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CONFORMALLY FLAT HYPERSURFACES IN A EUCLIDEAN SPACE

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Introduction. Let (M, g), or simply M, be a Riemannian n-manifold with Riemannian metric g. Throughout this paper manifolds under consideration are always assumed to be connected and smooth unless otherwise stated. M is called *conformally flat* if each point of M has a neighborhood where there exists a conformal diffeomorphism onto an open subset in a Euclidean space. It is well-known that every Riemannian 2-manifold is conformally flat because of the existence of isothermal coordinates.

In this paper, we shall study conformally flat n-manifolds, $n > 3$, which are also isometrically immersed in a Euclidean $(n+1)$-space as complete hypersurfaces, and determine the global form of such hypersurfaces under the following additional assumption:

(*) The Riemannian structure of (M, g) and the isometric immersion under consideration are both analytic.

In fact, we prove the following theorem, which is the main result of this paper.

THEOREM. *Let (M, g) be an analytic complete conformally flat Riemannian n-manifold, $n > 3$, and $f: M \rightarrow E^{n+1}$ an analytic isometric immersion of M into a Euclidean $(n+1)$-space E^{n+1}. Then $f(M)$ is one of the following forms:*

(i) *a flat hypersurface (i.e., a Euclidean n-space E^n, or a cylinder $E^{n-1} \times \gamma$ built over an analytic plane curve γ).*

(ii) *a tube (see §1, for definition; e.g., a Riemannian product manifold $S^{n-1} \times E^1$).*

(iii) *a surface of revolution (e.g., a Euclidean n-sphere S^n).*

It should be remarked that the corresponding local result is true without completeness of M and without analyticity condition (*) [1, 4, 5]. Roughly speaking, a general conformally flat hypersurface is obtained by smoothly glueing together pieces of hypersurfaces of the above three types, although arbitrary glueing is clearly not possible. Our additional assumption (*) makes it quite impossible to glue these pieces together.

Finally, we remark that a similar theorem has been announced by

Kulkarni [3] in the case where M is compact.

1. Preliminaries. In this section we recall some known results on conformally flat hypersurfaces in a Euclidean space. For details see [4].

Let $f: M \to E^{n+1}$ be an isometric immersion of a (not necessarily complete) Riemannian n-manifold, $n > 3$, into a Euclidean $(n + 1)$-space E^{n+1}. In the following, when the argument is local in nature, we may consider f as an imbedding and thus identify a point $x \in M$ with $f(x) \in E^{n+1}$.

Then the following has been known.

LEMMA 1 [1, 4, 5]. *Let A be the second fundamental form of M, which is considered as a symmetric linear transformation on each tangent space of M. Then M is conformally flat if and only if at each point of M, A is one of the following types:*

(I) *$A = \lambda I$, $I =$ the identity transformation.*

(II) *A has two distinct eigenvalues λ and μ of multiplicity $n - 1$ and 1 respectively.*

The proof of this lemma is done by a straightforward calculation.

A point of M is called an *umbilical* point if the second fundamental form A takes the form (I) at the point. Otherwise we call the point a *non-umbilical* point.

For further studying, we choose a local field of orthonormal frames e_A in E^{n+1} such that, restricted to M, the vectors e_i and e_n are tangent to M (and consequently, e_{n+1} is normal to M), where and throughout the rest of this paper, we agree on the following ranges of indices:

$$1 \leqq A, B, C, \cdots \leqq n + 1 ,$$
$$1 \leqq i, j, k, \cdots \leqq n - 1 .$$

With respect to the frame field chosen above, let ω_A and ω_{AB} be the field of dual frames and connection forms respectively. We restrict these forms to M. Then we have

$$\omega_{n+1} = 0 .$$

Now we assume that M is conformally flat. Then, on a (sufficiently small) neighborhood of a non-umbilical point, we can choose the above frame field e_A in such a way that

$$\omega_{i,n+1} = \lambda\omega_i ,$$
$$\omega_{n,n+1} = \mu\omega_n ,$$

due to Lemma 1 together with the continuity of the second fundamental form. Here λ as well as μ is a smooth function on the neighborhood. For later convenience, we call such a frame field an *adapted frame field* around a non-umbilical point.

Then we have

LEMMA 2 [4]. *With respect to each adapted frame field around a non-umbilical point, the following hold:*

(i) $\omega_{in} = [1/(\lambda - \mu)](\lambda_n \omega_i + \mu_i \omega_n)$,

(ii) $\lambda_j = 0$ *and* $\lambda_n \mu_j = 0$ *for all* j,

where we have put

$$d\lambda = \sum \lambda_i \omega_i + \lambda_n \omega_n ,$$
$$d\mu = \sum \mu_i \omega_i + \mu_n \omega_n .$$

For the proof of Lemma 2, see the literature.

Before going into the proof of the main theorem, we shall explain some examples of conformally flat hypersurfaces in a Euclidean space.

First, let γ be an arbitrary smooth curve in E^{n+1}. Then the total space of the normal sphere bundle of γ with (sufficiently small) fixed radius is, by definition, a *tube*. As is easily seen by Lemma 1, a tube is a conformally flat hypersurface in E^{n+1}.

Another example is a surface of revolution. Let S be the envelope of a one-parameter family of hyperspheres in E^{n+1}. S is called a *surface of revolution* if on a straight line there lies the locus of centers of hyperspheres of the family. From Lemma 1, we see that S is a conformally flat hypersurface in E^{n+1}. Note that a Euclidean n-sphere S^n is a special type of such hypersurfaces.

2. **Proof of the main theorem.** Let $f: M \to E^{n+1}$ be an analytic isometric immersion of an analytic complete conformally flat Riemannian n-manifold (M, g), $n > 3$, into a Euclidean $(n+1)$-space E^{n+1}. In the following, we always assume that M is simply connected. This assumption does cause no loss of generality of our argument. In fact, take the universal Riemannian covering manifold $\pi: M^* \to M$ of M. Then M^* is also an analytic complete conformally flat Riemannian manifold, and $f^* = f \circ \pi$ is an analytic isometric immersion of M^* into E^{n+1}. Moreover, we have $f(M) = f^*(M^*)$.

For later use, we put

$$\mathscr{U} = \{x \in M \,|\, x \text{ is an umbilical point}\} ,$$

and

$$\mathscr{N} = \{x \in M \,|\, x \text{ is a non-umbilical point}\} .$$

Then $\mathscr{U}$ is closed in M, and $\mathscr{N}$ is open in M, owing to the continuity of the second fundamental form.

First we divide the proof into the following two cases.

CASE I. $\mathscr{U}$ has an interior point.

Then the second fundamental form A of M takes the form

$$A = \lambda I, \ \lambda \text{ is a real number}, \tag{1}$$

on a non-empty open subset of M. Note that M admits an analytic field e_{n+1} of unit normal vectors defined on M due to the simple connectedness of M. Thus both sides of the equation (1) are analytic tensor fields defined globally on M. Hence we have $A = \lambda I$ on M.

Consequently $f(M)$ is either a Euclidean n-space E^n or a Euclidean n-sphere S^n according to whether λ is zero or not.

CASE II. $\mathscr{U}$ has no interior point.

Then the set of non-umbilical points of M is dense in M, i.e., $M = \mathrm{Cl}\,\mathscr{N}$, the closure of $\mathscr{N}$. Around each point of $\mathscr{N}$, we take an adapted frame field e_A so that

$$\omega_{i,n+1} = \lambda\omega_i , \tag{2}$$

$$\omega_{n,n+1} = \mu\omega_n , \tag{3}$$

as seen in §1. Here remark that λ and μ are both analytic functions defined on $\mathscr{N}$ by virtue of the simple connectedness of M. Then from Lemma 2 (ii), with respect to each adapted frame field, we have $\lambda_n\mu_j = 0$ for all j. Hence on $\mathscr{N}$ there can be the following three cases:

(A) For each adapted frame field, λ_n as well as μ_j for all j vanishes identically.

(B) For some adapted frame field, there exists a point $p \in \mathscr{N}$ such that $\mu_j(p) \neq 0$ for some j.

(C) For some adapted frame field, there exists a point $p \in \mathscr{N}$ such that $\lambda_n(p) \neq 0$.

REMARK. Note that these three cases cover every possibility on $\mathscr{N}$. Moreover, in the course of the proof, it will turn out that (B) and (C) cannot hold simultaneously.

First we consider

CASE II-A. For each adapted frame field, λ_n as well as μ_j for all j vanishes identically.

In this case, since $d\lambda = \sum \lambda_j\omega_j + \lambda_n\omega_n = 0$ at each point of $\mathscr{N}$, λ is constant on each connected component of $\mathscr{N}$ and hence on a dense subset

$\mathcal{N}$ itself by continuity of λ. Furthermore, from Lemma 2(i), we have

$$\omega_{in} = 0 , \tag{4}$$

for each adapted frame field. Consequently, taking exterior differentiation of (4), we get

$$\lambda\mu = 0 \quad \text{on} \quad \mathcal{N}. \tag{5}$$

If λ is a non-zero constant, then from (5), μ must vanish identically on $\mathcal{N}$. Therefore, $\mathcal{N}$ is a closed subset of M, which is also a non-empty open subset of M. Thus $\mathcal{N}$ coincides with M by connectedness of M. Hence we can conclude that $f(M)$ is of the form

$$S^{n-1} \times E^1 ,$$

the Riemannian product of a Euclidean $(n-1)$-sphere S^{n-1} and a straight line E^1.

In case λ vanishes identically on $\mathcal{N}$, we see immediately from the equation of Gauss that M is flat on $\mathcal{N}$. Then M itself is flat on account of the continuity of the Riemannian curvature tensor field, which is in fact an analytic tensor field on M. Thus, by a theorem of Hartman [2], $f(M)$ is a cylinder built over an analytic plane curve γ, i.e., of the form $E^{n-1} \times \gamma$.

CASE II–B. For some adapted frame field, there exists a point $p \in \mathcal{N}$ such that $\mu_j(p) \neq 0$ for some j.

First, choose and fix such an adapted frame field e_A and such a point $p_0 \in \mathcal{N}$ as well. Then, with respect to the e_A, we have an open connected neighborhood V of p_0 in $\mathcal{N}$ such that μ_j never vanishes on V. Since we have $\lambda_n\mu_j = 0$ on V from Lemma 2 (ii), λ_n must vanish identically on V. Hence λ is constant on V because λ_j always vanishes. Consequently it is observed that λ is constant on the connected component $\mathcal{N}_0$ of p_0 of $\mathcal{N}$ by analyticity of λ.

If λ vanishes identically on $\mathcal{N}_0$, then M is flat on $\mathcal{N}_0$. Therefore, M itself is flat because of the analyticity of the Riemannian curvature tensor field. Thus $f(M)$ is cylindrical over an analytic plane curve as seen in the previous case.

So from now on we assume that λ is a non-zero constant on the component $\mathcal{N}_0$ of $\mathcal{N}$. Furthermore, we may assume $\lambda > 0$ on $\mathcal{N}_0$ by replacing the unit normal vector field e_{n+1} with $-e_{n+1}$ if necessary.

From Lemma 2 (i), we have for each adapted frame field on $\mathcal{N}_0$

$$\omega_{in} = [\mu_i/(\lambda - \mu)]\omega_n , \tag{6}$$

because λ_n vanishes identically on $\mathscr{N}_0$.

Taking exterior differentiation of (6), we get, with respect to each adapted frame field on $\mathscr{N}_0$, the following partial differential equation for each i:

$$(\lambda - \mu)\mu_{ii} + 2(\mu_i)^2 + (\lambda - \mu)^2\lambda\mu = 0 , \tag{7}$$

where we have put

$$\mu_{ii} = d\mu_i(e_i) .$$

Let $M^{n-1}(p)$ denote the maximal integral submanifold through $p \in \mathscr{N}_0$ of the distribution defined by the space spanned by the principal vectors corresponding to λ. Then $M^{n-1}(p)$ is a totally geodesic submanifold of M, since (6) holds for each adapted frame field on $\mathscr{N}_0$ (cf. [4]). Therefore, we can restrict the above differential equation (7) to each geodesic of M issuing from $p \in \mathscr{N}_0$ and tangent to $M^{n-1}(p)$ at p. Then along the geodesic we get the following ordinary differential equation

$$(\lambda - \mu)\frac{d^2\mu}{ds^2} + 2\left(\frac{d\mu}{ds}\right)^2 + (\lambda - \mu)^2\lambda\mu = 0 , \tag{8}$$

where s is the arc length from p.

Putting $\phi = 1/(\lambda - \mu)$, (8) reduces to

$$\frac{d^2\phi}{ds^2} + \lambda^2\phi - \lambda = 0 . \tag{9}$$

By solving (9), we get

$$\phi = a \cos \lambda s + b \sin \lambda s + 1/\lambda , \tag{10}$$

where a and b are some constants of integration. Thus we obtain

$$\lambda - \mu = 1/(a \cos \lambda s + b \sin \lambda s + 1/\lambda) , \tag{11}$$

from which we have

$$\mu = \frac{\lambda(a \cos \lambda s + b \sin \lambda s)}{a \cos \lambda s + b \sin \lambda s + 1/\lambda} . \tag{12}$$

It is immediately verified from (11) that there does not exist any umbilical point on each geodesic issuing from $p \in \mathscr{N}_0$ and tangent to $M^{n-1}(p)$ at p as well.

Furthermore, we have

LEMMA 3. *In this case, M is umbilic free, i.e., there exists no umbilical point on M.*

PROOF. It suffices to show that $\mathscr{N}_0$ is a closed subset of M, since

$\mathscr{N}_0$ is a non-empty open subset of a connected M. For this purpose, let p_0 be a point on $\partial\mathscr{N}_0$, the boundary of $\mathscr{N}_0$, and $\{p_n\}$ a sequence of points of $\mathscr{N}_0$ converging to p_0: $\lim_{n\to\infty} p_n = p_0$. Let γ_n be a geodesic issuing from p_n whose initial vector coincides with the vector $e_i(p_n)$ for a fixed i.

Note that γ_n always belongs to $\mathscr{N}_0$, since there is no umbilical point on γ_n as remarked just before the lemma. Furthermore, it follows from (12) that for each n there exists a point q_n on γ_n for which $\mu(q_n) = 0$ holds.

Since M is complete, we may assert that the sequence $\{\gamma_n\}$ and the sequence $\{q_n\}$ converge respectively to γ_0, a geodesic issuing from p_0, and q_0, a point on γ_0, by choosing respective subsequences suitably if necessary. Then we have $\mu(q_0) = \lim_{n\to\infty} \mu(q_n) = 0$ by continuity of μ, and hence $q_0 \in \mathscr{N}_0$. Furthermore, since $\gamma_0 = \lim_{n\to\infty} \gamma_n$, γ_0 is tangent to $M^{n-1}(q_0)$ at q_0. Therefore, the remark just before the lemma implies that p_0 belongs to $\mathscr{N}_0$. This completes the proof. q.e.d.

REMARK. In this case, $\lambda \equiv \text{constant} > \mu$ holds everywhere on M. In fact, we assume the contrary. Namely, assume that $\mu > \lambda \equiv \text{constant} > 0$ holds everywhere. From (12), along any geodesic in M, we have

$$\mu = \frac{\lambda\sqrt{a^2 + b^2}\sin(\lambda s + \Phi)}{\sqrt{a^2 + b^2}\sin(\lambda s + \Phi) + 1/\lambda},$$

where $\sin\Phi = a/\sqrt{a^2 + b^2}$, $\cos\Phi = b/\sqrt{a^2 + b^2}$, a and b are certain constants, and s is the arc length from some point on the geodesic in question. Hence, if μ does not change its sign, μ must be zero, since s can take all real numbers. This is a contradiction.

PROPOSITION 4. *If M is of CASE II–B with $\lambda \equiv \text{constant} > 0$, then $f(M)$ is a tube.*

PROOF. Define a mapping $C: M \to E^{n+1}$ by

$$C(p) = f(p) + (1/\lambda)e_{n+1}(p), \quad p \in M, \tag{13}$$

which is evidently well-defined. Then we have

$$\begin{aligned} dC &= \sum \omega_i \otimes e_i + \omega_n \otimes e_n + (1/\lambda)de_{n+1} \\ &= (1 - \mu/\lambda)\omega_n \otimes e_n, \end{aligned} \tag{14}$$

noticing (2), (3) and the constancy of λ. This shows that the image of C can be parametrized by the canonical parameter of some integral curve of e_n, that is, the image is a curve in E^{n+1}, which is also denoted by C. Furthermore, the curve C is regular because $\lambda > \mu$ everywhere. Since λ is a positive constant, it is not difficult to see from these facts that

$f(M)$ is nothing but the total space of the normal sphere bundle of the curve C with radius $1/\lambda$. Therefore $f(M)$ is a tube. q.e.d.

Finally we deal with

CASE II-C. For some adapted frame field, there exists a point $p \in \mathcal{N}$ such that $\lambda_n(p) \neq 0$.

In this case, we may further assume that μ_j does vanish identically for each adapted frame field. In fact, otherwise Case II-C reduces to Case II-B so that λ is constant on M. This is a contradiction.

REMARK. Thus, from Lemma 2(i), we have for each adapted frame field

(15) $$\omega_{in} = [\lambda_n/(\lambda - \mu)]\omega_i ,$$

from which we get

$$d\omega_n = 0 .$$

Therefore, for each adapted frame field, we can locally put $\omega_n = ds$, where s is the canonical parameter of some integral curve (which is in fact a geodesic segment) of e_n.

We set

$$\mathcal{N}' = \{p \in \mathcal{N} \mid \lambda(p) \neq 0\} .$$

Note that $\mathcal{N}'$ is dense in $\mathcal{N}$ and hence in M as well, i.e., $M = \mathrm{Cl}\,\mathcal{N}'$, because λ is a non-constant analytic function.

We define a mapping $C: \mathcal{N}' \to E^{n+1}$ by

(16) $$C(p) = f(p) + (1/\lambda)(p)e_{n+1}(p), \ p \in \mathcal{N}' ,$$

which is obviously well-defined. Then we have

(17) $$\begin{aligned} dC &= \sum \omega_i \otimes e_i + \omega_n \otimes e_n \\ &\quad + d\Big(\frac{1}{\lambda}\Big) \otimes e_{n+1} + \frac{1}{\lambda} de_{n+1} \\ &= \Big(1 - \frac{\mu}{\lambda}\Big)\omega_n \otimes e_n + \Big(\frac{1}{\lambda}\Big)'\omega_n \otimes e_{n+1} \\ &= \Big\{\Big(1 - \frac{\mu}{\lambda}\Big)e_n + \Big(\frac{1}{\lambda}\Big)' e_{n+1}\Big\} ds , \end{aligned}$$

where the prime denotes the differentiation with respect to s. This shows, by the same argument as in Case II-B, that the image of C is a union of regular curves in E^{n+1}, which is also denoted by C.

Since, for each adapted frame field, λ_j as well as μ_j does vanish identically for all j, we can easily observe that $\mathcal{N}'$ and hence its closure M itself are (possibly a part of) the envelope of a one-parameter family of hyperspheres in E^{n+1}, and the curve C is nothing but (possibly a part of) the locus of centers of such hyperspheres (cf. [4]).

Now we prove

LEMMA 5. *Each component of C is a segment in E^{n+1}.*

PROOF. We put

$$\xi = dC/ds = [(\lambda - \mu)/\lambda]e_n + (1/\lambda)'e_{n+1} .$$

It suffices to show that at each point of C, two vectors ξ and $d\xi/ds$ are parallel. By making use of (15), we see

$$
\begin{aligned}
(18)\qquad d\xi &= \left(\frac{\lambda-\mu}{\lambda}\right)'\omega_n \otimes e_n + \left(\frac{1}{\lambda}\right)''\omega_n \otimes e_{n+1} \\
&\quad + \frac{\lambda-\mu}{\lambda}de_n + \left(\frac{1}{\lambda}\right)'de_{n+1} \\
&= \left[\left\{\left(\frac{\lambda-\mu}{\lambda}\right)' - \left(\frac{1}{\lambda}\right)'\mu\right\}e_n + \left\{\left(\frac{1}{\lambda}\right)'' + \frac{(\lambda-\mu)\mu}{\lambda}\right\}e_{n+1}\right]ds .
\end{aligned}
$$

On the other hand, taking exterior differentiation of (15), we get

$$(19)\qquad \left(\frac{\lambda'}{\lambda-\mu}\right)' - \left(\frac{\lambda'}{\lambda-\mu}\right)^2 - \lambda\mu = 0 ,$$

from which we obtain the following relation

$$(20)\qquad \left(\frac{1}{\lambda}\right)'\left\{\left(\frac{\lambda-\mu}{\lambda}\right)' - \left(\frac{1}{\lambda}\right)'\mu\right\} = \frac{\lambda-\mu}{\lambda}\left\{\left(\frac{1}{\lambda}\right)'' + \frac{(\lambda-\mu)\mu}{\lambda}\right\} ,$$

because (19) and (20) are both equivalent to

$$(21)\qquad \lambda''(\lambda-\mu) - \lambda'(2\lambda' - \mu') - \lambda\mu(\lambda-\mu)^2 = 0 .$$

Here the relation (20) shows that at each point of C, two vectors ξ and $d\xi/ds$ are parallel. This completes the proof. q.e.d.

Consequently, we have

PROPOSITION 6. *If M is of CASE II-C, then $f(M)$ is a surface of revolution.*

PROOF. We have only to prove that the curve C lies on a straight line. However, it is almost obvious now, since $f(M)$ is an analytic hypersurface in E^{n+1}, and the set $\mathcal{N}'$ is dense in M, i.e., $M = \mathrm{Cl}\,\mathcal{N}'$. q.e.d.

REMARK. It should be noted here that $f(M)$ is called a surface of revolution in the following sense: $f(M)$ is obtained by the analytic glueing of some surfaces of revolution defined in §1 such that each of the loci of their centers lies on the same straight line.

Summerizing the above results, we arrive at

THEOREM 7. *Let* (M, g) *be an analytic complete conformally flat Riemannian n-manifold,* $n > 3$, *and* $f: M \longrightarrow E^{n+1}$ *an analytic isometric immersion of M into a Euclidean* $(n+1)$*-space. Then* $f(M)$ *is one of the following*: (i) *a flat hypersurface,* (ii) *a tube and* (iii) *a surface of revolution.*

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