory are fictitious.

By using the geometric covariant derivative associated to this conformal metric, we can write down invariant actions for matter fields that non-linearly realize the conformal group. In order to get the action for the Goldstone we want to consider curvature invariants, which pick out the dynamics of the conformal mode π. To see that this method is completely equivalent to the coset construction, first note that because the metric (4.4.1) is obviously conformal to de Sitter — and thus conformally flat — all of the curvature information is contained in the Ricci tensor

$$R_{\mu\nu} = 3H^2 \bar{g}_{\mu\nu} - 2\bar{\nabla}_\mu \bar{\nabla}_\nu \pi - \bar{g}_{\mu\nu} \bar{\Box} \pi + 2\partial_\mu \pi \partial_\nu \pi - 2\bar{g}_{\mu\nu}(\partial\pi)^2 \ , \tag{4.4.2}$$

where all derivatives and contractions are with respect to the background de Sitter metric $\bar{g}_{\mu\nu}^{\text{eff}}$. It is possible to write $R_{\mu\nu}$ in terms of (4.2.18) as

$$R_{\mu\nu} = 4\mathcal{D}_\mu \xi_\nu + 2\mathcal{D}_\alpha \xi^\alpha g_{\mu\nu} + 3H^2 g_{\mu\nu} \ . \tag{4.4.3}$$

Tracing over this, it is possible to express the Ricci scalar as

$$R = 12\mathcal{D}_\mu \xi^\mu + 12H^2 \ . \tag{4.4.4}$$

Additionally, we know that the covariant derivative associated to $g_{\mu\nu}$ is a building block

in both cases. Therefore we see that the building blocks for the curvature invariant story $\{g_{\mu\nu},\ R_{\mu\nu},\ \nabla_\mu\}$, are equivalent to the ingredients of the coset construction $\{g_{\mu\nu},\ \mathcal{D}_\mu \xi_\nu,\ \nabla_\mu\}$. The curvature invariant prescription therefore provides an equivalent, and less technically demanding, route to build invariant actions.

It should be noted that while we have focused in this Chapter on the coset construction—as it is best suited for the problem of constructing non-linear realizations—there exist other powerful techniques for the construction of conformally-invariant actions. Perhaps the most elegant of these is the formalism of tractor calculus. Most simply, tractors play the same role in conformal geometry that tensors play in Riemannian geometry. Tractor calculus was first introduced in [121], building on earlier ideas from the 1920's [122, 123]. Tractors live in $\mathbb{R}^{4,2}$, where the conformal group $SO(4,2)$ acts naturally. A nice introduction to these ideas is given in [124]. Tractors provide a powerful formalism for handling conformal invariance; by contracting tractors and tractor covariant derivatives to construct scalars, one automatically obtains Weyl-invariant theories in four dimensions, analogous to how one ordinarily builds diffeomorphism invariant actions with tensors. Tractor calculus has been applied to physical systems in many ways, most notably to address the origins of mass [125, 126] and to view Einstein gravity from a six-dimensional viewpoint [127]. Although not included in our discussion, we have verified explicitly that the conformal actions constructed with apparatus of tractor calculus agree with those descending from the coset construction.

Another method of constructing field theories with non-linearly realized symmetries is the embedded-brane technique of [105, 107, 128], in which the physical space is imagined as a 3-brane floating in a non-dynamical bulk. The fields in the physical space-time then inherit non-linear symmetries from the Killing vectors of the higher-dimensional bulk. In [105], this approach was used to construct effective field theories realizing various patterns of symmetry breaking to maximal subalgebras.

Chapter 5

Aside: Wess–Zumino terms and Lie algebra cohomology

As was alluded to, the construction of the effective lagrangian for the Goldstone field of Chapter 4 is not quite complete. The coset construction manifestly generates terms which are *strictly* invariant under the non-linearly realized symmetries. This still leaves open the possibility of there being terms which shift under the symmetries *by a total derivative*, leaving the action invariant. The most familiar term of this type is the Wess–Zumino–Witten term of the Chiral lagrangian [129, 130]. As was shown by Witten [130], this term is topologically non-trivial—in $4d$, it corresponds to a nontrivial 5-form in de Rham cohomology of the group SU(3). This analysis was extended in [131] for general internal symmetry groups.

A similar story holds for space-time symmetry breaking. The relevant 5-forms are associated with non-trivial cocycles in an appropriate Lie algebra cohomology [112, 132, 133], which is a cohomology theory on forms which are left-invariant under vector fields that generate the symmetry algebra.[35] This is related to the internal symmetry case—for compact groups, de Rham and Lie algebra cohomology are isomorphic [135].

In this Chapter, we consider this story for the symmetry breaking pattern of interest. We argue that possible Wess–Zumino (WZ) terms are indexed by what is known as *relative*

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[35]A similar viewpoint was conveyed in [134], where the low-energy effective actions for non-relativistic strings and branes were obtained as Wess–Zumino terms.

Lie algebra cohomology or *Chevalley–Eilenberg cohomology*, which is associated to a given symmetry breaking pattern. We first briefly introduce the necessary cohomological tools and then apply them to a toy example—the free point particle in $(0+1)$-dimension—and show how the kinetic term arises as a WZ term. Finally, we turn to the symmetry breaking pattern of interest, $\mathfrak{so}(4,2) \to \mathfrak{so}(4,1)$ and construct the WZ term.

5.1 Cohomology

In this section, we introduce the necessary concepts and definitions of Lie algebra cohomology and relative Lie algebra cohomology needed for classifying Wess–Zumino terms for spacetime symmetries. For a more comprehensive introduction, including applications, see [133].

5.1.1 Lie algebra cohomology

Given a Lie algebra $\mathfrak{g}$, an n-cochain, $n = 0, 1, 2, \ldots$, is a totally anti-symmetric multi-linear mapping $\omega_n : \bigwedge^n \mathfrak{g} \to \mathbb{R}$, taking values in the reals.[36] The space of n-cochains is denoted $\Omega^n(\mathfrak{g})$. One then forms a coboundary operator $\delta_n : \Omega^n(\mathfrak{g}) \to \Omega^{n+1}(\mathfrak{g})$ whose action is defined by [133]

$$\delta\omega(X_1, X_2, \ldots, X_{n+1}) = \sum_{\substack{j,k=1 \\ j<k}}^{n+1} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{n+1}), \quad (5.1.1)$$

for $X_1, X_2, \ldots \in \mathfrak{g}$ and where $\hat{X}$ means the argument is omitted, and $[\,,\,]$ is the Lie algebra commutator. The first few instances are

$$\delta\omega_0(X_1) = 0,$$
$$\delta\omega_1(X_1, X_2) = -\omega_1([X_1, X_2]),$$
$$\delta\omega_2(X_1, X_2, X_3) = -\omega_2([X_1, X_2], X_3) + \omega_2([X_1, X_3], X_2) - \omega_2([X_2, X_3], X_1),$$
$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.1.2)$$

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[36]In general, one can consider the case in which the cochains take values in an arbitrary vector space on which acts a non-trivial representation of $\mathfrak{g}$, but we do not need that here.

One can show, using the Jacobi identity $[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$, that the coboundary operator is nilpotent

$$\delta^2 = 0 \ . \tag{5.1.3}$$

Thus we have $\mathrm{Im}_{\delta_{n-1}} \left(\Omega^{n-1} \right) \subset \mathrm{Ker}_{\delta_n} \left(\Omega^n \right)$, and we can define the cohomology spaces

$$H^n(\mathfrak{g}) = \frac{\mathrm{Ker}_{\delta_n} \left(\Omega^n(\mathfrak{g}) \right)}{\mathrm{Im}_{\delta_{n-1}} \left(\Omega^{n-1}(\mathfrak{g}) \right)} \ . \tag{5.1.4}$$

There is another way to represent the coboundary operator that is often more convenient when we have an explicit basis. Let $\{e_i\}$, $i = 1, \cdots, \dim(\mathfrak{g})$, be a basis for the Lie algebra $\mathfrak{g}$. The structure constants $c_{ij}{}^k$ are given by

$$[e_i, e_j] = c_{ij}{}^k e_k \ . \tag{5.1.5}$$

They are anti-symmetric in their first indices, $c_{ij}{}^k = -c_{ji}{}^k$. The Jacobi identity becomes $c_{il}{}^m c_{jk}{}^l + c_{jl}{}^m c_{ki}{}^l + c_{kl}{}^m c_{ij}{}^l = 0$. Let $\{\omega^i\}$ be a basis of the dual space $\mathfrak{g}^*$, dual to the basis $\{e_i\}$, so that $\omega^i(e_j) = \delta^i_j$. Then we can write any n-cochain ω_n as sums of wedge products of the ω^i,

$$\omega_n = \frac{1}{n!} \Omega_{i_1 i_2 \cdots i_n} \omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_n} \ , \tag{5.1.6}$$

where $\Omega_{i_1 i_2 \cdots i_n}$ is the totally anti-symmetric tensor of coefficients. The action of the coboundary operator on a single ω^i is given by

$$\delta \omega^i = -\frac{1}{2} c_{jk}{}^i \omega^j \wedge \omega^k \ , \tag{5.1.7}$$

and is extended to wedge products of multiple ω's by using linearity and the Leibniz product rule, where we are careful to include the addition of a minus sign every time δ has to pass through an ω.[37] For example, we have $\delta \left(\omega^i \wedge \omega^j \right) = -\frac{1}{2} c_{kl}{}^i \omega^k \wedge \omega^l \wedge \omega^j + \frac{1}{2} c_{kl}{}^j \omega^i \wedge \omega^k \wedge \omega^l$.

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[37]The coboundary operator, δ, is an anti-derivation on the algebra of cochains.

In terms of components, we have

$$(\delta\Omega)_{i_1\cdots i_{n+1}} = -\frac{n(n+1)}{2} c_{[i_1 i_2}{}^j \Omega_{|j|i_3\cdots i_{n+1}]}.\tag{5.1.8}$$

Lie algebra cohomology also has a geometric interpretation.[38] Consider the simply con-
nected Lie group G associated to the Lie algebra $\mathfrak{g}$. The space of p-forms on G which are
invariant under the left action of G on itself can be identified with the cochains of Lie algebra
cohomology. In fact, there is one left invariant 1-form for each generator of the Lie alge-
bra, and wedging them together in all ways generates all the invariant p-forms. The usual
exterior derivative operator on G, $\mathrm{d}_p : \Omega^p(G) \to \Omega^{p+1}(G)$ satisfies $\mathrm{d}\omega^i = -\frac{1}{2} c_{jk}{}^i \omega^j \wedge \omega^k$,
and can be identified with the operator δ of Lie algebra cohomology. Thus, Lie algebra
cohomology counts the number of left-invariant forms on G which cannot be written as the
exterior derivative of a form which is also left-invariant.

5.1.2 Relative Lie algebra cohomology

For characterizing symmetry breaking to a subalgebra, we will need a slightly more refined
notion of Lie algebra cohomology, known as relative Lie algebra cohomology. Consider a
subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We define the space of relative cochains $\Omega^n(\mathfrak{g}; \mathfrak{h})$, as the subspace of
cochains satisfying the following two conditions,

$$\Omega_n(V, X_2, \ldots, X_n) = 0 \;,\tag{5.1.9}$$

$$\Omega_n([V, X_1], X_2, \ldots, X_n) + \Omega_n(X_1, [V, X_2], \ldots, X_n) + \cdots + \Omega_n(X_1, X_2, \ldots, [V, X_n]) = 0 \;,$$

$$\text{for all} \quad V \in \mathfrak{h}, \text{ and } \quad X_2, \cdots, X_n \in \mathfrak{g} \;.\tag{5.1.10}$$

The first requirement says that if any of the arguments lie completely in $\mathfrak{h}$, then we get zero.
This means that the form is well defined on the quotient $\mathfrak{g}/\mathfrak{h}$. Equivalently, the n-cochains
are only constructed from wedging together one-forms which annihilate $\mathfrak{h}$. To see what this

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[38]In this geometric context, Lie algebra cohomology is known as Chevalley–Eilenberg Cohomology [132].

means in terms of components, choose a basis $\{h_I, f_a\}$ for $\mathfrak{g}$, where $\{h_I\}$, $I = 1, \ldots, \dim(\mathfrak{h})$ is a basis of $\mathfrak{h}$ and $\{f_a\}$, $a = 1, \ldots, \dim(\mathfrak{g}/\mathfrak{h})$ completes to a basis of $\mathfrak{g}$. Let the dual basis be $\{\eta^I, \omega^a\}$. To satisfy (5.1.9), forms are constructed by wedging together only the forms ω^a, so the components $\Omega_{i_1 \cdots i_n}$ of (5.1.6) are zero if any of the indices are in the $\mathfrak{h}$ directions.

The second condition, in terms of components (5.1.6), reads $c_{Ii_1}{}^j \Omega_{ji_2 \cdots i_n} + c_{Ii_2}{}^j \Omega_{i_1 j \cdots i_n} + \cdots + c_{Ii_n}{}^j \Omega_{i_1 i_2 \cdots j} = 0$. The combination of the two conditions (5.1.9) and (5.1.10) on the components, along with the fact that $c_{IJ}{}^a = 0$ since $\mathfrak{h}$ is a subgroup, gives our final conditions in terms of components for a cochain to be a relative cochain,

$$\Omega_{Ii_2 \cdots i_n} = 0 \ , \tag{5.1.11}$$

$$c_{Ia_1}{}^b \Omega_{ba_2 \cdots a_n} + c_{Ia_2}{}^b \Omega_{a_1 b \cdots a_n} + \cdots + c_{Ia_n}{}^b \Omega_{a_1 a_2 \cdots b} = 0 \ . \tag{5.1.12}$$

Given our basis, the matrices

$$\phi(h_I)_a{}^b = -c_{Ia}{}^b \tag{5.1.13}$$

form a representation of the subalgebra $\mathfrak{h}$,

$$\phi(h_I)\phi(h_J) - \phi(h_J)\phi(h_I) = c_{IJ}{}^K \phi(h_K) \ , \tag{5.1.14}$$

as can be straightforwardly shown using the Jacobi identity, as well as the condition $c_{IJ}{}^a = 0$ which follows from the fact that $\mathfrak{h}$ is a subalgebra. Thus, the indices $a, b, \ldots$ of the space $\mathfrak{g}/\mathfrak{h}$ furnish a representation of the subgroup $\mathfrak{h}$, and the condition (5.1.12) says that the cochain coefficients must be invariant tensors under the action of $\mathfrak{h}$ in this space.

The δ operator preserves the two conditions (5.1.9) and (5.1.10), so $\delta_n (\Omega^n(\mathfrak{g}; \mathfrak{h})) \subset \Omega^{n+1}(\mathfrak{g}; \mathfrak{h})$. Thus we may think of δ as acting on the spaces $\Omega^n(\mathfrak{g}; \mathfrak{h})$. The cohomology classes of this action are denoted by $H^p(\mathfrak{g}; \mathfrak{h})$ and the construction is known as *relative Lie*

algebra cohomology [133],

$$H^n(\mathfrak{g};\mathfrak{h}) = \frac{\text{Ker}_{\delta_n}\left(\Omega^n(\mathfrak{g};\mathfrak{h})\right)}{\text{Im}_{\delta_{n-1}}\left(\Omega^{n-1}(\mathfrak{g};\mathfrak{h})\right)} \, . \tag{5.1.15}$$

Each non-trivial element of $H^{d+1}(\mathfrak{g};\mathfrak{h})$ corresponds to a Wess–Zumino term for a d-dimensional space-time [112, 133].

Relative Lie algebra cohomology also has a geometric interpretation. Consider the connected Lie group G and subgroup H, corresponding to the algebra $\mathfrak{g}$ and subalgebra $\mathfrak{h}$. We can think of the group G as a fiber bundle, consisting of spaces H fibered over the base space G/H. The group G acts naturally on G/H (which is a homogeneous space with isotropy subgroup H). The relative cochains can be thought of as left invariant form on G which are projectable to G/H, *i.e.*, can be written as a pullback through the projection $G \rightarrow G/H$ of a unique form on G/H. Thus they can be identified with invariant forms on G/H. The operator δ can be identified with the usual exterior derivative d, so relative Lie algebra cohomology counts the number of left-invariant forms on G/H which cannot be written as the exterior derivative of a form which is also left-invariant.

5.2 Non-relativistic point particle moving in one dimension

We now proceed with the coset construction, first considering the simplest case of this construction: the one-dimensional non-relativistic free point particle. We can think of this as a $0+1$ dimensional brane probing a non-relativistic $1+1$ dimensional bulk. The Wess–Zumino nature of the kinetic term was pointed out in [136] and is elegantly treated using jet bundles in [135]. Here, instead, we will derive equivalent results from the coset perspective.

We denote the single degree of freedom as $q(t)$, where t is the one and only space-time coordinate. We want to construct Lagrangians which are invariant under the algebra $\mathfrak{Gal}(0+1,1)$, which is three dimensional and whose generators act on $q(t)$ as follows

$$\delta_C q = 1, \qquad\qquad \delta_B q = -t, \qquad\qquad \delta_P q = -\dot{q} \, . \tag{5.2.1}$$

Here δ_C is the shift symmetry on the field, δ_B is the analogue of the "galilean" shift symmetry (the galilean boost of the non-relativistic particle) and δ_P is time translation of the field. The algebra has only a single non-zero commutator[39]

$$[B, P] = C \ . \tag{5.2.2}$$

The only transformation among (5.2.1) which is linear is δ_P, the rigid translations of the line, so the breaking pattern is

$$\mathfrak{Gal}(0 + 1, 1) \longrightarrow \mathfrak{iso}(1). \tag{5.2.3}$$

To construct the most general Lagrangian which realizes these symmetries (5.2.1), we employ the coset construction for space-time symmetries reviewed in Section (4.1.2). The parametrization of the coset (4.1.10) is given by

$$\tilde{V} = e^{tP} e^{qC + \xi B} \ , \tag{5.2.4}$$

where q is the Goldstone field that will become the physical field associated with the shift symmetry, and ξ is the Goldstone field associated with the galilean boost symmetry. Since the momentum P is to be included in the coset, there is no subgroup H to be linearly realized. Thus the coset is the galilean group itself,

$$\mathrm{Gal}(0 + 1, 1) \ . \tag{5.2.5}$$

Next we compute the Maurer–Cartan form (4.1.12),

$$\omega = \tilde{V}^{-1} \mathrm{d}\tilde{V} = \mathrm{d}t P + (\mathrm{d}q - \xi \mathrm{d}t) \, C + \mathrm{d}\xi B \ , \tag{5.2.6}$$

---

[39]In relation to the d-dimensional algebra, we are defining $P \equiv P_0$, $B \equiv B_0$.

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and the component 1-forms used to build Lagrangians can then be read off as

$$\omega_P = \mathrm{d}t \;, \qquad\qquad \omega_C = \mathrm{d}q - \xi \mathrm{d}t \;, \qquad\qquad \omega_B = \mathrm{d}\xi \;. \qquad (5.2.7)$$

Now, it is important to note that there is an inverse Higgs constraint. Inspection of the only commutator of the algebra (5.2.2) shows that we can eliminate the ξ field in favor of q by setting $\omega_C = 0$, implying the relation

$$\xi = \dot{q} \;. \qquad (5.2.8)$$

Substitution into (5.2.7) then provides simplified expressions for the basis 1-forms

$$\omega_P = \mathrm{d}t \;, \qquad\qquad \omega_B = \ddot{q}\,\mathrm{d}t \;. \qquad (5.2.9)$$

Thus, all the ingredients available for constructing invariant Lagrangians involve at least two derivatives on each q. There is also the covariant derivative, but this turns out to be just $\mathrm{d}/\mathrm{d}t$, so taking higher covariant derivatives will only add more time derivatives. Lagrangians constructed in this way are all strictly invariant under the shift symmetries δ_B and δ_C.

This presents a puzzle, since we know that the free particle kinetic term, $\mathcal{L} = \frac{1}{2}\dot{q}^2$, is also galilean invariant. Although it is not invariant under δ_B, it is invariant up to a total derivative, so it represents a perfectly good Lagrangian, which is missed by the coset construction since it contains fewer than two derivatives per q. Another missed example is the tadpole term $\mathcal{L} = q$, which changes up to a total derivative under both δ_B and δ_C. How do we construct these missing terms?

The answer is that these terms will appear as particular shift and boost invariant 2-forms which are themselves constructible from the Maurer–Cartan form (5.2.7). These terms will live on the coset space, that is, the space in which q and ξ are considered as new coordinates

in addition to the t direction of space-time. These 2-forms will also be total derivatives in this higher dimensional space, writable as d of a 1-form. The Lagrangian will be obtained by integrating this 1-form on the 1 dimensional subspace where $q = q(t)$ and $\xi = \xi(t)$.

The symmetries on this space in our case are generated by the vector fields [135][40]

$$C = \partial_q \ , \qquad\qquad B = \partial_\xi + t\partial_q \ , \qquad\qquad P = \partial_t \ . \qquad (5.2.10)$$

The components of the Maurer–Cartan form (5.2.7), where we treat q and ξ as independent coordinates, are the (left) invariant 1-forms on the coset space parametrized by $\{q, \xi, t\}$; that is we have $\pounds_X \omega = 0$ where X is any of the vector fields (5.2.10) and ω is any of the forms (5.2.7).

Consider the invariant 2-forms, which are all obtained by wedging together all combinations of the invariant one-forms (5.2.7). There are three of these, with the first being

$$\omega_1^{\mathrm{wz}} = \omega_B \wedge \omega_C = \mathrm{d}\xi \wedge (\mathrm{d}q - \xi \mathrm{d}t) \ . \qquad (5.2.11)$$

We note that this can be written as the exterior derivative of a 1-form,

$$\omega_1^{\mathrm{wz}} = \mathrm{d}\beta_1^{\mathrm{wz}} \ , \qquad\qquad \beta_1^{\mathrm{wz}} = \xi \mathrm{d}q - \frac{1}{2}\xi^2 \mathrm{d}t \ . \qquad (5.2.12)$$

This 1-form can be used to construct an invariant action by pulling back to the surface space-time manifold ∂M, defined by $q = q(t)$, $\xi = \xi(t)$, and then integrating,

$$S_1^{\mathrm{wz}} = \int_{\partial M} \beta_1^{\mathrm{wz}} = \int \mathrm{d}t \ \xi\dot{q} - \frac{1}{2}\xi^2 \ . \qquad (5.2.13)$$

Imposing the inverse Higgs constraint $\xi = \dot{q}$ (or, equivalently, integrating out ξ), we recover the well-known kinetic term for the non-relativistic free point particle which was missed in

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[40]Note that the Lie bracket of left-invariant vector fields is minus the commutator of the algebra.

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the coset construction,

$$S_1^{\text{wz}} = \int_{\partial M} \beta^{\text{wz}} = \int \text{d}t \, \frac{1}{2} \dot{q}^2 \, . \tag{5.2.14}$$

The tadpole term may be constructed similarly from the two form

$$\omega_2^{\text{wz}} = \omega_C \wedge \omega_P = \text{d}q \wedge \text{d}t = \text{d}\beta_2^{\text{wz}} \, , \qquad\qquad \beta_2^{\text{wz}} = q\text{d}t \, . \tag{5.2.15}$$

$$S_2^{\text{wz}} = \int_{\partial M} \beta^{\text{wz}} = \int \text{d}t \, q \, . \tag{5.2.16}$$

The final possible invariant 2-form constructible from the invariant one forms (5.2.7) is $\omega_3^{\text{wz}} = \omega_B \wedge \omega_P = \text{d}\xi \wedge \text{d}t = \text{d}(\xi\text{d}t)$. This leads to an action which is a total derivative once the Higgs constraint is imposed, and so nothing new results. (This illustrates that the dimension of the relevant cohomology groups may not in general count the number of WZ terms exactly, but will only put an upper bound on the possible number.)

In all cases, the 2-form ω^{wz} is closed since it can be written as d of a one form β^{wz} (so that we may use it to construct an action). Furthermore, the 2-form ω^{wz} is by construction (left) invariant under the vector fields that generate the symmetries we are interested in (5.2.1). However, the 1-form β^{wz} is not invariant—it shifts by a total d (as it must since ω^{wz} is invariant, $\omega^{\text{wz}} = \text{d}\beta^{\text{wz}}$, and de Rham cohomology is trivial on all the spaces we're considering), but this still leaves the action invariant.

The interesting 2-forms are therefore those which are invariant under the action of the vector fields (5.2.10) but which *cannot* be written as the exterior derivative of a 1-form which is itself invariant [135] (since otherwise the corresponding 1-form on the boundary would be strictly invariant and would have already been captured by the coset construction). They can thus be identified with non-trivial elements of the Lie algebra cohomology

$$H^2\left(\mathfrak{Gal}(0+1,1)\right). \tag{5.2.17}$$

Lagrangians constructed in this manner are what we call Wess–Zumino terms. For a d-dimensional space-time, they are terms that correspond to non-trivial $d+1$ co-cycles in the cohomology of d acting on invariant vector fields on the coset space [132].

5.3 $\mathfrak{so}(4,2) \longrightarrow \mathfrak{so}(4,1)$ Wess–Zumino term

Starting with the conformal algebra in the basis (4.2.3), we wish to compute the relative Lie algebra cohomology

$$H^5(\mathfrak{so}(4,2); \mathfrak{so}(4,1)), \tag{5.3.1}$$

in order to catalog the possible Wess–Zumino terms. The forms which annihilate the vector subspace spanned by $\mathfrak{so}(4,1)$ are $\{\omega_D, \omega_K^\mu, \omega_{\hat{P}}^\mu\}$. These are used to create n-cochains for computing the relative Lie algebra cohomology. The coboundary operator d acts on the basis forms as

$$
\begin{aligned}
\mathrm{d}\omega_D &= 2\eta_{\mu\nu}\omega_K^\mu \wedge \omega_{\hat{P}}^\nu \ , \\
\mathrm{d}\omega_{\hat{P}}^\mu &= \omega_D \wedge \omega_{\hat{P}}^\mu + 2\eta_{\alpha\beta}\omega_{\hat{P}}^\beta \wedge \omega_J^{\alpha\mu} \ , \\
\mathrm{d}\omega_K^\mu &= -\frac{H^2}{2}\omega_D \wedge \omega_{\hat{P}}^\mu - \omega_D \wedge \omega_K^\mu + 2\eta_{\alpha\beta}\omega_K^\beta \wedge \omega_J^{\alpha\mu} \ .
\end{aligned}
\tag{5.3.2}
$$

We can construct the following six $\mathfrak{so}(4,1)$-invariant 5-cochains

$$
\begin{aligned}
\omega_1 &= \epsilon_{\mu\nu\rho\sigma}\omega_D \wedge \omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_{\hat{P}}^\sigma, \\[4pt]
\omega_2 &= \epsilon_{\mu\nu\rho\sigma}\omega_D \wedge \omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_K^\sigma, \\[4pt]
\omega_3 &= \epsilon_{\mu\nu\rho\sigma}\omega_D \wedge \omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma, \\[4pt]
\omega_4 &= \epsilon_{\mu\nu\rho\sigma}\omega_D \wedge \omega_{\hat{P}}^\mu \wedge \omega_K^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma, \\[4pt]
\omega_5 &= \epsilon_{\mu\nu\rho\sigma}\omega_D \wedge \omega_K^\mu \wedge \omega_K^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma, \\[4pt]
\omega_6 &= \eta_{\mu\nu}\eta_{\rho\sigma}\omega_D \wedge \omega_{\hat{P}}^\mu \wedge \omega_K^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_K^\sigma \ .
\end{aligned}
\tag{5.3.3}
$$

The cochains ω_1 to ω_5 are closed ($\mathrm{d}\omega = 0$), and we therefore have five possible non-trivial cocycles. However, we can write four linear combinations of these as coboundaries

$$\omega_1 = \mathrm{d}\left[\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_{\hat{P}}^\sigma\right],$$

$$\omega_2 = \mathrm{d}\left[\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} \wedge \omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_K^\sigma + \frac{H^2}{2}\epsilon_{\mu\nu\rho\sigma}\omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_{\hat{P}}^\rho \wedge \omega_{\hat{P}}^\sigma\right],$$

$$= \mathrm{d}\left[-\frac{2}{H^2}\omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma\right]$$

$$\frac{H^2}{2}\omega_3 - 2\omega_4 = \mathrm{d}\left[\epsilon_{\mu\nu\rho\sigma} \wedge \omega_{\hat{P}}^\mu \wedge \omega_K^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma\right],$$

$$-\frac{H^2}{2}\omega_4 - 4\omega_5 = -\frac{1}{4}\mathrm{d}\left[\epsilon_{\mu\nu\rho\sigma} \wedge \omega_K^\mu \wedge \omega_K^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma\right]. \tag{5.3.4}$$

However, there remains one linear combination which cannot be written as d(something), and is therefore a non-trivial cocycle. This is equivalent to $H^5(\mathfrak{so}(4,2),\mathfrak{so}(4,1))$ having a single element and correspondingly, there being a single Wess–Zumino term.

We are free to choose a representative 5-form cocycle, which we take to be

$$\omega_3 = \epsilon_{\mu\nu\rho\sigma}\omega_D \wedge \omega_{\hat{P}}^\mu \wedge \omega_{\hat{P}}^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma = \mathrm{d}\beta^{\mathrm{wz}} \tag{5.3.5}$$

Pulling back and imposing the inverse Higgs constraint (4.2.14), the final result is (4.2.26)

$$S_{\mathrm{wz}} = \int_{\partial M} \beta^{\mathrm{wz}} = \int \mathrm{d}^4 y\sqrt{-\bar{g}_{\mathrm{eff}}}\left[(\partial\pi)^4 + 2\bar{\Box}\pi(\partial\pi)^2 + 6H^2(\partial\pi)^2\right]. \tag{5.3.6}$$

Worth noting that is that in the limit $H \to 0$, this reproduces the standard WZ term for the conformal group broken to Poincaré [6]

$$\mathcal{L}_3 \sim (\partial\pi)^4 + 2\Box\pi(\partial\pi)^2, \tag{5.3.7}$$

which has been of some interest recently in connection with the a-theorem in four dimensions [137, 138]. This term for the 4 dimensional conformal group plays a similar role to that of the more well-known 2 dimensional Wess–Zumino term in the trace anomaly.

Chapter 6

Consistency relations for the conformal mechanism

In Chapter 4, we saw the power of non-linearly realized symmetries at the level of the action. Armed only with a symmetry breaking pattern, we were able to make very general statements about the conformal mechanism, and were led to a unique form for the action at lowest order.

We now turn to a related question, which is to understand how the non-linearly realized symmetries (4.2.35) act on correlation functions. Similar to the soft-pion theorems of the chiral lagrangian of QCD, these symmetries relate correlation functions with $(N+1)$ fields to those with only N fields. The consistency relations open up the possibility of strong observational tests of the conformal mechanism. In much the same way that observation of f_{NL} in the squeezed configuration would rule out all single-field models of the early universe, observation of a violation of one of these consistency relations would rule out the production of density perturbations by the conformal mechanism.

These are similar to the case in single-field inflation; where perturbations can be described most generally by the effective field theory of spontaneously broken time diffeomorphisms [79, 139]. Single-field inflation can also be understood in terms of *global* symmetries as the spontaneous breaking of the $\mathrm{SO}(4,1)$ conformal symmetry of $\mathbb{R}^3$ down to spatial translations and rotations [83, 140]. The corresponding Goldstone field is ζ, the curvature

perturbation of uniform-density hypersurfaces. Moreover, the well-known consistency re-
lations [64–66, 141, 142], which constrain the soft limit of correlation functions, arise as
Ward identities for the non-linearly realized symmetries [143–145]. Additionally, symmetry
considerations have proven to be a powerful tool in analyzing correlation functions of spec-
tator fields in inflation: both gravitons [146] and scalar field perturbations [147–151] are
constrained to have conformally-invariant correlators at late times.

6.1 $\mathrm{SO}(4,2) \longrightarrow \mathrm{SO}(4,1)$ consistency relation

The question we desire to answer is: *how do these non-linearly realized symmetries act on
correlation functions?* In this Section we show that the non-linear realization of conformal
symmetry constrains the form that correlation functions take in the limit that one of the π
external legs is taken to be very soft. Here we present a discussion complementary to [8],
where these same relations were derived using "background wave" arguments familiar from
inflation. In the following Sections, we derive equivalent results as field-theoretic Ward
identities using the machinery of [144].

6.1.1 Symmetries and charges

In the case of inflation in the decoupling limit, the isometries of de Sitter are spontaneously
broken by the inflaton's time-dependent background to the subgroup of rotations and trans-
lations. As a consequence of this spontaneous breaking, there are specific relations between
correlation functions of different order. In particular, the $(N+1)$-point correlation functions
in the squeezed limit are related to the variation of the N-point correlation functions under
the broken symmetries (dilations and special conformal transformations). These relations
go by the name of *consistency relations* [64–66, 140–142]. They are the Ward identities
resulting from the non-linearly realized symmetries in the broken phase of the theory [143–
145].

Our aim is to show that similar relations hold in the case of the nonlinearly-realized $\mathrm{SO}(4,2)$
symmetries. We again expect that the squeezed limit of an $(N+1)$-point correlation function
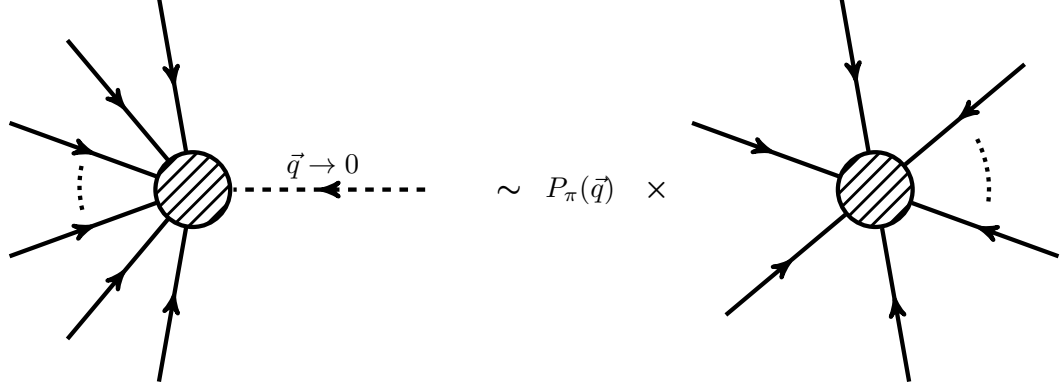
Figure 5: Schematic representation of the consistency relations: $(N + 1)$-point functions with a single soft external π leg are related to N-point functions.

is related to the action of the broken generators on the N-point function. Recall the transformation rules for the Goldstone field π (4.2.35):

$$
\begin{aligned}
\delta_{P_\mu}\pi &= -\partial_\mu\pi + \delta_\mu^0 \frac{1}{t}, \\
\delta_{J_{\mu\nu}}\pi &= (x_\mu\partial_\nu - x_\nu\partial_\mu)\pi + \left(\delta_\mu^0 \frac{x_\nu}{t} - \delta_\nu^0 \frac{x_\mu}{t}\right), \\
\delta_D\pi &= -x^\mu\partial_\mu\pi, \\
\delta_{K_\mu}\pi &= -(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\pi - \delta_\mu^0 \frac{x^2}{t}\ .
\end{aligned}
\tag{6.1.1}
$$

In this case the broken generators are time translations P_0, boosts J_{0i}, and the time component of a special conformal transformation K_0; correspondingly, there will be three Ward identities.

The conserved currents associated to these symmetries are given by

$$
j^0(x) = \frac{1}{2}\left\{\Pi(x), \delta\pi(x)\right\}\ ,
\tag{6.1.2}
$$

where $\Pi \equiv \delta\mathcal{L}/\delta\dot{\pi}$ is the momentum conjugate to π. These can be integrated to give the

Noether charges[41]

$$Q = \int d^3x \, j^0(x) = \frac{1}{2} \int d^3x \, \{\Pi(x), \delta\pi(x)\} \;, \qquad (6.1.4)$$

which generate the field transformations in the quantum theory

$$[Q, \pi] = -i\delta\pi \;. \qquad (6.1.5)$$

Time translation: The charge which generates (broken) time translations is given by

$$Q_{P_0} = W_{P_0} + \int d^3x \, \frac{1}{t}\Pi(x) \equiv W_{P_0} + \mathcal{Q}_{P_0} \;, \qquad (6.1.6)$$

where W is a piece that generates the part of the transformation linear in π. We will see that at sufficiently early times, the contribution from this part is irrelevant, so we only need to keep the non-linear part. Note that this charge is divergent, to regulate it, we interpret it as the $q \to 0$ limit of a Fourier transform

$$\mathcal{Q}_{P_0}(\vec{q}) = \int d^3 e^{-i\vec{q}\cdot\vec{x}}\frac{1}{t}\Pi(x) = \frac{1}{t}\Pi(\vec{q}) \;. \qquad (6.1.7)$$

Note that while the charge is Hermitian at zero momentum, at finite $\vec{q}$ we have

$$\mathcal{Q}_{P_0}^\dagger(\vec{q}) = \int d^3 e^{i\vec{q}\cdot\vec{x}}\frac{1}{t}\Pi(x) = \frac{1}{t}\Pi(-\vec{q}) = \mathcal{Q}_{P_0}(-\vec{q}) \;. \qquad (6.1.8)$$

Boosts: In a similar way, we can write the charge which generates boosts

$$Q_{J_{0i}} = W_{J_{0i}} + \int d^3x \, \frac{x^i}{t}\Pi(x) \equiv W_{J_{0i}} + \mathcal{Q}_{J_{0i}} \;. \qquad (6.1.9)$$

---

[41]Note that these charges generally suffer from an IR divergence, which can be seen by considering

$$\langle 0|Q^2|0\rangle = \int d^3x \langle 0|j^0(x)Q|0\rangle = \int d^3x \langle 0|e^{-i\vec{P}\cdot\vec{x}}j^0(0)e^{i\vec{P}\cdot\vec{x}}Q|0\rangle = \int d^3x \langle 0|j^0(0)Q|0\rangle \;, \qquad (6.1.3)$$

where we have used translational invariance of the current and charge. This integral diverges with the volume in the broken phase where $Q|0\rangle \neq 0$. Nevertheless, commutators of Q with local fields are well-defined.

Going to Fourier space in the same way to regulate the charge, we obtain

$$\mathcal{Q}_{J_{0i}}(\vec{q}) = \int \mathrm{d}^3x \ e^{-i\vec{q}\cdot\vec{x}} \frac{x^i}{t} \Pi(x) = -\frac{i}{t} \partial_{q^i} \Pi(\vec{q}) \ . \tag{6.1.10}$$

Special conformal transformation: Finally, we have the charge which generates the zero component of an SCT

$$Q_{K_0} = W_{K_0} + \int \mathrm{d}^3x \ \frac{x^2}{t} \Pi(x) \equiv W_{K_0} + \mathcal{Q}_{K_0} \ . \tag{6.1.11}$$

In Fourier space, this takes the form

$$\mathcal{Q}_{K_0}(\vec{q}) = \int \mathrm{d}^3x \ e^{-i\vec{q}\cdot\vec{x}} \frac{x^2}{t} \Pi(x) = \frac{1}{t} \partial_q^2 \Pi(\vec{q}) \ . \tag{6.1.12}$$

6.1.2 Derivation of the Ward identities

Recall that associated to each conserved current in a field theory, j^μ, is the Ward identity [152]

$$i\partial_\mu \langle T \left(j^\mu(\vec{x}, t') \mathcal{O}(\vec{y}, t) \right) \rangle = \delta(t - t') \delta^{(3)}(\vec{x} - \vec{y}) \langle \delta \mathcal{O} \rangle \ , \tag{6.1.13}$$

where $\mathcal{O}$ is an arbitrary product of operators, T denotes time-ordering of the operators and $\delta \mathcal{O}$ is the variation of $\mathcal{O}$ under the symmetry associated to j^μ. Integrating both sides leads to the identity [152]

$$\langle [Q, \phi_1 \dots \phi_N] \rangle = -i \sum_{a=1}^{N} \langle \phi_1 \dots \delta\phi_a \dots \phi_N \rangle \ , \tag{6.1.14}$$

where we have replaced $\mathcal{O}$ by an arbitrary product of fields in the theory. For notational simplicity, we will write $\phi_1 \dots \phi_N \equiv A(x_1, \dots, x_N)$ when it is convenient. This will be our starting point for deriving the Ward identities associated to each of the broken symmetries in (6.1.1)

Time translations: We start by considering time translations, for which, the general identity (6.1.14) reduces to (working in Fourier space)

$$\langle [Q_{P_0}, A(k_1, \ldots, k_N)] \rangle = -i\delta_{P_0} A(k_1, \ldots, k_N) \ . \tag{6.1.15}$$

We now carefully work to simplify both the left and right-hand sides of this expression.

Left hand side: Using some technical details of the charges (detailed in Appendix C), in particular imposing *weak convergence*, the left hand side is given by[42]

$$\langle \Omega | [Q_{P_0}, A] | \Omega \rangle = \lim_{t_i \to -\infty} \langle 0 | [\mathcal{Q}_{P_0}, A] | 0 \rangle = \lim_{t_i \to -\infty} \langle 0 | \mathcal{Q}_{P_0} A | 0 \rangle - \langle 0 | A \mathcal{Q}_{P_0} | 0 \rangle \ , \tag{6.1.16}$$

where $|0\rangle$ is the free field (Bunch–Davies) vacuum. We can compute the action of the charge most easily in Schrödinger picture, where the canonical momentum acts like a derivative [144, 153, 154]:

$$\pi_q |0\rangle \longmapsto \pi_q \Psi_{\mathrm{BD}}[\pi, t] \tag{6.1.17}$$

$$\Pi(\vec{q}) |0\rangle \longmapsto -i \frac{\delta}{\delta \pi_q} \Psi_{\mathrm{BD}}[\pi, t] \ . \tag{6.1.18}$$

The Bunch–Davies vacuum wavefunctional is a Gaussian

$$\Psi_{\mathrm{BD}}[\pi, t] \sim \exp\left(-\frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \pi_k D_\pi(k, t) \pi_{-k} \right) \ ; \tag{6.1.19}$$

the kernel D_π is related to the free theory power spectrum by considering

$$\langle 0 | \pi_k \pi_{k'} | 0 \rangle = P_\pi(\vec{k}, t) = \int \mathrm{d}\pi |\Psi_{\mathrm{BD}}|^2 \pi_k \pi_{k'} = (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{k}') \frac{1}{2\mathrm{Re}D_\pi} \ , \tag{6.1.20}$$

which implies

$$\mathrm{Re}D_\pi = \frac{1}{2P_\pi(\vec{k}, t)} \ . \tag{6.1.21}$$

---

[42]We're going to be a bit cavalier about specifying what picture we are working in, as it turns out not to matter much, see [144] for the details.

From this, we can work out the action of the charge $\mathcal{Q}_{P_0}$ on the vacuum:

$$\mathcal{Q}_{P_0}(\vec{q})|0\rangle = \frac{1}{t}\Pi(\vec{q})|0\rangle = -\frac{i}{t}\frac{\delta}{\delta\pi_q}\Psi_{\text{BD}}[\pi,t] = \frac{i}{t}D_\pi(\vec{q})\pi_q\Psi_{\text{BD}}[\pi,t] = \frac{i}{t}D_\pi(\vec{q})\pi_q|0\rangle \quad (6.1.22)$$

$$\langle 0|\mathcal{Q}_{P_0}(\vec{q}) = \langle 0|\frac{1}{t}Pi(\vec{q}) = \frac{i}{t}\frac{\delta}{\delta\pi_q}\Psi^\star_{\text{BD}}[\pi,t] = -\frac{i}{t}D_\pi^\star(\vec{q})\pi_{-q}\Psi^\star_{\text{BD}}[\pi,t] = -iD_\pi^\star(\vec{q})\langle 0|\pi_{-q}\ .$$

From this, it is straightforward to see

$$\langle\Omega|[Q_{P_0},A]|\Omega\rangle = \lim_{t_i\to-\infty} -\frac{i}{t}D_\pi^\star(\vec{q})\langle 0|\pi_{-q}A|0\rangle - \frac{i}{t}D_\pi(q)\langle 0|A\pi_q|0\rangle\ ; \qquad (6.1.23)$$

since we are only working at finite q to regulate the IR divergence, eventually we will take $q \to 0$ so we can replace $\pi_q = \pi_{-q}$. Also, A is just a product of fields, so we know $[A,\pi] = 0$. The above then reduces to

$$\langle\Omega|[Q_{P_0},A]|\Omega\rangle = \lim_{t_i\to-\infty}\lim_{q\to 0} -\frac{2i}{t}\text{Re}D_\pi(q)\langle 0|\pi_q A|0\rangle = -\frac{i}{t}\lim_{q\to 0}\frac{1}{P_\pi(q)}\langle\Omega|\pi_q A(k_1,\ldots,k_N)|\Omega\rangle\ .$$

$$(6.1.24)$$

Right hand side: Comparatively, the right hand side is simple We know that the transformation generated by Q_{P_0} is a time translation and therefore acts on fields as

$$\delta_{P_0}\phi_a = -\frac{\partial}{\partial t_a}\phi_a \qquad (6.1.25)$$

From which we deduce the final form of the Ward identity for broken time translations[43]

$$\lim_{q\to 0}\frac{1}{P_\pi(q)}\langle\pi_q\phi_{k_1}\ldots\phi_{k_N}\rangle' = -t\sum_{a=1}^{N}\frac{\partial}{\partial t_a}\langle\phi_{k_1}\ldots\phi_{k_N}\rangle'\ . \qquad (6.1.26)$$

Here the $'$ on the correlation functions indicate that we have removed the momentum-conserving delta functions from both sides.[44]

---

[43]An important subtlety, which we completely glossed over is that it is important that $\pi \to$ constant as $k \to 0$ in order to identify the free field power spectrum with that of the interacting theory and to translate back and forth between different pictures at late times [144]. Strictly speaking, we should define a new field $\hat{\pi} \equiv t\pi$, which *does* go to a constant at late times, and repeat the analysis, but this does not change anything.

[44]Explicitly, we have

$$\langle\mathcal{O}_{k_1}\ldots\mathcal{O}_{k_N}\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1 + \cdots + \vec{k}_N)\langle\mathcal{O}_1\ldots\mathcal{O}_N\rangle'. \qquad (6.1.27)$$

Boosts: We now consider the Ward identity associated to broken boosts:

$$\langle [Q_{J_{0i}}, A(k_1, \dots, k_N)] \rangle = -i\delta_{J_{0i}} A(k_1, \dots, k_N) \ . \tag{6.1.28}$$

Left hand side: We have already done most of the hard work related to simplifying the left hand side above,

$$\langle \Omega | [Q_{J_{0i}}, A] | \Omega \rangle = \lim_{t_i \to -\infty} \langle 0 | [\mathcal{Q}_{J_{0i}}, A] | 0 \rangle = \lim_{t_i \to -\infty} \langle 0 | \mathcal{Q}_{J_{0i}} A | 0 \rangle - \langle 0 | A \mathcal{Q}_{J_{0i}} | 0 \rangle \ , \tag{6.1.29}$$

where now we have $-\frac{i}{t}\partial_{q^i}\Pi(\vec{q})$ so that the left hand side is given by

$$\langle \Omega | [Q_{J_{0i}}, A] | \Omega \rangle = -\frac{1}{t} \lim_{q \to 0} \frac{\partial}{\partial q^i} \left(\frac{1}{P_\pi(q)} \langle \pi_q \phi_{k_1} \dots \phi_{k_N} \rangle \right) \tag{6.1.30}$$

Right hand side: Boosts act on fields in the theory as (in Fourier space)

$$\delta_{J_{0i}} \phi_{k_a} = -i \frac{\partial}{\partial k_a^i} \frac{\partial}{\partial t_a} \phi_{k_a} \ . \tag{6.1.31}$$

Putting these together, we obtain the Ward identity associated to broken boosts

$$\lim_{q \to 0} \frac{\partial}{\partial q^i} \left(\frac{1}{P_\pi(q)} \langle \pi_q \phi_{k_1} \dots \phi_{k_N} \rangle \right) = -t \sum_{a=1}^{N} \frac{\partial^2}{\partial t_a \partial k_a^i} \langle \phi_{k_1} \dots \phi_{k_N} \rangle \ . \tag{6.1.32}$$

Note that here the derivative with respect to q^i acts on the correlation function *with* the delta function, we may therefore write[45]

$$\frac{\partial}{\partial q^i} \left(\frac{1}{P_\pi(q)} \langle \pi_q \phi_{k_1} \dots \phi_{k_N} \rangle \right) = \frac{\partial}{\partial q^i} \left(\frac{1}{P_\pi(q)} \langle \pi_q \phi_{k_1} \dots \phi_{k_N} \rangle' \right)$$
$$+ \left(\frac{1}{P_\pi(q)} \langle \phi_{k_1} \dots \phi_{k_N} \rangle' \right) \partial_{\vec{P}} \delta^3(\vec{P} + \vec{q}) \ . \tag{6.1.33}$$

---

[45]Here we have defined $\vec{P} = \sum \vec{k}$.

Additionally, on the right hand side, the derivative with respect to momentum hits both the correlator and the delta function

$$-t\sum_{a=1}^{N}\frac{\partial^2}{\partial t_a \partial k_a^i}\langle\phi_{k_1}\ldots\phi_{k_N}\rangle = -t\sum_{a=1}^{N}\frac{\partial^2}{\partial t_a \partial k_a^i}\langle\phi_{k_1}\ldots\phi_{k_N}\rangle' - t\sum_{a=1}^{N}\frac{\partial}{\partial t_a}\langle\phi_{k_1}\ldots\phi_{k_N}\rangle\left(\partial_{\vec{P}}\delta^3(\vec{P})\right)$$

$$(6.1.34)$$

Upon using the lower-order Ward identity, the second term in (6.1.33) cancels the second term in (6.1.34) and we are left with

$$\frac{\partial}{\partial q^i}\left(\frac{1}{P_\pi(q)}\langle\pi(q)\phi_{k_1}\ldots\phi_{k_N}\rangle'\right) = -t\sum_{a=1}^{N}\frac{\partial^2}{\partial t_a \partial k_a^i}\langle\phi_{k_1}\ldots\phi_{k_N}\rangle' \, , \qquad (6.1.35)$$

where the prime indicates removal of the *same* delta function.

Special conformal transformation: Following similar steps, we see that the Ward identity associated to the broken special conformal transformation

$$\langle[Q_{K_0}, A(k_1,\ldots,k_N)]\rangle = -i\delta_{K_0}A(k_1,\ldots,k_N) \, , \qquad (6.1.36)$$

can be simplified to take the form[46]

$$\lim_{q\to 0}\frac{\partial^2}{\partial\vec{q}^2}\left(\frac{1}{P_\pi(q)}\langle\pi_q\phi_{k_1}\ldots\phi_{k_N}\rangle\right) = -t\sum_{a=1}^{N}\frac{\partial^2}{\partial\vec{k}_a^2}\frac{\partial}{\partial t_a}\langle\phi_{k_1}\ldots\phi_{k_N}\rangle \, . \qquad (6.1.37)$$

As before, we note that on the left hand side, the derivatives act on *both* the correlator and the delta function

$$\frac{\partial^2}{\partial\vec{q}^2}\left(\frac{1}{P_\pi(q)}\langle\pi_q\phi_{k_1}\ldots\phi_{k_N}\rangle\right) = \frac{\partial^2}{\partial\vec{q}^2}\left(\frac{1}{P_\pi(q)}\langle\pi_q\phi_{k_1}\ldots\phi_{k_N}\rangle'\right)\delta^3(\vec{P}+\vec{q})$$

$$+\left(\frac{1}{P_\pi(q)}\langle\pi_q\phi_{k_1}\ldots\phi_{k_N}\rangle'\right)\frac{\partial^2}{\partial\vec{q}^2}\delta^3(\vec{P}+\vec{q}) \qquad (6.1.38)$$

$$+2\frac{\partial}{\partial q^i}\left(\frac{1}{P_\pi(q)}\langle\pi_q\phi_{k_1}\ldots\phi_{k_N}\rangle'\right)\frac{\partial}{\partial q^i}\delta^3(\vec{P}+\vec{q}) \, .$$

---

[46]Note that we use $\delta_{K_0}\phi_{k_a} = \frac{\partial^2}{\partial k_a^2}\frac{\partial}{\partial t_a}\phi_{k_a}$.

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Similarly, on the right hand side,

$$\sum_{a=1}^{N} \partial_{k_a}^2 \partial_{t_a} \langle \phi_{k_1} \dots \phi_{k_N} \rangle = \sum_{a=1}^{N} \partial_{k_a}^2 \partial_{t_a} \left(\langle \phi_{k_1} \dots \phi_{k_N} \rangle' \right) \delta^3(\vec{P}) + \sum_{a=1}^{N} \partial_{t_a} \left(\langle \phi_{k_1} \dots \phi_{k_N} \rangle' \right) \partial_{k_a}^2 \delta^3(\vec{P})$$

$$+ 2 \sum_{a=1}^{N} \partial_{k_a^i} \partial_{t_a} \left(\langle \phi_{k_1} \dots \phi_{k_N} \rangle' \right) \partial_{k_a^i} \delta^3(\vec{P}) . \tag{6.1.39}$$

Again, using the lower-order Ward identities, we can cancel the terms where derivatives hit the delta functions and we are left with

$$\lim_{q \to 0} \frac{\partial^2}{\partial \vec{q}^2} \left(\frac{1}{P_\pi(q)} \langle \pi_q \phi_{k_1} \dots \phi_{k_N} \rangle' \right) = -t \sum_{a=1}^{N} \frac{\partial^2}{\partial \vec{k}_a^2} \frac{\partial}{\partial t_a} \langle \phi_{k_1} \dots \phi_{k_N} \rangle' . \tag{6.1.40}$$

6.1.3 Re-summed consistency relation

If we think of the various Ward identities as coefficients of a Taylor series, we can re-sum them into a *consistency relation*

$$\lim_{q \to 0} \frac{1}{P_\pi(q)} \langle \pi_q \phi_{k_1} \dots \phi_{k_N} \rangle' = -t \sum_{a=1}^{N} \left(1 + q^i \partial_{k_a^i} + \frac{1}{6} q^2 \partial_{k_a}^2 \right) \partial_{t_a} \langle \phi_{k_1} \dots \phi_{k_N} \rangle' . \tag{6.1.41}$$

The factor of $\frac{1}{6}$ comes from the fact that $\nabla^2 \vec{x}^2 = 6$. In the case of all the fields on the right hand side being the same, we can do the sum over the various times to get a total time derivative, provided we divide by N to obtain

$$\lim_{q \to 0} \frac{1}{P_\pi(q)} \langle \pi_q \phi_{k_1} \dots \phi_{k_N} \rangle' = - \left(1 + \frac{1}{N} q^i \sum_{a=1}^{N} \partial_{k_a^i} + \frac{1}{6N} \vec{q}^2 \sum_{a=1}^{N} \partial_{k_a}^2 \right) \frac{\mathrm{d}}{\mathrm{d} \log t} \langle \phi_{k_1} \dots \phi_{k_N} \rangle' . \tag{6.1.42}$$

This result was obtained in [8] using different techniques; see that article for an explicit demonstration that (6.1.42) is equivalent to (6.1.26), (6.1.35) and (6.1.40), along with many more checks of the identity. The intrepid reader who wishes to check these identities for themself will find explicit calculations of a myriad of correlation functions in Appendix B.2.

6.1.4 An explicit check of the Ward identities

Although we have derived the Ward identities (6.1.26), (6.1.35) and (6.1.40) systematically, it is still worthwhile to check them in an explicit example. To perform an explicit check, we work on-shell[47] on both sides at all times. Therefore, on the left-hand side, we express one of the momenta, say k_N, as a sum of the other momenta. We then take the squeezed limit, obtaining a function of q, the small momentum. In order to check the various relations, we can then take derivatives of this left hand side with respect to q and then set $q = 0$. On the right-hand side, we must also work on-shell. This means that we also write the k_N momentum in terms of the other $N - 1$ momenta (not including q).

Schematically, the procedure is as follows: consider checking the consistency relation

$$\partial_q^2 \left(\frac{1}{P_\pi(q)} \langle \pi_{\vec{q}} \phi_{\vec{k}_1} \ldots \phi_{\vec{k}_N} \rangle' \right) = -\frac{1}{N} t \sum_{a=1}^{N} \frac{\mathrm{d}}{\mathrm{d} k_a^2} \frac{\mathrm{d}}{\mathrm{d} t} \langle \phi_{\vec{k}_1} \ldots \phi_{\vec{k}_N} \rangle' . \tag{6.1.43}$$

We rewrite the left hand side so that it is a function of N different momenta, that is we take $\vec{k}_N = -\sum \vec{k}_a - \vec{q}$. We then take the squeezed limit $q \to 0$ and differentiate with respect to q. On the right hand side, we write $\vec{k}_N = -\sum \vec{k}_a$. This means that we actually only have to take $N - 1$ derivatives on the right hand side.

For illustrative purposes, we provide an explicit check of the consistency relation in differential form. Consider the soft limit of the three-point function involving only π fields, $\langle \pi^3 \rangle$. The three and two-point correlation functions are given by

$$\langle \pi_{\vec{q}} \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' = \frac{81 H^4}{4 M_\pi^4} \frac{\left(q^5 + k_1^5 + k_2^5 \right)}{q^5 k_1^5 k_2^5 t^4} \; ; \qquad \langle \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' = \frac{9 H^2}{2 M_\pi^2} \frac{1}{k_1^5 t^2} . \tag{6.1.44}$$

---

[47]This usage of on-shell is slightly non-standard. What we mean is that we re-write k_N on both sides in terms of the other momenta, enforcing the delta function constraint explicitly.

We take the limit $q \to 0$ to obtain the squeezed limit of the three-point function

$$\frac{1}{P_\pi(q)} \langle \pi_{\vec{q}} \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' = \frac{9H^2}{2M_\pi^2 t^2} \left[\frac{2}{k_1^5} + \frac{5(\vec{q} \cdot \vec{k}_1)}{k_1^7} + \frac{5 \left(7(\vec{q} \cdot \vec{k}_1)^2 - q^2 k_1^2 \right)}{2k_1^9} \right] + \mathcal{O}(q^3) \ . \quad (6.1.45)$$

From this, we immediately read off:

$$
\begin{aligned}
\frac{1}{P_\pi(q)} \langle \pi_{\vec{q}} \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle'_{q \to 0} &= \frac{9H^2}{M_\pi^2} \frac{1}{k_1^5 t^2} = -t \frac{\mathrm{d}}{\mathrm{d}t} \langle \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' \ ; \\
\frac{\partial}{\partial q^i} \left(\frac{1}{P_\pi(q)} \langle \pi_{\vec{q}} \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' \right) \bigg|_{q=0} &= \frac{45H^2}{M_\pi^2} \frac{k_1^i}{k_1^7 t^2} = -\frac{1}{2} t \frac{\mathrm{d}}{\mathrm{d}k_1^i} \frac{\mathrm{d}}{\mathrm{d}t} \langle \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' \ ; \quad (6.1.46) \\
\partial_q^2 \left(\frac{1}{P_\pi(q)} \langle \pi_{\vec{q}} \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' \right) \bigg|_{q=0} &= \frac{90H^2}{M_\pi^2} \frac{1}{k_1^7 t^2} = -\frac{1}{2} t \frac{\mathrm{d}^2}{\mathrm{d}k_1^2} \frac{\mathrm{d}}{\mathrm{d}t} \langle \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle' \ ,
\end{aligned}
$$

where the last step in each equation follows from differentiating $\langle \pi_{\vec{k}_1} \pi_{\vec{k}_2} \rangle'$. Thus the derivative form of the consistency relation checks out at each order.

6.2 Connection to observables: soft internal lines and anisotropy of the power spectrum

As we discussed, the breaking of $SO(4, 2)$ implies the existence of the Goldstone field π, and consequently the consistency relation we derived constrains correlation functions with soft external Goldstone lines. Unfortunately, the cosmological perturbations we observe come from a spectator field and not from π, so that it is not obvious how one can connect the previous results to observations. There are, however, two situations in which $SO(4, 2)$ is observationally relevant. The first is when diagrams of the spectator field contain soft *internal* π lines (for similar results in inflation see [140, 155, 156]). Internal soft π lines are expected to give the dominant contribution when a sum of external momenta becomes small, and they will dominate in comparison with soft internal lines of the spectator field, because of the very red spectrum of the Goldstone. The second possibility stems from the fact that, even if π is not directly measured, its value during the conformal phase is correlated with the modes of the spectator field and thus changes their statistics. In particular, very long modes of π induce an anisotropy in the spectator field power spectrum. These two
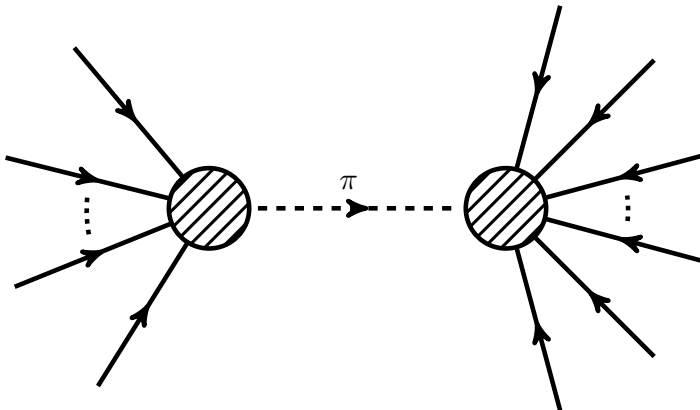
99

Figure 6: Factorization of an $(N + M)$-point function via an exchange of a single soft internal π field.

observational features were studied in [96–98]. Here we want to stress that these properties are a direct consequence of the non-linear realization of $SO(4,2)$ and not specific to a given model. We will also find an additional important contribution to the four-point function from a loop of π fields that has been overlooked in the literature. This contribution may be larger than the tree-level π exchange and it is phenomenologically quite different.

Let us start with soft internal π lines. In the limit in which the sum of N external momenta becomes small, the amplitude of an $(N + M)$-point function factorizes in the following way (see Fig. 6)

$$\langle \chi_{\vec{k}_1} \cdots \chi_{\vec{k}_{M+N}} \rangle'_{q \to 0} = \frac{1}{P_\pi(q)} \langle \pi_{-\vec{q}} \chi_{\vec{k}_1} \cdots \chi_{\vec{k}_M} \rangle'_{q \to 0} \langle \pi_{\vec{q}} \chi_{\vec{k}_{M+1}} \cdots \chi_{\vec{k}_{M+N}} \rangle'_{q \to 0} . \tag{6.2.1}$$

The $(N+1)$ and $(M+1)$-point functions are severely constrained by the $SO(4,1)$ symmetry and their squeezed limit is further constrained by the non-linear realization of $SO(4,2)$. In this way, the amplitude for the $(N + M)$-point function with a soft internal line can be expressed in terms N and M-point functions. The simplest case is the four-point function of massless spectator fields, which was studied in detail in [96–98]. Using the factorization (6.2.1) above and the squeezed limit (B.2.27) for the three-point function $\langle \pi \chi \chi \rangle$, in the

limit $\vec{k}_1 + \vec{k}_2 \equiv \vec{q} \to 0$ we get

$$\langle \chi_{\vec{k}_1} \cdots \chi_{\vec{k}_4} \rangle'_{q \to 0} = \frac{\pi^2}{144} \mathcal{P}_\pi \mathcal{P}_\chi^2 \frac{1}{q k_1^4 k_3^4} \left(3(\hat{k}_1 \cdot \hat{q})^2 - 1 \right) \left(3(\hat{k}_3 \cdot \hat{q})^2 - 1 \right) , \qquad (6.2.2)$$

where $\mathcal{P}_\pi \equiv 9H^2/2M_\pi^2$; $\mathcal{P}_\chi \equiv H^2/2M_\chi^2$ are the dimensionless power spectra. It is important to stress that the shape of the four-point function in the soft internal limit is completely specified by symmetries since the three-point function $\langle \pi \chi \chi \rangle$ is completely fixed by $SO(4,1)$ up to an overall constant. Notice that the squeezed limit of the three-point function is constrained, as we discussed in the previous Section, by $SO(4,2)$ as well. In the massless case we cannot obtain terms scaling as q^0 or q^1, and all terms scaling as q^2 must vanish when averaged over the angles. This is indeed what we have in (6.2.2).

The four-point function becomes very large in the $q \to 0$ limit, as it scales as $1/q$. This is a consequence of the very red spectrum of π and it can be contrasted, for example, with inflationary models with reduced speed of sound which are regular in the $q \to 0$ limit [157, 158] . We conclude that *a four-point function which becomes large in the soft internal (collapsed) limit, with the precise shape* (6.2.2)*, is a general prediction of the conformal mechanism.* Notice, however, that the overall multiplicative constant in (6.2.2) cannot be fixed by symmetry arguments.

If one assumes a linear relation between ζ and χ (non-linearities will give additional model-dependent contributions to local non-Gaussianity) we get that the four-point function above has an amplitude

$$\frac{\langle \zeta \zeta \zeta \zeta \rangle}{\mathcal{P}_\zeta^3} \simeq \frac{\pi^2}{144} \cdot \frac{\mathcal{P}_\pi}{\mathcal{P}_\zeta} . \qquad (6.2.3)$$

Although data analysis has not been performed for the particular momentum dependence of (6.2.2), one can get a rough constraint using limits on equilateral models of four-point function[48] obtained in [159]: $|t_{\mathrm{NL}}^{\mathrm{equil}}| \lesssim 7 \cdot 10^6$. This gives

$$\mathcal{P}_\pi \lesssim 500 . \qquad (6.2.4)$$

---

[48]This may be slightly conservative as equilateral shapes are regular in the limit $q \to 0$.

The four-point function we studied is obtained by averaging over the long wavelength modes of π. However, if we do not take the statistical average, we still have a realization-dependent effect: long modes of π induce an anisotropy in the power spectrum of the short modes, as pointed out in [96–98]. Notice that this is possible even though π does not contribute to the observed perturbations: its value during the conformal phase still affects the observable modes of the spectator field. This effect is also completely fixed, up to an overall constant, by the symmetries of the problem.

The effect of a long π mode on the observable 2-point function can be read from the three-point function $\langle \pi \chi \chi \rangle$, given by (B.2.27), in the squeezed limit

$$\langle \pi_{\vec{q}} \chi_{\vec{k}_1} \chi_{\vec{k}_2} \rangle'_{q \to 0} = \frac{\pi}{12} \mathcal{P}_\pi \mathcal{P}_\chi \frac{1}{q^5 k^6 t} q^2 k^2 \left(3 \cos^2 \theta - 1 \right) \ , \tag{6.2.5}$$

where θ is the relative angle between $\vec{k}$ and $\vec{q}$. We can write the variation of the power spectrum of χ in the presence of a given realization of the π field in a schematic way as

$$\delta \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle' = \frac{\langle \pi_{\vec{q}} \chi \chi \rangle'_{q \to 0}}{\langle \pi_{\vec{q}} \pi_{-\vec{q}} \rangle'_{q \to 0}} \pi_{\vec{q}} = \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle' \cdot \frac{\pi}{12} \frac{1}{k} (3 \cos^2 \theta - 1) t q^2 \pi_{\vec{q}} \ . \tag{6.2.6}$$

All modes $\pi_{\vec{q}}$ which are outside the present Hubble radius will contribute to the anisotropy of the χ power spectrum. The typical size of the effect is given by the square root of the variance calculated by summing over all super Hubble modes

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \langle t^2 q^4 \pi_{\vec{q}} \pi_{-\vec{q}} \rangle' = \frac{1}{2\pi^2} \int_0^{H_0} q^2 \mathrm{d}q \, t^2 q^4 \frac{\mathcal{P}_\pi}{q^5 t^2} \sim \frac{1}{4\pi^2} \mathcal{P}_\pi H_0^2 \ . \tag{6.2.7}$$

This gives

$$\langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle'_\pi = \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle' \left(1 + c_1 \frac{\sqrt{\mathcal{P}_\pi}}{2\pi} \frac{H_0}{k} (3 \cos^2 \theta - 1) \right) \ , \tag{6.2.8}$$

where c_1 is a number of order unity, which depends on our position in the Universe [96–98].

Another source of anisotropy in the power spectrum arises by considering a four-point function $\langle \pi \pi \chi \chi \rangle$ [96–98]. This induces a variation of the 2-point function $\langle \chi \chi \rangle$ in the

presence of two long modes of π. In this case the $SO(4,2)$ symmetry fixes both the shape and the normalization of the effect. The variation of the 2-point function $\langle \chi \chi \rangle$ in the presence of two long background modes π_1 and π_2 corresponds to the composition of the associated $SO(4,2)$ transformations. A possible issue is that the broken generators K_0, J_{0i} and P_0 do not commute, so that the overall transformation seems to depend on the ordering. Fortunately, in our case all the commutators of the broken generators of $SO(4,2)$ give *unbroken* generators. These do not change the 2-point function, so that we do not have to worry about non-commutativity in the case at hand. Since the 2-point function is time-independent, its variation at lowest order in gradients will come from a boost at second order. Without loss of generality, we consider a boost along the x-direction. The transformation of coordinates is given by

$$x' = \gamma(x - v_x t)\,, \ \ y' = y\,, \ \ z' = z\,, \ \ t' = \gamma(t - v_x x)\,, \tag{6.2.9}$$

where $\gamma \equiv (1 - v^2)^{-1/2}$. Neglecting parts proportional to t, the induced background field is, up to second order in v_x,

$$\pi = -\frac{\delta t}{t} = \frac{v_x x}{t}\,. \tag{6.2.10}$$

In momentum space, the parameter v_x is given by

$$v_x = it q_x \pi_{\vec{q}}\,. \tag{6.2.11}$$

The transformation (6.2.9) implies that in momentum space k_x component of the wave vector has to be multiplied by γ^{-1}, while k_y and k_z remain the same. Expanding k^{-3} in the denominator of the power spectrum, we find that the effect on the 2-point function of χ is:

$$\delta \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle' = \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle \frac{3}{2} \frac{(\vec{v} \cdot \vec{k})^2}{k^2} = -\langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle \frac{3}{2} t^2 q^2 \pi_{\vec{q}}^2 \cos^2 \theta\,. \tag{6.2.12}$$

We can calculate the typical value of $t^2 q^2 \pi_{\vec{q}}^2$ in a way similar to before:

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \langle t^2 q^2 \pi_{\vec{q}}^2 \rangle = \frac{1}{2\pi^2} \int_0^{H_0} q^2 \mathrm{d}q \, t^2 q^2 \frac{\mathcal{P}_\pi}{q^5 t^2} \sim \frac{1}{2\pi^2} \mathcal{P}_\pi \log \frac{H_0}{\Lambda} , \qquad (6.2.13)$$

where Λ is an IR cutoff. The contribution to the anisotropy is given by:

$$\delta \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle' = -\langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle \frac{3}{4\pi^2} \mathcal{P}_\pi \log \frac{H_0}{\Lambda} \cos^2 \theta . \qquad (6.2.14)$$

Combining with (6.2.8), the total anisotropy of the power spectrum is given by

$$\langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle'_{\pi_{\vec{q}}} = \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle' \left(1 + c_1 \frac{\sqrt{\mathcal{P}_\pi}}{2\pi} \frac{H_0}{k} (3 \cos^2 \theta - 1) + c_2 \frac{3 \mathcal{P}_\pi}{4\pi^2} \cos^2 \theta \log \frac{H_0}{\Lambda} \right) , \qquad (6.2.15)$$

where c_2 is another constant of order unity, which depends on the particular position in the Universe. The two sources of anisotropies are quite different. The first scales as $1/k$, and thus important only for long modes, while the second is scale invariant. Moreover, the first contribution averages to zero if summed over the possible orientations between long and short modes, while the second does not. Notice also that the first effect is dominated by π modes which are slightly longer than the present Hubble radius, while the second gets contributions from all scales as shown by the logarithmic dependence. The logarithmic enhancement can overcome the suppression due to the fact that the second effect is of order π^2 and not π.

As we have seen, the second contribution to the power-spectrum anisotropy is related to the correlator $\langle \pi\pi\chi\chi \rangle$. This suggests that we missed a potentially large contribution to the four-point function of χ's in the soft internal limit, coming from a loop of soft π particles (see Fig. 7). At first this looks worrisome as we expect a loop diagram to be small compared to a tree-level one. However, the situation is similar to the one we discussed for the anisotropy. When only one soft π is exchanged, the interaction with the χ's arises at order q^2 as we discussed above. When two soft π's are exchanged, on the other hand, each of them carries a single soft momentum, as the interaction arises from the non-linear realization of boosts.
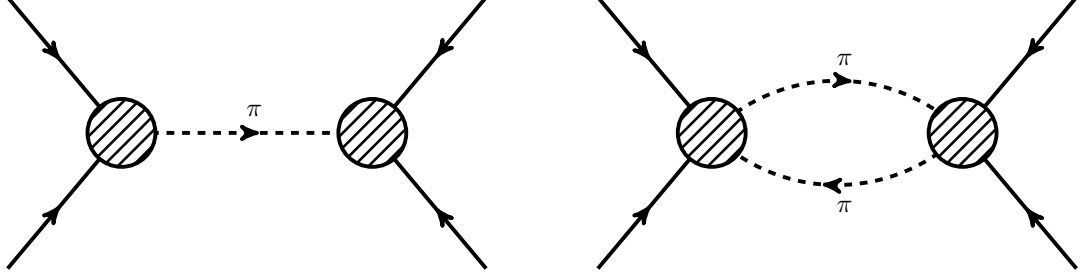
Figure 7: The four-point function of χ's with an exchange of both one and two soft internal π fields.

Therefore, in going from tree-level π exchange to a one-loop diagram the number of q's at the vertices remains the same, and we have the extra loop factor

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathcal{P}_\pi}{q^5} \sim \frac{\mathcal{P}_\pi}{q^2} \; . \tag{6.2.16}$$

If q is small enough compared with the external momenta, the loop diagram will dominate over the tree level exchange. Notice that this does not signal a breakdown of perturbation theory: it is straightforward to check that the exchange of extra π's is not further enhanced by powers of $1/q$, but only suppressed by powers of $\mathcal{P}_\pi$.

The loop diagram is straightforward to evaluate starting from (6.2.12)

$$\langle \chi_{\vec{k}_1} \chi_{\vec{k}_2} \chi_{\vec{k}_3} \chi_{\vec{k}_4} \rangle'_{q \to 0} = \frac{9}{2} \frac{\mathcal{P}_\chi}{k_1^3} \frac{\mathcal{P}_\chi}{k_3^3} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} (\hat{q}_1 \cdot \hat{k}_1)(\hat{q}_2 \cdot \hat{k}_1)(\hat{q}_1 \cdot \hat{k}_3)(\hat{q}_2 \cdot \hat{k}_3) \frac{\mathcal{P}_\pi}{q_1^3} \frac{\mathcal{P}_\pi}{q_2^3} \; , \tag{6.2.17}$$

where $\vec{q} \equiv \vec{k}_1 + \vec{k}_2$ and $\vec{q}_1 + \vec{q}_2 = \vec{q}$. In writing this expression we have assumed that both internal legs are soft so that their coupling is fixed by the non-linear realization of $SO(4, 2)$. Indeed we will see that the loop integral is dominated by having q_1 and q_2 both of order q. If we disregard the angular dependence and average over the direction of the short modes, we get

$$\frac{1}{2} \frac{\mathcal{P}_\chi}{k_1^3} \frac{\mathcal{P}_\chi}{k_3^3} \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} (\hat{q}_1 \cdot \hat{q}_2)^2 \frac{\mathcal{P}_\pi}{q_1^3} \frac{\mathcal{P}_\pi}{q_2^3} \; . \tag{6.2.18}$$

The loop integral is dominated by long modes and it is IR divergent, similarly to what

happened for the anisotropy of the power spectrum. We get

$$\langle \chi_{\vec{k}_1} \chi_{\vec{k}_2} \chi_{\vec{k}_3} \chi_{\vec{k}_4} \rangle'_{q \to 0} \sim \frac{1}{24\pi^2} \frac{\mathcal{P}_\chi}{k_1^3} \frac{\mathcal{P}_\chi}{k_3^3} \frac{\mathcal{P}_\pi^2}{q^3} \log \frac{q}{\Lambda} \; . \tag{6.2.19}$$

As promised this result contains, when compared with the tree-level calculation (6.2.2), a factor of $\mathcal{P}_\pi k^2 / q^2$ which may be large for sufficiently small q.

Notice that the momentum dependence of this result (after performing the angular average) is exactly the one of a $\tau_{\rm NL}$ non-Gaussianity. Again assuming a linear relation between ζ and χ we get

$$\tau_{\rm NL} \sim \frac{1}{96\pi^2} \frac{\mathcal{P}_\pi^2}{\mathcal{P}_\zeta} \log \frac{q}{\Lambda} \; . \tag{6.2.20}$$

Using the experimental limit $|\tau_{\rm NL}| \lesssim 2 \cdot 10^4$ [160] and neglecting the logarithmic enhancement, one gets a rough limit on $\mathcal{P}_\pi$

$$\mathcal{P}_\pi \lesssim \mathcal{P}_\zeta^{1/2} \cdot (96\pi^2 \cdot 2 \cdot 10^4)^{1/2} \simeq 1 \; . \tag{6.2.21}$$

This (rough) limit is stronger than the one obtained from the tree-level four-point function. The four-point function (6.2.19) will also contribute both to a stochastic scale-dependent bias [161] and to the power spectrum of μ-distortion [162]. It would be interesting to understand whether the angle dependence, which is different from a standard $\tau_{\rm NL}$ shape, affects these observables.

In this paper we only studied correlation functions in the absence of gravity. As discussed in [52, 99], this is a good approximation for sufficiently early times; π perturbations give a negligible contribution to the observable quantity ζ, while χ perturbations will source ζ by one of the standard conversion mechanisms.

An important concluding remark is in order. Our $SO(4,2)$ consistency relations are not as constraining as the ones for single-field inflation. In that case one can derive consistency relations directly in terms of the observed variable ζ which, if violated, would rule out any

single-field model. Here, on other hand, we can just single out the effects due to the emission of π, but their relation with observables is ultimately model-dependent: for instance, all the effects we discussed vanish in the limit $P_\pi \to 0$. This is ultimately due to the fact that we are discussing a multi-field model, where perturbations are sourced by an isocurvature field. Even though we cannot derive completely model-independent relations, the red spectrum of π makes the contributions discussed above sufficiently peculiar to be distinguishable from the other model-dependent effects.

Chapter 7

Violating the null energy condition

The null energy condition (NEC) is the most robust of all energy conditions. It states that, for any null vector n^μ,

$$T_{\mu\nu}n^\mu n^\nu \geq 0\,. \tag{7.0.1}$$

It has proven extremely difficult to violate this condition with well-behaved relativistic quantum field theories. Aside from being of purely theoretical interest, the NEC plays a fundamental role in our understanding of the early universe. In cosmology, (7.0.1) is equivalent to $\rho + P \geq 0$, which, combined with the equation for a spatially-flat universe,

$$M_{\rm Pl}^2 \dot{H} = -\frac{1}{2}(\rho + P)\,, \tag{7.0.2}$$

forbids a non-singular bounce from contraction to expansion. This means a contracting universe necessarily ends in a big crunch singularity, and an expanding universe must emerge from a big bang. Violating (7.0.1) is therefore central to any alternative to inflation relying either on a contracting phase before the big bang [5, 12, 13, 24, 32], or an expanding phase from an asymptotically static past [27, 52].

For theories with at most two derivatives, violating the NEC necessarily implies ghosts or gradient instabilities [163]. To evade this, one must therefore invoke higher derivatives, as in the *ghost condensate* [164]. Perturbations around the ghost condensate can violate the NEC in a stable manner [139], and this has been used in the New Ekpyrotic scenario [46, 48].

However, because the scalar field starts out with a wrong-sign kinetic term, the theory is unstable around its Poincaré-invariant vacuum.

Stable NEC violation can also be achieved with *conformal galileons* [103], a class of conformally-invariant scalar field theories with particular higher-derivative interactions. Remarkably, in spite of the fact that there are five independent galileon terms, only the kinetic term contributes to (7.0.1) [106]: violating the NEC requires a wrong-sign kinetic term, just like the ghost condensate. Another issue with conformal galileons is superluminal propagation around slight deformations of the NEC-violating background [52] (though this can be avoided by explicitly breaking special conformal transformations [106]).

In this Chapter, we show that the *DBI conformal galileons* [105, 107] can also violate the NEC in a stable manner, while avoiding nearly all of the aforementioned issues. Specifically, the coefficients of the five DBI galileons can be chosen such that:

1. There exists a stable, Poincaré-invariant vacuum.

2. The $2 \to 2$ scattering amplitude about this vacuum obeys standard analyticity conditions.

3. The theory admits a time-dependent, homogeneous and isotropic solution which violates the NEC in a stable manner.

4. Perturbations around the NEC-violating background, and around small deformations thereof, propagate subluminally.

5. This solution is stable against radiative corrections.

In other words, starting from a local relativistic quantum field theory defined around a Poincaré-invariant vacuum state, the theory allows consistent, stable, NEC-violating solutions. In fact, this NEC-violating background is an *exact* solution of the effective theory, including all possible higher-dimensional operators consistent with the assumed symmetries.

We will see that the above conditions can be satisfied for a broad region of parameter space. This represents a significant improvement over ghost condensation (which fails to satisfy 1 and 2) and the ordinary conformal galileons (which fail to satisfy 1, 2 and 4). Unfortunately, like conformal galileons, superluminal propagation around deformations of the Poincaré invariant solution is inevitable. As a result, the full S-matrix likely fails to be analytic. Additionally, one would like the theory to be consistent with black hole thermodynamics [165]. This is currently under investigation [166].

The geometric origin of the DBI conformal galileon as the theory of a 3-brane moving in an AdS$_5$ bulk makes contact with stringy scenarios, offering a promising avenue to search for NEC violations in string theory.

7.1 The theory

We return to the geometric construction of the conformal mechanism of Section 3.5 and generalize to higher order actions (see also Appendix A.1). We consider again a 3-brane, with worldvolume coordinates x^μ, probing an AdS$_5$ space-time with coordinates X^A and metric $G_{AB}(X)$ in the Poincaré patch

$$\mathrm{d}s^2 = G_{AB}\mathrm{d}X^A\mathrm{d}X^B = Z^{-2}\mathrm{d}Z^2 + Z^2\eta_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu\,, \tag{7.1.1}$$

where $Z \equiv X^5$, $0 < Z < \infty$. The dynamical variables are the embedding functions, $X^\mu(x)$, $Z(x) \equiv \phi(x)$. In unitary gauge, $X^\mu = x^\mu$, the brane induced metric is

$$\mathfrak{g}_{\mu\nu} = G_{AB}\partial_\mu X^A\partial_\nu X^B = \phi^2\eta_{\mu\nu} + \phi^{-2}\partial_\mu\phi\partial_\nu\phi\,. \tag{7.1.2}$$

The DBI conformal galileons are five geometric invariants consisting of 4D Lovelock terms ($\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_4$) and the boundary terms of 5D Lovelock terms ($\mathcal{L}_3$ and $\mathcal{L}_5$):

$$
\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{4}\phi^4\,, \\
\mathcal{L}_2 &= -\sqrt{-\mathfrak{g}} = -\gamma^{-1}\phi^4\,, \\
\mathcal{L}_3 &= \sqrt{-\mathfrak{g}}K = -6\phi^4 + \phi[\Phi] + \gamma^2\phi^{-3}\left(-[\phi^3] + 2\phi^7\right)\,, \\
\mathcal{L}_4 &= -\sqrt{-\mathfrak{g}}\mathfrak{R} \\
&= 12\gamma^{-1}\phi^4 + \gamma\phi^{-2}\left\{[\Phi^2] - \left([\Phi] - 6\phi^3\right)\left([\Phi] - 4\phi^3\right)\right\} \\
&+ 2\gamma^3\phi^{-6}\left\{-[\phi^4] + [\phi^3]\left([\Phi] - 5\phi^3\right) - 2[\Phi]\phi^7 + 6\phi^{10}\right\}\,, \\
\mathcal{L}_5 &= \frac{3}{2}\sqrt{-\mathfrak{g}}\left(-\frac{K^3}{3} + K^2_{\mu\nu}K - \frac{2}{3}K^3_{\mu\nu} - 2\mathfrak{G}_{\mu\nu}K^{\mu\nu}\right) \\
&= 54\phi^4 - 9\phi[\Phi] + \gamma^2\phi^{-5}\left\{9[\phi^3]\phi^2 + 2[\Phi^3] - 3[\Phi^2][\Phi]\right. \\
&+ 12[\Phi^2]\phi^3 + [\Phi]^3 - 12[\Phi]^2\phi^3 + 42[\Phi]\phi^6 - 78\phi^4\right\} \\
&+ 3\gamma^4\phi^{-9}\left\{-2[\phi^5] + 2[\phi^4]\left([\Phi] - 4\phi^3\right)\right. \\
&+ [\phi^3]\left([\Phi^2] - [\Phi]^2 + 8[\Phi]\phi^3 - 14\phi^6\right) \\
&+ 2\phi^7\left([\Phi]^2 - [\Phi^2]\right) - 8[\Phi]\phi^{10} + 12\phi^{13}\right\}\,. \tag{7.1.3}
\end{aligned}
$$

Here $\gamma \equiv 1/\sqrt{1 + (\partial\phi)^2/\phi^4}$, is the Lorentz factor for the brane motion, $\mathcal{L}_1$ measures the proper 5-volume between the brane and some fixed reference brane [107], and $\mathcal{L}_2$ is the world-volume action, *i.e.*, the brane tension.[49] The higher-order terms $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$ are functions of the extrinsic curvature tensor

$$
K_{\mu\nu} = \gamma\left(-\phi^{-1}\partial_\mu\partial_\nu\phi + \phi^2\eta_{\mu\nu} + 3\phi^{-2}\partial_\mu\phi\partial_\nu\phi\right) \tag{7.1.4}
$$

and the induced Ricci tensor $\mathfrak{R}_{\mu\nu}$ and scalar $\mathfrak{R}$, with $\mathfrak{G}_{\mu\nu} \equiv \mathfrak{R}_{\mu\nu} - \mathfrak{R}\mathfrak{g}_{\mu\nu}/2$ (and indices raised by $\mathfrak{g}^{\mu\nu}$). Following [103], Φ denotes the matrix of second derivatives $\partial_\mu\partial_\nu\phi$, $[\Phi^n] \equiv \mathrm{Tr}(\Phi^n)$, and $[\phi^n] \equiv \partial\phi \cdot \Phi^{n-2} \cdot \partial\phi$, with indices raised by $\eta^{\mu\nu}$.

---

[49]The brane tension, c_2, will turn out to be positive for the relevant region of parameter space

Each $\mathcal{L}$ is invariant up to a total derivative under the $\mathfrak{so}(4,2)$ conformal algebra, inherited from the isometries of AdS$_5$. Aside from Poincaré transformations, (7.1.3) is also invariant under dilation, $\delta_D \phi = -(1 + x^\mu \partial_\mu)\phi$, and special conformal transformations, $\delta_{K_\mu}\phi = (-2x_\mu - 2x_\mu x^\nu \partial_\nu + x^2 \partial_\mu + \phi^{-2}\partial_\mu)\phi$.

7.1.1 Around the Poincaré invariant vacuum

Expanding $\mathcal{L} = \sum_{i=1}^{5} c_i \mathcal{L}_i$ around a constant field profile, $\bar{\phi}_0$, up to quartic order in perturbations $\varphi = \phi - \bar{\phi}_0$, we obtain

$$
\begin{aligned}
\mathcal{L} = {} & -\frac{C_2}{2}(\partial\varphi)^2 + \frac{C_3}{12\bar{\phi}_0^3}(\partial\varphi)^2 \Box\varphi + \frac{(3C_2 - C_3)}{24\bar{\phi}_0^4}(\partial\varphi)^4 \\
& - \frac{C_3}{4\bar{\phi}_0^4}\varphi(\partial\varphi)^2 \Box\varphi + \frac{C_4}{24\bar{\phi}_0^6}(\partial\varphi)^2 \left[(\partial_\mu\partial_\nu\varphi)^2 - (\Box\varphi)^2\right] ;
\end{aligned}
\tag{7.1.5}
$$

where

$$
\begin{aligned}
C_2 &\equiv c_2 + 6c_3 + 12c_4 + 6c_5 , \quad C_3 \equiv 6c_3 + 36c_4 + 54c_5 , \\
C_4 &\equiv 12c_4 + 48c_5 , \qquad\qquad C_5 \equiv c_5 ,
\end{aligned}
\tag{7.1.6}
$$

where, in order for $\bar{\phi}_0$ to be a solution, we have imposed that the tadpole term vanish:

$$
C_1 \equiv -\frac{1}{4}c_1 - c_2 - 4c_3 + 12c_5 = 0 \quad \text{(Poincaré solution)} .
\tag{7.1.7}
$$

A necessary and sufficient condition for the stability of small fluctuations is

$$
C_2 > 0 \qquad \text{(stability)} .
\tag{7.1.8}
$$

Next, the scattering S-matrix derived from (7.1.5) should satisfy standard relativistic dispersion relations. Firstly, the $2 \to 2$ amplitude in the forward limit must display a positive s^2 contribution [92]. Only the $(\partial\varphi)^4$ vertex contributes in the forward limit—its coefficient must be strictly positive [92, 137]. There also exist constraints away from the forward

limit [167], which involve the $(\partial\varphi)^2\Box\varphi$ and $(\partial\varphi)^2(\partial_\mu\partial_\nu\varphi)^2$ vertices [104]. These analyticity conditions respectively impose

$$C_3 < 3C_2 \; ; \qquad C_3^2 > 6C_2C_4 \qquad\qquad \text{(analyticity)} . \qquad (7.1.9)$$

7.2 NEC-violating solution

We seek a time-dependent, isotropic background solution of the form

$$\bar{\phi} = \frac{\alpha}{(-t)} \; ; \qquad -\infty < t < 0 , \qquad\qquad (7.2.1)$$

where α is a constant. This profile, which is central to pseudo-conformal [7, 12, 13] and Galilean Genesis [52] cosmology, spontaneously breaks the $\mathfrak{so}(4,2)$ algebra down to an $\mathfrak{so}(4,1)$ subalgebra. Substituting (7.2.1) into the equation of motion for ϕ derived from (7.1.3), we obtain

$$C_2 + \frac{1}{2}C_3\beta + \frac{1}{2}C_4\beta^2 + 6C_5\beta^3 = 0 \qquad\qquad \text{(1/t solution)} , \qquad (7.2.2)$$

with $\beta \equiv \bar{\gamma} - 1 > 0$, $\bar{\gamma} = 1/\sqrt{1 - \alpha^{-2}}$. There is a solution for each real, positive root of (7.2.2).

We require this background to be stable against small perturbations. Expanding (7.1.3) to quadratic order in $\varphi \equiv \phi - \bar{\phi}$, we obtain

$$\mathcal{L}_{\text{quad, }1/t} = \frac{\mathcal{Z}}{2}\left(\dot{\varphi}^2 - \bar{\gamma}^{-2}(\vec{\nabla}\varphi)^2 + \frac{6}{t^2}\varphi^2\right) , \qquad (7.2.3)$$

where $\mathcal{Z} \equiv \bar{\gamma}^3(C_2 + C_3\beta + 3C_4\beta^2/2 + 24C_5\beta^3)$. Absence of ghosts therefore requires

$$C_2 + C_3\beta + \frac{3}{2}C_4\beta^2 + 24C_5\beta^3 > 0 \qquad\qquad \text{(stability)} . \qquad (7.2.4)$$

The sound speed is always subluminal, but for small deformations away from the solution

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to satisfy Condition 4, we want the sound speed $c_s = \bar{\gamma}^{-1}$ to be generously less than unity. Thus we demand

$$\beta \gtrsim 1 \qquad \text{(robust subluminality around } 1/t). \qquad (7.2.5)$$

To check for NEC violation, we calculate the stress tensor $T_{\mu\nu}$ by varying the covariant version of (7.1.3) with respect to the metric. The covariant theory is given uniquely by the brane construction [105], and is given by (7.1.3) with the replacements $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$, plus the following non-minimal couplings:

$$
\begin{aligned}
\delta\mathcal{L}_4 &= -\gamma^{-1}R\phi^2 + 2\gamma\phi^{-2}R^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\
\delta\mathcal{L}_5 &= (3/2)R\phi^{-5}\left\{\phi^4\left([\Phi]-4\phi^3\right) + \gamma^2\left(-[\phi^3]+2\phi^7\right)\right\} \\
&\quad - 3\phi^{-1}R^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\
&\quad + 3\gamma^2\phi^{-5}R^{\mu\nu}\left(\left(4\phi^3-[\Phi]\right)\nabla_\mu\phi + \nabla^\kappa\phi\nabla_\kappa\nabla_\mu\phi\right)\nabla_\nu\phi \\
&\quad + 3\gamma^2\phi^{-5}R^{\mu\kappa\nu\lambda}\nabla_\mu\phi\nabla_\nu\phi\nabla_\kappa\nabla_\lambda\phi,
\end{aligned} \qquad (7.2.6)
$$

where indices are now raised and lowered with $g_{\mu\nu}$, and we assume an overall $\sqrt{-g}$ factor. Since $\delta\mathcal{L}_{4,5}$ include non-minimal couplings, we must be precise about our definition of $T_{\mu\nu}$ and associated NEC. We couple this theory to Einstein-Hilbert gravity, and define $T_{\mu\nu}$ as the source of $G_{\mu\nu}$, i.e., $T_{\mu\nu} \equiv M_{\text{Pl}}^2 G_{\mu\nu}$. By matching this to a standard, radiation-dominated phase, below we will unambiguously ascertain whether the NEC violation is "genuine" or simply an artifact of non-minimal couplings.

Varying the action with respect to the metric, and setting $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\bar{\phi} = \alpha/(-t)$, yields an isotropic $T_{\mu\nu}$, with vanishing energy density and pressure scaling as t^{-4} (as it must by dilation invariance [5, 52]),

$$\rho = 0; \qquad\qquad P = \frac{\alpha^2}{t^4}\left(C_2 - C_4 + 12C_5\right), \qquad (7.2.7)$$

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where we have used (7.2.2) to simplify. To violate the NEC, the pressure must be negative,

$$C_2 - C_4 + 12C_5 < 0 \qquad \text{(NEC violation)} . \qquad (7.2.8)$$

7.2.1 Matching to standard cosmology

Integrating (7.0.2), we obtain a *DBI Genesis* cosmology, describing an expanding universe from an asymptotically static state:

$$H(t) = -(C_2 - C_4 + 12C_5)\frac{\alpha^2}{3M_{\text{Pl}}^2(-t)^3} . \qquad (7.2.9)$$

For this to represent a useful NEC violation, we verify that the DBI Genesis phase matches onto an *expanding* radiation-dominated phase. We remain agnostic about the reheating process; our main concern is whether the universe is expanding after the transition. In theories which admit an Einstein frame, the condition below implies continuity of the Einstein-frame H. Because of non-minimal couplings, we instead find that H is discontinuous [106]. Indeed, the pressure is of the form: $P = G(\phi, \dot{\phi}) + \mathrm{d}F(\phi, \dot{\phi})/\mathrm{d}t$. The G term is regular as ϕ is brought instantaneously to a halt, but the F term gives rise to a delta function. Explicitly, we have

$$F(t) \equiv \frac{\alpha^2}{6(-t)^3}\left(24C_5 - 2C_4 - (2C_4 - 60C_5)\beta - 18C_5\beta^2\right.$$
$$\left. - (C_3 - 3C_4 + 90C_5)\left(\frac{\bar{\gamma}\cosh^{-1}\bar{\gamma}}{\sqrt{1+\bar{\gamma}}\sqrt{\beta}} - 1\right)\right) . \qquad (7.2.10)$$

Integrating (7.0.2) around the delta-function singularity, we discover that $H + F/2M_{\text{Pl}}^2$ matches continuously at the transition. Hence we obtain the matching condition:

$$H_{\text{Genesis}} + \frac{F}{2M_{\text{Pl}}^2} = H_{\text{rad.-dom.}} . \qquad (7.2.11)$$

Combining (7.2.9) and (7.2.10), we find that the universe will be expanding in the radiation-

dominated phase if

$$2C_2 + (2C_4 - 60C_5)\beta + 18C_5\beta^2 + (C_3 - 3C_4 + 90C_5) \times \left(\frac{\bar{\gamma}\cosh^{-1}\bar{\gamma}}{\sqrt{1+\bar{\gamma}}\sqrt{\beta}} - 1 \right) < 0$$

$$\text{(matching)}. \qquad (7.2.12)$$

7.3 Summary of conditions

We started out with five coefficients, $C_1, \ldots, C_5$. Stability of the Poincaré-invariant vacuum sets $C_1 = 0$ and (without loss of generality) $C_2 = 1$. This leaves us with three coefficients, C_3, C_4 and C_5, which must be chosen such that the cubic equation (7.2.2) has a real root with $\beta \gtrsim 1$ (per (7.2.5)), and which must satisfy the inequalities (7.1.9), (7.2.4), (7.2.8) and (7.2.12).

All these conditions can be satisfied even with $C_5 = 0$. With $C_2 = 1$, the first inequality in (7.1.9) gives $C_3 < 3$, while (7.2.8) simplifies to $C_4 > 1$. The equation of motion (7.2.2) reduces to a quadratic equation, with roots $\beta_\pm = (\pm\sqrt{C_3^2 - 8C_4} - C_3)/2C_4$. It is easy to check that only β_+ can lead to a stable $1/t$ solution. In order for β_+ to be real and $\gtrsim 1$, we must require $C_3^2 > 8C_4$ and $C_3 \lesssim -(2 + C_4)$. With these conditions, (7.2.4) and the second inequality of (7.1.9) are automatically satisfied. The only remaining constraint is (7.2.12). Figure 8 shows (in white) the allowed region of (C_3, C_4) parameter space satisfying all of our constraints. Generalizing the analysis to $C_5 \neq 0$ only widens the allowed region.

7.4 Quantum stability

We now argue that the NEC-violating solution is robust against other allowed terms in the effective theory, *i.e.*, all diffeomorphism invariants of the induced metric and extrinsic curvature. Using the Gauss–Codazzi relation

$$\mathfrak{R}_{\mu\nu\rho\sigma} = \frac{2}{3}(\mathfrak{g}_{\mu\rho}\mathfrak{g}_{\nu\sigma} - \mathfrak{g}_{\mu\sigma}\mathfrak{g}_{\nu\rho}) + K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho} \qquad (7.4.1)$$
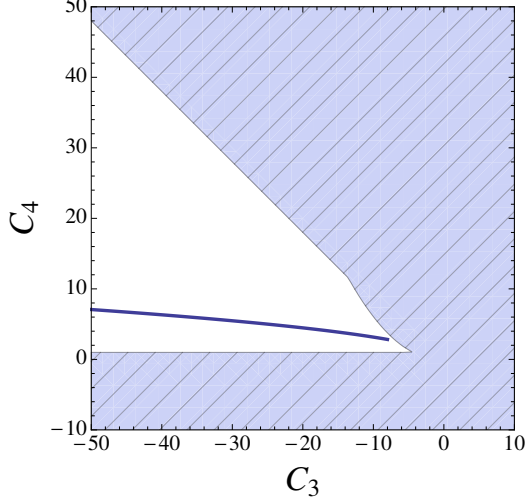
Figure 8: Allowed (white) region of (C_3, C_4) parameter space satisfying all of our conditions, with $C_1 = C_5 = 0$ and $C_2 = 1$. In the allowed region, $\beta \simeq -C_3/C_4$ for $|C_3| \gg 1$. On the solid curve, β grows without bound as $C_3 \to -\infty$, showing that all constraints can be satisfied for arbitrarily large β.

to eliminate all instances of $\mathfrak{R}_{\mu\nu\rho\sigma}$ in favor of $K_{\mu\nu}$, we see that the DBI galileons are particular polynomials in $K_{\mu\nu}$. As argued in the Appendix of [168], however, any polynomial in $K_{\mu\nu}$ can be brought to the galileon form through field redefinitions.

It remains to consider terms with covariant derivatives acting on $K_{\mu\nu}$, such as $K_{\mu\nu}\Box K^{\mu\nu}$. Since $\bar{K}_{\mu\nu} = -\bar{\gamma}\bar{\mathfrak{g}}_{\mu\nu}$ on the $1/t$ background, it is annihilated by ∇, so these higher-derivative terms do not contribute to the equation of motion for the $1/t$ ansatz. Hence the $1/t$ solution is an *exact* solution, including all possible higher-derivative terms in the effective theory.

These higher-derivative terms *do* contribute to perturbations, but it is technically natural to set their coefficients to zero if there is a hierarchy,

$$C_3 \sim \beta \; ; \qquad C_2 \sim C_4 \sim \mathcal{O}(1) \; ; \qquad C_5 \sim 1/\beta \; , \quad \text{where} \;\; \beta \gg 1 \; (\alpha \simeq 1). \qquad (7.4.2)$$

This corresponds to relativistic brane motion. The solid curve in Fig. 8, corresponding to $C_4 \simeq -C_3/\beta$ for $\beta \gg 1$, shows that all of our constraints can be satisfied for arbitrarily large β. In the limit of large $|t|$, the theory of perturbations is approximately the same

as that about a constant background. Consequently, the fluctuation lagrangian takes the form (7.1.5), where now $\bar{\phi}_0$ is (7.2.1), except that every spatial gradient is multiplied by a factor of the sound speed, $1/\bar{\gamma} \simeq 1/\beta$. A computation shows that the coefficient of an $\mathcal{O}(\varphi^n)$ term scales as β^{2n+1}. The (ordinary) galileon terms are suppressed by the lowest scale in the theory

$$\Lambda_{\mathrm{s}} \equiv \frac{\beta^{1/6}}{|t|} \simeq \beta^{1/6}\bar{\phi}(t)\,, \tag{7.4.3}$$

which we identify as the strong coupling scale. We now study the limit $\beta \to \infty$, $|t| \to \infty$, keeping Λ_{s} fixed. Only the ordinary galileon terms [103] survive, with spatial gradients suppressed by γ, so we scale them in taking the limit so that the limiting theory looks Lorentz invariant. Because of the galileon non-renormalization theorem [128, 169, 170], it follows that if we work at finite β, radiative corrections to $C_1, \ldots, C_5$ must be suppressed by powers of $1/\beta$, so the hierarchy we have set up is stable. Loop corrections also produce higher-derivative terms suppressed by Λ_{s}, but these are consistently small at low energy so we have a derivative expansion in $\partial/\Lambda_{\mathrm{s}}$.

Finally, we discuss the issue of superluminality around the Poincaré-invariant vacuum $\phi = \bar{\phi}_0$. With $C_3 \neq 0$, weak deformations of this background exhibit superluminal propagation [104]. (Our conditions cannot be simultaneously satisfied with $C_3 = 0$.) Following the arguments of [104], superluminal effects can be consistently ignored in the effective theory if the cutoff is sufficiently low: $\Lambda_0 \lesssim \bar{\phi}_0/\sqrt{|C_3|} \sim \bar{\phi}_0/\sqrt{\beta}$. By relativistic and conformal invariance, the cutoff around any background scales as $\Lambda \sim \phi/\gamma$. For consistency of our analysis, the lowest allowed cutoff around the NEC-violating solution is set by the mass of φ, namely $1/|t|$. This implies $\Lambda_0 \sim \beta\bar{\phi}_0$, hence superluminal effects lie within the effective theory.

Chapter 8

Conclusions

In this dissertation, we have investigated an alternative to the inflationary paradigm. In contrast to the violent superluminal expansion of inflation, the conformal mechanism posits that space-time is nearly static at early times, and highly symmetric. Inflation is also deeply rooted in symmetries, and the different symmetry-breaking patterns of the two theories lead to clear observational signatures of each scenario.

We have seen that generically, single-field alternatives to inflation become strongly coupled after producing a finite number of scale-invariant modes. This points us in the direction of multi-field cosmologies to search for alternatives to inflation. The conformal mechanism is one such alternative; we have seen that it is more general than any particular incarnation, the desired scale-invariant spectrum follows from the pattern of symmetry-breaking in the theory.

This pattern of symmetry breaking also leads to strong observational signatures of the conformal scenario; it is worth summarizing some of them:

- Absence of detectable gravitational waves.

- Model-dependent local non-Gaussianity from the conversion mechanism.

- Anisotropy of the power spectrum, see (6.2.15), [96–98].

- 4-point function in the soft internal limit due to tree-level π exchange, (6.2.2), [96–98].

This is relevant on large scales.

- 4-point function in the soft internal limit due to one-loop π exchange, (6.2.17). This dominates for sufficiently small internal momentum, and it shows up as stochastic bias and in the power spectrum of μ-distortion.

In the future, it will be interesting to confront these predictions with observational tests.

Finally, we have attempted to address a theoretical obstruction which all alternatives to inflation face. Inevitably, they must violate the null energy condition at some point in the evolution. As of writing, there is neither a fully consistent theory which violates the null energy condition within the context of QFT or string theory, nor a proof that such violation is impossible. Nonetheless, we have made significant progress in this direction, by constructing a field theory which possesses both a stable Poincaré-invariant solution and a solution which violates the NEC. It will be interesting to further pursue this direction— either to see where the tension between violating the NEC and fundamental physics lies or to construct a fully consistent violation.

Appendix A

Embedded brane construction

We begin with a D-dimensional bulk, $\mathcal{M}$, with coordinates X^A and metric $G_{AB}(X)$. The position of a 4-dimensional brane living in the bulk is given by embedding functions

$$X^A(x) : \mathbb{R}^{3,1} \longhookrightarrow \mathcal{M} , \tag{A.0.1}$$

where x^μ are coordinates on the brane; these are the dynamical variables. Tangent vectors to the brane have components $e_\mu^A = \frac{\partial X^A}{\partial x^\mu}$ and the induced metric on the brane is

$$\bar{g}_{\mu\nu} = e_\mu^A e_\nu^B G_{AB} . \tag{A.0.2}$$

There are also $N \equiv (D - 4)$ vectors normal to the brane indexed by I, with components n_I^A, which satisfy

$$n_I^A e_\mu^B G_{AB} = 0 , \qquad n_I^A n_J^B G_{AB} = \delta_{IJ}. \tag{A.0.3}$$

The normal and tangent vectors are used to construct the N extrinsic curvature tensors,

$$K_{\mu\nu}^I = e_\mu^A e_\nu^B \nabla_A n_B^I , \tag{A.0.4}$$

where ∇_A is the bulk covariant derivative, as well as the twist connection, which is the connection on the normal bundle,

$$\beta^I_{\mu J} = n^{BI} e^A_\mu \nabla_A n_{BJ} \; ; \tag{A.0.5}$$

it has an associated curvature $R^I{}_{J\mu\nu}$.

Requiring the action to be invariant under reparametrizations of the brane restricts the action to be a diffeomorphism scalar constructed from these geometric ingredients,

$$S = \int \mathrm{d}^4 x \, \sqrt{-g} \, \mathcal{L}(\bar{g}_{\mu\nu}, \bar{\nabla}_\mu, \bar{R}^\mu{}_{\nu\rho\sigma}, K^I_{\mu\nu}, R^I{}_{J\mu\nu}) \; . \tag{A.0.6}$$

Here $\bar{\nabla}_\mu$ is the world-volume connection, which acts on 4D spacetime indices with the Levi–Civita connection of the induced metric, and on normal indices with the twist connection. We fix the reparametrization symmetry of the brane world-volume coordinates by choosing Monge (static) gauge

$$X^\mu(x) = x^\mu \; , \qquad\qquad X^I(x) = \pi^I(x), \tag{A.0.7}$$

that is, we take the 4 world-volume coordinates to coincide with the first 4 coordinates used in the bulk. The N remaining functions π^I are the physical degrees of freedom for the brane.

Given a Killing vector K^A of the bulk metric G_{AB}, the induced metric and extrinsic curvature (and hence the action (A.0.6)) are invariant under $\delta_K X^A = K^A$. However, generically this destroys the gauge choice (A.0.7) by sending

$$x^\mu \longmapsto x^\mu + K^\mu, \tag{A.0.8}$$

and we must restore the desired gauge via a compensating brane reparametrization $\delta_\xi X^A(x) = \xi^\mu \partial_\mu X^A(x)$ with $\xi^\mu = -K^\mu$ so that the combined gauge-preserving π^I sym-

metry acts as

$$(\delta_K + \delta_{\text{comp}})\pi^I = -K^\mu \partial_\mu \pi^I + K^I, \tag{A.0.9}$$

and becomes a global symmetry of the gauge-fixed action. Symmetries that have a K^I component are nonlinearly realized and are thus symmetries of the bulk that are spontaneously broken due to the presence of the brane.

A.1 Conformal Dirac–Born–Infeld

We are interested in the case where the bulk space-time is five-dimensional Anti-de Sitter space. Here we apply the brane construction to this case; we consider the bulk space to be AdS$_5$ in Poincaré coordinates, which has line element

$$\mathrm{d}s^2_{\text{AdS}} = G_{AB}\mathrm{d}X^A\mathrm{d}X^B = \mathcal{R}^2 \left[\frac{1}{z^2}\mathrm{d}z^2 + z^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \right] , \tag{A.1.1}$$

where $0 < z < \infty$ is the radial AdS direction. In addition to the manifest Poincaré symmetries of the x^μ coordinates,

$$P_\mu = -\partial_\mu \ ; \qquad\qquad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \ , \tag{A.1.2}$$

AdS$_5$ has five additional Killing vectors,[50]

$$
\begin{aligned}
K_\mu &= 2x_\mu z\partial_z + \left(\frac{1}{z^2} + x^2 \right)\partial_\mu - 2x_\mu x^\nu\partial_\nu \ , \\
D &= -z\partial_z + x^\mu\partial_\mu \ .
\end{aligned}
\tag{A.1.4}
$$

---

[50]These can be obtained either by solving Killing's equation directly,

$$K^C\partial_C G_{AB} + G_{BC}\partial_A K^C + G_{AC}\partial_B K^C = 0 \ , \tag{A.1.3}$$

or by considering AdS as itself embedded in an ambient $\mathbb{R}^{4,2}$ and pulling back the bulk Killing vectors, as is done in [105].

There will be one transverse π^I field corresponding to the radial direction z, and we'll call this field ϕ. Accordingly, we fix the gauge

$$X^\mu(x) = x^\mu \; , \qquad\qquad X^5(x) = z = \phi(x). \qquad\qquad \text{(A.1.5)}$$

Using (A.0.9); the symmetries (A.1.4) generate the following global symmetries on ϕ in the gauge-fixed action,

$$
\begin{aligned}
\delta_D \phi &= -\left(\Delta_\phi + x^\nu \partial_\nu\right) \phi \,, \\
\delta_{K_\mu} \phi &= -2x_\mu \left(\Delta_\phi + x^\nu \partial_\nu\right) \phi + x^2 \partial_\mu \phi + \frac{1}{\phi^2} \partial_\mu \phi \,,
\end{aligned}
\qquad \text{(A.1.6)}
$$

where $\Delta_\phi = 1$. In addition, the manifest Poincaré symmetries of the x^μ coordinates generate the standard Poincaré transformations on ϕ

$$\delta_{P_\mu} \phi = -\partial_\mu \phi \; , \qquad\qquad \delta_{J_{\mu\nu}} \phi = (x_\mu \partial_\nu - x_\nu \partial_\mu)\phi \; , \qquad\qquad \text{(A.1.7)}$$

Together, the 5 symmetries (A.1.6) and the 10 Poincaré symmetries, (A.1.7), satisfy the algebra (3.1.13) and provide a non-linear realization of $\mathfrak{so}(4,2)$. Compared to the transformations (3.1.20) in the standard case, there is an extra term $\phi^{-2}\partial_\mu \phi$ in the expression for $\delta_{K_\mu} \phi$; in the DBI action, the special conformal transformations are realized non-linearly.

The induced metric on the brane (A.0.2) is, in the gauge (A.1.5),

$$\bar{g}_{\mu\nu}(x) = \mathcal{R}^2 \phi^2 \left(\eta_{\mu\nu} + \frac{\partial_\mu \phi}{\phi^2} \frac{\partial_\nu \phi}{\phi^2} \right) . \qquad\qquad \text{(A.1.8)}$$

To construct the leading order action for the brane, we combine a tadpole potential term with a kinetic term arising from the induced volume form on the brane as

$$S_{\text{DBI}} = \int \mathrm{d}^4 x \left[\left(1 + \frac{\lambda}{4}\right) \phi^4 - \frac{1}{\mathcal{R}^4}\sqrt{-g_{\text{ind}}} \right] = \int \mathrm{d}^4 x \, \phi^4 \left(1 + \frac{\lambda}{4} - \sqrt{1 + \frac{(\partial\phi)^2}{\phi^4}} \right) .$$

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where indices are contracted with $\eta_{\mu\nu}$. This is precisely the action (3.5.1). For convenience we have chosen the constant so that a Poincaré invariant solution, $\phi = $ constant, exists only when $\lambda = 0$. The action is normalized such that expanding around this solution we have a canonical, healthy scalar kinetic term.

A.1.1 AdS$_5 \times S^1$ brane construction

We consider the product space Ads$_5 \times S^1$ [7]. The line element for this space is

$$\mathrm{d}s^2 = G_{AB}\mathrm{d}X^A\mathrm{d}X^B = \mathcal{R}^2 \left[\frac{1}{z^2}\mathrm{d}z^2 + z^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \right] + \ell^2\mathrm{d}\Theta^2 \,, \qquad (\text{A.1.9})$$

where the A, B indices now run from 0 to 5, and $0 < \Theta < 2\pi$ is an angular coordinate for the S^1. Fixing unitary gauge, as we did in (A.1.5), there are now two fields, ϕ and θ, which represent the transverse position of the brane in the radial AdS direction and in the S^1, respectively:

$$X^\mu(x) = x^\mu \,, \qquad X^5(x) \equiv \phi(x) \,, \qquad X^6 \equiv \theta(x) \,. \qquad (\text{A.1.10})$$

With this choice of coordinates, the induced metric takes the form

$$\bar{g}_{\mu\nu}(x) = \mathcal{R}^2\phi^2 \left(\eta_{\mu\nu} + \frac{\partial_\mu\phi}{\phi^2}\frac{\partial_\nu\phi}{\phi^2} + \frac{\ell^2}{\mathcal{R}^2}\frac{\partial_\mu\theta}{\phi}\frac{\partial_\nu\theta}{\phi} \right) \,, \qquad (\text{A.1.11})$$

and the global symmetries of the gauge fixed action are given by (in addition to Poincareé symmetry, which acts in the normal way (A.1.7))

$$\delta_D\phi = -\left(\Delta_\phi + x^\nu\partial_\nu\right)\phi \,; \qquad \delta_{K_\mu}\phi = -2x_\mu\left(\Delta_\phi + x^\nu\partial_\nu\right)\phi + x^2\partial_\mu\phi + \frac{1}{\phi^2}\partial_\mu\phi \,;$$

$$\delta_D\theta = -\left(\Delta_\theta + x^\nu\partial_\nu\right)\theta \,; \qquad \delta_{K_\mu}\theta = -2x_\mu\left(\Delta_\theta + x^\nu\partial_\nu\right)\theta + x^2\partial_\mu\theta + \frac{1}{\phi^2}\partial_\mu\theta \,,$$

where $\Delta_\phi = 1$ and $\Delta_\theta = 0$. In addition, there is a 16th Killing vector, corresponding to a translation in the angular variable

$$C = \partial_\Theta \,. \qquad (\text{A.1.12})$$

The action of the S^1 generator on ϕ is trivial, $\delta_C \phi = 0$, while its action on θ,

$$\delta_C \theta = 1 \, , \tag{A.1.13}$$

corresponds to a shift symmetry. This is exactly the extra symmetry we will need to protect the scale invariance of θ perturbations. The 15 AdS$_5$ generators satisfy the algebra (3.1.13), while the S^1 generator δ_C commutes with itself and all of the AdS$_5$ generators.

The lowest order action involving θ and ϕ is given by the volume of the induced metric plus a tadpole term with an appropriately chosen coefficient

$$S_{\phi\theta} = \int \mathrm{d}^4 x \, \phi^4 \left(1 + \frac{\lambda}{4} - \sqrt{1 + \frac{(\partial \phi)^2}{\phi^4} + \frac{(\partial \theta)^2}{\phi^2} + \frac{(\partial \phi)^2 (\partial \theta)^2 - (\partial \phi \cdot \partial \theta)^2}{\phi^6}} \right) \, , \tag{A.1.14}$$

where we have canonically normalized θ so that it now ranges over $(0, \frac{2\pi \ell}{\mathcal{R}})$. Note that the shift symmetry $\theta \mapsto \theta + c$ implies that the tadpole does not depend on θ.

Appendix B

Some properties of field theory on de Sitter

B.1 Linearly realized SO(4,1) and 3d conformal transformations

The conformal mechanism relies on linearly realized SO(4,1) invariance, so in this appendix, we review some properties of scalar fields on de Sitter space. Throughout, we will work in the planar slicing of de Sitter space, where the line element takes the form[51]

$$\mathrm{d}s^2 = \frac{1}{H^2 t^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) \ , \tag{B.1.1}$$

where $t < 0$ is conformal time. This is identical to the situation in multi-field inflation, where spectator fields feel a de Sitter geometry and do not back-react appreciably. A key difference, worth reemphasizing, is that de Sitter space is a fake geometry in the conformal mechanism—the actual, Einstein-frame metric is slowly evolving. Nevertheless, for the purpose of this discussion we can remain agnostic as to whether or not the background de Sitter corresponds to the actual metric.

The de Sitter metric (B.1.1) corresponds to a maximally symmetric space-time and therefore enjoys 10 isometries. Six of these are the familiar translations and rotations of the flat spatial

---

[51]Here we use t as the conformal time coordinate on de Sitter space in order to emphasize the connection with models in which de Sitter arises as a fictitious background from broken conformal invariance.

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slices:

$$x^i \longmapsto x^i + \alpha^i \; ; \tag{B.1.2}$$

$$x^i \longmapsto J^i{}_j x^j \; . \tag{B.1.3}$$

Additionally, de Sitter space is invariant under a dilation of both spatial and time coordinates

$$x^\mu \longmapsto \lambda x^\mu \; . \tag{B.1.4}$$

Finally, it is invariant under the simultaneous transformation of space and time as

$$t \longmapsto t - 2t(\vec{b} \cdot \vec{x}) \; ; \tag{B.1.5}$$

$$x^i \longmapsto x^i + b^i(-t^2 + \vec{x}^2) - 2x^i(\vec{b} \cdot \vec{x}) \; , \tag{B.1.6}$$

where b^i is a real-valued 3-vector.

Next, consider a free scalar field on the de Sitter background:

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(-\frac{1}{2}(\partial \phi)^2 - \frac{m_\phi^2}{2}\phi^2 \right) \; . \tag{B.1.7}$$

The de Sitter isometries act on ϕ as follows: spatial rotations and translations (B.1.3) act in the usual way,

$$\delta_{P_i} \phi = -\partial_i \phi \; ;$$

$$\delta_{J_{ij}} \phi = (x_i \partial_j - x_j \partial_i)\phi \; , \tag{B.1.8}$$

while the remaining four isometries (B.1.4) and (B.1.6) act as

$$\delta_D \phi = -(-t\partial_t + \vec{x} \cdot \vec{\partial})\phi \; ;$$

$$\delta_{K_i} \phi = -\left(-2x_i t\partial_t + 2x_i \vec{x} \cdot \vec{\partial} - (-t^2 + x^2)\partial_i \right) \phi \; . \tag{B.1.9}$$

We are interested in how these transformations act at late times $(t \to 0)$. In Fourier space, the equation of motion that follows from the above action in the coordinates (B.1.1) is

$$\ddot{\phi}_k - \frac{2}{t}\dot{\phi}_k + \left(k^2 + \frac{m_\phi^2}{H^2 t^2}\right)\phi_k = 0 , \tag{B.1.10}$$

with the well-known solution given by Hankel functions. In the long-wavelength $(k \to 0)$ limit, the time dependence of the mode functions simplifies to

$$\phi_k \sim t^{\Delta_\pm} , \qquad \text{with } \Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{m_\phi^2}{H^2}} . \tag{B.1.11}$$

Assuming $m_\phi^2 \le 9H^2/4$, the growing mode corresponds to $\Delta_- \equiv \Delta$, and the time depen-dence of the field is $\phi \sim t^\Delta$ as $t \to 0$. In this limit, we can therefore replace $t\partial_t \to \Delta$ in the transformation rules (B.1.9) and neglect $\mathcal{O}(t^2)$ terms to obtain

$$
\begin{aligned}
\delta_D \phi &= \left(\Delta - \vec{x} \cdot \vec{\partial}\right)\phi ; \\
\delta_{K_i} \phi &= \left(2\Delta x_i - 2x_i \vec{x} \cdot \vec{\partial} + x^2 \partial_i\right)\phi .
\end{aligned} \tag{B.1.12}
$$

These are recognized respectively as spatial dilations and special conformal transforma-tions for a field of conformal weight Δ. Combined with the spatial rotations and trans-lations (B.1.8), they form the conformal group on $\mathbb{R}^3$. Therefore, correlation functions of fields on de Sitter must be invariant under conformal transformations of Euclidean 3-space on the future boundary [146–150], which is of course the basis of the proposed dS/CFT correspondence [171]. Here we assumed that the free evolution (B.1.11) dominates at late times. If this is not the case, one cannot trade the time dependence of correlation functions for Δ's.

B.1.1 Conformal transformations on correlation functions

Here we derive the action in Fourier space of the linearly-realized dilation and spatial special conformal transformations on correlation functions.

Dilation

We will work in an arbitrary number of dimensions, d. The dilation operator acts linearly on fields in position space as

$$\delta_D \phi = (\Delta - x^A \partial_A)\phi \ . \tag{B.1.13}$$

We note that the field ϕ can be written in Fourier space using

$$\phi(x) = \int d^d k \, e^{ik \cdot x} \phi_k \ , \tag{B.1.14}$$

we may therefore write

$$\delta_D \phi = \int d^d k \, \phi_k (\Delta + \vec{k} \cdot \vec{\partial}_k) e^{ik \cdot x} \ . \tag{B.1.15}$$

Now, we can integrate by parts to obtain two terms

$$\delta_D \phi = \int d^d k \, e^{ik \cdot x} \left(\Delta - d - \vec{k} \cdot \vec{\partial}_k \right) \phi_k \ . \tag{B.1.16}$$

From this, we deduce the Fourier space transformation rule

$$\delta_D \phi_k = - \left((d - \Delta) + \vec{k} \cdot \vec{\partial}_k \right) \phi_k \tag{B.1.17}$$

Now, we want to obtain the action of dilation on a correlation function. A correlation function has two parts, the amplitude and the delta function, schematically it is of the form

$$\delta_D \mathcal{A} = \delta_D \left(\delta^3(\vec{P}) \mathcal{A}' \right) \ , \tag{B.1.18}$$

where the prime indicates removal of the delta function and $\vec{P}$ is the sum of the momenta $\vec{P} = \sum \vec{k}$. We may then write

$$
\begin{aligned}
\delta_D \left(\delta^3(\vec{P}) \mathcal{A}' \right) &= -\sum_{a=1}^{N} \left((d - \Delta_a) + \vec{k}_a \cdot \vec{\partial}_{k_a} \right) \left[\delta^3(\vec{P}) \mathcal{A}' \right] \\
&= -\mathcal{A}' \vec{P} \cdot \vec{\partial}_P \delta^3(\vec{P}) - \sum_{a=1}^{N} \delta^3(\vec{P}) \left((d - \Delta_a) + \vec{k}_a \cdot \vec{\partial}_{k_a} \right) \mathcal{A}' \; .
\end{aligned}
$$

The term outside the sum may be integrated by parts to obtain a factor of d. The term where the derivative $\vec{\partial}_P$ hits $\mathcal{A}'$ vanishes because it multiplies $\vec{P} \delta^3(\vec{P}) = 0$. We then have

$$
\delta_D \left(\delta^3(\vec{P}) \mathcal{A}' \right) = \delta^3(\vec{P}) \left[d - \sum_{a=1}^{N} \left((d - \Delta_a) + \vec{k}_a \cdot \vec{\partial}_{k_a} \right) \mathcal{A}' \right] \; .
\tag{B.1.19}
$$

From this, we deduce that the dilation operator acts on the amplitude without the delta function as

$$
\delta_D \mathcal{A}' = \left[-d(N-1) + \sum_{a=1}^{N} \left(\Delta_a - \vec{k}_a \cdot \vec{\partial}_{k_a} \right) \right] \mathcal{A}'
\tag{B.1.20}
$$

Special conformal transformations

Special conformal transformations act in real space as

$$
\delta_{K^A} = (2\Delta x^A - 2x_A x^B \partial_B + x^2 \partial_A) \phi \; .
\tag{B.1.21}
$$

Following the same steps as above, we may write this in Fourier space acting on the primed correlator as

$$
\delta_{K^A} \mathcal{A}' = i \sum_{a=1}^{N} \left(2(\Delta_a - d) \partial_{k_a^A} + k_a^A \vec{\partial}_{k_a}^2 - 2\vec{k}_a \cdot \vec{\partial}_{k_a} \partial_{k_a^A} \right) \mathcal{A}'
\tag{B.1.22}
$$

B.2 Correlation functions

Here we collect some results for correlation functions involving spectator fields coupled to the Goldstone field π.

B.2.1 Mode functions for massive fields

In this Appendix, we derive the expression for the mode functions of a massive scalar field on de Sitter space in terms of Hankel functions. This expressions are needed to compute the correlation functions we need to check the consistency relations. Consider the general quadratic action for a massive scalar

$$S_{2,\phi} = M_\phi^2 \int \mathrm{d}^4 x \sqrt{-g} \left(-\frac{1}{2}(\partial\phi)^2 - \frac{m_\phi^2}{2}\phi^2 \right) . \tag{B.2.1}$$

Where m_ϕ^2 is an arbitrary mass. The equation of motion following from this action is

$$\Box\phi + \frac{2}{t}\dot{\phi} - \frac{m_\phi^2}{H^2 t^2}\phi = 0 . \tag{B.2.2}$$

We define the canonically-normalized variable

$$v = \frac{M_\phi}{H(-t)}\phi , \tag{B.2.3}$$

whose mode functions satisfy

$$v_k'' + \left[k^2 - \left(2 - \frac{m_\phi^2}{H^2} \right)\frac{1}{t^2} \right] v_k = 0 . \tag{B.2.4}$$

Defining $x \equiv -kt$ and $\nu \equiv \sqrt{\frac{9}{4} - \frac{m_\phi^2}{H^2}}$, after changing variables to $f_k \equiv v_k/\sqrt{x}$ this can be cast as Bessel's equation

$$x^2 \frac{\mathrm{d}^2 f_k}{\mathrm{d}x^2} + x\frac{\mathrm{d}f_k}{\mathrm{d}x} + (x^2 - \nu^2)f_k = 0 , \tag{B.2.5}$$

which is well-known to be solved by (we choose Hankel functions as our basis)

$$f_k(x) = c_1(k) H_\nu^{(1)}(x) + c_2(k) H_\nu^{(2)}(x) \,. \tag{B.2.6}$$

We fix the coefficients by demanding that in the far past $(-kt \to \infty)$, the mode functions of the canonically normalized variable, v_k, have their Minkowski space form. This is the so-called adiabatic vacuum (Bunch–Davies) choice. That is, we demand

$$v_k(t) \xrightarrow[-kt\to\infty]{} \frac{1}{\sqrt{2k}} e^{-ikt} \tag{B.2.7}$$

Then, using the asymptotic expansion for the Hankel functions as $-kt \to \infty$

$$H_\nu^{(1)}(-kt) \sim -e^{\frac{i\pi}{2}\left(\frac{3}{2}-\nu\right)} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{-kt}} e^{-ikt}$$

$$H_\nu^{(2)}(-kt) \sim e^{\frac{i\pi}{2}\left(\frac{1}{2}+\nu\right)} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{-kt}} e^{ikt}$$

This implies that we need to take $c_1(k) = -e^{-\frac{i\pi}{2}\left(\frac{3}{2}-\nu\right)} \sqrt{\frac{\pi}{4}} \frac{1}{\sqrt{k}}$ and $c_2(k) = 0$ in (B.2.6). This leads to the expression for the ϕ_k mode functions

$$\phi_k(t) = -e^{-\frac{i\pi}{2}\left(\frac{3}{2}-\nu\right)} \sqrt{\frac{\pi}{4}} \frac{H(-t)^{3/2}}{M_\phi} H_\nu^{(1)}(-kt) \qquad \text{with} \quad \nu = \sqrt{\frac{9}{4} - \frac{m_\phi^2}{H^2}} \,, \tag{B.2.8}$$

where $H_\nu^{(1)}(-kt)$ is a Hankel function of the first kind. Note that for $m^2 > \frac{9H^2}{4}$ the solution is a Hankel function of imaginary order.

B.2.2 In-in integrals

In order to compute correlation functions, we employ the Schwinger–Keldysh or in-in formalism (see [64, 172] for an exposition). In this formalism, rather than computing in-out S-matrix elements, we compute correlation functions sandwiched between the same vacuum.

The correlation function for an operator, $\mathcal{O}(t)$ is given by [64, 172]

$$\langle \mathcal{O}(t) \rangle = \langle 0 | \bar{T} e^{i \int_{t_0}^{t} dt' H_{\text{int}}(t')} \mathcal{O}(t) T e^{-i \int_{t_0}^{t} dt' H_{\text{int}}(t')} | 0 \rangle \; . \tag{B.2.9}$$

Here H_{int} is the interaction Hamiltonian, T denotes time-ordering while $\bar{T}$ denotes anti-time-ordering and t_0 is an early time. Generally we will only work to leading order (tree-level) where the correlation function is given by

$$\langle \mathcal{O}(t) \rangle = -i \int_{-\infty}^{t} dt' \, \langle 0 | \left[\mathcal{O}(t), H_{\text{int}}(t') \right] | 0 \rangle \; . \tag{B.2.10}$$

B.2.3 Correlation functions of π

Here we compute the two and three-point correlators for the Goldstone field π. We consider the action (4.2.28). The quadratic equations of motion lead to the following mode function for the field π

$$\pi_k(t) = -i \frac{H(-t)^{3/2}}{M_\pi} \sqrt{\frac{\pi}{4}} H_{5/2}^{(1)}(-kt) = \frac{-3H}{\sqrt{2k^5}(-t)M_\pi} \left(1 + ikt - \frac{k^2 t^2}{3} \right) e^{-ikt} \; . \tag{B.2.11}$$

From this the two-point function can straightforwardly be computed:

$$P_\pi(k) \equiv \langle \pi_{\vec{k}} \pi_{-\vec{k}} \rangle' = \frac{9H^2}{2M_\pi^2 k^5 t^2} \left(1 + \frac{k^2 t^2}{3} + \frac{k^4 t^4}{9} \right) \; . \tag{B.2.12}$$

Note that this field has an extremely red spectrum, peaking strongly as $k \to 0$.

From the action (4.2.28), we can also compute the three-point function, $\langle \pi^3 \rangle$. The interaction Hamiltonian, H_{int}, at this order is minus the lagrangian

$$H_{\text{int}} = - \int d^3 x \mathcal{L}_{\text{int}} = M_\pi^2 \int d^3 x \left[\frac{1}{H^2 t^2} \pi (\partial \pi)^2 - \frac{4}{H^2 t^4} \pi^3 \right] \; . \tag{B.2.13}$$

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Applying the formula (B.2.10), we obtain (at late times)

$$\langle \pi_{\vec{k}_1} \pi_{\vec{k}_2} \pi_{\vec{k}_3} \rangle' = \frac{81 H^4}{4 M_\pi^4} \frac{\left(k_1^5 + k_2^5 + k_3^5 \right)}{k_1^5 k_2^5 k_3^5 t_*^4} \ , \tag{B.2.14}$$

where t_* is a cutoff introduced to regulate the divergence as $t \to 0$.

B.2.4 Massive spectator field, $\Delta = 1$

The simplest case of a spectator field coupled to π is a massive field with $m_\phi^2 \equiv m_\varphi^2 = 2H^2$, corresponding to $3d$ conformal weight $\Delta = 1$. We take the action (4.2.43) with this choice of mass:

$$S_\varphi = M_\varphi^2 \int \mathrm{d}^4 x \sqrt{-g} \left(-\frac{1}{2}(\partial \varphi)^2 - H^2 \varphi^2 - 4H^2 \pi \varphi^2 - \pi(\partial \varphi)^2 - \lambda \varphi^3 - 4\lambda \pi \varphi^3 \right) \ , \tag{B.2.15}$$

The mode functions for the field are given by

$$\varphi_k(t) = \frac{iH(-t)}{\sqrt{2k} M_\varphi} e^{ikt} \ , \tag{B.2.16}$$

which leads to the two-point function for the spectator

$$P_\varphi(k) \equiv \langle \varphi_{\vec{k}} \varphi_{-\vec{k}} \rangle' = \frac{H^2}{2M_\varphi^2} \frac{t^2}{k} \ . \tag{B.2.17}$$

We can also compute various higher-point correlation functions involving this spectator. The simplest is the three-point function involving only φ, the tree-level correlation function is given by

$$\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle' = \frac{3\pi H^2 \lambda}{4 M_\varphi^4} \frac{t^3}{k_1 k_2 k_3} \ . \tag{B.2.18}$$

Additionally, we can compute the $\langle \pi \varphi \varphi \rangle$ three-point function for these fields. There are two contributions to the correlation function, one from each of the $\pi \varphi \varphi$ vertex and the $\pi(\partial \varphi)^2$

vertex; the final result is given by

$$\langle \pi_{\vec{q}} \varphi_{\vec{k}_2} \varphi_{\vec{k}_2} \rangle' = -\frac{9H^4}{4M_\varphi^2 M_\pi^2} \frac{1}{q^5 k_1 k_2} (k_1 + k_2) \ . \tag{B.2.19}$$

This correlation function is invariant under $(4d)$ dilations and under δ_{K^i} with $\Delta_a = \{-1, 1, 1\}$, agreeing with our general arguments for when conformal weights may be consistently defined, in spite of the fact that this correlation function does not scale in the naïve way with time.

Finally, we compute a four-point function, involving three φ fields and one Goldstone; this computation is slightly more involved. There are two contributions to this four-point function, one coming from a contact diagram involving the $\pi \varphi^3$ vertex and one coming from an exchange diagram at second order in the vertices involving a single π and two φ's. The interaction Hamiltonian is given by[52]

$$H_{\text{int}}^{(3)} = M_\varphi^2 \int \mathrm{d}^3x \left(-\frac{1}{H^2 t^2} \pi \dot{\varphi}^2 + \frac{4}{H^2 t^4} \pi \varphi^2 + \frac{\lambda}{H^4 t^4} \varphi^3 \right)$$
$$H_{\text{int}}^{(4)} = M_\varphi^2 \int \mathrm{d}^3x \left(\frac{4\lambda}{H^4 t^4} \pi \varphi^3 \right) \ . \tag{B.2.20}$$

The correlation function is then a sum of three terms

$$\langle \pi_{\vec{q}} \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle = -i \int_{-\infty}^{t} \mathrm{d}t' \langle 0 | [\pi_{\vec{q}} \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3}(t), H_{\text{int}}^{(4)}(t')] | 0 \rangle$$
$$+ \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t} \mathrm{d}t'' \langle 0 | H_{\text{int}}^{(3)}(t') \pi_{\vec{q}} \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3}(t) H_{\text{int}}^{(3)}(t'') | 0 \rangle \tag{B.2.21}$$
$$- 2\text{Re} \left(\int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \langle 0 | \pi_{\vec{q}} \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3}(t) H_{\text{int}}^{(3)}(t') H_{\text{int}}^{(3)}(t'') | 0 \rangle \right) \ .$$

When the dust settles, the four-point function is given by

$$\langle \pi_{\vec{q}} \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle' = -\frac{27\pi H^4 \lambda}{8M_\pi^2 M_\varphi^4} \frac{t}{q^5 k_1 k_2 k_3} \left(\frac{k_1}{|\vec{q} + \vec{k}_1|} + \frac{k_2}{|\vec{q} + \vec{k}_2|} + \frac{k_3}{|\vec{q} + \vec{k}_3|} \right) \ . \tag{B.2.22}$$

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[52]Note that at this order, we must be careful in deriving the interaction Hamiltonian, in this case it is still minus the interaction lagrangian, but in general this will not be true at quartic order.

B.2.5 Massless spectator field, $\Delta = 0$

We now consider a massless spectator field, corresponding to (4.2.43) with with $m_\chi^2 = \lambda = 0$.

The cubic action for this field is given by

$$S_\chi = M_\chi^2 \int \mathrm{d}^4 x \sqrt{-g} \left(-\frac{1}{2}(\partial\chi)^2 - \pi(\partial\chi)^2 \right) . \tag{B.2.23}$$

The mode functions for χ are the well-known result for massless fields

$$\chi_{\vec{k}}(t) = \frac{H}{\sqrt{2k^3}M_\chi} (1 - ikt) \, e^{ikt} . \tag{B.2.24}$$

Using this, the two point function for a massless field is the standard result

$$P_\chi(k) \equiv \langle \chi_{\vec{k}} \chi_{-\vec{k}} \rangle' = \frac{H^2}{2k^3 M_\chi^2}(1 + k^2 t^2) . \tag{B.2.25}$$

Additionally, we can compute the three-point function $\langle \pi\chi\chi \rangle$ using the standard techniques, summarized above. At late times, we obtain

$$\langle \pi_{\vec{q}} \chi_{\vec{k}_1} \chi_{\vec{k}_2} \rangle' = \frac{3\pi H^4}{16 M_\pi^2 M_\chi^2} \frac{1}{q^5 k_1^3 k_2^3 t} \left(q^4 + 2q^2(k_1^2 + k_2^2) - 3(k_1^2 - k_2^2)^2 \right) \tag{B.2.26}$$
$$- \frac{9H^4}{8 M_\chi^2 M_\pi^2} \frac{1}{q^5 k_1^3 k_2^3} \left(q^2(k_1^3 + k_2^3) - (k_1^5 + k_2^5) + 3(k_1^3 k_2^2 + k_1^2 k_2^3) \right) .$$

This correlation function is invariant under δ_{K^i} with $\Delta_a = \{-1, 0, 0\}$. Additionally, it has the leading scaling behavior with respect to time that is expected. Note that the squeezed limit $(q \to 0)$ is given by

$$\langle \pi_{\vec{q}} \chi_{\vec{k}_1} \chi_{\vec{k}_2} \rangle'_{q \to 0} = \frac{3\pi H^4}{16 M_\pi^2 M_\chi^2} \frac{1}{q^5 k^6 t} \left(3(\vec{k} \cdot \vec{q})^2 - k^2 q^2 + \mathcal{O}(q^3) \right) , \tag{B.2.27}$$

Appendix C

Some charge identities

Here we collect some important identities and results involving the Noether charges which generate the broken symmetries in Section 6.1.1. In general, a symmetry is just a map from one solution to another, enforcing this relation both before and after time-evolving a state implies in Schrödinger picture

$$Q_S(t) = U(t, t_0)Q(t_0)U^\dagger(t, t_0) \ , \tag{C.0.1}$$

where at t_0 all pictures coincide. In the case where Q is a time-independent operator, this reduces to the fact that Q commutes with U. The Heisenberg picture Q is given by

$$Q_H(t) = U^\dagger Q_S U = Q(t_0) \ , \tag{C.0.2}$$

which is time-independent. Next, we consider the interaction picture operator

$$Q_I(t) = U_I(t, t_0)Q(t_0)U_I^\dagger(t, t_0) \ , \tag{C.0.3}$$

where $U(t, t_0) \equiv U_0^\dagger(t, t_0)U(t, t_0)$ and U_0 is the free-field time evolution operator. The thing that appears in correlation functions is $U_I^\dagger(t, t_i)Q_I(t)U(t, t_i)$. Unpacking the definitions, we

find that this satisfies

$$U^\dagger(t, t_i)Q_I(t)U(t, t_i) = U_I^\dagger(t, t_i)U_I(t, t_0)Q(t_0)U^\dagger(t, t_0)U_I(t, t_i)$$

$$= U_I(t_i, t_0)Q(t_0)U_I^\dagger(t_i, t_0) = Q_I(t_i) . \tag{C.0.4}$$

Now, we split the charge into a piece that generates non-linear transformations and a piece that generates linear transformations as

$$Q_S = \mathcal{Q}_S(t) + W_S(t) . \tag{C.0.5}$$

Since $\mathcal{Q}_S$ generates a non-linear transformation, it is a symmetry the free Hamiltonian so we have

$$\mathcal{Q}_S(t) = U_0(t, t_0)Q(t_0)U_0^\dagger(t, t_0) . \tag{C.0.6}$$

This implies that $\mathcal{Q}_I$ is time-independent

$$\mathcal{Q}_I(t) = U^\dagger(t, t_0)\mathcal{Q}_S(t)U_0(t, t_0) = \mathcal{Q}(t_0) , \tag{C.0.7}$$

so we have

$$U^\dagger(t, t_i)\mathcal{Q}_I(t)U(t, t_i) = \mathcal{Q}_I + W_I(t_i) . \tag{C.0.8}$$

In deriving the Ward identities, we make the assumption that terms involving the Ws vanish as $t_i \to -\infty$. This is "weak convergence":

$$\lim_{t_i \to -\infty} U^\dagger(t, t_i)\mathcal{Q}_I(t)U(t, t_i) = \mathcal{Q}_I . \tag{C.0.9}$$

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