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Conforming and divergence-free Stokes elements in three dimensions

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Conforming finite element pairs for the three-dimensional Stokes problem on general simplicial triangulations are constructed. The pressure space simply consists of piecewise constants, where as the velocity space consists of cubic polynomials augmented with rational functions. We show the existence of a bounded Fortin projection and therefore the necessary LBB condition is satisfied. In addition the divergence operator maps the velocity space into the space of piecewise constants. Consequently, the method produces exactly divergence-free velocity approximations.

Keywords: Stokes, finite elements, divergence-free

1. Introduction

In this paper we construct three dimensional Stokes finite elements $\boldsymbol{V}_h \times W_h$ with no-slip boundary conditions on general triangulations with the following desirable properties:

1) The pair $\boldsymbol{V}_h \times W_h$ is inf-sup stable, i.e., the LBB condition

$$\inf_{w \in W_h} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{(\operatorname{div} \boldsymbol{v}, w)}{\|\boldsymbol{v}\|_{H^1(\Omega)} \|w\|_{L^2(\Omega)}} \geqslant \beta \tag{1.1}$$

   is satisfied for a constant $\beta > 0$ independent of h.

2) The finite element pair is conforming. In particular, the discrete velocity space satisfies the inclusion $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega) := [H_0^1(\Omega)]^3$.

3) Discretely divergence-free functions are exactly divergence-free point wise:

$$Z_{h,0} := \{\boldsymbol{v} \in \boldsymbol{V}_h : \ (\operatorname{div} \boldsymbol{v}, q) = 0 \ \forall q \in W_h\} = \{\boldsymbol{v} \in \boldsymbol{V}_h : \ \operatorname{div} \boldsymbol{v} = 0\}. \tag{1.2}$$

Here, $(\cdot,\cdot)$ denotes the L^2 inner product over the domain $\Omega \subset \mathbb{R}^3$.

It is easy to see that property (3) implies that the resulting velocity approximations are exactly incompressible, i.e., conservation of mass is maintained. Numerical conservation of mass has been shown to be a crucial factor with respect to stability and accuracy for a variety of problems (Linke, 2008; Diening *et al.*,

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2012; Auricchio *et al.*, 2010). In particular, it is well-known that rotation-free forcing terms can be constructed such that the velocity approximations suffer from significantly large divergence errors, leading to large spurious oscillations of the solution (Galvin *et al.*, 2012; Olshanskii & Reusken, 2004). Recent numerical experiments in (Linke, 2008; Galvin *et al.*, 2012) show that this phenomenon is not limited to simple academic examples, but occurs in physically-relevant flow problems.

The theoretical justification of the importance of mass conservation is simple to formulate. Namely, if a stable finite element pair satisfies (1)–(2) but not (3), then the resulting abstract error estimates show that the velocity and pressure errors are coupled

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} \leqslant C\Big(\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{H^1(\Omega)} + \nu^{-1}\inf_{p\in W_h}\|p-p_h\|_{L^2(\Omega)}\Big).$$

Here, ν denotes the kinematic viscosity of the fluid and C is a positive constant depending only on the parameter β appearing in the inf-sup condition. Note that velocity error deteriorates globally as the viscosity tends to zero and if the pressure gradient is of order $\mathcal{O}(1)$. Translating this result to the Navier-Stokes problem, we see that such finite element pairs are not robust for problems with large Reynolds numbers and large pressure gradients. On the other hand, if a finite element pair satisfies the three conditions above, then the resulting velocity error is decoupled and simply reads

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{H^1(\Omega)} = \inf_{\boldsymbol{v}\in\boldsymbol{Z}_{h,0}}\|\boldsymbol{u}-\boldsymbol{v}\|_{H^1(\Omega)},$$

where the discrete space of divergence-free functions $\boldsymbol{Z}_h$ is defined in condition (3). Although the above estimate may still depend on the viscosity (since high-order norms of $\boldsymbol{u}$ may depend on ν), the dependence is local.

The construction of elements satisfying conditions (1)-(3) has been a challenge for general triangulations in both two and three dimensions. Starting with the classical work of Scott and Vogelius (Scott & Vogelius, 1985), several elements have been constructed assuming certain restrictions on the triangulations (Scott & Vogelius, 1985; Arnold & Qin, 1992; Zhang, 2005, 2008). For example, the two-dimensional Scott-Vogelius element $\boldsymbol{P}_k^c - P_{k-1}^{dc}$ is stable only if $k \geqslant 4$ and the triangulation does not contain any singular vertices, i.e., vertices that lie on exactly two straight lines. Arnold & Qin (1992) later showed that the Scott-Vogelius element is stable for $k \geqslant 2$ if the triangulation is obtained by a global barycenter refinement, and Zhang (2005) has recently extended this result to three dimensions (where the restriction is then $k \geqslant 3$). A full theory of the Scott-Vogelius elements in three dimensions is still open. Recently, we have constructed finite element pairs that satisfy conditions (1)–(3) on general triangulations in two dimensions (Guzmán & Neilan, 2013). These spaces are obtained by enriching $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming elements locally with rational shape-functions. Falk and the second author have constructed two dimensional elements satisfying (1)-(3) using purely polynomial basis functions (Falk & Neilan, 2013). For problems with Dirichlet boundary conditions, the elements in (Falk & Neilan, 2013) impose a mild restriction only on triangles that touch the corners of the polygonal domain. Several papers have constructed elements satisfying (1)-(3) using splines and IsoGeometric analysis on rectangular meshes (Evans & Hughes, 2012a,b; Buffa *et al.*, 2011a,b).

Finally, several elements have been developed for the Stokes-Brinkman problem that satisfy (1) and (3) but relax the conformity condition (2) (Tai & Winther, 2006; Guzmán & Neilan, 2012; Xie *et al.*, 2008; Mardal *et al.*, 2002). These elements are $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming but are not $\boldsymbol{H}^1(\Omega)$-conforming. In order to impose *some* degree of continuity in the tangential direction on interfaces, standard polynomial basis are augmented on each simplex by divergence-free polynomials with vanishing normal components. In fact, the elements constructed here and in (Guzmán & Neilan, 2013) are similar, but we instead augment the local spaces with divergence-free *rational* functions that have vanishing normal components. A related idea is to use $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming elements and penalize their tangential jumps (see (Wang & Ye, 2007; Cockburn *et al.*, 2007)) in the variational formulation.

As far as we are aware, the Stokes elements constructed in this paper are the first pair satisfying conditions (1)–(3) on general simplicial triangulations in three dimensions. The pressure space W_h is simply the space of piecewise constants and the velocity space $\boldsymbol{V}_h$ consists of piecewise cubic polynomials enriched with rational functions. The local dimension of the velocity space is 60, i.e, 60 degrees of freedom are needed on each tetrahedron to unique determine a velocity function. However, we show that there exists a reduced element with the same pressure space and a local velocity space with only sixteen degrees of freedom per simplex. The degrees of freedom of the lowest order element are simply vertex degrees of freedom (three per vertex) and the average of normal components on faces (one degree of freedom per face). Note that these are the same degrees of freedom as the Bernardi-Raugel elements (Bernardi & Raugel, 1985), and therefore the two finite element spaces have the same dimension. Similar to the two-dimensional setting, we use rational divergence-free basis functions associated to each face of a tetrahedron, but it addition, we also require rational divergence-free functions associated to each edge. The basis functions associated to faces is an exact generalization of basis functions used in our two-dimensional elements. The reduced elements clearly have a relatively small number of degrees of freedom. However, due to their complexity, the practical significance of the proposed elements may be questionable. Nonetheless, we believe that their construction and theory may shed new light onto the derivation of simpler conforming and divergence-free Stokes elements.

Here is the organization of the paper. In the next section, we set the notation and assumptions and define the rational functions. In Section 3 we describe the local finite element pair, and in Section 4 we define the global pair and their approximation properties. In Section 5 we construct the reduced elements and derive analogous approximation results. Also in Section 5, we present different pairs of Stokes elements in any dimension by using the Bogovskii operator locally to construct the basis functions. We apply the elements to the Stokes problem in Section 6, where standard techniques are used to derive optimal order convergence results. An appendix is given at the end of the paper where some technical results are reported.

2. Notation and Preliminaries

Given a set $D \subset \Omega$, we denote by $H^m(D)$ $(m \geqslant 0)$ the Sobolev space consisting of all $L^2(D)$ functions whose distributional derivatives up to order m are in $L^2(\Omega)$, and $H_0^m(D)$ to denote the set of functions whose traces vanish up to order $m-1$ on ∂D. We then set the corresponding vector Sobolev spaces as $\boldsymbol{H}^m(D) = (H^m(D))^3$ and $\boldsymbol{H}_0^m(D) = (H_0^m(D))^3$, and define the space of square integrable with vanishing mean as $L_0^2(D)$. The L^2 inner product over a three dimensional (resp., one or two dimensional) set D is denoted by $(\cdot,\cdot)_D$ (reps., $\langle \cdot,\cdot \rangle_D$). In the case $D = \Omega$, we set $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$ and $\langle \cdot,\cdot \rangle := \langle \cdot,\cdot \rangle_{\partial\Omega}$. We shall also use the Sobolev spaces

$$\begin{aligned} \boldsymbol{H}(\mathrm{div};D) &= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(D) :\ \mathrm{div}\,\boldsymbol{v} \in L^2(D)\right\}, \\ \boldsymbol{H}_0(\mathrm{div};D) &= \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};D) :\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial D} = 0\right\}, \\ \boldsymbol{H}(\mathbf{curl};D) &= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(D) :\ \mathbf{curl}\,\boldsymbol{v} \in L^2(D)\right\}, \end{aligned}$$

where $\boldsymbol{n}$ denotes the outward normal of the boundary ∂D.

For a given tetrahedron S and $m \geqslant 0$, the vector-valued polynomials are defined as $\boldsymbol{P}_m(S) = [P_m(S)]^3$, where $P_m(S)$ is the space of polynomials defined on S of degree less than or equal to m. We also set $P_m(S)$ and $\boldsymbol{P}_m(S)$ to be the empty set for any negative valued m. Let $\mathcal{T}_h$ be a shape-regular tetrahedral decomposition of Ω (Ciarlet, 1978; Brenner & Scott, 2008) with $h_T = \mathrm{diam}(T)$ for all $T \in \mathcal{T}_h$ and $h = \max_{T\in\mathcal{T}_h} h_T$. Given $T \in \mathcal{T}_h$ the barycentric coordinates are given by $\{\lambda_i\}_{i=1}^4$ and are labeled such that λ_i vanishes on face $F_i \subset \partial T$.

We also denote by $\{x_i\}_{i=1}^4$ the four vertices of T with $\lambda_i(x_j) = \delta_{ij}$, and by $e_{i,j}$ the edge with $e_{i,j} = \partial F_i \cap \partial F_j$. We note that $e_{i,j} = e_{j,i}$. The volume bubble, face bubbles, and edge bubbles are defined respectively as

$$b_T := \lambda_1\lambda_2\lambda_3\lambda_4 \in P_4(T), \quad b_i := \prod_{j\neq i} \lambda_j \in P_3(T), \quad b_{i,j} := \prod_{\substack{k\neq i\\ k\neq j}} \lambda_k \in P_2(T). \tag{2.1}$$

By construction, the bubble functions satisfy the following properties:

$$b_T\big|_{\partial T} = 0, \qquad \frac{\partial b_T}{\partial \boldsymbol{n}_i}\Big|_{F_i} = a_{F_i} b_i, \qquad b_i\big|_{\partial T\setminus F_i} = 0, \tag{2.2a}$$

$$b_i\big|_{F_i} > 0, \qquad b_{i,j}\big|_{\partial T\setminus(F_i\cup F_j)} = 0, \quad b_{i,j}\big|_{F_i\cup F_j} > 0, \tag{2.2b}$$

where

$$a_{F_i} := -|\nabla\lambda_i| \neq 0, \tag{2.3}$$

and $\boldsymbol{n}_i$ denotes the outward unit normal of F_i.

We will also define the rational face bubble functions ($i = 1,2,3,4$)

$$B_i = \begin{cases} \dfrac{b_T b_i}{(\lambda_i+\lambda_{i+1})(\lambda_i+\lambda_{i+2})(\lambda_i+\lambda_{i+3})} & \text{for } 0 \leqslant \lambda_i \leqslant 1,\ 0 \leqslant \lambda_{i+1},\lambda_{i+2},\lambda_{i+3} < 1, \\ 0 & \text{otherwise} \end{cases}$$

Here and throughout an index j of a barycentric coordinate will be calculated using the formula $1+(j-1 \bmod 4)$. The rational face bubble functions are a three dimensional analogue of the rational edge bubbles used in the construction of the singular Zienkiewicz triangle (Zienkiewicz, 1971; Ciarlet, 1978), and they inherit similar properties as shown in the following lemma.

LEMMA 2.1 There holds

$$B_i \in C^2(\overline{T}), \qquad B_i\big|_{\partial T} = 0, \qquad \nabla B_i(x_j) = 0\ (j = 1,2,3,4), \tag{2.4a}$$

$$\nabla B_i\big|_{\partial T\setminus F_i} = 0, \qquad \frac{\partial B_i}{\partial \boldsymbol{n}_i}\Big|_{F_i} = a_{F_i} b_i, \qquad \nabla B_i\big|_{F_i} \in \boldsymbol{P}_3(F_i). \tag{2.4b}$$

Proof. We first show $B_{F_i} \in C^2(\overline{T})$. Since this property is invariant under affine transformations it suffices to show that the function

$$\hat{B}(x) := \begin{cases} \dfrac{x_1x_2^2x_3^2(1-x_1-x_2-x_3)^2}{(x_1+x_2)(x_1+x_3)(1-x_2-x_3)} & x\neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is $C^2(\overline{T})$ in the case T is the unit simplex with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. This property will follow if we show that $\hat{B}(x)$ is C^2 at the origin. Write $\hat{B}(x) = \hat{g}(x)\hat{s}(x)$, with

$$\hat{g}(x) = \begin{cases} \dfrac{x_1x_2^2x_3^2}{(x_1+x_2)(x_1+x_3)} & x\neq 0, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \hat{s}(x) = \frac{(1-x_1-x_2-x_3)^2}{(1-x_2-x_3)}.$$

Since $\hat{s}(x)$ is well-behaved at the origin, it suffices to show that $\hat{g}(x)$ is C^2 at the origin. A direct calculation gives us

$$\frac{\partial \hat{g}}{\partial x_1}(x) = -\frac{x_2^2 x_3^2 (x_1^2 - x_2 x_3)}{(x_1 + x_2)^2 (x_1 + x_3)^2},$$

and therefore since $x_i \geqslant 0$, we have

$$\lim_{x \to 0} \left| \frac{\partial \hat{g}}{\partial x_1}(x) \right| \leqslant \lim_{x \to 0} |x_1^2 - x_2 x_3| = 0.$$

Similar arguments show that $\lim_{x \to 0} \left| \frac{\partial \hat{g}}{\partial x_2}(x) \right| = 0$ and $\lim_{x \to 0} \left| \frac{\partial \hat{g}}{\partial x_3}(x) \right| = 0$ as well. It then follows that $\hat{g}(x) \in C^1(\overline{T})$. Next, we have

$$\frac{\partial^2 \hat{g}}{\partial x_1^2}(x) = \frac{2x_2^2 x_3^2 (x_1^3 - 3x_1 x_2 x_3 - x_2 x_3^2 - x_2^2 x_3)}{(x_1 + x_2)^3 (x_1 + x_3)^3}.$$

Noting that

$$\frac{x_2^2 x_3^2 x_1^3}{(x_1 + x_2)^3 (x_1 + x_3)^3} \leqslant \frac{4x_2^2 x_3^2 x_1^3}{(x_1^3 + x_2^3)(x_1^3 + x_2^3)} \leqslant \frac{4x_2^2 x_3^2 x_1^3}{x_1^3 (x_2^3 + x_3^3)} \leqslant 2(x_2 x_3)^{1/2},$$
$$\frac{x_1 x_2^3 x_3^3}{(x_1 + x_2)^3 (x_1 + x_3)^3} \leqslant x_1, \quad \text{and} \quad \frac{x_2^4 x_3^3}{(x_1 + x_2)^3 (x_1 + x_3)^3} \leqslant x_2,$$

we obtain $\lim_{x \to 0} \frac{\partial^2 \hat{g}}{\partial x_1^2}(x) = 0$. Similar arguments show $\lim_{x \to 0} \frac{\partial^2 \hat{g}}{\partial x_i \partial x_j}(x) = 0$ $(i, j = 1, 2)$. It then follows that $\hat{g}(x) \in C^2(\overline{T})$ and therefore $B_i \in C^2(\overline{T})$.

Next, since $b_T\big|_{\partial T}$, we have $B_i\big|_{\partial T} = 0$ and

$$\nabla B_i\big|_{\partial T} = \frac{\nabla b_T b_i}{(\lambda_i + \lambda_{i+1})(\lambda_i + \lambda_{i+2})(\lambda_i + \lambda_{i+3})}\Big|_{\partial T}.$$

Since $b_i\big|_{\partial T \setminus F_i} = 0$ and $\partial b_T / \partial \boldsymbol{n}_i\big|_{F_i} = a_{F_i} b_i$, we obtain $\nabla B_i\big|_{\partial T \setminus F_i} = 0$, $\partial B_i / \partial \boldsymbol{n}_i\big|_{F_i} = a_{F_i} b_i$ and $\nabla B_i\big|_{F_i} \in \boldsymbol{P}_3(F_i)$. Finally the property $\nabla B_i(x_j) = 0$ follows from the inclusion $B_i \in C^1(\overline{T})$ and $\nabla B_i\big|_{\partial T \setminus F_i} = 0$. □

REMARK 2.1 The rational face bubbles in three dimensions inherit better regularity properties than the analogous two dimensional edge bubbles. In particular, the rational edge bubbles in two dimensions are not $C^2(\overline{T})$ (cf. Ciarlet (1978); Guzmán & Neilan (2013)).

In addition to the rational face bubbles, we will also include rational edge bubbles to form the local space of the velocity elements. To describe these functions we let $i, j = \{1, 2, \ldots, 4\}$ with $i < j$ and let $e_{i,j} = \partial F_i \cap \partial F_j$. We then set

$$\boldsymbol{s}_{i,j} = \frac{b_T b_{i,j}}{2(\lambda_i \lambda_j + b_{i,j}(\lambda_i + \lambda_j))(\lambda_i + \lambda_j)} \big(\nabla(\lambda_j^2 - \lambda_i^2) + 4(\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) \big). \tag{2.5}$$

The following lemma states the crucial properties of $\boldsymbol{s}_{i,j}$. The proof is contained in the appendix.

LEMMA 2.2 For $i, j \in \{1,2,\ldots,6\}$ and $i < j$. Then we have

$$\mathbf{curl}\, \boldsymbol{s}_{i,j} \in \boldsymbol{C}^0(\overline{T}) \cap \boldsymbol{W}^{1,\infty}(T), \tag{2.6a}$$
$$\mathbf{curl}\, \boldsymbol{s}_{i,j}|_{\partial T} = b_{i,j}(\nabla \lambda_i \times \nabla \lambda_j), \tag{2.6b}$$
$$\boldsymbol{s}_{i,j}|_{\partial T} = 0. \tag{2.6c}$$

To close this section, we recall some well-known results about some auxiliary finite element spaces which we use in the construction and analysis in the subsequent section. First, the classical $\boldsymbol{H}(\mathbf{curl};\Omega)$ Nedelec spaces of index $k-1$ (Nedelec, 1980) is given by

$$\boldsymbol{N}_{k-1}(T) = \boldsymbol{P}_{k-2}(T) + \{\boldsymbol{w} \in \boldsymbol{P}_{k-1}(T) : \boldsymbol{w}\cdot\boldsymbol{x} = 0\}.$$

The dimension of $\boldsymbol{N}_{k-1}(T)$ is $\frac{1}{2}(k+1)k(k-1)$. Furthermore, any vector polynomial $\boldsymbol{v} \in \boldsymbol{P}_k(T)$ is uniquely determined by the following degrees of freedom (Nedelec, 1986):

$$\langle \boldsymbol{v}\cdot\boldsymbol{n}_i, \kappa\rangle_{F_i} \qquad \text{for all } \kappa \in P_k(F_i)\ (i = 1,2,3,4), \tag{2.7a}$$
$$(\boldsymbol{v}, \boldsymbol{\rho})_T \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{N}_{k-1}(T). \tag{2.7b}$$

3. The Local Space

Our local space will be a sum of three spaces: an $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming space, a space containing rational face bubbles, and a space containing rational edge bubbles. We introduce each space and give their corresponding degrees of freedom. We then combine the spaces and give the degrees of freedom of the local $H^1(\Omega)$ space in Theorem 3.1 below.

First we define the following local space of divergence-free functions.

$$\boldsymbol{Q}_m(T) = \{v \in \boldsymbol{P}_m(T) : (\boldsymbol{v},\boldsymbol{\rho})_T = 0 \text{ for all } \boldsymbol{\rho} \in \boldsymbol{N}_{m-1}(T) \text{ and } \langle \boldsymbol{v}\cdot\boldsymbol{n}_i,\kappa\rangle_{F_i} = 0 \text{ for all } \kappa \in \boldsymbol{P}_{m-1}(F_i)\ (i=1,2,3,4)\}.$$

It is easy to see that the dimension of $\boldsymbol{Q}_m(T)$ is $4(m+1)$. Indeed, since there are $\frac{1}{2}(m+2)(m+1)(m-1)+2m(m+1)$ constraints imposed in the space $\boldsymbol{Q}_m(T)$, we have $\dim \boldsymbol{Q}_m(T) \geqslant \dim \boldsymbol{P}_m(T) - \frac{1}{2}(m+2)(m+1)(m-1) - 2m(m+1) = 4(m+1)$. Also, since functions in $\boldsymbol{P}_m(T)$ are uniquely determined by the degrees of freedom (2.7), we see that functions $\boldsymbol{q} \in \boldsymbol{Q}_m(T)$ are determined by the values $\langle \boldsymbol{q}\cdot\boldsymbol{n}_i,\kappa\rangle_{F_i}$ with $\kappa \in \tilde{\boldsymbol{P}}_m(F_i) := \boldsymbol{P}_m(F_i)\backslash \boldsymbol{P}_{m-1}(F_i)$. Since $\dim \tilde{\boldsymbol{P}}_m(F_i) = m+1$, we deduce that $\dim \boldsymbol{Q}_m(T) = 4(m+1)$. Furthermore by the definition of $\boldsymbol{Q}_m(T)$ and $\boldsymbol{N}_{m-1}(T)$, we have

$$\int_T \mathrm{div}\, \boldsymbol{q} v\, dx = -\int_T \boldsymbol{q}\cdot\nabla v\, dx + \int_{\partial T} \boldsymbol{q}\cdot\boldsymbol{n} v\, ds = 0 \qquad \forall \boldsymbol{q} \in \boldsymbol{Q}_m(T),\ \forall v \in P_{m-1}(T),$$

and therefore $\mathrm{div}\, \boldsymbol{Q}_m(T) = \{0\}$.

Next we define a local $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming space

$$\boldsymbol{M}(T) = \boldsymbol{P}_1(T) + \boldsymbol{Q}_2(T) + \boldsymbol{Q}_3(T). \tag{3.1}$$

The associated degrees of freedom of $\boldsymbol{M}(T)$ are given by

$$\boldsymbol{v}(x_i) \qquad \text{for all vertices } x_i, \tag{3.2a}$$
$$\langle \boldsymbol{v}\cdot\boldsymbol{n}_k, s\rangle_{e_{i,j}} \qquad \text{for all } s \in P_1(e_{i,j})\ (i,j = 1,\ldots,4,\ \ k = i,j) \tag{3.2b}$$
$$\langle \boldsymbol{v}\cdot\boldsymbol{n}_i, \kappa\rangle_{F_i} \qquad \text{for all } \kappa \in P_0(F_i)\ \ (i = 1,\ldots,4). \tag{3.2c}$$

LEMMA 3.1 The degrees of freedom (3.2) are unisolvent on $\boldsymbol{M}(T)$.

Proof. From the definition of $\boldsymbol{Q}_2(T)$ and $\boldsymbol{Q}_3(T)$, we see that the two sums in (3.1) are direct. It then follows that $\dim \boldsymbol{M}(T) = \dim \boldsymbol{P}_1(T) + 12 + 16 = 40$ which is exactly the number of degrees of freedom given in (3.2). Hence to prove unisolvency we assume that $\boldsymbol{v} \in \boldsymbol{M}(T)$ and has vanishing degrees of freedom (3.2) and show $\boldsymbol{v} \equiv 0$.

Using the degrees of freedom (3.2a)–(3.2c) we have that $\boldsymbol{v} \cdot \boldsymbol{n}$ vanishes on ∂T. Write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{q}$ where $\boldsymbol{v}_0 \in \boldsymbol{P}_1(T)$ and $\boldsymbol{q} \in \boldsymbol{Q}_2(T) + \boldsymbol{Q}_3(T)$. Using the defintion of the space $\boldsymbol{Q}_m(T)$ we have

$$0 = \left\langle \boldsymbol{v} \cdot \boldsymbol{n}, \boldsymbol{v}_0 \cdot \boldsymbol{n} \right\rangle_{\partial T} = \left\langle \boldsymbol{v}_0 \cdot \boldsymbol{n}, \boldsymbol{v}_0 \cdot \boldsymbol{n} \right\rangle_{\partial T}.$$

This shows that $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ on ∂T and hence we conclude that $\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on ∂T. Again using the definition of $\boldsymbol{Q}_m(T)$ we have that $\boldsymbol{q} \equiv 0$. To finish the proof, we see that $\boldsymbol{v}_0 \equiv 0$ since $\boldsymbol{v}_0 \cdot \boldsymbol{n}$ vanishes on ∂T. Thus, $\boldsymbol{v} \equiv 0$ and so the degrees of freedom (3.2) are unisolvent on $\boldsymbol{M}(T)$. □

As a next step to develop local conforming elements for the 3D Stokes problem, we introduce and study a space consisting of rational face bubbles. The local rational face bubble space is defined by

$$\boldsymbol{U}(T) = \sum_{i=1}^{4} \boldsymbol{U}^{(i)}(T), \qquad \boldsymbol{U}^{(i)}(T) = \mathbf{curl}\left(B_{F_i} \boldsymbol{P}_0(T) \times \boldsymbol{n}_i\right). \tag{3.3}$$

LEMMA 3.2

(i) The dimension of $\boldsymbol{U}(T)$ is eight.

(ii) Any $\boldsymbol{z} \in \boldsymbol{U}(T)$ is uniquely determined by

$$\left\langle \boldsymbol{z} \times \boldsymbol{n}_{F_i}, \boldsymbol{q} \times \boldsymbol{n}_i \right\rangle_{F_i} \qquad \text{for all } \boldsymbol{q} \in \boldsymbol{P}_0(F_i) \text{ and } i = 1,2,3,4. \tag{3.4}$$

(iii) Functions in $\boldsymbol{U}(T)$ vanish at the degrees of freedom (3.2).

Proof. It is easy to see from (3.3) that the dimension of $\boldsymbol{U}^{(i)}(T)$ is two. Thus in order show that $\dim \boldsymbol{U}(T) = 8$, it suffices to show that the sum in (3.3) is direct.

Suppose that $0 = \boldsymbol{z} = \mathbf{curl}\,(\sum_{i=1}^{4} \boldsymbol{z}_i)$ with $\boldsymbol{z}_i \in \boldsymbol{U}^{(i)}$, $\boldsymbol{z}_i = B_{F_i} \boldsymbol{p}_i \times \boldsymbol{n}_{F_i}$ and $\boldsymbol{p}_i \in \boldsymbol{P}_0(T)$. We show that $\boldsymbol{z}_i \equiv 0$ $(i = 1,2,3,4)$. Since B_{F_i} vanishes on ∂T and ∇B_{F_i} vanishes on $\partial T \backslash F_i$, we deduce that $0 = \boldsymbol{z}\big|_{F_i} = \nabla B_{F_i} \times (\boldsymbol{p}_i \times \boldsymbol{n}_i)\big|_{F_i} = -a_{F_i} b_{F_i} \boldsymbol{n}_i \times (\boldsymbol{p}_i \times \boldsymbol{n}_i)\big|_{F_i}$. We then have

$$\boldsymbol{z} \times \boldsymbol{n}_i\big|_{F_i} = -a_{F_i} b_{F_i} (\boldsymbol{p}_i \times \boldsymbol{n}_i)\big|_{F_i}, \quad \text{and} \quad \boldsymbol{z} \cdot \boldsymbol{n}_i\big|_{F_i} = 0 \tag{3.5}$$

on each face $F_i \subset \partial T$. Since b_{F_i} is strictly positive on F_i and $\boldsymbol{z} = 0$, we conclude that $\boldsymbol{p}_i \times \boldsymbol{n}_i = 0$ and so $\boldsymbol{z} \equiv 0$. Therefore $\dim \boldsymbol{U}_{k-1}(T) = 8$.

Next, suppose that $\boldsymbol{z} \in \boldsymbol{U}(T)$ vanishes at the degrees of freedom given in (3.4). As before, write $\boldsymbol{z} = \mathbf{curl}\,(\sum_{i=1}^{4} \boldsymbol{z}_i)$ with $\boldsymbol{z}_i \in \boldsymbol{U}_{k-1}^{(i)}$, $\boldsymbol{z}_i = B_{F_i} \boldsymbol{p}_i \times \boldsymbol{n}_{F_i}$ and $\boldsymbol{p}_i \in \boldsymbol{P}_0(T)$. Then the left-hand side equality in (3.5) holds and therefore by (3.4), we have $\boldsymbol{p}_i \times \boldsymbol{n}_i\big|_{F_i} = 0$. Using the same arguments as above, we conclude that $\boldsymbol{p}_i \equiv 0$ and therefore $\boldsymbol{z} \equiv 0$. Since there are exactly eight conditions in (3.4), it follows that the degrees of freedom are unisolvent on $\boldsymbol{U}_{k-1}(T)$.

Finally, by Lemma 2.1 and (3.5) we see that any $\boldsymbol{z} \in \boldsymbol{U}(T)$ vanishes at the degrees of freedom (3.2). □

The last ingredient of the local velocity space is the space of rational edge bubbles. The addition of this space will enforce tangental continuity on each edge of the tetrahedron. To start, we define the two dimensional space $M^{(i,j)}(T) = \text{span}\{\lambda_k, \lambda_\ell\}$, where the indices $k, \ell \in \{1,2,3,4\}$ are chosen such that $k, \ell \neq i$, $k, \ell \neq j$, $k \neq \ell$. By properties of the barycentric coordinates, there holds $M^{(i,j)}(T)\big|_{e_{i,j}} = P_1(e_{i,j})$. We then set the space of rational edge bubbles to be

$$\boldsymbol{W}(T) = \{\sum_{\substack{i,j=1\\ i>j}}^{4} \mathbf{curl}\,(p_{i,j}\boldsymbol{s}_{i,j}) : p_{i,j} \in M^{(i,j)}(T)\}, \tag{3.6}$$

where $\boldsymbol{s}_{i,j}$ is given by (2.5).

LEMMA 3.3 Let $\boldsymbol{v} \in \boldsymbol{W}(T)$ have the form $\boldsymbol{v} = \sum_{i,j=1_{i>j}}^{4} \mathbf{curl}\,(p_{i,j}\boldsymbol{s}_{i,j})$. We then have

$$\boldsymbol{v}|_{e_{i,j}} = p_{i,j} b_{i,j} (\nabla\lambda_i \times \nabla\lambda_j), \qquad \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial T} = 0. \tag{3.7}$$

Moreover, the dimension of $\boldsymbol{W}(T)$ is 12, and a set of degrees of freedom that uniquely determines a function in $\boldsymbol{W}(T)$ are given by

$$\langle \boldsymbol{v}\cdot\boldsymbol{t}_{i,j}, s\rangle_{e_{i,j}} \qquad \text{for all } s \in P_1(e_{i,j}). \tag{3.8}$$

Here, $\boldsymbol{t}_{i,j}$ is a unit vector tangent to the edge $e_{i,j}$.

Proof. By the product rule and Lemma 2.2 we have

$$\boldsymbol{v}|_{\partial T} = \sum_{\substack{i,j=1\\ i>j}}^{4} (p_{i,j}\mathbf{curl}\,(\boldsymbol{s}_{i,j}) + \nabla p_{i,j} \times \boldsymbol{s}_{i,j})|_{\partial T} = \sum_{\substack{i,j=1\\ i>j}}^{4} p_{i,j} b_{i,j} (\nabla\lambda_i \times \nabla\lambda_j)|_{\partial T}.$$

Since $b_{i,j}$ has support on $e_{i,j}$ and vanishes at the other five edges, we obtain $\boldsymbol{v}|_{e_{i,j}} = p_{i,j} b_{i,j}(\nabla\lambda_i \times \nabla\lambda_j)$. Furthermore, since $(\nabla\lambda_i \times \nabla\lambda_j)$ is orthogonal to $\boldsymbol{n}_i$ and $\boldsymbol{n}_j$, and since $b_{i,j}$ vanishes on $\partial T\backslash(F_i \cup F_j)$, there holds $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial T} = 0$.

To show that the dimension of $\boldsymbol{W}(T)$ is twelve, we show that if $\boldsymbol{v} = 0$ in T, then all of the functions $p_{i,j} \in M^{(i,j)}(T)$ must vanish. The assumption that $\boldsymbol{v} = 0$ implies that $p_{i,j} b_{i,j}(\nabla\lambda_i \times \nabla\lambda_j) = 0$ on $e_{i,j}$ by the arguments above. Since $(\nabla\lambda_i \times \nabla\lambda_j) \neq 0$ and $b_{i,j} > 0$ on $e_{i,j}$ we conclude that $p_{i,j} = 0$ on $e_{i,j}$. Since $M^{(i,j)}(T)$ restricted to $e_{i,j}$ spans $P_1(e_{i,j})$, this condition implies $p_{i,j} = 0$ on T. Since the dimension of $\mathbf{curl}\,\big(\boldsymbol{s}_{i,j} M^{(i,j)}(T)\big)$ is two, we conclude that the dimension of $\boldsymbol{W}(T)$ is twelve.

Finally since $(\nabla\lambda_i \times \nabla\lambda_j)$ is parallel to $\boldsymbol{t}_{i,j}$ and since $\boldsymbol{v}|_{e_{i,j}} = p_{i,j} b_{i,j}(\nabla\lambda_i \times \nabla\lambda_j)|_{i,j}$, we easily see that the degrees of freedom (3.8) are unisolvent on $\boldsymbol{W}(T)$. □

With the three separate spaces established, we are now ready to define the local velocity space. The 60 dimensional space is defined to be the sum of the three previously defined spaces, namely,

$$\boldsymbol{V}(T) = \boldsymbol{M}(T) + \boldsymbol{U}(T) + \boldsymbol{W}(T). \tag{3.9}$$

LEMMA 3.4 There holds for any $\boldsymbol{v} \in \boldsymbol{V}(T)$,

$$\operatorname{div} \boldsymbol{v} \in P_0(T), \quad \boldsymbol{v} \in \boldsymbol{C}^0(\overline{T}) \cap \boldsymbol{W}^{1,\infty}(T), \quad \boldsymbol{v}|_{\partial T} \in \boldsymbol{P}_3(\partial T). \tag{3.10}$$

Proof. By (3.9) and (3.1), and since $\boldsymbol{U}(T)$ and $\boldsymbol{W}(T)$ consists of divergence-free functions, we have $\operatorname{div} \boldsymbol{V}(T) = \operatorname{div} \boldsymbol{M}(T) = \operatorname{div} \boldsymbol{P}_1(T) + \operatorname{div} \boldsymbol{Q}_2(T) + \operatorname{div} \boldsymbol{Q}_3(T)$. Functions in $\boldsymbol{Q}_2(T)$ and $\boldsymbol{Q}_3(T)$ are also divergence-free. Therefore since the divergence operator maps $\boldsymbol{P}_1(T)$ onto $P_0(T)$, we obtain the first identity in (3.10).

The regularity result in (3.10) immediately follows from (3.9), (3.1), (3.3), (3.6), Lemma 2.1 and Lemma 2.2. Similarly, the identity $\boldsymbol{v}|_{\partial T} \in \boldsymbol{P}_3(\partial T)$ follows from Lemmas 2.1 and 2.2 and the definition of the local finite element space. □

THEOREM 3.1 We have $\dim \boldsymbol{V}(T) = \dim \boldsymbol{M}(T) + \dim \boldsymbol{U}(T) + \dim \boldsymbol{W}(T) = 60$. A set of unisolvent degrees of freedom for a function $\boldsymbol{v} \in \boldsymbol{V}(T)$ is given by

$$\boldsymbol{v}(x_i) \qquad \text{for all vertices } x_i, \tag{3.11a}$$
$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{e_{i,j}} \qquad \text{for all } \boldsymbol{w} \in \boldsymbol{P}_1(e_{i,j}) \text{ and } i,j = 1,\ldots,4 \tag{3.11b}$$
$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{F_i} \qquad \text{for all } \boldsymbol{w} \in \boldsymbol{P}_0(F_i) \text{ and } i = 1,\ldots,4. \tag{3.11c}$$

Proof. The dimension count immediately follows from the direct sum $\boldsymbol{V}(T) = \boldsymbol{M}(T) \oplus \boldsymbol{U}(T) \oplus \boldsymbol{W}(T)$ and Lemmas 3.1, 3.2 and 3.3.

Next, let $\boldsymbol{v} \in \boldsymbol{V}(T)$ such that the degrees of freedom (3.11) vanish. We will argue that $\boldsymbol{v}$ vanishes on T to prove unisolvency. Write

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3,$$

where $\boldsymbol{v}_1 \in \boldsymbol{M}(T), \boldsymbol{v}_2 \in \boldsymbol{U}(T), \boldsymbol{v}_3 \in \boldsymbol{W}(T)$. From Lemmas 3.2 and 3.3, both $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ vanish on the degrees of freedom (3.2). Using the degrees of freedom (3.11), we have that $\boldsymbol{v}_1$ must vanish on the degrees of freedom (3.2). Since $\boldsymbol{v}_1 \in \boldsymbol{M}(T)$ we conclude that $\boldsymbol{v}_1 \equiv 0$ in light of Lemma 3.1. Next, since $\boldsymbol{v}_2|_{F_i} = -a_{F_i} b_{F_i} (\boldsymbol{p}_i \times \boldsymbol{n}_i)|_{F_i}$ for some $\boldsymbol{p}_i \in \boldsymbol{P}_0(T)$ (cf. (3.5)), we see that $\boldsymbol{v}_2$ vanishes on each edge of T (this follows from the properties of the cubic face bubble). In particular, $\boldsymbol{v}_2 \cdot \boldsymbol{t}_{i,j}|_{e_{i,j}} = 0$ for each edge $e_{i,j}$ of T. By (3.11b) we conclude that $\boldsymbol{v}_3$ vanishes at the degrees of freedom (3.8). By Lemma 3.3, $\boldsymbol{v}_3$ must be identically zero. Finally by (3.11b), we have that $\boldsymbol{v}_2$ vanishes on the degrees of freedom (3.4). Hence, $\boldsymbol{v}_2 \equiv 0$ by Lemma 3.2 and therefore $\boldsymbol{v} \equiv 0$. □

4. The Global Spaces and Their Approximation Properties

Now that we have constructed the local spaces we can glue them together to form a global H^1 space.

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \ \boldsymbol{v}|_T \in \boldsymbol{V}(T), \text{for all } T \in \mathcal{T}_h\}.$$

The pressure space is the space of piecewise constants with vanishing mean, i.e,

$$W_h = \{w \in L_0^2(\Omega) : w|_T \in P_0(T) \text{ for all } T \in \mathcal{T}_h\}.$$

The degrees of freedom (3.11) naturally lead us to define the (Fortin) projection $\hat{\Pi} : \boldsymbol{C}(\Omega) \to \boldsymbol{V}_h$ determined by the conditions

$$\hat{\Pi}\boldsymbol{v}(x) = \boldsymbol{v}(x) \qquad \text{for all vertices } x \text{ of } \mathcal{T}_h, \tag{4.1a}$$
$$\langle \hat{\Pi}\boldsymbol{v}, \boldsymbol{w} \rangle_e = \langle \boldsymbol{v}, \boldsymbol{w} \rangle_e \qquad \text{for all edges } e \text{ of } \mathcal{T}_h \text{ and } \boldsymbol{w} \in \boldsymbol{P}_1(e), \tag{4.1b}$$
$$\langle \hat{\Pi}\boldsymbol{v}, \boldsymbol{w} \rangle_F = \langle \boldsymbol{v}, \boldsymbol{w} \rangle_F \qquad \text{for all faces } F \text{ of } \mathcal{T}_h \text{ and } \boldsymbol{w} \in \boldsymbol{P}_0(F). \tag{4.1c}$$

Theorem 3.1 ensures that $\hat{\Pi}$ is well-defined. The projection $\hat{\Pi}$ is not defined for all $H_0^1(\Omega)$ since requires well-defined traces on edges and vertices. We instead use a standard modification. Let $\Pi_S : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{L}_h$ denote the Scott-Zhang interpolant (Scott & Zhang, 1990), where $\boldsymbol{L}_h$ is the space of continuous piecewise linear polynomials. We then define $\Pi : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ to be the unique operator satisfying

$$\Pi\boldsymbol{v}(x) = \Pi_S\boldsymbol{v}(x) \qquad \text{for all vertices } x \text{ of } \mathcal{T}_h, \tag{4.2a}$$
$$\langle \Pi\boldsymbol{v}, \boldsymbol{w} \rangle_e = \langle \Pi_S\boldsymbol{v}, \boldsymbol{w} \rangle_e \qquad \text{for all edges } e \text{ of } \mathcal{T}_h \text{ and } \boldsymbol{w} \in \boldsymbol{P}_1(e), \tag{4.2b}$$
$$\langle \Pi\boldsymbol{v}, \boldsymbol{w} \rangle_F = \langle \boldsymbol{v}, \boldsymbol{w} \rangle_F \qquad \text{for all faces } F \text{ of } \mathcal{T}_h \text{ and } \boldsymbol{w} \in \boldsymbol{P}_0(F). \tag{4.2c}$$

The following commuting property then easily follows from integration by parts and (4.2c):

$$\operatorname{div} \Pi \boldsymbol{v} = P_0 \operatorname{div} \boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4.3}$$

where P_0 is the L^2 projection onto W_h.

LEMMA 4.1 There exists a constant C such that for any $\boldsymbol{v} \in \boldsymbol{H}_0^s(\Omega)$

$$\|\boldsymbol{v} - \Pi \boldsymbol{v}\|_{H^1(\Omega)} \leqslant C h^{s-1} \|\boldsymbol{v}\|_{H^s(\Omega)}, \tag{4.4}$$

where $1 \leqslant s \leqslant 2$.

Proof. The proof follows the same arguments as those found in (Guzmán & Neilan, 2013).

First, denote by $\Pi_T := \Pi|_T$ the local Fortin operator restricted to a single element $T \in \mathcal{T}_h$. We then decompose the local Fortin operator as $\boldsymbol{I} - \Pi_T = (\boldsymbol{I} - \Pi_{U,T})(\boldsymbol{I} - \Pi_{W,T})(\boldsymbol{I} - \Pi_{M,T})$ where $\boldsymbol{I}$ is the identity operator on $\boldsymbol{H}_0^1(T)$, and $\Pi_{M,T}$, $\Pi_{W,T}$ and $\Pi_{U,T}$ are the canonical projections associated with the local spaces $\boldsymbol{M}(T)$, $\boldsymbol{W}(T)$ and $\boldsymbol{U}(T)$. In particular, the projections $\Pi_{M,T} : \boldsymbol{H}_0^1(T) \to \boldsymbol{M}(T)$, $\Pi_{W,T} : \boldsymbol{H}_0^1(T) \to \boldsymbol{W}(T)$ and $\Pi_{U,T} : \boldsymbol{H}_0^1(T) \to \boldsymbol{U}(T)$ satisfy

$$\begin{aligned}
&\Pi_{M,T}\boldsymbol{v}(x_i) = \Pi_S \boldsymbol{v}(x_i) && \text{for all vertices } x_i \text{ of } T,\\
&\left\langle (\Pi_{M,T}\boldsymbol{v}) \cdot \boldsymbol{n}_k, s \right\rangle_{e_{i,j}} = \left\langle (\Pi_S \boldsymbol{v}) \cdot \boldsymbol{n}_k, s \right\rangle_{e_{i,j}} && \text{for all edges } e_{i,j} \text{ of } T \text{ and } s \in P_1(e_{i,j}) \ (k = i, j),\\
&\left\langle (\Pi_{M,T}\boldsymbol{v}) \cdot \boldsymbol{n}_i, \kappa \right\rangle_{F_i} = \left\langle \boldsymbol{v} \cdot \boldsymbol{n}_i, \kappa \right\rangle_{F_i} && \text{for all faces } F_i \text{ of } T \text{ and } \kappa \in P_0(F_i),\\
&\left\langle (\Pi_{U,T}\boldsymbol{v}) \times \boldsymbol{n}_i, \kappa \right\rangle_{F_i} = \left\langle \boldsymbol{v} \times \boldsymbol{n}_i, \kappa \right\rangle_{F_i} && \text{for all faces } F_i \text{ of } T \text{ and } \kappa \in P_0(F_i),\\
&\left\langle (\Pi_{W,T}\boldsymbol{v}) \cdot \boldsymbol{t}_{i,j}, s \right\rangle_{e_{i,j}} = \left\langle (\Pi_S \boldsymbol{v}) \cdot \boldsymbol{t}_{i,j}, s \right\rangle_{e_{i,j}} && \text{for all edges } e_{i,j} \text{ of } T \text{ and } s \in P_1(e_{i,j}).
\end{aligned}$$

Using arguments found in (Arnold & Winther, 2002; Guzmán & Neilan, 2013), we find

$$\|\boldsymbol{v} - \Pi_{M,T}\boldsymbol{v}\|_{H^m(T)} \leqslant C h_T^{s-m} \|\boldsymbol{v}\|_{H^s(\omega(T))} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^s(\omega(T)), \tag{4.5}$$

where $\omega(T) := \cup_{T' \in \mathcal{T}_h \bar{T} \cap \bar{T}' \neq \emptyset} T'$. Moreover, by using the same arguments in (Guzmán & Neilan, 2013, Lemma 3.3) we have

$$\|\Pi_{W,T}\boldsymbol{v}\|_{H^m(T)} \leqslant C h_T^{1/2-m} \|\boldsymbol{v} \times \boldsymbol{n}\|_{L^2(\partial T)} \ \ (m = 0, 1). \tag{4.6}$$

Next, denote by $F_T : \hat{T} \to T$ with $F(\hat{x}) = A\hat{x} + b$ to be the standard affine transformations, where $\hat{T}$ is the reference element with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Given $\boldsymbol{v} \in \boldsymbol{U}(T)$, we define $\hat{\boldsymbol{v}}(\hat{x}) = A^T \boldsymbol{v}(x)$, where $x = F_T(\hat{x})$. Under this transformation, we have $\left\langle \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{t}}_{i,j}, \hat{s} \right\rangle_{\hat{e}_{i,j}} = \left\langle \boldsymbol{v} \cdot \boldsymbol{t}_{i,j}, s \right\rangle_{e_{i,j}}$ (Monk, 2003). Therefore by Lemma 3.3, the function $\hat{\boldsymbol{v}}$ is uniquely determined by the moments $\left\langle \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{t}}_{i,j}, \hat{s} \right\rangle_{\hat{e}_{i,j}}$ over all edges $\hat{e}_{i,j}$ of $\hat{T}$. Since these values uniquely determine $\hat{\boldsymbol{v}}$ and since all norms are equivalent in a finite dimensional setting, we have $(m \geqslant 0)$

$$\|\hat{\boldsymbol{v}}\|_{H^m(\hat{T})}^2 \leqslant C \Bigg| \sum_{\substack{i,j=1 \\ i>j}}^{4} \sup_{\substack{\hat{s} \in P_1(\hat{e}_{i,j}) \\ \|\hat{s}\|_{L^2(\hat{e}_{i,j})} = 1}} \left\langle \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{t}}_{i,j}, \hat{s} \right\rangle_{\hat{e}_{i,j}} \Bigg|^2.$$

Let $\hat{s}_{i,j}^* \in P_1(\hat{e}_{i,j})$ with $\|s_{i,j}^*\|_{L^2(\hat{e}_{i,j})} = 1$ be a function satisfying

$$\left\langle \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{t}}_{i,j}, \hat{s}_{i,j}^* \right\rangle_{e_{i,j}} = \sup_{\substack{\hat{s} \in P_1(\hat{e}_{i,j}) \\ \|\hat{s}\|_{L^2(\hat{e}_{i,j})} = 1}} \left\langle \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{t}}_{i,j}, \hat{s} \right\rangle_{\hat{e}_{i,j}}.$$

Setting $s^*_{i,j}(x) = \sqrt{|\hat{e}_{i,j}|/|e_{i,j}|}\hat{s}^*_{i,j}(\hat{x})$ so that $\|s^*_{i,j}\|_{L^2(e_{i,j})} = 1$, we have

$$\Big|\sum_{\substack{i,j=1\\ i>j}}^{4} \sup_{\substack{\hat{s}\in P_1(\hat{e}_{i,j})\\ \|\hat{s}\|_{L^2(\hat{e}_{i,j})}=1}} \big\langle \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{t}}_{i,j},\hat{s}\big\rangle_{\hat{e}_{i,j}}\Big|^2 = \Big|\sum_{\substack{i,j=1\\ i>j}}^{4} \sqrt{|e_{i,j}|/|\hat{e}_{i,j}|}\big\langle \boldsymbol{v}\cdot\boldsymbol{t}_{i,j},s^*_{i,j}\big\rangle_{e_{i,j}}\Big|^2.$$

Combining these results, and replacing $\boldsymbol{v}$ by $\Pi_U\boldsymbol{v}$, we obtain

$$\begin{aligned}\|\widehat{\Pi_U\boldsymbol{v}}\|^2_{H^m(T)} &\leqslant C\Big|\sum_{\substack{i,j=1\\ i>j}}^{4}\sqrt{|e_{i,j}|/|\hat{e}_{i,j}|}\big\langle \Pi_U\boldsymbol{v}\cdot\boldsymbol{t}_{i,j},s^*_{i,j}\big\rangle_{e_{i,j}}\Big|^2\\ &\leqslant Ch_T\sum_{i=1}^{4}\|\Pi_S\boldsymbol{v}\|^2_{L^2(\partial F_i)} \leqslant Ch_T^{-1}\|\Pi_S\boldsymbol{v}\|^2_{L^2(T)},\end{aligned}$$

where a trace inequality was used to derive the last inequality. Consequently, by a scaling argument and the stability properties of the Scott-Zhang interpolant we have

$$\|\Pi_U\boldsymbol{v}\|^2_{H^1(T)} \leqslant Ch_T^{-1}\|\widehat{\Pi_U\boldsymbol{v}}\|^2_{L^2(\hat{T})} \leqslant Ch_T^{-1}\|\Pi_S\boldsymbol{v}\|^2_{L^2(T)} \leqslant Ch_T^{-2}\|\boldsymbol{v}\|^2_{L^2(T)} \qquad \forall\boldsymbol{v}\in\boldsymbol{H}^1_0(\Omega). \tag{4.7}$$

Finally, recalling the decomposition $\boldsymbol{I}-\Pi_T = (\boldsymbol{I}-\Pi_{U,T})(\boldsymbol{I}-\Pi_{W,T})(\boldsymbol{I}-\Pi_{M,T})$ and applying the estimates (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned}\|\boldsymbol{v}-\Pi_T\boldsymbol{v}\|_{H^1(T)} &\leqslant \big\|(\boldsymbol{I}-\Pi_{W,T})(\boldsymbol{I}-\Pi_{M,T})\boldsymbol{v}\big\|_{H^1(T)} + Ch_T^{-1}\big\|(\boldsymbol{I}-\Pi_{W,T})(\boldsymbol{I}-\Pi_{M,T})\boldsymbol{v}\big\|_{L^2(T)}\\ &\leqslant C\Big(\|\boldsymbol{v}-\Pi_{M,T}\boldsymbol{v}\|_{H^1(T)} + h_T^{-1}\|\boldsymbol{v}-\Pi_{M,T}\boldsymbol{v}\|_{L^2(T)} + h_T^{-1/2}\|\boldsymbol{v}-\Pi_{M,T}\boldsymbol{v}\|_{L^2(\partial T)}\Big)\\ &\leqslant C\big(\|\boldsymbol{v}-\Pi_{M,T}\boldsymbol{v}\|_{H^1(T)} + h_T^{-1}\|\boldsymbol{v}-\Pi_{M,T}\boldsymbol{v}\|_{L^2(T)}\big) \leqslant Ch_T^{s-m}\|\boldsymbol{v}\|_{H^s(\omega(T))}.\end{aligned}$$

□

THEOREM 4.1 The LBB condition (1.1) holds for a constant $\beta > 0$ independent of h. In addition, the space of discretely divergence-free functions are divergence-free pointwise; that is, property (1.2) is satisfied.

Proof. The first assertion follows from (4.3) and (4.4) (with $s = 1$) and using standard arguments (cf. Brezzi & Fortin (1991); Boffi *et al.* (2008)).

Next, recall the definition of the local spaces $\boldsymbol{V}(T) = \boldsymbol{M}(T)+\boldsymbol{U}(T)+\boldsymbol{W}(T)$ and $\boldsymbol{M}(T) = \boldsymbol{P}_1(T)+\boldsymbol{Q}_2(T)+\boldsymbol{Q}_3(T)$. Since $\boldsymbol{U}(T)$, $\boldsymbol{W}(T)$ and $\boldsymbol{Q}_m(T)$ consists of divergence-free functions, we have $\operatorname{div}\boldsymbol{V}(T) = \operatorname{div}\boldsymbol{M}(T) = \operatorname{div}\boldsymbol{P}_1(T) \subseteq P_0(T)$. Therefore $\operatorname{div}\boldsymbol{V}_h \subseteq Q_h$ and (1.2) immediately follows. □

5. Reduced Elements

Theorem 3.1 states that the local velocity space $\boldsymbol{V}(T)$ has sixty degrees of freedom. Moreover, it is easy to see from the degrees of freedom (3.11) that the global dimension of the velocity space is $\dim\boldsymbol{V}_h = 3N_v+6N_e+3N_f$, where N_v, N_f, N_e denote, respectively, the number of interior vertices, faces and edges in the triangulation. In this section, we show that the dimension of the velocity space can be significantly reduced, with only sixteen degrees of freedom per element, while not affecting the stability and approximation properties.

To define the reduced velocity space, we first define the local space and its degrees of freedom. To this end, let T be an arbitrary element in the triangulation. For each $\ell = 1,2,3,4$, define $\boldsymbol{v}_\ell^R$ to be the unique function in $\boldsymbol{V}(T)$ satisfying

$$\begin{aligned}
&\boldsymbol{v}_\ell^R(x_i) = 0 && \text{for all vertices } x_i,\\
&\left\langle \boldsymbol{v}_\ell^R, \boldsymbol{w}\right\rangle_{e_{i,j}} = 0 && \text{for all } \boldsymbol{w} \in \boldsymbol{P}_1(e_{i,j}) \text{ and } i,j = 1,\ldots,4,\\
&\left\langle \boldsymbol{v}_\ell^R \cdot \boldsymbol{n}_i, w\right\rangle_{F_i} = \delta_{i\ell} && \text{for all } \boldsymbol{w} \in \boldsymbol{P}_0(F_i) \text{ and } i = 1,\ldots,4,\\
&\left\langle \boldsymbol{v}_\ell^R \times \boldsymbol{n}_i, \boldsymbol{w}\right\rangle_{F_i} = 0 && \text{for all } \boldsymbol{w} \in \boldsymbol{P}_0(F_i) \text{ and } i = 1,\ldots,4.
\end{aligned}$$

Theorem 3.1 ensures that $\boldsymbol{v}_\ell^R$ is well-defined.

Notice that the tangential component of each $\boldsymbol{v}_\ell^R$ is zero on all the faces and the normal component vanishes on faces F_i with $i \neq \ell$. In other words, on the boundary of T one has

$$\boldsymbol{v}_\ell^R|_{\partial T} = c_\ell \lambda_{\ell+1}\lambda_{\ell+2}\lambda_{\ell+3}\nabla\lambda_\ell$$

for some non-zero constant c_ℓ. We note that the right hand side is exactly the Bernardi-Raugel face bubble (Bernardi & Raugel, 1985). Of course, the difference between the two functions is that $\operatorname{div} \boldsymbol{v}_\ell^R$ is a constant on T while $\operatorname{div}(\lambda_{\ell+1}\lambda_{\ell+2}\lambda_{\ell+3}\nabla\lambda_\ell)$ is a quadratic polynomial.

The reduced local space is defined as

$$\boldsymbol{V}^R(T) = \boldsymbol{P}_1(T) + \operatorname{span}\{\boldsymbol{v}_1^R, \boldsymbol{v}_2^R, \boldsymbol{v}_3^R, \boldsymbol{v}_4^R\}.$$

It is easy to see that the dimension of the space is 16 with a unisolvent set of degrees of freedom given by

$$\boldsymbol{v}(x_i) \qquad \text{for all vertices } x_i, \tag{5.1a}$$

$$\left\langle \boldsymbol{v} \cdot \boldsymbol{n}_i, \kappa\right\rangle_{F_i} \qquad \text{for all } \kappa \in P_0(F_i) \text{ and } i = 1,\ldots,4. \tag{5.1b}$$

Again, these degrees of freedom are the same as the Bernardi-Gaugel element (Bernardi & Raugel, 1985).

The global space simply consists of continuous functions with vanishing trace that are locally in $\boldsymbol{V}^R(T)$, i.e.,

$$\boldsymbol{V}_h^R = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) : \ \boldsymbol{v}|_T \in \boldsymbol{V}^R(T), \forall T \in \mathcal{T}_h\}.$$

The global dimension is thus $\dim \boldsymbol{V}_h^R = 3N_v + N_f$.

It is not difficult to show that $\boldsymbol{V}_h^R \times W_h$ also satisfies properties (1)-(3) stated in the introduction. Indeed, it is easy to see that the divergence operator maps $\boldsymbol{V}_h^R$ into the space of piecewise constants (i.e., $\operatorname{div} \boldsymbol{V}_h^R \subseteq W_h$) and therefore the divergence-free property (1.2) is trivially satisfied. Moreover, we can define the operator $\Pi_R : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h^R$ satisfying

$$\Pi_R \boldsymbol{v}(x) = \Pi_S \boldsymbol{v}(x) \qquad \text{for all vertices } x \text{ of } \mathcal{T}_h, \tag{5.2}$$

$$\left\langle \Pi_R \boldsymbol{v} \cdot \boldsymbol{n}, \kappa\right\rangle_F = \left\langle \boldsymbol{v} \cdot \boldsymbol{n}, \kappa\right\rangle_F \qquad \text{for all faces } F \text{ of } \mathcal{T}_h \text{ and } \kappa \in P_0(F). \tag{5.3}$$

The following commuting property follows from integration by parts and (5.3):

$$\operatorname{div} \Pi_R \boldsymbol{v} = P_0 \operatorname{div} \boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{5.4}$$

where we call $P_0 : L^2(\Omega) \to W_h$ is the L^2-projection onto the space of piecewise constants. We also have the following approximation properties of the Fortin projection.

LEMMA 5.1 There exists a constant C such that for any $\boldsymbol{v} \in \boldsymbol{H}_0^s(\Omega)$

$$\|\boldsymbol{v} - \Pi_R \boldsymbol{v}\|_{H^1(\Omega)} \leqslant Ch^{s-1} \|\boldsymbol{v}\|_{H^s(\Omega)}, \tag{5.5}$$

where $1 \leqslant s \leqslant 2$.

The proof of Lemma 5.1 is nearly identical to the proof of Lemma 4.1 so we omit it. Combining Lemma 5.1 with the commuting property (5.4) we easily see that the LBB condition (1.1) is satisfied, and therefore $\boldsymbol{V}_h^R \times W_h$ is a stable finite element pair for the Stokes problem.

5.1 *Elements using Bogovskii's operator*

In this section we give a different approach to construct Stokes elements that produce divergence-free approximations and have the same number of degrees of freedom as the reduced elements constructed above. The argument presented below is valid in any dimension $d \geqslant 2$. The basis function of the velocity space may or may not be rational functions.

The central idea is to start with the Bernardi-Raugel face bubbles and then correct them so that the divergence of the resulting functions are constant while not affecting the values on the boundary. To this end, let T be a d-dimensional simplex and let F_i $(i = 1, \ldots, d+1)$ denote the $(d-1)$-dimensional subsimplices of T. For each, $i = 1, \ldots, d+1$ the ith Bernardi-Raugel bubble of T is given by (Bernardi & Raugel, 1985)

$$\boldsymbol{b}_i := \lambda_{i+1}\lambda_{i+2}\cdots\lambda_{i+d}\nabla\lambda_i,$$

where we recall λ_i is the barycentric coordinate of T that vanishes on F_i. The following properties of $\boldsymbol{b}_i$ are evident from its definition:

- The tangential components of $\boldsymbol{b}_i$ vanish on F_j for all $j = 1, \ldots, d+1$.
- The normal component of $\boldsymbol{b}_i$ vanishes on F_j if $j \neq i$.
- $\boldsymbol{b}_i \cdot \boldsymbol{n}|_{F_i} \neq 0$.
- $\operatorname{div} \boldsymbol{b}_i$ is a non-zero polynomial of degree $d-1$.

We note that due to the last property, the Bernardi-Raugel element (where the pressure space consists of piecewise constants) does not produce exactly divergence-free velocity approximations.

We now seek a continuous functions $\boldsymbol{v}_i$ with the following properties:

(P1) $\boldsymbol{v}_i|_{\partial T} = \boldsymbol{b}_i|_{\partial T}$

(P2) $\operatorname{div} \boldsymbol{v}_i \in P_0(T)$.

To construct such a function, we set

$$g_i := \operatorname{div}(\boldsymbol{b}_i) - \frac{1}{|T|}\int_T \operatorname{div} \boldsymbol{b}_i \, dx,$$

and note that g_i has zero mean. We then define $\boldsymbol{w}_i$ by the Bogovskii operator (Bogovskii, 1979; Durán, 2012) with source g_i:

$$\boldsymbol{w}_i(x) := \int_T G(x,y) g_i(y) dy.$$

Here, the kernel G is given by

$$G(x,y) := \int_0^1 \frac{(x-y)}{t} \omega(y + \frac{x-y}{t}) \frac{dt}{t^d},$$

and $\omega \in C_0^\infty(B)$ with $\int_B \omega\,dx = 1$ for some ball $B \subset T$. The function $\boldsymbol{w}_i$ then satisfies $\operatorname{div}\boldsymbol{w}_i = g_i$ and $\boldsymbol{w}_i \in \boldsymbol{W}_0^{1,p}(T)$ for any $1 < p < \infty$ (cf. (Durán *et al.*, 2001; Durán, 2012)). Therefore, $\boldsymbol{w}_i \in \boldsymbol{C}^0(\overline{T})$ by a Sobolev embedding.

We then set

$$\boldsymbol{v}_i := \boldsymbol{b}_i - \boldsymbol{w}_i.$$

Since $\boldsymbol{w}_i$ has vanishing trace, we have $\boldsymbol{v}_i|_{\partial T} = \boldsymbol{b}_i|_{\partial T}$. Moreover, $\operatorname{div}\boldsymbol{v}_i = \operatorname{div}\boldsymbol{b}_i - \operatorname{div}\boldsymbol{w}_i = \operatorname{div}\boldsymbol{b}_i - g_i = -\frac{1}{|T|}\int_T \operatorname{div}\boldsymbol{b}_i\,dx \in P_0(T)$. Therefore the function $\boldsymbol{v}_i$ satisfies the desired properties (P1)–(P2).

Next, we define the local space and global spaces

$$\begin{aligned} \boldsymbol{V}^B(T) &:= \boldsymbol{P}^1(T) + \operatorname{span}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{d+1}\}, \\ \boldsymbol{V}_h^B &:= \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega):\ \boldsymbol{v}|_T \in \boldsymbol{V}^B(T), \forall T \in \mathcal{T}_h\}. \end{aligned}$$

This local space has all of the properties of the reduced elements constructed above. For example, any function $\boldsymbol{v} \in \boldsymbol{V}^B(T)$ is uniquely determined by the degrees of freedom (5.1). Moreover, the degrees of freedom naturally lead to a projection $\Pi_B : \boldsymbol{H}^1(\Omega) \to \boldsymbol{V}_h^B$ satisfying the commuting property $\operatorname{div}\Pi_B = P_0\operatorname{div}$.

6. Stokes Problem

In this section we apply the elements constructed in Sections 3 and 5 toward the Stokes problem with no-slip boundary conditions:

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{6.1a}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{6.1b}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega. \tag{6.1c}$$

In (6.1a). $\boldsymbol{f}$ is a given $\boldsymbol{L}^2(\Omega)$ function and $\nu > 0$ is the kinematic viscosity. The finite element approximation reads: find a pair $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ such that

$$\nu(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}) - (p_h, \operatorname{div}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{6.2a}$$

$$(\operatorname{div}\boldsymbol{u}_h, q) = 0 \qquad \forall q \in W_h. \tag{6.2b}$$

Using standard arguments (Brezzi & Fortin, 1991; Boffi *et al.*, 2008) we obtain the following convergence results.

THEOREM 6.1 There exists a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ to problem (6.2). Moreover, there holds

$$\begin{aligned} \|\nabla(\boldsymbol{u}_h - \boldsymbol{u})\|_{L^2(\Omega)} &\leqslant \|\nabla(\Pi\boldsymbol{u} - \boldsymbol{u})\|_{L^2(\Omega)}, \\ \|p - p_h\|_{L^2(\Omega)} &\leqslant C(\|p - P_0 p\|_{L^2(\Omega)} + \nu\|\nabla(\Pi\boldsymbol{u} - \boldsymbol{u})\|_{L^2(\Omega)}). \end{aligned}$$

Therefore, by Lemma 4.1 and approximation properties of the L^2-projection, we have

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{H^1(\Omega)} &\leqslant Ch^{s-1}\|\boldsymbol{u}\|_{H^s(\Omega)}, \\ \|p - p_h\|_{L^2(\Omega)} &\leqslant Ch^{s-1}\big(\|p\|_{H^{s-1}(\Omega)} + \nu\|\boldsymbol{u}\|_{H^s(\Omega)}\big) \qquad (s = 1, 2). \end{aligned}$$

REMARK 6.1 Due to property (1.2) the velocity error is decoupled from the pressure error and is independent of the viscosity coefficient.

REMARK 6.2 The same conclusions hold in Theorem 6.1 if the velocity space is replaced by the reduced space $\boldsymbol{V}_h^R$ constructed in Section 5.

7. Conclusions

In this paper we constructed a stable finite element pair for the three dimensional Stokes problem that produces exactly divergence-free velocity approximations. As far as we are aware, this is the first Stokes pair with these properties on general shape-regular simplicial triangulations.

We end the paper by remarking that the proposed elements can be generalized to arbitrary order. Similar to the construction in Section 3, this is achieved by enriching an $\boldsymbol{H}(\mathrm{div}\,;\Omega)$-conforming finite element space with rational face bubbles and rational edge bubbles. In this case the local space of the $\boldsymbol{H}(\mathrm{div}\,;\Omega)$ element is given by $\boldsymbol{M}(T) = \boldsymbol{P}_k(T) + \boldsymbol{Q}_{k+1}(T) + \boldsymbol{Q}_{k+2}$ with $k \geqslant 1$, and the enriching local spaces are given by (3.3) and (3.6) with $\boldsymbol{U}^{(i)} = \mathbf{curl}\,(B_{F_i}\boldsymbol{A}^{(i)}_{k-1} \times \boldsymbol{n}_i)$ and $M^{(i,j)}(T) = \mathrm{span}\{\lambda_k^{\alpha_1}\lambda_\ell^{\alpha_2} : |\alpha| \leqslant k\}$. Here, $\boldsymbol{A}^{(i)}_{k-1}$ is the space of functions $\boldsymbol{p} \in \boldsymbol{P}_{k-1}(T)$ satisfying $(\boldsymbol{p} \times \boldsymbol{n}_i, B_{F_i}\boldsymbol{q} \times \boldsymbol{n}_i)_T = 0$ for all $\boldsymbol{q} \in \boldsymbol{P}_{k-2}(T)$. The pressure space in the general setting is the space of piecewise polynomials of degree $k-1$. Using similar arguments as in the proceeding sections, it can be shown that the resulting spaces form a stable pair for the Stokes problem and the approximations converge with order k.

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A. Appendix

A.1 *The proof of Lemma 2.2*

We break the proof of Lemma 2.2 into several steps. First, noting that $\nabla(\lambda_j^2-\lambda_i^2)+4(\lambda_i\nabla\lambda_j-\lambda_j\nabla\lambda_i) = 3\nabla(\lambda_j-\lambda_i)(\lambda_i+\lambda_j)-\nabla(\lambda_i+\lambda_j)(\lambda_j-\lambda_i)$, we may write

$$\boldsymbol{s}_{i,j} = p_{i,j}\boldsymbol{g}_{i,j}$$

with

$$p_{i,j} = \frac{b_{i,j}^2(\lambda_i+\lambda_j)}{\lambda_i\lambda_j + b_{i,j}(\lambda_i+\lambda_j)}, \quad \boldsymbol{g}_{i,j} = \frac{3}{2}\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}\nabla(\lambda_j-\lambda_i) - \frac{1}{2}\frac{\lambda_i\lambda_j(\lambda_j-\lambda_i)}{(\lambda_i+\lambda_j)^2}\nabla(\lambda_i+\lambda_j).$$

We now analyze the functions $p_{i,j}$ and $\boldsymbol{g}_{i,j}$ individually.

LEMMA A.1 There holds

$$|\boldsymbol{g}_{i,j}| \leqslant C\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}, \qquad |\nabla \boldsymbol{g}_{i,j}| \leqslant C, \qquad |D^2\boldsymbol{g}_{i,j}| \leqslant \frac{C}{\lambda_i+\lambda_j},$$
$$|\nabla p_{i,j}| \leqslant C, \qquad |D^2 p_{i,j}| \leqslant C + \frac{C}{\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}} + \frac{C}{\lambda_i+\lambda_j}.$$

where the constants $C>0$ are independent of the barycentric coordinates (but depend on h). Moreover,

$$\mathbf{curl}\,\boldsymbol{g}_{i,j} = \nabla\lambda_i \times \nabla\lambda_j. \tag{A.1}$$

Proof. The first estimate easily follows from the inequality $|\lambda_j-\lambda_i| \leqslant |\lambda_j+\lambda_i| = \lambda_j+\lambda_i$ and the definition of $\boldsymbol{g}_{i,j}$. The second and third inequalities follow from the estimates

$$\frac{\lambda_i\lambda_j}{(\lambda_i+\lambda_j)^2} \leqslant C, \quad \left|\nabla\left(\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}\right)\right| \leqslant C, \quad \left|D^2\left(\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}\right)\right| \leqslant \frac{C}{\lambda_i+\lambda_j}. \tag{A.2}$$

Next we write

$$p_{i,j} = \frac{b_{i,j}^2}{r_{i,j}} \qquad \text{with } r_{i,j} := \frac{\lambda_i\lambda_j}{\lambda_j+\lambda_j} + b_{i,j}, \tag{A.3}$$

and note that (cf. (A.2))

$$r_{i,j}^{-1} \leqslant \frac{1}{\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}}, \qquad r_{i,j}^{-1} \leqslant \frac{1}{b_{i,j}}, \qquad |\nabla r_{i,j}| \leqslant C, \qquad |D^2 r_{i,j}| \leqslant \frac{C}{\lambda_i+\lambda_j}. \tag{A.4}$$

We then have

$$|\nabla p_{i,j}| = \Big|\frac{2b_{i,j}\nabla b_{i,j}}{r_{i,j}} - \frac{b_{i,j}^2}{r_{i,j}^2}\nabla r_{i,j})\Big| \leqslant 2|\nabla b_{i,j}| + |\nabla r_{i,j}| \leqslant C. \tag{A.5}$$

Furthermore by (A.4) we have

$$\begin{aligned}
|D^2 p_{i,j}| &= \Big|\frac{2\nabla b_{i,j}\nabla b_{i,j}^t + 2b_{i,j}D^2 b_{i,j}}{r_{i,j}} - \frac{2b_{i,j}(\nabla b_{i,j}\nabla r_{i,j}^t + \nabla r_{i,j}\nabla b_{i,j}^t) + b_{i,j}^2 D^2 r_{i,j}}{r_{i,j}^2} + \frac{2b_{i,j}^2\nabla r_{i,j}\nabla r_{i,j}^t}{r_{i,j}^3}\Big| \\
&\leqslant \frac{2|\nabla b_{i,j}|^2}{r_{i,j}} + 2|D^2 b_{i,j}| + \frac{4b_{i,j}|\nabla b_{i,j}||\nabla r_{i,j}|}{r_{i,j}^2} + |D^2 r_{i,j}| + \frac{2b_{i,j}^2|\nabla r_{i,j}|^2}{r_{i,j}^3} \\
&\leqslant \frac{2|\nabla b_{i,j}|^2}{r_{i,j}} + 2|D^2 b_{i,j}| + \frac{4|\nabla b_{i,j}||\nabla r_{i,j}|}{r_{i,j}} + |D^2 r_{i,j}| + \frac{2|\nabla r_{i,j}|^2}{r_{i,j}} \leqslant C + \frac{C}{\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}} + \frac{C}{\lambda_i+\lambda_j}.
\end{aligned}$$

Next, since $\nabla\lambda_i$ and $\nabla\lambda_j$ are constant vectors we have

$$\begin{aligned}\mathbf{curl}\,\boldsymbol{g}_{i,j} &= \frac{3}{2}\frac{\nabla(\lambda_i\lambda_j)}{\lambda_i+\lambda_j}\times\nabla(\lambda_j-\lambda_i) - \frac{3}{2}\frac{\lambda_i\lambda_j\nabla(\lambda_i+\lambda_j)\times\nabla(\lambda_j-\lambda_i)}{(\lambda_i+\lambda_j)^2}\\ &\quad - \frac{1}{2}\frac{(\lambda_j-\lambda_i)\nabla(\lambda_i\lambda_j)+\lambda_i\lambda_j\nabla(\lambda_j-\lambda_i)}{(\lambda_i+\lambda_j)^2}\times\nabla(\lambda_i+\lambda_j)\\ &= \frac{3}{2}(\nabla\lambda_i\times\nabla\lambda_j) - \frac{1}{2}\frac{(6\lambda_i\lambda_j+(\lambda_j-\lambda_i)(\lambda_i+\lambda_j)-2\lambda_i\lambda_j)}{(\lambda_i+\lambda_j)^2}(\nabla\lambda_i\times\nabla\lambda_j) = \nabla\lambda_i\times\nabla\lambda_j.\end{aligned}$$

□

COROLLARY A.1 There holds

$$|\nabla(\nabla p_{i,j}\times\boldsymbol{g}_{i,j})| \leqslant C.$$

Proof. This result directly follows from the estimates stated in Lemma A.1:

$$|\nabla(\nabla p_{i,j}\times\boldsymbol{g}_{i,j})| \leqslant |D^2 p_{i,j}||\boldsymbol{g}_{i,j}| + |\nabla p_{i,j}||\nabla\boldsymbol{g}_{i,j}| \leqslant C.$$

□

LEMMA A.2 There holds

$$p_{i,j}\mathbf{curl}\,\boldsymbol{g}_{i,j}|_{\partial T} = b_{i,j}(\nabla\lambda_i\times\nabla\lambda_j), \tag{A.6}$$

$$\nabla p_{i,j}\times\boldsymbol{g}_{i,j}|_{\partial T} = 0. \tag{A.7}$$

Consequently, $p_{i,j}\mathbf{curl}\,\boldsymbol{g}_{i,j}\in\boldsymbol{C}^0(\overline{T})$ and $\nabla p_{i,j}\times\boldsymbol{g}_{i,j}\in\boldsymbol{C}^0(\overline{T})$.

Proof. By Lemma A.1 we have

$$p_{i,j}\mathbf{curl}\,\boldsymbol{g}_{i,j}|_{\partial T} = p_{i,j}(\nabla\lambda_i\times\nabla\lambda_j).$$

Since $b_{i,j}|_{\partial T\setminus(F_i\cup F_j)} = 0$ we have $p_{i,j}|_{\partial T\setminus(F_i\cup F_j)} = 0$. Moreover, since $\lim_{(\lambda_i,\lambda_j)\to(0,0)} r_{i,j} = b_{i,j}$, we have (cf. (A.3)) $\lim_{(\lambda_i,\lambda_j)\to(0,0)} p_{i,j} = b_{i,j}$. Thus, $p|_{\partial T} = b_{i,j}$ and therefore the identity (A.6) holds.

Next, using the estimates in Lemma A.1 we obtain

$$|\nabla p_{i,j}\times\boldsymbol{g}_{i,j}| \leqslant |\nabla p_{i,j}||\boldsymbol{g}_{i,j}| \leqslant C|\boldsymbol{g}_{i,j}| \leqslant C\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}.$$

Consequently,

$$\lim_{(\lambda_i,\lambda_j)\to(0,0)} |\nabla p_{i,j}\times\boldsymbol{g}_{i,j}| = \lim_{(\lambda_i,\lambda_j)\to(0,0)} C\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j} = 0.$$

On the other hand, the edge bubble $b_{i,j}$ vanishes on $\partial T\setminus(F_i\cup F_j)$. Hence by (A.5) we have $\nabla p|_{\partial T\setminus(F_i\cup F_j)} = 0$, and therefore $\nabla p\times\boldsymbol{g}_{i,j}|_{\partial T\setminus(F_i\cup F_j)} = 0$. We then conclude that $\nabla p\times\boldsymbol{g}_{i,j}|_{\partial T} = 0$. □

We are now in position to prove Lemma 2.2. First, by Lemma A.2 we have

$$\mathbf{curl}\,\boldsymbol{s}_{i,j}|_{\partial T} = p_{i,j}\mathbf{curl}\,\boldsymbol{g}_{i,j}|_{\partial T} + \nabla p_{i,j}\times\boldsymbol{g}_{i,j}|_{\partial T} = b_{i,j}(\nabla\lambda_i\times\nabla\lambda_j),$$

and $\mathbf{curl}\,\boldsymbol{s}_{i,j}\in\boldsymbol{C}^0(\overline{T})$.

We also have by Lemma A.1,

$$|\nabla\mathbf{curl}\,\boldsymbol{s}_{i,j}| \leqslant |\nabla(p_{i,j}\mathbf{curl}\,\boldsymbol{g}_{i,j})| + |\nabla(\nabla p_{i,j}\times\boldsymbol{g}_{i,j})| \leqslant C|\nabla p_{i,j}| + C \leqslant C.$$

Thus, $\mathbf{curl}\,\boldsymbol{s}_{i,j}\in\boldsymbol{W}^{1,\infty}(T)$.