

Conformally symmetric manifolds and quasi conformally recurrent Riemannian manifolds

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1 **Abstract.** In order to give a new proof of a theorem concerned with
2 conformally symmetric Riemannian manifolds due to Roter and Derdzin-
3 sky [8], [9] and Miyazawa [15], we have adopted the technique used in a
4 theorem about conformally recurrent manifolds with harmonic conformal
5 curvature tensor in [3]. In this paper, we also present a new proof of a suc-
6 cessive refined version of a theorem about conformally recurrent manifolds
7 with harmonic conformal curvature tensor. Moreover, as an extension of
8 theorems mentioned above we prove some theorems related to quasi con-
9 formally recurrent Riemannian manifolds with harmonic quasi conformal
10 curvature tensor.

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12 **Key words:** conformal curvature tensor; quasi conformal curvature tensor; confor-
13 mally symmetric; conformally recurrent; Ricci recurrent; Riemannian manifolds.

14 1 Introduction

Let M be a non flat $n(\geq 4)$ dimensional Riemannian manifold with metric g_{ij} and Riemannian connection ∇ . It is said to be conformally recurrent if the conformal curvature tensor satisfies $\nabla_i C_{jk\downarrow}^m = \lambda_i C_{jk\downarrow}^m$ (See [1], [3] and [11]), where λ_i is some non null covector and the components of the conformal curvature tensor [16] are given by :

$$(1.1) \quad C_{jk\downarrow}^m = R_{jk\downarrow}^m + \frac{1}{n-2}(\delta_j^m R_{k\downarrow} - \delta_k^m R_{j\downarrow} + R_j^m g_{k\downarrow} - R_k^m g_{j\downarrow}) \\ - \frac{R}{(n-1)(n-2)}(\delta_j^m g_{k\downarrow} - \delta_k^m g_{j\downarrow}).$$

15 Here we have defined the Ricci tensor to be $R_{k\downarrow} = -R_{mk\downarrow}^m$ [23] and the scalar
16 curvature $R = g^{ij} R_{ij}$. The recurrence properties of Weyl's tensor has been analyzed
17 also in [13]. If $\nabla_i C_{jk\downarrow}^m = 0$, the manifold is said to be conformally symmetric (See [5],
18 [8],[10] and [18]). If $\nabla_m C_{jk\downarrow}^m = 0$, the manifold is said to have harmonic curvature

19 tensor (See [4]). If $C_{jk\uparrow}^m = 0$, the manifold is called conformally flat (See [16]). In
 20 [13] the properties of some class of conformally flat manifolds are pointed out. It may
 21 be scrutinized that the conformal curvature tensor vanishes identically if $n = 3$ and if
 22 M is a space of constant curvature. A manifold is said to be Ricci recurrent if its non
 23 null Ricci tensor is recurrent, i.e. if $\nabla_k R_{ij} = \beta_k R_{ij}$ (See [11]) where β_k is another
 24 non null covector.

25 Recently a theorem concerning conformally recurrent Riemannian or semi-Riemannian
 26 manifolds with harmonic curvature tensor was introduced in [3] (Theorem 3.4) and [19].
 27 We refer to it as :

28 **Theorem 1.1.** *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian
 29 connection ∇ . Assume that M is conformally recurrent and has the harmonic
 30 conformal curvature tensor. If the scalar curvature is constant ($\nabla_j R = 0$), then M is
 31 conformally symmetric, conformally flat or Ricci recurrent.*

32 This theorem was used in [3] to give a complete classification of conformally re-
 33 current Riemannian manifolds with harmonic curvature tensor. In the same reference
 34 it was stated another Theorem ([3], Theorem 3.6) that refines Theorem 1.1. We refer
 35 to it as:

36 **Theorem 1.2.** *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian
 37 connection ∇ . Assume that M is conformally recurrent and has the harmonic
 38 conformal curvature tensor. If the scalar curvature is non zero constant, then M is
 39 conformally flat or locally symmetric.*

40 In [19] the authors extended Theorem 1.2 to the case of semi-Riemannian man-
 41 ifolds. Moreover they also pointed out that the assumption of a constant scalar
 42 curvature may be dropped in the case of a definite metric and stated the following
 43 (see [19] Remark 3.3) :

44 **Theorem 1.3.** *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian
 45 connection ∇ . Assume that M is conformally recurrent and has the harmonic
 46 conformal curvature tensor. Then M is conformally symmetric.*

47 In this paper we give a new proof of a classical theorem about conformally symmetric
 48 Riemannian manifolds using a technique adopted in [3] for Theorem 1.1. Now we
 49 assert the following :

50 **Theorem 1.4.** *An $n(\geq 4)$ dimensional conformally symmetric manifold is confor-
 51 mally flat or locally symmetric.*

52 This result is fulfilled on a manifold with positive definite metrics. Miyazawa proved
 53 this statement with the extra assumption of $n > 4$ in [15]. A proof of the general
 54 case $n > 3$ was pointed out by Derdzinski and Roter in [9]. In section 2 of this paper
 55 we reobtain Theorem 1.4 by a correction of the procedure employed in the proof of
 56 Theorem 1.1 used in [3]. In section 3 we give an alternative proof of Theorem 1.3 and
 57 provide extensions of Theorems 1.1, 1.3 and 1.4 related to quasi-conformal symmetric
 58 or quasi-conformal recurrent Riemannian manifold.

59 Moreover, combining the results of Theorems 1.3 and 1.4, we can state another the-
 60 orem as follows:

61 **Theorem 1.5.** *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian connection ∇ . Assume that M is conformally recurrent and has the harmonic*
 62 *conformal curvature tensor. Then M is conformally flat or locally symmetric.*
 63

64 2 The proof of Theorem B

In this section the procedure adopted in [3] is pursued to obtain a proof of Theorem 1.4. It is worth to notice that the assumption of constant scalar curvature mentioned in Theorem 1.1 and employed in [3] is not used here in the proof of Theorem 1.4. Let M be an n dimensional conformally symmetric manifold. Then the following relation is fulfilled:

$$(2.1) \quad \begin{aligned} \nabla_i R_{jk\downarrow}^m &= -\frac{1}{n-2}(\delta_j^m \nabla_i R_{k\downarrow} - \delta_k^m \nabla_i R_{j\downarrow} + \nabla_i R_j^m g_{k\downarrow} - \nabla_i R_k^m g_{j\downarrow}) \\ &\quad + \frac{\nabla_i R}{(n-1)(n-2)}(\delta_j^m g_{k\downarrow} - \delta_k^m g_{j\downarrow}). \end{aligned}$$

65 From the previous result we can state the following

66 **Remark 2.1.** *Any conformally symmetric manifold with parallel Ricci tensor is symmetric in the sense of Cartan, that is, $\nabla_i R_{jk\downarrow}^m = 0$ (See [12], [16] and [18]).*
 67

From the notion of conformally symmetric manifold one easily gets $(\nabla_b \nabla_a - \nabla_a \nabla_b)C_{jk\downarrow}^m = 0$. Then by the Ricci identity [23], we can write the following equation:

$$(2.2) \quad R_{ba\downarrow}^p C_{pk\downarrow}^m + R_{bak}^p C_{jp\downarrow}^m + R_{ba\downarrow}^p C_{jkpm} + R_{bam}^p C_{jk\downarrow}^p = 0.$$

Performing the covariant derivative of equation (2.2) and taking account that $\nabla_i C_{jk\downarrow}^m = 0$, one obtains:

$$(2.3) \quad \nabla_i R_{ba\downarrow}^p C_{pk\downarrow}^m + \nabla_i R_{bak}^p C_{jp\downarrow}^m + \nabla_i R_{ba\downarrow}^p C_{jkpm} + \nabla_i R_{bam}^p C_{jk\downarrow}^p = 0.$$

From (2.3) and the fact that the manifold is conformally symmetric we obtain:

$$(2.4) \quad \begin{aligned} &(\nabla_i R_{bj} C_{ak\downarrow}^m + \nabla_i R_{bk} C_{ja\downarrow}^m + \nabla_i R_{b\downarrow} C_{jkam} + \nabla_i R_{bm} C_{jk\downarrow}^a) \\ &\quad - (\nabla_i R_{aj} C_{bk\downarrow}^m + \nabla_i R_{ak} C_{jb\downarrow}^m + \nabla_i R_{a\downarrow} C_{jkbm} + \nabla_i R_{am} C_{jk\downarrow}^b) \\ &\quad + \frac{1}{n-1} \nabla_i R (g_{aj} C_{bk\downarrow}^m + g_{ak} C_{jb\downarrow}^m + g_{a\downarrow} C_{jkbm} + g_{am} C_{jk\downarrow}^b) \\ &\quad - \frac{1}{n-1} \nabla_i R (g_{bj} C_{ak\downarrow}^m + g_{bk} C_{ja\downarrow}^m + g_{b\downarrow} C_{jkam} + g_{bm} C_{jk\downarrow}^a) \\ &\quad - \nabla_i R_b^p (g_{aj} C_{pk\downarrow}^m + g_{ak} C_{jp\downarrow}^m + g_{a\downarrow} C_{jkpm} + g_{am} C_{jk\downarrow}^p) \\ &\quad + \nabla_i R_a^p (g_{bj} C_{pk\downarrow}^m + g_{bk} C_{jp\downarrow}^m + g_{b\downarrow} C_{jkpm} + g_{bm} C_{jk\downarrow}^p) = 0. \end{aligned}$$

Now transvecting the last equation with g^{jb} taking account of the first Bianchi identity for the conformal curvature tensor we have :

$$(2.5) \quad (n-2)\nabla_i R_{ab} C_{m\downarrow k}{}^b + \nabla_i R_{bk} C_{m\downarrow a}{}^b + \nabla_i R_{b\downarrow} C_{mak}{}^b + \nabla_i R_{bm} C_{a\downarrow k}{}^b \\ - (g_{a\downarrow} C_{mpk}{}^b + g_{am} C_{p\downarrow k}{}^b) \nabla_i R_b^p = 0.$$

Again the previous equation is transvected with g^{im} to obtain :

$$(2.6) \quad (n-2)\nabla^m R_{ab} C_{m\downarrow k}{}^b + \nabla^m R_{bk} C_{m\downarrow a}{}^b + \nabla^m R_{b\downarrow} C_{mak}{}^b + \frac{1}{2}(\nabla_b R) C_{a\downarrow k}{}^b \\ - g_{a\downarrow} C_{mpk}{}^b \nabla^m R_b^p - C_{p\downarrow k}{}^b \nabla_a R_b^p = 0.$$

Now it is well known that the divergence of the conformal curvature is given by ([8] and [9]) :

$$(2.7) \quad \nabla_m C_{jk\downarrow}{}^m = \frac{n-3}{n-2} \left[\nabla_k R_{j\downarrow} - \nabla_j R_{k\downarrow} + \frac{1}{2(n-1)} \{ (\nabla_j R) g_{k\downarrow} - (\nabla_k R) g_{j\downarrow} \} \right].$$

So if the manifold is conformally symmetric, it is easily seen that :

$$(2.8) \quad \nabla_j R_{k\downarrow} - \nabla_k R_{j\downarrow} = \frac{1}{2(n-1)} \{ (\nabla_j R) g_{k\downarrow} - (\nabla_k R) g_{j\downarrow} \}.$$

This result allows us to examine the last two terms contained in equation (2.6). The first term vanishes; in fact :

$$(2.9) \quad g_{a\downarrow} C_{mpk}{}^b \nabla^m R_b^p = \frac{1}{2} g_{a\downarrow} C_{mpk}{}^b (\nabla^m R_b^p - \nabla^p R_b^m) \\ = \frac{1}{2} g_{a\downarrow} C^{mp}{}_{k}{}^b (\nabla_m R_{pb} - \nabla_p R_{mb}) \\ = \frac{1}{4(n-1)} g_{a\downarrow} C^{mp}{}_{k}{}^b \{ (\nabla_m R) g_{pb} - (\nabla_p R) g_{mb} \} \\ = 0.$$

Moreover with similar procedure the last term results to be :

$$(2.10) \quad C_{p\downarrow k}{}^b \nabla_a R_b^p = C^p{}_{\downarrow k}{}^b \nabla_a R_{pb} \\ = C^p{}_{\downarrow k}{}^b \left[\nabla_p R_{ab} + \frac{1}{2(n-1)} \{ (\nabla_a R) g_{pb} - (\nabla_p R) g_{ab} \} \right] \\ = C^p{}_{\downarrow k}{}^b \nabla_p R_{ab} - \frac{1}{2(n-1)} C^p{}_{\downarrow k}{}^b (\nabla_p R) g_{ab} \\ = C_{m\downarrow k}{}^b \nabla^m R_{ab} - \frac{1}{2(n-1)} (\nabla_m R) C^m{}_{\downarrow ka}.$$

So equation (2.6) can be rewritten in the following form :

$$(2.11) \quad (n-3)\nabla^m R_{ab} C_{m\downarrow k}{}^b + \nabla^m R_{bk} C_{m\downarrow a}{}^b + \nabla^m R_{b\downarrow} C_{mak}{}^b + \frac{1}{2}(\nabla_m R) C_{a\downarrow k}{}^m \\ + \frac{1}{2(n-1)} (\nabla_m R) C^m{}_{\downarrow ka} = 0.$$

⁶⁸ Now in [3] an interesting Lemma is pointed out (See also [9]) :

Lemma 2.2. *Let M be an n dimensional conformally symmetric manifold. Then the following equations hold :*

$$(2.12) \quad \begin{aligned} R_{ab}C_{m\downarrow k}{}^b + R_{mb}C_{\downarrow ak}{}^b + R_{\downarrow b}C_{amk}{}^b &= 0, \\ \nabla_s R_{ab}C_{m\downarrow k}{}^b + \nabla_s R_{mb}C_{\downarrow ak}{}^b + \nabla_s R_{\downarrow b}C_{amk}{}^b &= 0. \end{aligned}$$

Transvecting the last of the previous relations with g^{sm} one obtains :

$$(2.13) \quad \nabla^m R_{ab}C_{m\downarrow k}{}^b - \nabla^m R_{\downarrow b}C_{mak}{}^b = \frac{1}{2}(\nabla_m R)C_{a\downarrow k}{}^m.$$

The equivalent relation $-2\nabla^m R_{ba}C_{m\downarrow k}{}^b = -2\nabla^m R_{b\downarrow}C_{mak}{}^b - (\nabla_m R)C_{a\downarrow k}{}^m$ is then substituted in equation (2.11) to obtain :

$$(2.14) \quad \begin{aligned} (n-1)\nabla^m R_{ab}C_{m\downarrow k}{}^b + \nabla^m R_{bk}C_{m\downarrow a}{}^b - \nabla^m R_{b\downarrow}C_{mak}{}^b - \frac{1}{2}(\nabla_m R)C_{a\downarrow k}{}^m \\ + \frac{1}{2(n-1)}(\nabla_m R)C_{\downarrow ka}{}^m = 0. \end{aligned}$$

Again employing Lemma 2.2 with indices k and a exchanged gives :

$$(2.15) \quad \nabla^m R_{bk}C_{m\downarrow a}{}^b - \nabla^m R_{b\downarrow}C_{mka}{}^b = \frac{1}{2}(\nabla_m R)C_{k\downarrow a}{}^m.$$

So equation (2.14) takes the form :

$$(2.16) \quad \begin{aligned} (n-1)\nabla^m R_{ab}C_{m\downarrow k}{}^b + \nabla^m R_{b\downarrow}(C_{mka}{}^b - C_{mak}{}^b) + \frac{1}{2}(\nabla_m R)(C_{k\downarrow a}{}^m + C_{\downarrow ak}{}^m) \\ + \frac{1}{2(n-1)}(\nabla_m R)C_{\downarrow ka}{}^m = 0. \end{aligned}$$

Recalling that $C_{mka}{}^b + C_{kam}{}^b + C_{amk}{}^b = 0$, the previous equation may be written in the following form :

$$(2.17) \quad (n-1)\nabla^m R_{ab}C_{m\downarrow k}{}^b + \nabla^m R_{b\downarrow}C_{akm}{}^b + \frac{1}{2}(\nabla_m R)C_{ka\downarrow}{}^m + \frac{1}{2(n-1)}(\nabla_m R)C_{\downarrow ka}{}^m = 0.$$

Now recalling that $\nabla_m R_{b\downarrow} - \nabla_b R_{m\downarrow} = \frac{1}{2(n-1)}\{(\nabla_m R)g_{b\downarrow} - (\nabla_b R)g_{m\downarrow}\}$, the second term of the previous equation satisfies the following identities chain :

$$(2.18) \quad \begin{aligned} \nabla^m R_{b\downarrow}C_{akm}{}^b &= \nabla_m R_{b\downarrow}C_{ak}{}^{mb} = \frac{1}{2}C_{ak}{}^{mb}(\nabla_m R_{b\downarrow} - \nabla_b R_{m\downarrow}) \\ &= \frac{1}{4(n-1)}C_{ak}{}^{mb}(\nabla_m Rg_{b\downarrow} - \nabla_b Rg_{m\downarrow}) \\ &= \frac{1}{2(n-1)}\left[(\nabla^m R)C_{akm\downarrow} - (\nabla_b R)C_{ak\downarrow}{}^b\right] \\ &= \frac{1}{4(n-1)}(\nabla^m R)\left[C_{akm\downarrow} - C_{ak\downarrow}{}^m\right] = \frac{(\nabla^m R)}{2(n-1)}C_{akm\downarrow}. \end{aligned}$$

So equation (2.16) takes the form :

$$(2.19) \quad \begin{aligned} (n-1)\nabla^m R_{ab}C_{m\downarrow k}^b + \frac{1}{2(n-1)}(\nabla^m R)C_{akm\downarrow} \\ + \frac{1}{2}(\nabla_m R)C_{ka\downarrow}^m + \frac{1}{2(n-1)}(\nabla^m R)C_{m\downarrow ka} = 0, \end{aligned}$$

or better :

$$(2.20) \quad (n-1)\nabla^m R_{ab}C_{m\downarrow k}^b + \frac{1}{2}(\nabla^m R)C_{ka\downarrow m} = 0.$$

Now one can observe that $\nabla_m R_{ab} = \nabla_a R_{mb} + \frac{1}{2(n-1)}\{(\nabla_m R)g_{ab} - (\nabla_a R)g_{mb}\}$ and thus we can write :

$$(2.21) \quad \begin{aligned} \nabla^m R_{ab}C_{m\downarrow k}^b &= \nabla_m R_{ab}C_{m\downarrow k}^b \\ &= \nabla_a R_{mb}C_{m\downarrow k}^b + \frac{1}{2(n-1)}\left[(\nabla_m R)C_{m\downarrow k}^b g_{ab} - (\nabla_a R)C_{m\downarrow k}^b g_{mb}\right]. \end{aligned}$$

This fact implies that :

$$(2.22) \quad \nabla_m R_{ab}C_{m\downarrow k}^b = \nabla_a R_{mb}C_{m\downarrow k}^b + \frac{1}{2(n-1)}(\nabla^m R)C_{m\downarrow ka}.$$

If the equivalent relation $(n-1)\nabla_m R_{ab}C_{m\downarrow k}^b = (n-1)\nabla_a R_{mb}C_{m\downarrow k}^b + \frac{1}{2}(\nabla^m R)C_{m\downarrow ka}$ is substituted in equation (2.20), one obtains that the following holds :

$$(2.23) \quad (n-1)\nabla_a R_{mb}C_{m\downarrow k}^b = 0.$$

At last equation (2.5) takes the form :

$$(2.24) \quad (n-2)\nabla_i R_{ab}C_{m\downarrow k}^b + \nabla_i R_{bk}C_{m\downarrow a}^b + \nabla_i R_{b\downarrow}C_{mak}^b + \nabla_i R_{bm}C_{a\downarrow k}^b = 0.$$

Now Lemma 2.2 is again employed in the form $\nabla_i R_{mb}C_{a\downarrow k}^b + \nabla_i R_{ab}C_{\downarrow mk}^b + \nabla_i R_{\downarrow b}C_{mak}^b = 0$ to equation (2.24) to obtain :

$$(2.25) \quad (n-1)\nabla_i R_{ab}C_{m\downarrow k}^b = -\nabla_i R_{bk}C_{m\downarrow a}^b.$$

Now exchanging the indices k and a in the previous result gives immediately :

$$(2.26) \quad (n-1)\nabla_i R_{kb}C_{m\downarrow a}^b = -\nabla_i R_{ab}C_{m\downarrow k}^b.$$

This implies that $(n-1)^2\nabla_i R_{ab}C_{m\downarrow k}^b = \nabla_i R_{ab}C_{m\downarrow k}^b$ and so as in [3] and [19] that :

$$(2.27) \quad \nabla_i R_{bk}C_{m\downarrow a}^b = 0.$$

Transvecting the previous result with g^{ik} it follows immediately that :

$$(2.28) \quad \frac{1}{2}\nabla_b R C_{m\downarrow a}^b = 0.$$

Transvecting (2.4) with $\nabla_i R_{bj}$ or with $C_{ak\downarrow m}$ and applying (2.27), one can obtain the following results :

$$\nabla_i R_{bj} \nabla^i R^{bj} C_{ak\downarrow m} = 0 \quad \text{or} \quad \nabla_i R_{bj} C_{ak\downarrow m} C^{ak\downarrow m} = 0.$$

In fact if equation (2.4) is transvected with $\nabla^i R^{bj}$ one obtains:

$$(2.29) \quad (\nabla^i R^{bj} \nabla_i R_{bj} - \frac{1}{n-1} \nabla^i R \nabla_i R) C_{ak\downarrow m} = 0.$$

On the other hand if equation (2.4) is transvected with $C^{ak\downarrow m}$ one easily obtains:

$$(2.30) \quad \nabla_i R_{bj} C_{ak\downarrow m} C^{ak\downarrow m} + \frac{1}{n-1} \nabla_i R \{g_{aj} C_{bk\downarrow m} - g_{bj} C_{ak\downarrow m} - g_{bk} C_{ja\downarrow m} - g_{b\downarrow} C_{jkam} - g_{bm} C_{jk\downarrow a}\} C^{ak\downarrow m} = 0.$$

Transvecting this last result with g^{ij} and making use of equation (2.28) one comes to the following:

$$(2.31) \quad \frac{n-3}{2(n-1)} \nabla_b R C_{akm\downarrow} C^{akm\downarrow} = 0.$$

69 Thus we obtain that the manifold is conformally flat or the manifold has constant
70 scalar curvature and employing (2.30) it is Ricci symmetric. In this way we have
71 proved that the following Theorem holds :

72 **Theorem 2.3.** *Let M be an n dimensional conformally symmetric manifold. Then*
73 *it is Ricci symmetric or conformally flat.*

74 Now recalling Remark 2.1 and Theorem 2.3, we have just proved that $\nabla_i R_{jk\downarrow}^m = 0$
75 or $C_{jk\downarrow}^m = 0$.

76 **Remark 2.4.** *It is worth to notice that from Theorem 2.3 we recover a result of*
77 *Tanno ([20], Theorem 6): any non conformally flat conformally symmetric manifold*
78 *has constant scalar curvature. This result was used in [9] for the proof of Theorem 1.4.*
79 *In the present paper it has been recovered in our main argument.*

80 **3 An alternative proof of Theorem 1.3 and gener-** 81 **alizations of Theorems 1.1, 1.3 and 1.4**

82 In this section we provide an alternative proof of Theorem 1.3 given in [19] and
83 consider a possible generalization of Theorems 1.1, 1.3 and 1.4.

84 **Theorem 1.3.** *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian*
85 *connection ∇ . Assume that M is conformally recurrent and has the harmonic*
86 *conformal curvature tensor. Then M is conformally symmetric or conformally flat.*

Proof. It is well known ([1] eq. 3.7) that the second Bianchi identity for the conformal curvature tensor may be written in the following form :

$$(3.1) \quad \begin{aligned} & \nabla_i C_{jk\downarrow}^m + \nabla_j C_{ki\downarrow}^m + \nabla_k C_{ij\downarrow}^m \\ &= \frac{1}{n-3} \left[\delta_j^m \nabla_p C_{ki\downarrow}^p + \delta_k^m \nabla_p C_{ij\downarrow}^p + \delta_i^m \nabla_p C_{jk\downarrow}^p \right. \\ & \quad \left. + g_{k\downarrow} \nabla_p C_{ji}^{mp} + g_{i\downarrow} \nabla_p C_{kj}^{mp} + g_{j\downarrow} \nabla_p C_{ik}^{mp} \right]. \end{aligned}$$

Thus on a manifold with harmonic conformal curvature tensor [4], the second Bianchi identity reduces to :

$$(3.2) \quad \nabla_i C_{jk\downarrow}^m + \nabla_j C_{ki\downarrow}^m + \nabla_k C_{ij\downarrow}^m = 0.$$

If the manifold is also conformally recurrent, i.e. $\nabla_i C_{jk\downarrow}^m = \lambda_i C_{jk\downarrow}^m$, the last equation takes the form :

$$(3.3) \quad \lambda_i C_{jk\downarrow}^m + \lambda_j C_{ki\downarrow}^m + \lambda_k C_{ij\downarrow}^m = 0.$$

We note also that if the manifold has the harmonic conformal curvature tensor, i.e. $\nabla_m C_{jk\downarrow}^m = 0$, then $\lambda_m C_{jk\downarrow}^m = 0$. Now equation (3.3) is multiplied by λ^i to obtain the following result :

$$(3.4) \quad \lambda^i \lambda_i C_{jk\downarrow}^m + \lambda^i \lambda_j C_{ki\downarrow}^m + \lambda^i \lambda_k C_{ij\downarrow}^m = 0.$$

In the previous equation the second and the last terms vanish. In fact for example one easily obtains $\lambda^i \lambda_j C_{ki\downarrow}^m = g^{mp} \lambda_j \lambda^i C_{ki\downarrow}^p = g^{mp} \lambda_j \lambda^i C_{\downarrow pki} = 0$. Then equation (3.4) give the following result :

$$(3.5) \quad \lambda^i \lambda_i C_{jk\downarrow}^m = 0.$$

We have thus obtained that the manifold is conformally flat. In the same manner equation (3.3) is multiplied by $C^{jk\downarrow}_m$ and the following is fulfilled :

$$(3.6) \quad \lambda_i C_{jk\downarrow}^m C^{jk\downarrow}_m + \lambda_j C_{ki\downarrow}^m C^{jk\downarrow}_m + \lambda_k C_{ij\downarrow}^m C^{jk\downarrow}_m = 0.$$

Thus following the same procedure employed previously we have that the relation :

$$(3.7) \quad \lambda_i C_{jk\downarrow}^m C^{jk\downarrow}_m = 0.$$

87 So we have $\lambda_i = 0$ and the manifold is conformally symmetric. □

88 It is worth to notice that the class of conformally symmetric spaces includes the
89 class of conformally flat spaces. The version of Theorem 1.3 proved in the present
90 paper is slightly different from [19].

Now we consider a possible generalization of Theorems 1.1, 1.3 and 1.4 in the direction of quasi-conformal symmetric or quasi-conformal recurrent Riemannian manifold. In order to do this, first we need the definition of the concircular curvature tensor (See [17] and [21]), that is :

$$(3.8) \quad \tilde{C}_{jk\downarrow}^m = R_{jk\downarrow}^m + \frac{R}{n(n-1)} (\delta_j^m g_{k\downarrow} - \delta_k^m g_{j\downarrow}).$$

Contracting m with j gives the so called Z tensor, i.e. $Z_{k\downarrow} = -\tilde{C}_{mk\downarrow}{}^m$, that is:

$$(3.9) \quad Z_{k\downarrow} = R_{k\downarrow} - \frac{R}{n}g_{k\downarrow}.$$

91 It may be noted from (3.8) that the vanishing of the concircular tensor implies the
 92 manifold to be a space of constant curvature and from (3.9) that the vanishing of the
 93 Z tensor implies the manifold to be an Einstein space. So the concircular tensor is a
 94 measure of the deviation of a manifold from a space of constant curvature and the Z
 95 tensor is a measure of the deviation from an Einstein space (See [14]).

In 1968 Yano and Sawaki [22] defined and studied a tensor $W_{jk\downarrow}{}^m$ on a Riemannian
 manifold of dimension n , which includes both the conformal curvature tensor $C_{jk\downarrow}{}^m$
 and the concircular curvature tensor $\tilde{C}_{jk\downarrow}{}^m$ as particular cases. This tensor is known
 as quasi conformal curvature tensor and its components are given by:

$$(3.10) \quad W_{jk\downarrow}{}^m = -(n-2)bC_{jk\downarrow}{}^m + [a + (n-2)b]\tilde{C}_{jk\downarrow}{}^m.$$

96 In the previous equation $a \neq 0$, $b \neq 0$ are constants and $n > 3$ since the conformal
 97 curvature tensor vanishes identically for $n = 3$. A non flat manifold is said to be
 98 quasi-conformally recurrent if $\nabla_i W_{jk\downarrow}{}^m = \alpha_i W_{jk\downarrow}{}^m$ for a non null covector α_i . It
 99 is said to be quasi-conformally symmetric if $\nabla_i W_{jk\downarrow}{}^m = 0$ and has the harmonic quasi
 100 conformal curvature tensor if $\nabla_m W_{jk\downarrow}{}^m = 0$. Z recurrency or Z symmetry are de-
 101 fined in analogous ways. Clearly the class of quasi conformally recurrent Riemannian
 102 manifolds includes all the class of quasi conformally symmetric and quasi conformally
 103 flat manifolds. In [2] Amur and Maralabhavi proved that a quasi conformally flat
 104 Riemannian manifold is either conformally flat or Einstein. A similar remark can be
 105 proved for quasi conformally symmetric manifolds.

106 **Remark 3.1.** *Let M be an $n(\geq 4)$ dimensional quasi conformally symmetric Rie-*
 107 *mannian manifold. Then it is either conformally symmetric or Ricci symmetric.*

Proof. In fact the condition $\nabla_i W_{jk\downarrow}{}^m = 0$ implies:

$$(3.11) \quad (n-2)b\nabla_i C_{jk\downarrow}{}^m = [a + (n-2)b]\nabla_i \tilde{C}_{jk\downarrow}{}^m.$$

108 Contracting m with j in the previous equation gives $[a + (n-2)b] = 0$ or $\nabla_i Z_{k\downarrow} = 0$,
 109 that is, by the equation (3.11) the manifold is conformally symmetric or Z symmetric.
 110 Now Z symmetric implies $\nabla_i R_{k\downarrow} = \frac{1}{n}(\nabla_i R)g_{k\downarrow}$ and transvecting with g^{ik} one gets
 111 $\nabla_l R = 0$ and thus $\nabla_i R_{k\downarrow} = 0$ \square

112 We note that the class of Z symmetric spaces includes the class of Einstein spaces.
 113 The previous remark allows us to state a modified version of Theorem 1.4 whose proof
 114 follows immediately from Remark 3.1 and Theorem 1.4 itself:

115 **Theorem 3.2.** *Let M be an $n(\geq 4)$ dimensional quasi-conformally symmetric man-*
 116 *ifold. Then it is conformally flat or locally symmetric.*

117 The statement of the previous theorem is due to the fact that local symmetry
 118 implies Ricci symmetry. We can also state the following modified version of Theo-
 119 rem 1.1.

120 **Theorem 3.3.** *Let M be an $n(\geq 4)$ dimensional Riemannian manifold of with Rie-*
 121 *mannian connection ∇ . Assume that M is quasi-conformally recurrent and has the*
 122 *harmonic quasi conformal curvature tensor. Then M is conformally symmetric, con-*
 123 *formally flat or generalized Ricci recurrent [6].*

Proof. If $\nabla_i W_{jk\uparrow}^m = \alpha_i W_{jk\uparrow}^m$, then one has :

$$(3.12) \quad -(n-2)b\nabla_i C_{jk\uparrow}^m + [a+(n-2)b]\nabla_i \tilde{C}_{jk\uparrow}^m = -(n-2)b\alpha_i C_{jk\uparrow}^m + [a+(n-2)b]\alpha_i \tilde{C}_{jk\uparrow}^m.$$

Contracting m with j in the previous equation gives :

$$(3.13) \quad [a+(n-2)b]\nabla_i Z_{k\uparrow} = [a+(n-2)b]\alpha_i Z_{k\uparrow}.$$

That is, the manifold is Z recurrent or $[a+(n-2)b] = 0$. In this case we get from (3.10) that :

$$(3.14) \quad \nabla_m W_{jk\uparrow}^m = -(n-2)b\nabla_m C_{jk\uparrow}^m + [a+(n-2)b]\nabla_m \tilde{C}_{jk\uparrow}^m.$$

This fact implies that $\nabla_m W_{jk\uparrow}^m = -(n-2)b\nabla_m C_{jk\uparrow}^m$ and hence that $\nabla_m C_{jk\uparrow}^m = 0$ because $\nabla_m W_{jk\uparrow}^m = 0$. From (3.12) we have also in the same case $[a+(n-2)b] = 0$ that :

$$(3.15) \quad -(n-2)b\nabla_i C_{jk\uparrow}^m = -(n-2)b\alpha_i C_{jk\uparrow}^m.$$

124 That is the manifold is conformally recurrent.

On the other hand, if the covariant derivative with respect to the index m is applied on the definition of quasi conformal curvature tensor, one obtains straightforwardly

$$(3.16) \quad \nabla_m W_{jk\uparrow}^m = [a+b]\nabla_m R_{jk\uparrow}^m + \frac{2a-b(n-1)(n-4)}{2n(n-1)} [(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}].$$

Now if $\nabla_m W_{jk\uparrow}^m = 0$, transvecting the previous equation with $g^{k\uparrow}$ after some calculations it follows that

$$(3.17) \quad (n-2)\frac{a+b(n-2)}{n}\nabla_j R = 0.$$

This means that $\nabla_j R = 0$ if $a+(n-2)b \neq 0$ or $a+(n-2)b = 0$. Inserting the latter case in (3.16) we obtain the following

$$(3.18) \quad \nabla_m R_{jk\uparrow}^m = \frac{1}{2(n-1)} [(\nabla_k R)g_{jl} - (\nabla_j R)g_{kl}].$$

125 From this, we recover obviously $\nabla_m C_{jk\uparrow}^m = 0$. Now if the conditions $\nabla_i W_{jk\uparrow}^m = \alpha_i W_{jk\uparrow}^m$
 126 and $\nabla_m W_{jk\uparrow}^m = 0$ are taken in conjunction, we have two cases. One is obtained from
 127 (3.12) that $\nabla_i C_{jkl}^m = \alpha_i C_{jkl}^m$ with $\nabla_m C_{jkl}^m = 0$ when $a+b(n-2) = 0$. The other case
 128 can be given by (3.13) that $\nabla_i Z_{kl} = \alpha_i Z_{kl}$ with $\nabla_j R = 0$ when $a+(n-2)b \neq 0$.

129 In the first case we are in the hypothesis of Theorem 1.3. Accordingly, M is
 130 conformally symmetric or conformally flat.

131 In the second case, we have a Z -recurrent manifold with $\nabla_j R = 0$ and thus
 132 $\nabla_i R_{kl} = \alpha_i(R_{kl} - \frac{R}{n}g_{kl})$, that is, a generalized Ricci recurrent manifold [6]. \square

133 Combining the results of Theorems 3.3 and 1.5, we can state the following modified
134 version of Theorem 1.3 :

135 **Theorem 3.4.** *Let M be an $n(\geq 4)$ dimensional Riemannian manifold of with Rie-*
136 *mannian connection ∇ . Assume that M is quasi-conformally recurrent and has the*
137 *harmonic quasi conformal curvature tensor. Then M is conformally flat, locally sym-*
138 *metric, or generalized Ricci recurrent.*

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