

CONFORMALLY INVARIANT POWERS OF THE LAPLACIAN  
— A COMPLETE NONEXISTENCE THEOREM

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1. INTRODUCTION

Conformally invariant operators and the equations they determine play a central role in the study of manifolds with pseudo-Riemannian, Riemannian, conformal and related structures. This observation dates back to at least the very early part of the last century when it was shown that the equations of massless particles on curved space-time exhibit conformal invariance. In this setting a key operator is the conformally invariant wave operator which has leading term a pseudo-Laplacian. The Riemannian signature variant of this operator is a fundamental tool in the Yamabe problem on compact manifolds. Here one seeks to find a metric, from a given conformal class, that has constant scalar curvature. Recently it has become clear that higher order analogues of these operators, viz., conformally invariant operators on weighted functions (i.e., conformal densities) with leading term a power of the Laplacian, have a central role in generating and solving other curvature prescription problems as well as other problems in geometric spectral theory and mathematical physics [2, 5, 15].

In the flat setting, the existence of such operators dates back to [16], where it is shown that, on 4-dimensional Minkowski space, for  $k \in \mathbb{N} = \{1, 2, \dots\}$ , the  $k^{\text{th}}$  power of the flat wave operator  $\Delta^k$ , acting on densities of the appropriate weight, is invariant under the action of the conformal group. More generally, if  $\mathcal{E}[w]$  denotes the space of conformal densities of weight  $w \in \mathbb{R}$ , then on a flat conformal manifold of dimension  $n \geq 3$  (and any signature) there exists, for each  $k \in \mathbb{N}$ , a unique conformally invariant operator

$$\square_{2k} : \mathcal{E}[k - n/2] \rightarrow \mathcal{E}[-k - n/2]$$

and the leading part of  $\square_{2k}$  is  $\Delta^k$ . Furthermore, this set of operators is complete in the sense that it contains all natural conformally invariant differential operators (see Section 2) between densities. These facts are easily recovered from the general results in [6] and references therein.

Many of these operators can be generalised to curved conformal manifolds; Graham, Jenne, Mason and Sparling [14] constructed natural conformally invariant

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operators

$$P_{2k}: \mathcal{E}[k - n/2] \rightarrow \mathcal{E}[-k - n/2]$$

with leading term  $\Delta^k$  for all  $k \in \mathbb{N}$  and  $n \geq 3$  except for the cases of  $n$  even and  $k > n/2$ . (See the references in [14] for earlier constructions of some low order examples.) They also conjectured that their result is sharp, based partially on the fact, proved by Graham [13], that  $\square_6$  in dimension 4 does not admit curved analogue. Recently [19] added weight to this conjecture by establishing the nonexistence of a curved analogue for  $\square_8$  in dimension 6. In this paper we prove the conjecture. We state this as a theorem.

**Theorem.** *If  $n \geq 4$  is even and  $k > n/2$ , there is no conformally invariant natural differential operator between densities with the same principal part as  $\Delta^k$ .*

In [13] Graham explains that “the basic reason for the nonexistence of an invariant curved modification of  $\Delta^3$  in dimension 4 is the conformal invariance of the classical Bach tensor.” An analogue of this reasoning still holds true for the proof of our theorem, although the proof is completely different from that of Graham. In higher even dimensions we replace the Bach tensor by its analogue, the Fefferman-Graham obstruction tensor  $B_{ab}$ , which arises in the ambient metric construction of [7]; see (2.9) in the next section. Our strategy for the proof is to construct a curvature expression that is shown to be nonzero for a class of conformal metrics for which  $B_{ab} = 0$ , while it is also shown to vanish for the same class of metrics under the assumption of the existence of the curved analogue of  $\square_{2k}$ . This is a contradiction. The former is done by a direct computation using the tractor calculus, which will be review in Section 2; the latter is a consequence of some classical invariant theory.

This explanation is somewhat of a simplification. Nevertheless the proof of the theorem in Section 3 can be viewed as a careful elaboration of this idea. The proof is greatly simplified by the use of a special class of metrics and Section 4 is concerned with showing that this class is nontrivial.

Finally we should point out that there are many other settings where similar nonexistence issues remain to be resolved. In [6] Eastwood and Slovák use and develop some semiholonomic Verma module theory to prove that in odd dimensions every conformally invariant operator between irreducible bundles on (locally) conformally flat manifolds (including spin manifolds) has a curved analogue. In even dimensions they show that the same is true, save for an exceptional class of operators. The class consists of the operators corresponding dually to those nonstandard nonsingular homomorphisms which go between the generalised Verma modules at either extreme of generalised Bernstein-Gelfand-Gelfand resolutions. This includes the  $\square_{2k}$  for  $k \geq n/2$ , as discussed above, but also many other operators. Some operators in the exceptional class do have curved analogues, in particular the  $\square_{2k}$  for  $k = n/2$ . However we suspect that otherwise the result of Eastwood and Slovák is sharp. Similar questions can be asked for many other similar geometries such as CR structures. In [11] there is a construction of CR invariant powers of the sub-Laplacian that generates curved analogues for most but not all the invariant operators from the CR flat setting. Once again there is the question of whether this result is sharp. For more general CR operators the existence theory is much less developed than in the conformal case.

## 2. CONFORMAL GEOMETRY AND TRACTOR CALCULUS

We collect here the minimal background materials from conformal geometry and tractor calculus as required for the proof of the theorem. The initial development of tractor calculus in conformal geometry dates back to the work of T. Y. Thomas [18] and was reformulated and further developed in a modern setting in [1]. It is intimately related to the Cartan conformal connection; for a comprehensive treatment exposing this connection and relating the conformal case to the wider setting of parabolic structures; see [4, 3]. The calculational techniques, conventions and notation used here follow [12] and [10].

Let  $(M, [g])$  be a conformal manifold of dimension  $n \geq 3$  and of signature  $(p, q)$ . A conformal structure is equivalent to a ray subbundle  $\mathcal{Q}$  of  $S^2T^*M$ ; points of  $\mathcal{Q}$  are pairs  $(g_x, x)$  where  $x \in M$  and  $g_x$  is a metric at  $x$ , each section of  $\mathcal{Q}$  gives a metric  $g$  on  $M$  and the metrics from different sections agree up to multiplication by a positive function. The bundle  $\mathcal{Q}$  is a principal bundle with group  $\mathbb{R}_+$ , and we denote by  $\mathcal{E}[w]$  the vector bundle induced from the representation of  $\mathbb{R}_+$  on  $\mathbb{R}$  given by  $t \mapsto t^{-w/2}$ . Sections of  $\mathcal{E}[w]$  are called *conformal densities of weight  $w$*  and may be identified with functions on  $\mathcal{Q}$  that are homogeneous of degree  $w$ , i.e.,  $f(s^2g_x, x) = s^w f(g_x, x)$  for any  $s \in \mathbb{R}_+$ . We will often use the same notation  $\mathcal{E}[w]$  for the space of sections of the bundle. Note that for each choice of a metric  $g$  (i.e., section of  $\mathcal{Q}$ , which we term a *choice of conformal scale*), we may identify a section  $f \in \mathcal{E}[w]$  with a function  $f_g$  on  $M$  by  $f_g(x) = f(g_x, x)$ . In particular,  $\mathcal{E}[0]$  is canonically identified with  $C^\infty(M)$ . Finally we emphasise that for  $w \neq 0$  the bundle  $\mathcal{E}[w]$ , by its definition, depends on the conformal structure.

The operators of our main interest are defined as maps between densities  $P: \mathcal{E}[w] \rightarrow \mathcal{E}[w']$ . For each choice of a scale  $g \in [g]$ ,  $P$  induces a map  $P_g: C^\infty(M) \rightarrow C^\infty(M)$  via the identifications  $\mathcal{E}[w] \cong C^\infty(M)$ . We say that  $P$  is a *natural differential operator* if  $P_g$  can be written as a universal polynomial in covariant derivatives with coefficients depending polynomially on the metric, its inverse, the curvature tensor and its covariant derivatives. The coefficients of natural operators are called *natural tensors*. In the case that they are scalar they are often also called *Riemannian invariants*. We say  $P$  is a *conformally invariant differential operator* if it is a natural operator in this way and is well defined on conformal structures (i.e., is independent of a choice of conformal scale).

We embrace Penrose's abstract index notation [17] throughout this paper and indices should be assumed abstract unless otherwise indicated. We write  $\mathcal{E}^a$  to denote the tangent bundle on  $M$ , and  $\mathcal{E}_a$  the cotangent bundle. We use the notation  $\mathcal{E}_a[w] = \mathcal{E}_a \otimes \mathcal{E}[w]$ ,  $\mathcal{E}_{ab}[w] = \mathcal{E}_a \otimes \mathcal{E}_b \otimes \mathcal{E}[w]$  and so on. An index which appears twice, once raised and once lowered, indicates a contraction. Each symmetric tensor product of the cotangent bundle is written as  $\mathcal{E}_{(ab\dots c)}$  and  $\mathcal{E}_{(ab\dots c)_0}$  indicates the completely trace-free subbundle. Similarly,  $\mathcal{E}_{[ab\dots c]}$  means the skew tensor product, that is, the bundle of differential forms. We also use this notation to indicate the projection onto these bundles, e.g.,  $2T_{[ab]} = T_{ab} - T_{ba}$ . These conventions will be extended in an obvious way to the tractor bundles described below.

Note that there is a tautological function  $\mathbf{g}$  on  $\mathcal{Q}$  taking values in  $\mathcal{E}_{(ab)}$ . It is the function which assigns to the point  $(g_x, x) \in \mathcal{Q}$  the metric  $g_x$  at  $x$ . This is homogeneous of degree 2 since  $\mathbf{g}(s^2g_x, x) = s^2g_x$ . If  $\xi$  is any positive function on  $\mathcal{Q}$  homogeneous of degree  $-2$ , then  $\xi\mathbf{g}$  is independent of the action of  $\mathbb{R}_+$  on the fibres of  $\mathcal{Q}$ , and so  $\xi\mathbf{g}$  descends to give a metric from the conformal class. Thus  $\mathbf{g}$

determines and is equivalent to a canonical section of  $\mathcal{E}_{ab}[2]$  (called the conformal metric) that we also denote  $\mathbf{g}$  (or  $\mathbf{g}_{ab}$ ). This in turn determines a canonical section  $\mathbf{g}^{ab}$  (or  $\mathbf{g}^{-1}$ ) of  $\mathcal{E}^{ab}[-2]$  with the property that  $\mathbf{g}_{ab}\mathbf{g}^{bc} = \delta_a^c$  (where  $\delta_a^c$  is the Kronecker delta, i.e., the section of  $\mathcal{E}_a^c$  corresponding to the identity endomorphism of the tangent bundle). The conformal metric (and its inverse  $\mathbf{g}^{ab}$ ) will be used to raise and lower indices. Given a choice of metric  $g \in [g]$ , we write  $\nabla_a$  for the corresponding Levi-Civita connection. For each choice of metric there is also a canonical connection on  $\mathcal{E}[w]$  determined by the identification of  $\mathcal{E}[w]$  with  $C^\infty(M)$ , as described above, and the exterior derivative on functions. We will also call this the Levi-Civita connection and thus for tensors with weight, e.g.,  $v_a \in \mathcal{E}_a[w]$ , there is a connection given by the Leibniz rule. With these conventions the Laplacian  $\Delta$  is given by  $\Delta = \mathbf{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$ .

The Riemannian curvature  $R_{ab}{}^c{}_d$ , determined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = R_{ab}{}^c{}_d v^d, \quad \text{where } v^c \in \mathcal{E}^c,$$

can be decomposed into the totally trace-free Weyl curvature  $C_{abcd}$  and the symmetric Schouten tensor  $P_{ab}$  according to

$$(2.1) \quad R_{abcd} = C_{abcd} + 2\mathbf{g}_{c[a}P_{b]d} + 2\mathbf{g}_{d[b}P_{a]c}.$$

This defines  $P_{ab}$  as a trace modification of the Ricci tensor  $R_{ab} = R_{ca}{}^c{}_b$ :

$$R_{ab} = (n - 2)P_{ab} + P_c{}^c \mathbf{g}_{ab}.$$

Note that the Weyl tensor has the symmetries

$$(2.2) \quad C_{abcd} = C_{[ab][cd]} = C_{cdab}, \quad C_{[abc]d} = 0.$$

Moreover, it follows from the Bianchi identity that

$$(2.3) \quad \nabla^c C_{abcd} = 2(n - 3)\nabla_{[a}P_{b]d}$$

and

$$(2.4) \quad (n - 3)\nabla_{[a}C_{bc]de} = \mathbf{g}_{d[a}\nabla^s C_{bc]se} - \mathbf{g}_{e[a}\nabla^s C_{bc]sd}.$$

Under a conformal transformation, we replace our choice of metric  $g$  by the metric  $\widehat{g} = e^{2\Upsilon}g$ , where  $\Upsilon$  is a smooth function. The Levi-Civita connection then transforms as follows:

$$(2.5) \quad \widehat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + \mathbf{g}_{ab}\Upsilon^c u_c, \quad \widehat{\nabla}_a \sigma = \nabla_a \sigma + w\Upsilon_a \sigma.$$

Here  $u_b \in \mathcal{E}_b$ ,  $\sigma \in \mathcal{E}[w]$ , and  $\Upsilon_a = \nabla_a \Upsilon$ . The Weyl curvature is conformally invariant, that is,  $\widehat{C} = C$ , and the Schouten tensor transforms by

$$(2.6) \quad \widehat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + O(\Upsilon^2),$$

where  $O(\Upsilon^2)$  denotes nonlinear terms in  $\Upsilon$ .

We define  $P^{(\ell)} \in \mathcal{E}_{(a_1 \dots a_\ell)}$  for  $\ell \geq 2$  by

$$P^{(\ell)} = P_{a_\ell \dots a_1} := \nabla_{(a_\ell} \dots \nabla_{a_3} P_{a_2 a_1)}.$$

From (2.1) and (2.3) it follows easily that if  $n > 3$ , then the jets of  $R$  at  $p$  can be expressed in terms of  $P^{(\ell)}$  and the jets of  $C$  at  $p$ . Note that we can always choose, for each point  $p \in M$ , a representative  $g$  from a conformal class such that

$$(2.7) \quad P^{(\ell)}(p) = 0, \quad \ell \geq 2;$$

following [10] we call  $g$  a *normal scale*. This is an easy consequence of the conformal variational formula:

$$(2.8) \quad \widehat{P}_{a_\ell \dots a_1} = P_{a_\ell \dots a_1} - \nabla_{(a_\ell} \dots \nabla_{a_1)} \Upsilon + O(\Upsilon^2),$$

since the terms in  $O(\Upsilon^2)$  involve at most  $\ell - 1$  derivatives of  $\Upsilon$ . In a normal scale, the jets of  $R$  at  $p$  can be expressed in terms of the Weyl curvature  $C$  and its covariant derivatives at  $p$ .

In dimension 4, it is well known that

$$B_{ab} = \nabla^d \nabla^c C_{cadb} + P^{dc} C_{cadb}$$

is a conformally invariant tensor, called the *Bach tensor*. The existence of a natural conformally invariant tensor, taking values in  $\mathcal{E}_{(ab)_0}[2 - n]$  and which generalises the Bach tensor to even dimensions, is deduced in [7] where it arises as the obstruction to the existence of a formal power series solution to their ambient metric construction. We will also denote this *Fefferman-Graham obstruction tensor* by  $B_{ab}$ . While no general explicit expression for  $B_{ab}$  has been given, it is easily shown from its origins as an obstruction that it contains linear terms when we consider perturbations from the flat metric. Using this, its naturality and conformal invariance as well as the symmetries and identities satisfied by the Weyl curvature, it is straightforward to deduce that its linear in curvature term is given (up to nonzero constant multiple) by

$$(2.9) \quad \Delta^{n/2-2} \nabla^c \nabla^d C_{cadb}.$$

We next define the standard tractor bundle over  $(M, [g])$ . It is a vector bundle of rank  $n + 2$  defined for each  $g \in [g]$  by  $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ . If  $\widehat{g} = e^{2\Upsilon}g$ , we identify  $(\sigma, \mu_a, \tau) \in [\mathcal{E}^A]_g$  with  $(\widehat{\sigma}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$  by the transformation

$$(2.10) \quad \begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a^b & 0 \\ -\frac{1}{2}\Upsilon_c \Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu_b \\ \tau \end{pmatrix}.$$

It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle*  $\mathcal{E}^A$  over the conformal manifold. (Alternatively the standard tractor bundle may be constructed as a canonical quotient of a certain 2-jet bundle or as an associated bundle to the normal conformal Cartan bundle [3].) The bundle  $\mathcal{E}^A$  admits an invariant metric  $h_{AB}$  of signature  $(p + 1, q + 1)$  and an invariant connection, which we shall also denote by  $\nabla_a$ , preserving  $h_{AB}$ . In a conformal scale  $g$ , these are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathbf{g}_{ab} \tau + P_{ab} \sigma \\ \nabla_a \tau - P_{ab} \mu^b \end{pmatrix}.$$

It is readily verified that both of these are conformally well defined, i.e., independent of the choice of a metric  $g \in [g]$ . Note that  $h_{AB}$  defines a section of  $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$ , where  $\mathcal{E}_A$  is the dual bundle of  $\mathcal{E}^A$ . Hence we may use  $h_{AB}$  and its inverse  $h^{AB}$  to raise or lower indices of  $\mathcal{E}_A, \mathcal{E}^A$  and their tensor products.

In computations, it is often useful to introduce the “projectors” from  $\mathcal{E}^A$  to the components  $\mathcal{E}[1], \mathcal{E}_a[1]$  and  $\mathcal{E}[-1]$  which are determined by a choice of scale. They are respectively denoted by  $X_A \in \mathcal{E}_A[1], Z_{Aa} \in \mathcal{E}_{Aa}[1]$  and  $Y_A \in \mathcal{E}_A[-1]$ , where

	$Y^A$	$Z^{Ac}$	$X^A$
$Y_A$	0	0	1
$Z_{Ab}$	0	$\delta_b^c$	0
$X_A$	1	0	0

FIGURE 1. Tractor inner product

$\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$ , etc. Using the metrics  $h_{AB}$  and  $g_{ab}$  to raise indices, we define  $X^A, Z^{Aa}, Y^A$ . Then we immediately see that

$$Y_A X^A = 1, \quad Z_{Ab} Z^A{}_c = g_{bc}$$

and that all other quadratic combinations that contract the tractor index vanish. This is summarised in Figure 1.

It is clear from (2.10) that the first component  $\sigma$  is independent of the choice of a representative  $g$  and hence  $X^A$  is conformally invariant. For  $Z^{Aa}$  and  $Y^A$ , we have the transformation laws:

$$(2.11) \quad \widehat{Z}^{Aa} = Z^{Aa} + \Upsilon^a X^A, \quad \widehat{Y}^A = Y^A - \Upsilon_a Z^{Aa} + O(\Upsilon^2).$$

Given a choice of conformal scale, we have the corresponding Levi-Civita connection on tensor and density bundles. In this setting we can use the coupled Levi-Civita tractor connection to act on sections of the tensor product of a tensor bundle with a tractor bundle. This is defined by the Leibniz rule in the usual way. For example if  $u^b V^C \sigma \in \mathcal{E}^b \otimes \mathcal{E}^C \otimes \mathcal{E}[w] =: \mathcal{E}^{bC}[w]$ , then  $\nabla_a u^b V^C \sigma = (\nabla_a u^b) V^C \sigma + u^b (\nabla_a V^C) \sigma + u^b V^C \nabla_a \sigma$ . Here  $\nabla$  means the Levi-Civita connection on  $u^b \in \mathcal{E}^b$  and  $\sigma \in \mathcal{E}[w]$ , while it denotes the tractor connection on  $V^C \in \mathcal{E}^C$ . In particular with this convention we have

$$(2.12) \quad \nabla_a X_A = Z_{Aa}, \quad \nabla_a Z_{Ab} = -P_{ab} X_A - Y_A g_{ab}, \quad \nabla_a Y_A = P_{ab} Z_A{}^b.$$

Note that if  $V$  is a section of  $\mathcal{E}_{A_1 \dots A_\ell}[w]$ , then the coupled Levi-Civita tractor connection on  $V$  is not conformally invariant but transforms just as the Levi-Civita connection transforms on densities of the same weight:  $\widehat{\nabla}_a V = \nabla_a V + w \Upsilon_a V$ .

Given a choice of conformal scale, the *tractor-D operator*

$$D_A: \mathcal{E}_{A_1 \dots A_\ell}[w] \rightarrow \mathcal{E}_{AA_1 \dots A_\ell}[w - 1]$$

is defined by

$$(2.13) \quad D_A V := (n + 2w - 2)w Y_A V + (n + 2w - 2)Z_{Aa} \nabla^a V - X_A \square V,$$

where  $\square V := \Delta V + w P_b{}^b V$ . This also turns out to be conformally invariant as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of  $D$ ; see, e.g., [9]).

The curvature  $\Omega$  of the tractor connection is defined by

$$(2.14) \quad [\nabla_a, \nabla_b] V^C = \Omega_{ab}{}^C{}_E V^E$$

for  $V^C \in \mathcal{E}^C$ . Using (2.12) and the usual formulae for the curvature of the Levi-Civita connection, we calculate (cf. [1])

$$(2.15) \quad \Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - \frac{2}{n-3} X_{[C} Z_{E]}{}^e \nabla^d C_{abde}.$$

Here, and in the remainder of this section, to simplify the formulae we have assumed  $n \geq 4$ . Since our later discussions are all set in even dimensions, there is no need here for the results in dimension 3. We also set

$$\Omega_{ABCE} = Z_A^a Z_B^b \Omega_{abCE}, \quad \Omega_{BsCE} = Z_B^b \Omega_{bsCE}.$$

It is easily verified that  $[D_B, D_C]$  vanishes on densities. For tractors  $V \in \mathcal{E}_{A_1 A_2 \dots A_k}[w]$ , it is straightforward to use (2.14) and (2.12) to show

$$(2.16) \quad [D_B, D_C]V_{A_1 A_2 \dots A_k} = E_{BCA_1}{}^Q V_{QA_2 \dots F} + \dots + E_{BCA_k}{}^Q V_{A_1 \dots A_{k-1} Q},$$

where

$$(2.17) \quad E_{ABCE} = (n + 2w - 2) \left( (n + 2w - 4) \Omega_{ABCE} - 2X_{[A} Z_{B]}{}^b \nabla^p \Omega_{pbCE} + 4X_{[A} \Omega_{B]}{}^s{}_{CE} \nabla_s \right).$$

For our forthcoming calculations, we need to express  $Y^A Y^C E_{ABCE}$  in terms of  $C$ . The first term of  $E$  is killed by contraction with  $Y^A$  and the last term gives

$$4Y^A Y^C X_{[A} \Omega_{B]}{}^p{}_{CE} \nabla^p = \frac{-2}{n - 3} Z_B^b Z_E^e (\nabla^a C_{aebc}) \nabla^c.$$

For the middle term, using (2.12), we have

$$-2Y^A Y^C X_{[A} Z_{B]}{}^b \nabla^p \Omega_{pbCE} = \frac{1}{n - 3} Z_B^b Z_E^e \nabla^a \nabla^c C_{abce} + O(R^2),$$

where the  $O(R^2)$  indicates nonlinear terms in the curvature. Thus using (2.2), we get

$$(2.18) \quad Y^A Y^C E_{ABCE} = \frac{n + 2w - 2}{n - 3} \times Z_B^b Z_E^e \left( \nabla^c \nabla^d C_{cedb} + 2(\nabla^c C_{cedb}) \nabla^d \right) + O(R^2).$$

### 3. PROOF OF THE THEOREM

For the remainder of the paper we restrict to manifolds of even dimension  $n$ .

Our key tool for the proof is the natural differential operator

$$L_g : \mathcal{E}[k - n/2] \rightarrow \mathcal{E}[-k - n/2]$$

defined by

$$L_g f := Y^{A_k} \dots Y^{A_1} D_{A_k} \dots D_{A_1} f.$$

This is a composition of the conformally invariant operator

$$D_{A_k} \dots D_{A_2} D_{A_1} : \mathcal{E}[k - n/2] \rightarrow \mathcal{E}_{A_k \dots A_2 A_1}[-n/2]$$

and the projector, determined by  $g$ ,

$$Y^{A_k} \dots Y^{A_1} : \mathcal{E}_{A_k \dots A_1}[-n/2] \rightarrow \mathcal{E}[-k - n/2].$$

It follows easily from (2.13) and the relations in Figure 1 that  $L_g$  has leading part  $(-\Delta)^k$ . In view of (2.11) we do not expect  $L_g$  to exhibit invariance under conformal rescaling. However if  $g$  is conformally flat, it turns out that  $L_g$  is the unique conformally invariant operator between densities whose leading part is  $(-\Delta)^k$ ; see e.g., Proposition 2.1 of [12] or [9].

Our strategy for proving the theorem is as follows. We study the dependence of  $L_g f$  on deformations from the flat metric  $g_0$ . For a smooth family of Riemannian

(or pseudo-Riemannian) metrics  $\{g_t\}_{t \in \mathbb{R}}$  such that  $g_0$  is the flat metric and  $\Upsilon \in C^\infty(M \times \mathbb{R})$ , let

$$L[s, t] := L_{g[s,t]}f(p), \quad \text{where } g[s, t](x) := e^{2s\Upsilon(x,t)}g_t(x),$$

for each  $s \in \mathbb{R}$ . Here we fix  $f \in \mathcal{E}[k-n/2]$  on  $(M, [g_0])$  and then regard it as a density in  $\mathcal{E}[k-n/2]$  for each conformal structure  $[g_t]$  by the identification  $f_{g_t} = f_{g_0}$ . With a view to a contradiction we suppose that there exists a natural conformally invariant operator  $P_{2k}$  between density bundles with leading term  $\Delta^k$ , where  $k > n/2$ . Such an operator in particular gives an operator on conformally flat spaces and so must appear in the classification of such operators described in the introduction. Thus we have

$$(3.1) \quad P_{2k} : \mathcal{E}[k - n/2] \rightarrow \mathcal{E}[-k - n/2].$$

(In particular, if  $g$  is conformally flat, then we have  $(-1)^k P_{2k} = L_g$ .) Then we choose  $p \in M$  and set

$$P[t] := (-1)^k P_{2k}f(p)$$

in the metric  $g[s, t]$ . Note that, since  $P_{2k}$  is conformally invariant, the right-hand side is independent of  $s$ . We compute  $\partial_t \partial_s L[0, 0]$  by two methods, which give different answers. With some assumptions on the family  $g[s, t]$  and on  $f$ , we show, by one set of calculations, that

$$\partial_t \partial_s L[0, 0] \neq 0.$$

On the other hand, with the other approach, we obtain

$$L[s, t] = P[t] + O(s^2) + O(t^2),$$

where  $O(\cdot)$  is used in the sense of ideals in the ring of formal powers series; the notation  $+O(\cdot)$  means modulo the addition of elements from the ideal generated by the indicated argument. Hence this implies  $\partial_t \partial_s L[0, 0] = 0$ . Since this is a contradiction, we conclude that the operator in (3.1) cannot exist.

In what follows, we use the notation

$$D_{A_1 \dots A_k} := D_{A_1} \dots D_{A_k}, \quad \nabla_{a_1 \dots a_k} := \nabla_{a_1} \dots \nabla_{a_k}.$$

Let us write  $\nabla^{(\ell)}C$  as shorthand for the tensor  $\nabla_{a_1 \dots a_\ell} C_{bcde}$ , and set  $\nabla^{(0)}C = C$ . Similarly we will write  $\nabla^{(j)}\Delta^\ell f$  as shorthand for  $\nabla_{a_1 \dots a_j} \Delta^\ell f$ . Unless otherwise stated,  $\nabla, C, P$  are assumed to be defined with respect to  $g_t$ . Finally we set

$$w = k - n/2,$$

which is a positive integer.

We will work with a one parameter family of metrics  $g_t$  such that

$$(3.2) \quad P^{(\ell)}(p) = O(t^2), \quad \ell \geq 2,$$

$$(3.3) \quad \nabla^{(\ell)}C(p) = O(t^2), \quad 0 \leq \ell \leq w + n - 5,$$

$$(3.4) \quad \nabla^{(\ell)}\Delta^{n/2-2}\nabla_{bc}C_a^b{}_d{}^c(p) = O(t^2), \quad \ell \geq 0,$$

and, for  $n \geq 6$ ,

$$(3.5) \quad \Delta^{n/2-3}\nabla_{bc(a_{w+2} \dots a_3}C_{a_2}{}^b{}_{a_1)}{}^c(p) = O(t^2).$$

We next take a scaling function  $\Upsilon$  such that

$$(3.6) \quad \nabla_{(a_\ell \dots a_1)}\Upsilon(p, t) = 0, \quad \ell \geq 2.$$



Finally, for the density  $f$ , we impose

$$(3.7) \quad \nabla^{(\ell)} f(p) = 0, \quad 0 \leq \ell \leq w.$$

From the conformal invariance of the Weyl curvature  $C$  and (2.5) it follows that condition (3.3) is conformally invariant — in the sense that if a family of metrics  $g_t$  satisfies a set of conditions, then so does  $e^{2\Upsilon} g_t$  for any scaling function  $\Upsilon(x, t)$ . For  $g_t$  satisfying (3.3), it is also clear from (2.5) that (3.5), which is a condition on  $\nabla^{(w+n-4)} C$ , is conformally invariant. Condition (3.4) can be rewritten in terms of the conformally invariant Fefferman-Graham obstruction tensor  $B_{ab}$ ,

$$\nabla^{(\ell)} B(p) = O(t^2), \quad \ell \geq 0,$$

because of (2.9). Hence it is also conformally invariant. Condition (3.7) is exactly equivalent to requiring that  $f$  is a density such that its  $w$ -jet vanishes at  $p$ . Thus this condition is independent of the choice of  $g_t$  and, in particular, is conformally invariant. Finally from (2.8) it is clear that (3.2) is not a conformally invariant condition. Whereas the point of conditions (3.3), (3.4) and (3.5) is to specialise the class of conformal structures we allow, the role of (3.2) is rather as a scale normalisation condition which restricts metrics allowed from within a given conformal class  $[g_t]$ . Nevertheless it is crucial to our arguments that with (3.2) some conformal scaling freedom remains. In particular if we assume (3.6), then we have (3.11) below.

Under these assumptions, we will show the following two results.

**Lemma 3.1.** *Assume that (3.2)–(3.7) hold. Then*

$$L[s, t] = P[t] + O(s^2) + O(t^2).$$

**Lemma 3.2.** *Assume that (3.2)–(3.7) hold and further assume*

$$(3.8) \quad \nabla^{(\ell)} \Delta f(p) = O(t), \quad \ell \geq 0.$$

*Then*

$$\partial_s L[0, t] = c \Upsilon_b (\Delta^{n/2-2} \nabla_{c a_{w+1} \dots a_3} C_{a_2}{}^b{}_{a_1}{}^c) \nabla^{a_{w+1} \dots a_1} f(p) + O(t^2),$$

where  $c$  is a nonzero constant.

With these lemmas established, the theorem is a consequence of the existence of a density satisfying (3.8) and the following proposition which will be proved in the next section.

**Proposition 3.3.** *There is a deformation  $\{g_t\}_{t \in \mathbb{R}}$  of the flat metric  $g_0$  that satisfies (3.2)–(3.5) yet with*

$$(3.9) \quad F^b_{(a_{w+1} \dots a_1)_0}(p) \neq 0,$$

where

$$F^b_{a_{w+1} \dots a_1} = \partial_t \Big|_{t=0} \Delta^{n/2-2} \nabla_{c a_{w+1} \dots a_3} C_{a_2}{}^b{}_{a_1}{}^c.$$

*Proof of Theorem.* Let  $g_t$  be a family of metrics satisfying conditions (3.2)–(3.5) and (3.9). The existence of such a family is guaranteed by Proposition 3.3 above. Then by (3.9) we may find  $\mu_b \in \mathcal{E}_b|_p$  and  $\xi_{a_1 a_2 \dots a_{w+1}} \in \mathcal{E}_{(a_1 \dots a_{w+1})_0}[w]|_p$  such that

$$(3.10) \quad \mu_b \xi^{a_1 a_2 \dots a_{w+1}} F^b_{a_1 \dots a_{w+1}} \neq 0.$$

Denoting by  $x^i$  some fixed choice of normal coordinates for  $g_0$  centered at  $p$ , we set

$$f(x) = \xi_{i_1 i_2 \dots i_{w+1}} x^{i_1} \dots x^{i_{w+1}}.$$

Then  $f$  clearly satisfies the assumptions (3.7). In the metric  $g_0$  we also have  $\nabla^{(\ell)}\Delta f(p) = 0$  for all  $\ell \geq 0$ . Thus, since the contorsion tensor distinguishing the metric connections of  $g_t$  and  $g_0$  is  $O(t)$ , it follows immediately that  $f$  satisfies (3.8). Next we construct  $\Upsilon(x, t)$  by setting

$$\Upsilon(x, 0) = \mu_i x^i$$

and then we obtain a function on  $M \times \mathbb{R}$  satisfying (3.6) by solving the equation  $\nabla_{(a_1 \dots a_\ell)} \Upsilon(0, t) = 0$ ,  $\ell \geq 2$ , for each  $t$ . From standard theory this can be achieved within  $C^\infty(M \times \mathbb{R})$ . Then Lemma 3.1 implies  $L[s, t] = P[t] + O(s^2) + O(t^2)$  so that  $\partial_s \partial_t L[0, 0] = 0$ , while Lemma 3.2 shows

$$\partial_s \partial_t L[0, 0] = c \mu_b \xi^{a_1 a_2 \dots a_{w+1}} F^b_{(a_1 \dots a_{w+1})_0} \neq 0,$$

which is a contradiction. □

We now prove the lemmas used in the proof above. Throughout the proofs we will work at  $p \in M$ . In all final expressions the tensors involved are evaluated at  $p$  and we write simply  $\nabla_a C_{bcde}$  to mean  $\nabla_a C_{bcde}(p)$  and so forth.

*Proof of Lemma 3.1.* We first prove

$$S[t] := L[0, t] - P[t] = O(t^2).$$

Since  $L_g$  and  $(-1)^k P_{2k}$  are natural operators which agree for conformally flat metrics, it follows that there is an expression for  $S[t]$  as a sum of terms where each term is homogeneous of degree at least one in the jets of the curvature  $R$  at  $p$ . Next via (2.1) and (2.3) we may express the jets of  $R$  in terms of  $P^{(\ell)}$  and the jets of  $C$  and obtain a new expression for  $S[t]$  which is polynomial in these tensors. By (3.2) and since  $C = O(t)$ , the terms containing  $P^{(\ell)}$  and those which are nonlinear in  $C$  are  $O(t^2)$  and so can be neglected. Thus using standard classical invariant theory and elementary weight considerations, we can express the Riemannian invariant  $S[t] \bmod O(t^2)$  as a linear combination of complete contractions of  $(\nabla^{(\ell)} C) \nabla^{(m)} f$  with  $\ell + m = 2k - 2$ . In view of conditions (3.3) and (3.7), to obtain a nontrivial term we must have  $\ell \geq w + n - 4$  and  $m \geq w + 1$ . Thus a possible nonvanishing term should be a complete contraction of one of the following two tensors:

$$(\nabla^{(w+n-3)} C) \nabla^{(w+1)} f \quad \text{or} \quad (\nabla^{(w+n-4)} C) \nabla^{(w+2)} f.$$

Now consider the possible ways such a complete contraction could be made. First note that since the tensor field  $C$  is completely trace-free, it is clear that in such a complete contraction we can assume, without loss of generality, that the indices of  $C_{abcd}$  are paired with indices on  $\nabla$ 's. Next observe that the tensor field  $C$  has the symmetry  $C_{abcd} = C_{[ab][cd]}$  while  $\nabla^{a_\ell \dots a_1} f = \nabla^{(a_\ell \dots a_1)} f + O(t)$ . Thus  $(\nabla^{(\ell)} C_{abcd}) \nabla^{(m)} \nabla^{abc} f = O(t^2)$  for  $\ell, m \geq 0$ . Also  $\nabla^{(\ell)} \nabla^{abc} C_{abcd} = 0$  and for both similar results hold for any permutation of the indices on  $C$ . Thus from the symmetries of the Weyl tensor  $C$ , the possible nonzero complete contractions of the displayed terms must be contractions of the tensors

$$(\nabla^{(w-1)} \Delta^{n/2-2} \nabla^{ab} C_{abcd}) \nabla^{(w-1)} \nabla^{cd} f$$

and,

$$(\nabla^{(w)} \Delta^{n/2-3} \nabla^{ab} C_{abcd}) \nabla^{(w)} \nabla^{cd} f, \quad \text{if } n \geq 6,$$

or

$$(\nabla^{(w-2)} \nabla^{ab} C_{abcd}) \nabla^{(w-2)} \Delta \nabla^{cd} f, \quad \text{if } n = 4 \text{ and } w \geq 2.$$

But these are  $O(t^2)$  by (3.4) and (3.5). Thus  $S[t] = O(t^2)$ .

To prove the general case, we first consider the tensors  $P^{(\ell)}$  in the metric  $g[s, t]$ . By the conformal transformation law (2.8) of  $P^{(\ell)}$ , we have

$$[P_{a_\ell \dots a_1}]_{g[s,t]} = P_{a_\ell \dots a_1} - s \nabla_{(a_\ell \dots a_1)} \Upsilon + O(s^2).$$

Thus (3.2) and (3.6) imply

$$(3.11) \quad [P_{a_\ell \dots a_1}]_{g[s,t]} = O(s^2) + O(t^2).$$

The other conditions imposed on, and properties of,  $C$ ,  $f$  and their covariant derivatives, as used in the argument above at a metric  $g_t$ , are all conformally invariant and so hold for  $g[s, t]$ . Thus replacing  $g_t$  with  $g[s, t]$ , the argument above that led to the conclusion  $S[t] = O(t^2)$  can be repeated exactly with the single exception that  $P^{(\ell)}$  can now be neglected with error  $O(s^2) + O(t^2)$  (rather than  $O(t^2)$  as above). Thus with  $S[s, t] := L[s, t] - P[t]$ , we obtain  $S[s, t] = O(s^2) + O(t^2)$ .  $\square$

*Proof of Lemma 3.2.* Since  $D_{A_k \dots A_1}$  is conformally invariant, the conformal variation of  $L_g f$  is determined entirely by the variation (2.11) of  $Y^A$ . Thus we have

$$\partial_s L[0, t] = -\Upsilon_b \sum_{j=1}^k Z^{A_j b} Y^{A_k \dots \widehat{A}_j \dots A_1} D_{A_k \dots A_1} f,$$

where  $\widehat{A}_j$  indicates an absent index. Now (with  $s$  still set to 0) we work with the metrics  $g_t$ . From the definition of  $D_{A_k \dots A_1}$  we may re-express it as

$$D_{A_k \dots A_1} f = 2D_{A_k \dots [A_{j+1} A_j] \dots A_1} f + 2D_{A_k \dots [A_{j+2} A_j] A_{j+1} \widehat{A}_j \dots A_1} f + \dots + 2D_{[A_k A_j] A_{k-1} \dots \widehat{A}_j \dots A_1} f + D_{A_j A_k \dots \widehat{A}_j \dots A_1} f.$$

Using this, we have at once that

$$\partial_s L[0, t] = - \sum_{j=1}^k j \Upsilon_b F_{(j)}^b,$$

where

$$F_{(j)}^b := 2Z^{Bb} Y^{A_k \dots \widehat{A}_j \dots A_1} D_{A_k \dots [A_{j+1} B] A_{j-1} \dots A_1} f$$

for  $j \leq k - 1$ , and

$$F_{(k)}^b := Z^{Bb} Y^{A_{k-1} \dots A_1} D_{B A_{k-1} \dots A_1} f.$$

We first show that  $F_{(j)}^b = O(t^2)$  if  $j \neq 2$ . Note that  $F_{(1)}^b = 0$  because  $D_{[A_2 B]}$  vanishes on densities. Next we recall that  $f$  is a density of weight  $k - n/2$  and each  $D$  lowers weight by 1. So from (2.13) we have

$$Z^{Bb} D_{B A_{k-1} \dots A_1} f = -Z^{Bb} X_B \square D_{A_{k-1} \dots A_1} f = 0,$$

which implies  $F_{(k)}^b = 0$ . To prove the other cases, we recall (2.16):

$$(3.12) \quad D_{[CB] A_{j-1} \dots A_1} f = (k - j) \sum_{\ell=1}^{j-1} E_{BC A_\ell}{}^Q D_{A_{j-1} \dots A_{\ell+1} Q A_{\ell-1} \dots A_1} f,$$

where  $E_{ABCD}$  is given by (2.17) and it is  $O(t)$ . If we commute the indices for  $D_{A_{j-1} \dots A_{\ell+1} Q A_{\ell-1} \dots A_1} f$ , we get another  $O(t)$  term. Thus we see that each summand

of the right-hand side of (3.12) is independent of  $\ell$  up to permutations of  $A_1 \cdots A_{j-1}$  and modulo  $O(t^2)$ . Hence  $F_{(j)}^b \pmod{O(t^2)}$  for  $j > 2$  is a multiple of

$$(3.13) \quad Z^{Bb} Y^{A_{k-1} \cdots A_1} D_{A_{k-1} \cdots A_{j+1}} (E_{A_j B A_{j-1}})^Q D_{Q A_{j-2} \cdots A_1} f.$$

This is  $O(t^2)$ , which we see as follows. From

$$(3.14) \quad [Y^A, \nabla_b] = O(t)$$

and formula (2.13) for  $D_B$ , we conclude that there is a (weight dependent) operator  $E_B$  such that  $Y^A D_B = E_B Y^A + O(t)$  and so in (3.13) we may pass  $Y^{A_1}$  to the right where we finally observe that

$$Y^{A_1} D_{A_1} f = -\Delta f + O(t) = O(t),$$

from (3.8).

We now focus on the computation of

$$F_{(2)}^b = 2Z^{Bb} Y^{A_{k-1} \cdots A_1} D_{A_{k-1} \cdots A_3} (E_{A_2 B A_1})^Q D_Q f.$$

Using (3.14) and  $Y^A D_A = -\Delta + O(t)$ , we simplify  $F_{(2)}^b$  to

$$(3.15) \quad 2Z^{Bb} (-\Delta)^{k-3} (Y^{A_2 A_1} E_{A_2 B A_1})^Q D_Q f,$$

modulo  $O(t^2)$ . Substituting (2.18), we may reduce the above formula to a nonzero multiple of

$$(3.16) \quad Z^{Bb} \Delta^{k-3} \left( Z_B^a Z^{Qq} ((\nabla^{cd} C_{cqda}) + 2(\nabla^c C_{cqda}) \nabla^d) D_Q f \right).$$

Next using the identities of (2.12) and Figure 1, we have

$$Z^{Bb} \Delta^{k-3} Z_B^a = g^{ab} \Delta^{k-3} + O(t).$$

Similarly, (2.12) and (2.13) imply

$$(\nabla^d Z^{Qq}) D_Q f = -g^{dq} Y^Q D_Q f = g^{dq} \Delta f + O(t) = O(t)$$

and

$$Z^{Qq} D_Q f = (n + 2w - 2) \nabla^q f.$$

Thus (3.16) is reduced, up to a nonzero multiple and modulo  $O(t^2)$ , to

$$(3.17) \quad \Delta^{k-3} ((\nabla^{cd} C_{cqd}{}^b) \nabla^q f) + 2\Delta^{k-3} ((\nabla^c C_{cqd}{}^b) \nabla^{dq} f).$$

Finally we expand  $\Delta^{k-3}$  by using the Leibniz rule. Observe that, in each term of the result, the total number of  $\nabla$  is  $2k - 3 = n + 2w - 3$ , while in order to get a nonvanishing term, we need to apply at least  $(n + w - 4)$   $\nabla$ 's to  $C$ , by (3.3), and at least  $(w + 1)$   $\nabla$ 's to  $f$ , by (3.7). Such a partition of  $n + 2w - 3$  is unique and, using (3.8), we see (3.17) is reduced to

$$2^w \binom{k-3}{w} I^b + 2^w \binom{k-3}{w-1} J^b,$$

where

$$I^b = (\Delta^{n/2-3} \nabla_{c d a_{w+1} \cdots a_2} C_{a_1}{}^{c db}) \nabla^{a_{w+1} \cdots a_1} f,$$

$$J^b = (\Delta^{n/2-2} \nabla_{c a_{w+1} \cdots a_3} C_{a_2 a_1}{}^b) \nabla^{a_{w+1} \cdots a_1} f.$$

If  $n = 4$ , the first term does not appear and we immediately see that  $\partial_s L[0, t] \pmod{O(t^2)}$  is a nonzero multiple of  $J^b \Upsilon_b$ . If  $n \geq 6$ , we have

$$(3.18) \quad (w + 2) I^b = w J^b + O(t^2),$$

and, since  $w > 0$ , we are led to the same conclusion. Therefore it remains only to prove this equation for  $n \geq 6$ .

Corresponding to the curvature terms of  $I^b$  and  $J^b$ , we set

$$\begin{aligned} \tilde{I} &= \Delta^{n/2-3} \nabla_{cd(a_{w+1} \dots a_2} C^c_{a_1}{}^d{}_b, \\ \tilde{J} &= \Delta^{n/2-2} \nabla_{c(a_{w+1} \dots a_3} C^c_{a_2 a_1) b}, \end{aligned}$$

where we have suppressed the indices on  $\tilde{I}$  and  $\tilde{J}$  to simplify the notation. Then noting that  $\nabla_{cd a_{w+1} \dots a_2} C^c_{a_1}{}^d{}_b = \nabla_{cd(a_{w+1} \dots a_2)} C^c_{(a_1}{}^d{}_b) + O(t^2)$ , we may rewrite (3.5) as

$$(3.19) \quad 2\tilde{I} + wK = O(t^2),$$

where

$$K = \Delta^{n/2-3} \nabla_{cdb(a_{w+1} \dots a_3} C^c_{a_2}{}^d{}_{a_1}).$$

On the other hand, note that from (2.3),

$$\begin{aligned} \nabla_{[b} \nabla^d C_{ca_2] da_1} &= 2(n-3) \nabla_{[b} \nabla_c P_{a_2] a_1} \\ &= O(t^2), \end{aligned}$$

and hence  $\nabla^{cd}{}_{[b} C_{ca_2] da_1} = O(t^2)$  which further implies

$$\Delta^{n/2-3} \nabla_{a_{w+1} \dots a_3} \nabla^{cd}{}_{[b} C_{ca_2] da_1} = O(t^2).$$

Symmetrising the left-hand side of this last expression over  $a_{w+1}, \dots, a_1$  gives

$$-\tilde{I} + \tilde{J} + K = O(t^2).$$

Comparing this equation with (3.19), we finally get

$$(w+2)\tilde{I} = w\tilde{J} + O(t^2),$$

which implies (3.18) because  $\nabla^{a_{w+1} \dots a_1} f = \nabla^{(a_{w+1} \dots a_1)} f + O(t)$ . □

#### 4. CONSTRUCTION OF THE METRIC

It is clear that the issue of existence/nonexistence of invariant operators is independent of signature (and could equally be treated in the complex setting). To simplify the proof below, we shall be satisfied with constructing a Riemannian signature metric. With very slight modification the same argument yields a proof of Proposition 3.3 in any other desired signature.

*Proof of Proposition 3.3.* We first linearise the problem. For a symmetric two form  $\psi = \psi_{ab} \in \mathcal{E}_{(ab)}$  and each  $t \in \mathbb{R}$ , we write  $R_{abcd}[t]$ ,  $C_{abcd}[t]$  and  $P_{ab}[t]$ , respectively, for the Riemannian curvature, the Weyl curvature and the Schouten tensor of  $g_t = g_0 + t\psi$ . Then set

$$R_{abcd} := \left. \frac{d}{dt} \right|_{t=0} R_{abcd}[t], \quad C_{abcd} := \left. \frac{d}{dt} \right|_{t=0} C_{abcd}[t], \quad P_{ab} := \left. \frac{d}{dt} \right|_{t=0} P_{ab}[t].$$

It follows from the definition of curvature that

$$(4.1) \quad R_{abcd} = \frac{1}{2} (\nabla_{c[a} \psi_{b]d} - \nabla_{d[a} \psi_{b]c}).$$

Then  $C_{abcd}$  is the trace-free part of this, while  $P_{ab}$  is a scaled trace adjustment of a single trace of (4.1). Here  $\nabla$  is defined with respect to the flat metric  $g_0$ . In terms of these tensors, Proposition 3.3 is reduced to the following lemma.

**Lemma 4.1.** *For each  $w \in \mathbb{N}$  there exists a symmetric two form  $\psi_{ab} \in \mathcal{E}_{(ab)}$  on  $\mathbb{R}^n$  such that*

$$(4.2) \quad \nabla_{(a_\ell \dots a_3} P_{a_2 a_1)}(0) = 0, \quad \ell \geq 2,$$

$$(4.3) \quad \nabla^{(\ell)} C(0) = 0, \quad 0 \leq \ell \leq w + n - 5,$$

$$(4.4) \quad \nabla^{(\ell)} \Delta^{n/2-2} \nabla_{bc} C_a{}^b{}_d{}^c(0) = 0, \quad \ell \geq 0,$$

and, for  $n \geq 6$

$$(4.5) \quad \Delta^{n/2-3} \nabla_{bc(a_w+2 \dots a_3} C_{a_2}{}^b{}_{a_1)}{}^c(0) = 0$$

yet with

$$(4.6) \quad \Delta^{n/2-2} \nabla_{c(a_w+1 \dots a_3} C_{a_2}{}^b{}_{a_1)}{}^c(0) \neq 0.$$

Before we prove this, we need some background on the representation theory used. The irreducible finite dimensional representations of  $SL(n)$  can be classified by Young diagrams. We use the notation  $(\ell_1, \ell_2, \dots, \ell_{n-1})$  to indicate the representation corresponding to a Young diagram with rows (beginning from the top) of length  $\ell_1 \geq \ell_2 \geq \dots \geq \ell_{n-1} \geq 0$ . We identify  $SL(n)$  with its defining representation, and via this standard action of  $SL(n)$  on  $\mathbb{R}^n$  there are tensorial realisations of these representations. For example a tensorial realisation of  $(\ell_1, \ell_2, \dots, \ell_{n-1})$  is given by  $Y(\ell_1, \ell_2, \dots, \ell_{n-1})$  which denotes the space of (covariant)  $\mathbb{R}^n$  tensors, of rank  $\sum \ell_i$ , with the manifest symmetries

$$F_{a_1 \dots a_{\ell_1} b_1 \dots b_{\ell_2} \dots d_1 \dots d_{\ell_{n-1}}} = F_{(a_1 \dots a_{\ell_1})(b_1 \dots b_{\ell_2}) \dots (d_1 \dots d_{\ell_{n-1}})}$$

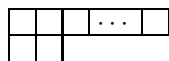
and so-called “hidden” symmetries which can be described as follows: first a complete symmetrisation over any  $\ell_1 + 1$  of the indices annihilates  $F$ ; if we exclude the set  $a_1 \dots a_{\ell_1}$ , then a complete symmetrisation over any  $\ell_2 + 1$  of the remaining indices annihilates  $F$ ; if we exclude the sets  $a_1 \dots a_{\ell_1}$  and  $b_1 \dots b_{\ell_2}$ , then a complete symmetrisation over any  $\ell_3 + 1$  of the remaining indices annihilates  $F$  and so on. To simplify the notation, we will omit terminal strings of zeros. Thus we write  $(\ell_1, \ell_2)$  as shorthand for  $(\ell_1, \ell_2, 0, \dots, 0)$  and similarly  $Y(\ell_1, \ell_2)$  for the described tensorial realisation of this.

On the space of tensors of rank  $\sum \ell_i$ , there are different projections onto a space  $Y(\ell_1, \ell_2, \dots, \ell_{n-1})$  according to different orderings of the indices. (These are easily described explicitly [8].) There are identities between these projections but we do not need these details. Any such projection will (also) be denoted by  $Y(\ell_1, \ell_2, \dots, \ell_{n-1})$  and is termed a Young symmetriser.

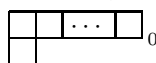
The finite dimensional  $SO(n)$ -representations (where as usual  $n$  is even) are also classified by strings of integers, in this case just  $n/2$  of these,  $[\ell_1, \dots, \ell_{n/2}]$ , where  $\ell_1 \geq \dots \geq \ell_{n/2-1} \geq \lfloor \ell_{n/2} \rfloor$  and if  $n/2$  is odd, then  $\ell_{n/2} \geq 0$ . We omit terminal strings of zeros in this case too. Via the defining representation, where we view  $SO(n)$  as the subgroup of  $SL(n)$  preserving the standard metric  $\delta$ , these also have tensorial realisations:  $Y_0[\ell_1, \dots, \ell_{n/2-1}]$  (i.e.,  $Y_0[\ell_1, \dots, \ell_{n/2-1}, 0]$ ) is the subspace of  $Y(\ell_1, \dots, \ell_{n/2-1})$  consisting of completely trace-free tensors. Continuing this notation, if  $\ell_{n/2} > 0$ , then the subspace of  $Y(\ell_1, \dots, \ell_{n/2})$  consisting of completely trace-free tensors will be denoted  $Y_0[\ell_1, \dots, \ell_{n/2}]$ . This is an irreducible  $O(n)$ -module but upon restriction to  $SO(n)$  is either irreducible or further decomposes depending on the parity of  $n/2$ : if  $n/2$  is odd, then  $Y_0[\ell_1, \dots, \ell_{n/2}]$  is irreducible, while if  $n/2$  is even,  $Y_0[\ell_1, \dots, \ell_{n/2}]$  decomposes into a direct sum of irreducible

representations, each an eigenspace of an action of the volume form. The latter are realisations of representations usually denoted  $[\ell_1, \dots, \ell_{n/2}]$  and  $[\ell_1, \dots, -\ell_{n/2}]$ . (In the corresponding complex theory of  $SO(n, \mathbb{C})$  one obtains such a decomposition regardless of the parity of  $n/2$ .)

*Proof of Lemma 4.1.* Consider the irreducible  $SL(n)$  representation  $(n + w - 2, 2)$ :



Viewed as a representation of  $SO(n)$ , by restriction, this decomposes into a direct sum of  $SO(n)$  irreducible representations. Using elementary representation theory [8], it is easily verified that, since  $n$  is even, the representation  $[w + 1, 1]$ :



(and in dimension four  $[w + 1, 1] \oplus [w + 1, -1]$ ) occurs exactly once as a summand in this decomposition.

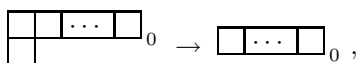
At the level of tensor realisations this means that  $Y_0[w + 1, 1]$  is a summand in the orthogonal decomposition of  $Y(n + w - 2, 2)$  and so for  $K$  any nonzero tensor from the space  $Y_0[w + 1, 1]$  there is a Young symmetriser  $Y(n + w - 2, 2)$  which does not annihilate

$$K \otimes \underbrace{\delta \otimes \dots \otimes \delta}_{n/2-1}.$$

Applying this Young symmetriser to the displayed tensor, let us denote the image by  $\Psi$ . Then  $\Psi \in Y(n + w - 2, 2)$ , and so letting  $m := n + w - 2$ , we have, in particular, that  $\Psi = \Psi_{a_1 \dots a_m b_1 b_2} = \Psi_{(a_1 \dots a_m)(b_1 b_2)}$ .

We can view  $\Psi$  as a (constant) covariant tensor on  $\mathbb{R}^n$ , as an affine space, and in this setting we define  $\psi_{j_1 j_2} := \Psi_{i_1 \dots i_m j_1 j_2} x^{i_1} \dots x^{i_m}$ , where  $x^i$  are the standard coordinates. Let  $p$  be the origin in  $\mathbb{R}^n$  (with  $n \geq 4$  even as usual) and take  $g_0 := \sum_1^n dx^i \cdot dx^i$  so the component matrix of  $g_0$  is  $\delta$ . Then we claim that  $\psi$  is a solution to (4.2)–(4.6).

First note that, since  $\psi$  is homogeneous of degree  $m$ , (4.2)–(4.4) are satisfied except possibly for  $\ell = m$  in (4.2) and  $\ell = w - 2$  in (4.4). In both cases the tensors on the left-hand sides are obtained by algebraic operations of symmetrisation and tracing from the tensor  $\Psi$ . These operations are  $SO(n)$ -equivariant and, since also the map from  $K$  to  $\Psi$  is  $SO(n)$ -equivariant, we see that the maps from  $K$  to these tensors are  $SO(n)$ -equivariant. The same comment applies to the left-hand side of (4.5) (for  $n \geq 6$ ). Consider first (4.2) with  $\ell = m$ . Note that  $\nabla_{(b_m \dots b_3} P_{b_2 b_1)}(0)$  takes values in  $Y(m)$  and that, as an  $SO(n)$ -module,  $(m)$  decomposes into a direct sum  $[m] \oplus [m - 2] \oplus \dots$  (terminating in  $[1]$  or  $[0]$  according to the parity of  $m$ ). Thus if  $\nabla_{(b_m \dots b_3} P_{b_2 b_1)}(0)$  were nonzero for any  $K$ , then, by the composition of the linear equivariant operation mentioned with the projection to  $SO(n)$ -irreducible components, this would imply, when  $n \geq 6$ , the existence of a nontrivial  $SO(n)$ -module homomorphism  $[w + 1, 1] \rightarrow [q]$  for some  $q \in \mathbb{N}$ ,



or similarly in dimension 4 it would imply the existence of a nontrivial  $SO(n)$ -module homomorphism  $([w + 1, 1] \oplus [w + 1, -1]) \rightarrow [q]$  for some  $q \in \mathbb{N}$ . These equivariant mappings are impossible since for any  $w \in \mathbb{N}$ ,  $[w + 1, 1]$  and  $[q]$  (and

$[w + 1, -1]$  if  $n = 4$ ) are distinct irreducible  $\mathrm{SO}(n)$ -modules. An almost identical argument shows that (4.5) holds. As an  $\mathrm{SO}(n)$ -space,  $(w + 2)$  decomposes orthogonally to  $[w + 2] \oplus [w] \oplus \cdots$  and so if  $\Delta^{n/2-3} \nabla_{bc(a_w+2 \cdots a_3} C_{a_2}{}^b{}_{a_1}{}^c(0)$  were nonzero for any  $K$ , then we would once again arrive at a contradiction by deducing a nontrivial linear  $\mathrm{SO}(n)$ -mapping  $[w + 1, 1] \rightarrow [q]$  (or  $([w + 1, 1] \oplus [w + 1, -1]) \rightarrow [q]$  if  $n = 4$ ) for some  $q \in \mathbb{N}$ . Finally we consider (4.4) with  $\ell = w - 2 \geq 0$ . Note first that  $\nabla^{(w-2)} \Delta^{n/2-2} \nabla_{bc} C_a{}^b{}_{d^c}(0)$  has values in  $Y(w - 2) \otimes Y(2)$ . For the  $\mathrm{SL}(n)$  tensor product we have the decomposition  $(w - 2) \otimes (2) = (w) \oplus (w - 1, 1) \oplus (w - 2, 2)$ . The  $\mathrm{SO}(n)$ -branch components of  $(w)$ ,  $(w - 1, 1)$  and  $(w - 2, 2)$  are all modules of the form  $[a, b]$  where  $a + b \leq w$  (where  $b$  may be 0). In particular  $[w + 1, 1]$  (and  $[w + 1, -1]$  for the case  $n = 4$ ) are not summands in the  $\mathrm{SO}(n)$ -decomposition of  $(w - 2) \otimes (2)$  and so arguing as in the previous cases, we immediately conclude that  $\nabla^{(w-2)} \Delta^{n/2-2} \nabla_{bc} C_a{}^b{}_{d^c}(0)$  must vanish.

It remains to establish (4.6). First we observe that there is a nontrivial  $(n/2 - 1)$ -fold trace of  $\Psi$  which takes values in  $Y_0[w + 1, 1]$ . Up to scale this inverts the map which inserts  $Y_0[w + 1, 1]$  as an orthogonal summand in  $Y(m, 2)$ . Next observe that if we skew over the pairs  $a_1 b_1$  and  $a_2 b_2$  of  $\Psi_{a_1 a_2 \cdots a_m b_1 b_2}$  and then on the result symmetrise over the indices  $a_1 \cdots a_m$  and also over the indices  $b_1 b_2$ , then the result is a nonzero multiple of  $\Psi$ . In fact this composition of mappings can be taken as (up to scale) the definition of the Young symmetriser projection onto the space  $Y(m, 2)$  containing  $\Psi$ . Since, up to a nonzero scale,  $\Psi_{a_1 \cdots a_m b_1 b_2}$  is  $\nabla_{a_m \cdots a_1} \psi_{b_1 b_2}$ , it follows from (4.1) that  $\Psi$  is a symmetry adjustment of  $\nabla^{(m-2)} \mathbf{R}(0)$ . On the other hand from (4.2) and the linearisations of (2.1) and (2.3), it follows that  $\nabla^{(m-2)} \mathbf{R}(0)$  is a trace adjustment of  $\nabla^{(m-2)} \mathbf{C}(0)$  (cf. the discussion of normal scale in Section 2). Combining these observations, it follows that there is a symmetry and trace operation on  $\nabla^{(m-2)} \mathbf{C}(0)$  with a nontrivial outcome taking values in  $Y_0[w + 1, 1]$ . Using the symmetries of  $\mathbf{C}$  and the linearisation of the Bianchi identity (2.4), it is easily established that, up to scale, (4.6) is the unique possibility.  $\square$

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