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Congruence classes of minimal ruled real hypersurfaces in a nonflat complex space form

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Abstract. In this paper we study congruency of minimal ruled real hypersurfaces in a nonflat complex space form with respect to the action of its isometry group. We show that those in a complex hyperbolic space are classified into 3 classes and show that those in a complex projective space are congruent to each other hence form just one class.

Key words: Minimal ruled real hypersurfaces, complex space forms, circles.

1. Introduction

In a nonflat complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c ($\neq 0$) and of complex dimension n (≥ 2), we have two typical classes of real hypersurfaces. One is the class of Hopf hypersurfaces and the other is the class of ruled real hypersurfaces. A real hypersurface M of $\mathbb{C}M^n$ is said to be *Hopf* if its characteristic vector field ξ is a principal curvature vector at each point. Here, the characteristic vector field is defined by $\xi = -J\mathcal{N}$ with complex structure J on $\mathbb{C}M^n$ and a unit normal $\mathcal{N}$ of M in $\mathbb{C}M^n$. It is well known that Hopf hypersurfaces in $\mathbb{C}M^n$ all of whose principal curvatures are constant functions must be homogeneous, that is to say, such hypersurfaces are orbits of some subgroups of the isometry group $\mathrm{Iso}(\mathbb{C}M^n)$ of the ambient space $\mathbb{C}M^n$. Moreover, such hypersurfaces are classified by Takagi [10] in the case $c > 0$ and by Berndt [3] in the case $c < 0$. A real hypersurface M of $\mathbb{C}M^n$ is said to be *ruled* if its holomorphic distribution $T^0M = \bigcup_{p\in M}\{v \in T_pM \mid v \perp \xi_p\}$ is integrable and each

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of its maximal integral manifolds is a totally geodesic complex hypersurface which is locally congruent to $\mathbb{C}M^{n-1}$. It is known that the characteristic vector field of a ruled real hypersurface M is not principal on some open dense subset of M. Therefore, Hopf hypersurfaces and ruled real hypersurfaces are quite contrastive. Besides, in their paper [4] Berndt-Tamaru classified all homogeneous real hypersurfaces in a complex hyperbolic space. There are many non-homogeneous Hopf hypersurfaces as well as many homogeneous Hopf hypersurfaces in this ambient space. We note that there exists a minimal homogeneous ruled real hypersurface in a complex hyperbolic space, which is a typical example of a non-Hopf homogeneous hypersurface. We are hence interested in minimal ruled real hypersurfaces in nonflat complex space forms.

In [6], Lohnherr-Reckziegel studied ruled real hypersurfaces by parameterizing them by maps of the form $\mathbb{R} \times \mathbb{C}M^{n-1} \to \mathbb{C}M^n$ and show properties on these maps. We here focus our mind on minimal ones and represent them explicitly through Hopf fibrations. By use of representations we can make clear their congruency. In a complex hyperbolic space minimal ruled real hypersurfaces are classified into three classes, which are called axial, parabolic and elliptic, and in a complex projective space all minimal ruled real hypersurfaces are congruent to each other.

We emphasize that the classifications of minimal ruled real hypersurfaces correspond to the classifications of totally real circles on nonflat complex space forms. On a complex projective space $\mathbb{C}P^n(c)$, all totally real circles are closed. On a complex hyperbolic space $\mathbb{C}H^n(c)$, totally real circles are closed if their curvatures are greater than $\sqrt{|c|}/2$, are horocyclic if their curvatures are equal to $\sqrt{|c|}/2$, and are unbounded and have two distinct points at infinity if their curvatures are less than $\sqrt{|c|}/2$.

2. Ruled real hypersurfaces in $\mathbb{C}M^n$

On a real hypersurface M in a Kähler manifold $\widetilde{M}$ with complex structure J and Riemannian metric $\langle\ ,\ \rangle$, an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ is naturally induced as $\xi = -J\mathcal{N}$ with a unit normal local vector field $\mathcal{N}$ of M in $\widetilde{M}$ and as $\eta(v) = \langle v, \xi\rangle$, $\phi(v) = Jv - \eta(v)\mathcal{N}$ for each tangent vector $v \in TM$. We call ξ and ϕ the characteristic vector field and the characteristic tensor field, respectively.

Let $\mathbb{C}M^n(c)$ denote a complex space form of constant holomorphic sectional curvature c, which is a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ and a complex hyperbolic space $\mathbb{C}H^n(c)$ according as c is positive, zero and negative. We say a real hypersurface M in $\mathbb{C}M^n(c)$ to be *ruled* if $T^0M = \bigcup_{p\in M}\{v \in T_pM \mid v \perp \xi_p\}$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hypersurface. A ruled real hypersurface in $\mathbb{C}M^n$ is hence a real hypersurface which is foliated by totally geodesic complex hypersurfaces in $\mathbb{C}M^n$. More exactly, every ruled real hypersurface in $\mathbb{C}M^n$ is constructed by attaching complex hyperplanes on a smooth curve in the following manner (see [5]). We take an arbitrary regular (real) curve $\gamma : I \to \mathbb{C}M^n$ defined on some open interval I. At each point $\gamma(t)$ $(t \in I)$ we attach a totally geodesic complex hypersurface M_t which is locally congruent to $\mathbb{C}M^{n-1}$ and is orthogonal to the real plane spanned by $\{\dot\gamma(t),\ J\dot\gamma(t)\}$ at that point $\gamma(t)$. We then get a ruled real hypersurface $M = \bigcup_{t\in I} M_t$ in $\mathbb{C}M^n$. We shall call this a ruled real hypersurface associated with γ.

Ruled real hypersurfaces are characterized by the property of their shape operators in the following manner (see [9]). For a real hypersurface M we define two functions $\mu, \nu : M \to \mathbb{R}$ by $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$, where A denotes the shape operator on M. Then M is ruled if and only if the following two conditions hold:

i) the set $M_* = \{p \in M \mid \nu(p) \neq 0\}$ is an open dense subset of M;
ii) there is a unit vector field U on M_*, which is orthogonal to ξ and satisfies

$$A\xi = \mu\xi + \nu U, \quad AU = \nu\xi, \quad Av = 0$$

for an arbitrary tangent vector v orthogonal to both ξ and U.

The reason why we consider the set M_* is that we define the vector field U clearly by $U = (A\xi - \mu\xi)/\nu$. By this definition, the vector field U can not be extended to a smooth vector field on M in general. But in some cases, considering $-U$ instead of U and $-\nu$ instead of ν on some components of M_*, we can define a smooth vector field U and a smooth function ν on M satisfying the equalities in the condition ii). In the following, we shall apply this rule to U and ν.

The above characterization on ruled real hypersurfaces shows that their characteristic vector fields are not principal, that is, not to be eigenvectors of their shape operators, at each point of open dense subsets. We hence find that ruled real hypersurfaces are contrastive of Hopf hypersurfaces, which are real hypersurfaces whose characteristic vector fields are principal at everywhere on them. For the function ν and integral curves of the vector field ϕU we have the following.

Lemma 1 ([8]) *For a ruled real hypersurface M in a nonflat complex space form $\mathbb{C}M^n(c)$, the following hold.*

(1) *Every integral curve of ϕU is a geodesic on M.*
(2) *The function ν satisfies the differential equation $\phi U \nu = \nu^2 + (c/4)$.*

By this Lemma, on an integral curve σ of ϕU, which is a geodesic, the function ν is given as follows:

i) When $c > 0$, it satisfies $\nu(\sigma(s)) = \pm(\sqrt{c}/2)\tan((\sqrt{c}/2)s + a)$ with some constant a;
ii) When $c < 0$, it satisfies

$$\nu(\sigma(s)) = \begin{cases} \pm(\sqrt{|c|}/2)\tanh\big((\sqrt{|c|}/2)s + a\big), \\ \pm(\sqrt{|c|}/2), \\ \pm(\sqrt{|c|}/2)\coth\big((\sqrt{|c|}/2)s + a\big) \end{cases}$$

with some constant a, according as initial condition $|\nu(\sigma(0))|$ is less than $\sqrt{|c|}/2$, equal to $\sqrt{|c|}/2$ or greater than $\sqrt{|c|}/2$.

As we have a choice of directions for the vector field U, we put double signs in the above expressions on ν.

In order to go into our study on minimal ruled real hypersurfaces, we here recall a characterization of them by a property on integral curves of characteristic vector fields. We say a smooth curve γ on a Riemannian manifold N to be a *circle* if it satisfies the differential equation $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -k^2\dot\gamma$ with some nonnegative constant k. We call this $k = k_\gamma$ the *curvature* of γ. When N is a Kähler manifold and k_γ is positive, we can see that $\langle\dot\gamma, J\nabla_{\dot\gamma}\dot\gamma\rangle/k_\gamma$ is constant along γ. We denote this constant by τ_γ and call it the *complex torsion* of γ. We call this circle Kähler when $\tau_\gamma = \pm 1$, and call it totally real when $\tau_\gamma = 0$. On $\mathbb{C}M^n(c)$, every Kähler circle lies on some totally geodesic $\mathbb{C}M^1(c)$ and every totally real circle lies on some totally geodesic real 2-dimensional submanifold $\mathbb{R}M^2(c/4)$ of constant sectional curvature $c/4$. Moreover, two circles γ_1, γ_2 on $\mathbb{C}M^n(c)$ are congruent to each other (i.e. $\gamma_2(t) = \varphi \circ \gamma_1(t)$ for all t with some isometry φ of $\mathbb{C}M^n(c)$) if and only if they satisfies one of the following:

i) They are geodesics (i.e. $k_{\gamma_1} = k_{\gamma_2} = 0$);
ii) They have the same positive curvatures ($k_{\gamma_1} = k_{\gamma_2} > 0$) and they have the same absolute values of their complex torsions ($|\tau_{\gamma_1}| = |\tau_{\gamma_2}|$).

For a Hopf hypersurface in $\mathbb{C}M^n(c)$, every integral curve of its characteristic vector field is a Kähler circle if we regard it as a curve in $\mathbb{C}M^n(c)$. On the contrary, we know the following for ruled real hypersurfaces.

Proposition 1 ([7]) *Let M be a ruled real hypersurface in a nonflat complex space form $\mathbb{C}M^n(c)$.*

(1) *If M is minimal, the function ν satisfies $\xi\nu = 0$ for its characteristic vector field ξ, and every integral curve γ of ξ is a totally real circle if we regard it as a curve in $\mathbb{C}M^n(c)$. More precisely, it satisfies*

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma(\gamma(t)) = \nu(\gamma(t))\phi U(\gamma(t)), \quad \widetilde{\nabla}_{\dot\gamma}(\phi U)(\gamma(t)) = -\nu(\gamma(t))\dot\gamma(t),$$

where $\widetilde{\nabla}$ denotes the Riemannian connection on $\mathbb{C}M^n(c)$.

(2) *If M is not minimal, it has an integral curve of its characteristic vector field which does not lies on some totally geodesic real 2-dimensional submanifold $\mathbb{R}M^2(c/4)$ in $\mathbb{C}M^n(c)$.*

This result also shows that minimal ruled real hypersurfaces and Hopf hypersurfaces are quite contrastive. This result also shows that minimal ruled real hypersurfaces in a nonflat complex space form $\mathbb{C}M^n(c)$ are ruled real hypersurfaces associated with totally real circles.

3. Classification of minimal ruled real hypersurfaces in $\mathbb{C}H^n$

In this section we study minimal ruled real hypersurfaces in a complex hyperbolic space $\mathbb{C}H^n$. On $\mathbb{C}H^n(c)$ we take a totally real circle γ of curvature $k_\gamma \geq 0$. On $\mathbb{C}H^n(c)$ totally real circles are classified into 3 kinds of classes (see [1]). When the curvature k_γ of a totally real circle γ is greater than $\sqrt{|c|}/2$, it is closed of length $4\pi/\sqrt{4k_\gamma^2 + c}$, hence is bounded. When $k_\gamma \leq \sqrt{|c|}/2$, it is unbounded and has limit points at infinity $\gamma(\infty) = \lim_{t\to\infty}\gamma(t)$, $\gamma(-\infty) = \lim_{t\to-\infty}\gamma(t)$ in the ideal boundary $\partial\mathbb{C}H^n$ of a Hadamard manifold $\mathbb{C}H^n$. When $k_\gamma < \sqrt{|c|}/2$, it has two distinct points at infinity. When $k_\gamma = \sqrt{|c|}/2$, it is horocyclic, that is, it has a single point at infinity and if it crosses to a geodesic σ which satisfies $\sigma(\infty) = \gamma(\infty)$ then they cross orthogonally. Thus we classify totally real circles according as $k_\gamma > \sqrt{|c|}/2$, $k_\gamma = \sqrt{|c|}/2$ and $0 \leq k_\gamma < \sqrt{|c|}/2$. We consider a minimal ruled real hypersurface M which corresponds to a totally real circle γ. Corresponding to the classification of totally real circles, we call this ruled real hypersurface M of elliptic type, parabolic type and axial type according as the curvature of γ satisfies $k_\gamma > \sqrt{|c|}/2$, $k_\gamma = \sqrt{|c|}/2$ and $0 \leq k_\gamma < \sqrt{|c|}/2$. Since an open subset of a ruled real hypersurface is also called a ruled real hypersurface, when we say minimal ruled real hypersurfaces of axial type, of parabolic type and of elliptic type, we assume that they are not contained properly into other minimal ruled real hypersurfaces. Our goal of this section is to show that there are essentially three congruence classes of minimal ruled real hypersurfaces in $\mathbb{C}H^n(c)$.

Theorem 1 *Minimal ruled real hypersurfaces in a complex hyperbolic space $\mathbb{C}H^n(c)$ satisfy the following properties.*

(1) *Minimal ruled real hypersurfaces of axial type (resp. of parabolic type, of elliptic type) are congruent to each other with respect to the action of the isometry group of $\mathbb{C}H^n(c)$.*
(2) *If two minimal ruled real hypersurfaces are not of the same type, they are not congruent to each other.*
(3) *Minimal ruled real hypersurfaces of parabolic type and axial type are complete, but not ruled real hypersurfaces of elliptic type.*
(4) *Every minimal ruled real hypersurface is congruent to an open subset of one of minimal ruled real hypersurfaces of axial type, of parabolic type and elliptic type.*

We are enough to study in $\mathbb{C}H^n(-4)$. Let $\varpi : H_1^{2n+1} \to \mathbb{C}H^n(-4)$ be a canonical fibration of an anti-de Sitter space $H_1^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid \langle\langle z, z\rangle\rangle = -1\}$. Here, the Hermitian product $\langle\langle\ ,\ \rangle\rangle$ on $\mathbb{C}^{n+1}$ is given as $\langle\langle z, w\rangle\rangle = -z_0\bar{w}_0 + z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$ for $z = (z_0, \ldots, z_n)$, $w = (w_0, \ldots, w_n) \in \mathbb{C}^{n+1}$.

We first study axial minimal ruled real hypersurfaces. We take a point $z_A(0) = (1, 0, \ldots, 0) \in H_1^{2n+1} \subset \mathbb{C}^{n+1}$ and a horizontal vector

$$u_A(0) = \big(z_A(0), (0, 1, 0, \ldots, 0)\big) \in T_{z_A(0)}H_1^{2n+1} \subset T_{z_A(0)}\mathbb{C}^{n+1} \cong \mathbb{C}^{n+1},$$

and consider a geodesic γ_0 in $\mathbb{C}H^n(-4)$ with initial condition $\gamma_0(0) =$

$\varpi(z_A(0))$, $\dot\gamma_0(0) = d\varpi\big(u_A(0)\big)$. Its horizontal lift $\hat\gamma_0$ with $\hat\gamma_0(0) = z_A(0)$ is of the form $\hat\gamma_0(t) = \cosh t\, z_A(0) + \sinh t\, u_A(0)$. Here, we identify $u_A(0) \in T_{z_A(0)}H_1^{2n+1}$ with $(0,1,0,\dots,0) \in \mathbb{C}^{n+1}$. We frequently use such a convention from now on. The axial ruled real hypersurface M associated with γ_0 is expressed as $\varpi(\widehat{M})$ with

$$\widehat{M} = \left\{ e^{\sqrt{-1}\psi}\big(\cosh s\cosh t,\ \cosh s\sinh t,\ \sinh s\, w_2,\dots,\sinh s\, w_n\big) \in H_1^{2n+1} \;\middle|\; \begin{array}{l} w=(w_2,\dots,w_n)\in\mathbb{C}^{n-1},\ \sum_{j=2}^n |w_j|^2 = 1, \\ s,t\in\mathbb{R}, 0\le\psi<2\pi \end{array} \right\} \subset \mathbb{C}^{n+1}.$$

By this expression, it is clear that this ruled real hypersurface is complete.

For κ with $-1<\kappa<1$ and $\kappa\neq 0$, by choosing a point

$$z_A(\kappa) = \big(1/\sqrt{1-\kappa^2}, 0, \kappa/\sqrt{1-\kappa^2}, 0,\dots,0\big) \in H_1^{2n+1} \subset \mathbb{C}^{n+1}$$

and horizontal orthonormal vectors

$$\begin{aligned} u_A(\kappa) &= \big(z_A(\kappa), (0,1,0,\dots,0)\big), \\ v_A(\kappa) &= \big(z_A(\kappa), (-\kappa/\sqrt{1-\kappa^2}, 0, -1/\sqrt{1-\kappa^2}, 0,\dots,0)\big) \in T_{z_A(\kappa)}H_1^{2n+1}, \end{aligned}$$

we take a totally real circle γ_κ of curvature $|\kappa|$ with initial condition

$$\gamma_\kappa(0) = \varpi\big(z_A(\kappa)\big), \quad \dot\gamma_\kappa(0) = d\varpi\big(u_A(\kappa)\big), \quad \nabla_{\dot\gamma_\kappa}\dot\gamma_\kappa(0) = |\kappa| d\varpi\big(v_A(\kappa)\big).$$

According to [1] we find that its horizontal lift $\hat\gamma_\kappa$ with $\hat\gamma_\kappa(0) = z(\kappa)$ is represented as

$$\begin{aligned} \hat\gamma_\kappa(t) &= \frac{1}{1-\kappa^2}\big(\cosh\sqrt{1-\kappa^2}\,t - \kappa^2\big) z_A(\kappa) + \frac{1}{\sqrt{1-\kappa^2}}\sinh\sqrt{1-\kappa^2}\,t\, u_A(\kappa) \\ &\quad + \frac{\kappa}{1-\kappa^2}\big(\cosh\sqrt{1-\kappa^2}\,t - 1\big) v_A(\kappa). \\ &= \left(\frac{1}{\sqrt{1-\kappa^2}}\cosh\sqrt{1-\kappa^2}\,t,\ \frac{1}{\sqrt{1-\kappa^2}}\sinh\sqrt{1-\kappa^2}\,t,\ \frac{\kappa}{\sqrt{1-\kappa^2}}, 0,\dots,0\right). \end{aligned}$$

We shall show that the axial minimal ruled real hypersurface associated with γ_κ coincides with the axial ruled real hypersurface M associated with

γ_0. We take a leaf M_T of the axial ruled real hypersurface M which is attached to γ_0 at $\gamma_0(T)$. By the expression of M we see $\widehat{M_T} = \varpi^{-1}(M_T)$ is given by

$$\widehat{M_T} = \Big\{ e^{\sqrt{-1}\psi}\big(\cosh s \cosh T,\ \cosh s \sinh T,\ \sinh s\, w_2, \dots, \sinh s\, w_n\big) \in \widehat{M} \\ \Big|\ w \in \mathbb{C}^{n-1},\ \|w\| = 1,\ s \in \mathbb{R},\ 0 \le \psi < 2\pi \Big\}.$$

It is a totally geodesic $\mathbb{C}H^{n-1}(-4)$ in $\mathbb{C}H^n(-4)$. Moreover, two distinct leaves do not meat each other. It is clear that the horizontal lift $\hat{\gamma}_\kappa$ of the circle γ_κ crosses to $\widehat{M_T}$ at $\hat{\gamma}_\kappa\big(T/\sqrt{1-\kappa^2}\,\big)$. Since we have

$$\dot{\hat{\gamma}}_\kappa(t) = \big(\hat{\gamma}_\kappa(t), (\sinh \sqrt{1-\kappa^2}\, t, \cosh \sqrt{1-\kappa^2}\, t, 0, \dots, 0)\big),$$

we find γ_κ and M_T cross Hermitian orthogonally at $\gamma_\kappa(T/\sqrt{1-\kappa^2})$. Here, we shall say a curve σ on $\mathbb{C}H^n$ crosses Hermitian orthogonally to a submanifold N in $\mathbb{C}H^n$ at $\sigma(t_0)$ if the complex line spanned by $\dot{\sigma}(t_0)$ and the tangent space $T_{\sigma(t_0)}N$ cross orthogonally. Thus, we find M_T coincides with the leaf of the ruled real hypersurface associated with γ_κ which is attached at $\gamma_\kappa(T/\sqrt{1-\kappa^2})$. Therefore we can conclude that M coincides with the axial ruled real hypersurface associated with γ_κ.

If we make mention of the function ν of this manifold, we can say that $\nu(\sigma(s)) = \tanh s$ on an integral curve σ of ϕU with $\sigma(0) = \gamma_0(T)$, which is given by $\sigma(s) = \varpi((\cosh s \cosh T, \cosh s \sinh T, \sinh s, 0, \dots, 0))$.

Next we study elliptic minimal ruled real hypersurfaces. Given a constant κ with $|\kappa| > 1$, we take a point $z_E(\kappa) = (\kappa/\sqrt{\kappa^2-1}, 0, 1/\sqrt{\kappa^2-1}, 0, \dots, 0) \in H_1^{2n+1} \subset \mathbb{C}^{n+1}$ and horizontal orthonormal vectors

$$u_E(\kappa) = \big(z_E(\kappa), (0,1,0,\dots,0)\big),$$
$$v_E(\kappa) = \big(z_E(\kappa), (-1/\sqrt{\kappa^2-1}, 0, -\kappa/\sqrt{\kappa^2-1}, 0, \dots, 0)\big) \in T_{z_E(\kappa)} H_1^{2n+1}.$$

We consider a totally real circle γ_κ on $\mathbb{C}H^n(-4)$ with initial condition

$$\gamma_\kappa(0) = \varpi(z_E(\kappa)), \quad \dot{\gamma}_\kappa(0) = d\varpi(u_E(\kappa)), \quad \nabla_{\dot{\gamma}_\kappa}\dot{\gamma}_\kappa(0) = |\kappa| d\varpi(v_E(\kappa)).$$

It is a closed curve of length $2\pi/\sqrt{\kappa^2-1}$. Its horizontal lift $\hat{\gamma}_\kappa$ with $\hat{\gamma}_\kappa(0) = z_E(\kappa)$ is of the form

$$\begin{aligned}\hat{\gamma}_\kappa(t) &= \frac{1}{\kappa^2-1}\big(\kappa^2 - \cos\sqrt{\kappa^2-1}\,t\big)z_E(\kappa) + \frac{1}{\sqrt{\kappa^2-1}}\sin\sqrt{\kappa^2-1}\,t\; u_E(\kappa)\\ &\quad + \frac{\kappa}{\kappa^2-1}\big(1-\cos\sqrt{\kappa^2-1}\,t\big)v_E(\kappa)\\ &= \bigg(\frac{\kappa}{\sqrt{\kappa^2-1}}, \frac{1}{\sqrt{\kappa^2-1}}\sin\sqrt{\kappa^2-1}\,t, \frac{1}{\sqrt{\kappa^2-1}}\cos\sqrt{\kappa^2-1}\,t, 0,\dots,0\bigg).\end{aligned}$$

We here choose an arbitrary κ_0 with $\kappa_0 > 1$. The elliptic minimal ruled real hypersurface M associated with γ_{κ_0} is expressed as $\varpi(\widehat{M})$ with

$$\widehat{M} = \left\{ e^{\sqrt{-1}\psi}\big(\cosh s, \sinh s\sin\theta w_2, \sinh s\cos\theta w_2, \sinh s w_3,\dots,\sinh s w_n\big) \in H_1^{2n+1} \;\middle|\; \begin{array}{l} w=(w_2,\dots,w_n)\in\mathbb{C}^{n-1},\ \sum_{j=2}^n |w_j|^2 = 1,\\ 0<s<\infty,\ 0\le\theta,\psi<2\pi \end{array} \right\} \subset \mathbb{C}^{n+1}.$$

We shall show that the elliptic minimal ruled real hypersurface associated with γ_κ satisfying $|\kappa| > 1$ coincides with this elliptic minimal ruled real hypersurface M. We take a leaf M_Θ which is attached to γ_{κ_0} at $\gamma_{\kappa_0}(\Theta/\sqrt{\kappa_0^2-1})$. By the expression of M we see $\widehat{M_\Theta} = \varpi^{-1}(M_\Theta)$ is given by

$$\widehat{M_\Theta} = \left\{ e^{\sqrt{-1}\psi}\big(\cosh s, \sinh s\sin\Theta w_2, \sinh s\cos\Theta w_2, \sinh s w_3,\dots,\sinh s w_n\big) \in \widehat{M} \;\middle|\; \|w\|=1,\ 0<s<\infty,\ 0\le\psi<2\pi \right\}.$$

Every leaf M_Θ is an open subset of a totally geodesic $\mathbb{C}H^{n-1}$. It is clear that the horizontal lift $\hat{\gamma}_\kappa$ of a totally real circle γ_κ ($|\kappa|>1$) crosses to this $\widehat{M_\Theta}$ at $\hat{\gamma}_\kappa(\Theta/\sqrt{\kappa^2-1})$. Since we have

$$\dot{\hat{\gamma}}_\kappa(t) = \big(\hat{\gamma}_\kappa(t), (0, \cos\sqrt{\kappa^2-1}\,t, -\sin\sqrt{\kappa^2-1}\,t, 0,\dots,0)\big),$$

we find γ_κ and M_Θ cross Hermitian orthogonally at $\gamma_\kappa(\Theta/\sqrt{\kappa^2-1})$. Thus, we can conclude that M is the elliptic ruled real hypersurface associated with γ_κ.

For the function ν we find that $\nu(\sigma(s)) = \coth s$ on an integral curve σ :

$(0,\infty) \to M$ of ϕU with $\sigma((1/2)\log|(\kappa_0+1)/(\kappa_0-1)|) = \gamma_{\kappa_0}\big(\Theta/\sqrt{\kappa_0^2-1}\big)$, which is given by $\sigma(s) = \varpi((\cosh s, \sinh s \sin\Theta, \sinh s \cos\Theta, 0, \dots, 0))$. As we see $\lim_{s\downarrow 0}\nu(s) = \infty$, we find that the geodesic σ can not be extended. Therefore this elliptic minimal ruled real hypersurface is not complete.

We finally study parabolic minimal ruled hypersurfaces. We here write down the parabolic minimal ruled hypersurface M. Take a point $z_H = (1, 0, \dots, 0) \in H_1^{2n+1} \subset \mathbb{C}^{n+1}$ and horizontal orthonormal vectors

$$u_H = \big(z_H, (0,1,0,\dots,0)\big), \quad v_H = \big(z_H, (0,0,-1,0,\dots,0)\big) \in T_{z_H}H_1^{2n+1}.$$

We consider a totally real circle γ_1 with initial condition

$$\dot\gamma_1(0) = \varpi(z_H), \quad \dot\gamma_1(0) = d\varpi(u_H), \quad \nabla_{\dot\gamma_1}\dot\gamma_1(0) = d\varpi(v_H).$$

Its horizontal lift $\hat\gamma_1$ with $\hat\gamma_1(0) = z_H$ is of the form

$$\hat\gamma_1(t) = \frac{1}{2}(t^2+2)z_H + tu_H + \frac{t^2}{2}v_H = \left(\frac{t^2+2}{2}, t, -\frac{t^2}{2}, 0, \dots, 0\right).$$

Hence the parabolic minimal ruled real hypersurface M is expressed as $\varpi(\widehat{M})$ with

$$\widehat{M} = \left\{ e^{\sqrt{-1}\psi}\left(\frac{t^2+2}{2}\cosh s - \frac{t^2}{2}\sinh s w_2, t\cosh s - t\sinh s w_2, \right.\right.$$
$$\left. -\frac{t^2}{2}\cosh s + \frac{t^2-2}{2}\sinh s w_2, \sinh s w_3, \dots, \sinh s w_n\right) \in H_1^{2n+1}$$
$$\left.\left|\ \begin{array}{l} w = (w_2, \dots, w_n) \in \mathbb{C}^{n-1},\ \|w\| = 1, \\ s, t \in \mathbb{R},\ 0 \le \psi < 2\pi \end{array}\right.\right\} \subset \mathbb{C}^{n+1}.$$

It is clear that this parabolic ruled real hypersurface is complete. For the function ν we have $\nu(\sigma(s)) \equiv 1$ on an integral curve σ of ϕU with $\sigma(0) = \gamma_1(T)$, which is give by

$$\sigma(s) = \varpi\big(((T^2/2)(\cosh s - \sinh s) + \cosh s, T(\cosh s - \sinh s),$$
$$- (T^2/2)(\cosh s - \sinh s) - \sinh s, 0, \dots, 0)\big).$$

It is clear that minimal ruled real hypersurfaces we give above are "maximal" in the sense that they are not properly contained in other ruled real hypersurfaces. Since two totally real circles on $\mathbb{C}H^n(4)$ are congruent to each other if and only if they have the same curvatures, with above argument, we can conclude that there are 3 and just 3 congruence classes of "maximal" minimal ruled real hypersurfaces in $\mathbb{C}H^n$; ruled real hypersurfaces of axial type, parabolic type and elliptic type. We can also conclude that every ruled minimal real hypersurface is congruent an open subset of one of these minimal ruled real hypersurfaces.

For the sake of readers' convenience we here give images of ruled real hypersurfaces of axial, parabolic and elliptic types in a ball model $D^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\}$ of a complex hyperbolic space $\mathbb{C}H^n$. In figures, dotted lines show leaves of ruled real hypersurfaces and lines inside of the balls show totally real circles on a totally geodesic $\mathbb{R}H^2$, which are denoted by γ_κ in the above. If we regard these figures as models of $\mathbb{R}H^2$ containing the totally real circles, the dotted lines show integral curves of ϕU. We note that all the totally real circles have the same pairs of points at infinity as we can see in Figure 1. In Figure 2, two lines show horocyclic totally real circles having the same points at infinity. One may easily guess that only parabolic one is homogeneous. In this case, our minimal ruled real hypersurface is expressed as an orbit under the action of the direct product of the isometry group $\mathrm{Iso}(\mathbb{C}H^{n-1})$ of totally geodesic $\mathbb{C}H^{n-1}(c)$ and a one-parameter subgroup $\{\varphi_s\}_{s\in\mathbb{R}}$ generating a horocycle in $\mathbb{C}H^n(c)$.

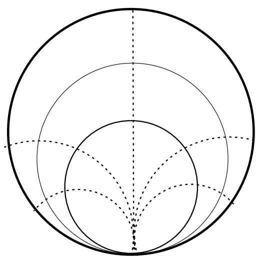

Figure 1. axial type.

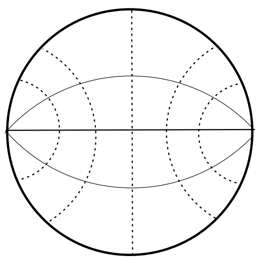

Figure 2. parabolic type.

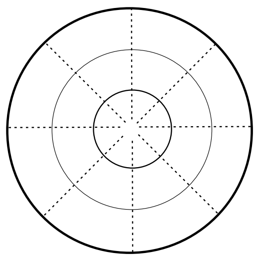

Figure 3. elliptic type.

4. Ruled real hypersurfaces in $\mathbb{C}P^n$

Next we study minimal ruled real hypersurfaces in a complex projective space $\mathbb{C}P^n$. On $\mathbb{C}P^n(c)$ every totally real circle γ is closed and is of length

$4\pi/\sqrt{4k_\gamma^2+c}$. We consider minimal ruled real hypersurfaces associated with totally real circles which are "maximal" in the sense that they are not properly contained in other minimal ruled real hypersurfaces. We shall show the following.

Theorem 2 *In a complex projective space $\mathbb{C}P^n(c)$, all "maximal" minimal ruled real hypersurfaces are not complete and they are congruent to each other with respect to the action of its isometry group.*

Remark 1 Theorem 2 shows that every minimal ruled real hypersurface is congruent to an open subset of the "maximal" minimal ruled real hypersurface associated with a geodesic.

In order to show Theorem 2 we are enough to study the case $c=4$. Let $\varpi : S^{2n+1}\big(\subset \mathbb{C}^{n+1}\big) \to \mathbb{C}P^n(4)$ be a Hopf fibration. Given a constant κ, we take a point

$$z_E(\kappa) = \big(1/\sqrt{\kappa^2+1}, 0, \kappa/\sqrt{\kappa^2+1}, 0, \dots, 0\big) \in S^{2n+1} \subset \mathbb{C}^{n+1}$$

and horizontal orthonormal vectors

$$\begin{aligned} u_E(\kappa) &= \big(z_E(\kappa), (0,1,0,\dots,0)\big),\\ v_E(\kappa) &= \big(z_E(\kappa), (-\kappa/\sqrt{\kappa^2+1}, 0, 1/\sqrt{\kappa^2+1}, 0, \dots, 0)\big) \in T_{z_E(\kappa)}S^{2n+1}. \end{aligned}$$

We consider a totally real circle γ_κ on $\mathbb{C}P^n(4)$ with initial condition

$$\gamma_\kappa(0) = \varpi(z_E(\kappa)), \quad \dot\gamma_\kappa(0) = d\varpi(u_E(\kappa)), \quad \nabla_{\dot\gamma_\kappa}\dot\gamma_\kappa(0) = |\kappa| d\varpi(v_E(\kappa)).$$

It is a closed curve of length $2\pi/\sqrt{\kappa^2+1}$. Its horizontal lift $\hat\gamma_\kappa$ with $\hat\gamma_\kappa(0) = z_E(\kappa)$ is of the form

$$\begin{aligned} \hat\gamma_\kappa(t) &= \frac{1}{\kappa^2+1}\big(\kappa^2 + \cos\sqrt{\kappa^2+1}\,t\big) z_E(\kappa) + \frac{1}{\sqrt{\kappa^2+1}} \sin\sqrt{\kappa^2+1}\,t\; u_E(\kappa)\\ &\quad + \frac{\kappa}{\kappa^2+1}\big(1-\cos\sqrt{\kappa^2+1}\,t\big) v_E(\kappa)\\ &= \left(\frac{1}{\sqrt{\kappa^2+1}}\cos\sqrt{\kappa^2+1}\,t, \frac{1}{\sqrt{\kappa^2+1}}\sin\sqrt{\kappa^2+1}\,t, \frac{\kappa}{\sqrt{\kappa^2+1}}, 0, \dots, 0\right). \end{aligned}$$

The minimal ruled real hypersurface M associated with a geodesic γ_0 is expressed as $\varpi(\widehat{M})$ with

$$\widehat{M} = \Big\{ e^{\sqrt{-1}\psi}\big(\cos s\cos t, \cos s\sin t, \sin s w_2, \ldots, \sin s w_n\big) \in S^{2n+1} \;\Big|\; \begin{array}{l} w = (w_2, \ldots, w_n) \in \mathbb{C}^{n-1},\ \|w\| = 1, \\ -\pi/2 < s < \pi/2,\ 0 \le t, \psi < 2\pi \end{array} \Big\} \subset \mathbb{C}^{n+1}.$$

We shall show that the minimal ruled real hypersurface associated with γ_κ coincides with M. We take a leaf M_T of M which is attached to γ_0 at $\gamma_0(T)$. The horizontal lift $\hat{\gamma}_\kappa$ of a circle γ_κ crosses to $\widehat{M}_T$ at $\hat{\gamma}_\kappa(T/\sqrt{\kappa^2+1})$. Since we have

$$\dot{\hat{\gamma}}_\kappa(t) = \big(\hat{\gamma}_\kappa(t), (-\sin\sqrt{\kappa^2+1}\,t, \cos\sqrt{\kappa^2+1}\,t, 0, \ldots, 0)\big),$$

we find γ_κ and M_T cross Hermitian orthogonally at $\gamma_\kappa(T/\sqrt{\kappa^2+1})$. Thus, we can conclude that M is the ruled real hypersurface associated with γ_κ. Since two totally real circles on $\mathbb{C}P^n(4)$ are congruent to each other if and only if they have the same curvatures, we find minimal ruled real hypersurfaces in $\mathbb{C}P^n(4)$ are congruent to each other.

For the function ν of this real hypersurface we have $\nu\big(\sigma(s)\big) = \tan s$ on an integral curve σ of ϕU with $\sigma(0) = \gamma_0(T)$, which is given as $\sigma(s) = \varpi((\cos s\cos T, \cos s\sin T, \sin s, 0, \ldots, 0))$. Since $\lim_{s\uparrow\pi/2}\nu(\sigma(s)) = \infty$, we find that the geodesic σ can not be extended. Therefore we see minimal ruled real hypersurfaces in $\mathbb{C}P^n$ are not complete.

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