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CONGRUENCE FOR RATIONAL POINTS OVER FINITE FIELDS AND CONIVEAU OVER LOCAL FIELDS

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ABSTRACT. If the ℓ-adic cohomology of a projective smooth variety, defined over a local field K with finite residue field k, is supported in codimension ≥ 1, then every model over the ring of integers of K has a k-rational point. For K a p-adic field, this is proved in (Esnault, 2007, Theorem 1.1). If the model $\mathcal{X}$ is regular, one has a congruence $|\mathcal{X}(k)| \equiv 1$ modulo $|k|$ for the number of k-rational points (Esnault, 2006, Theorem 1.1). The congruence is violated if one drops the regularity assumption.

1. INTRODUCTION

Let X be a projective variety defined over a local field K with finite residue field $k = \mathbb{F}_q$. Let R be the ring of integers of K. A *model* of X/K is a flat projective morphism $\mathcal{X} \to \mathrm{Spec}(R)$, with $\mathcal{X}$ an integral scheme, such that tensored with K over R, it coincides with $X \to \mathrm{Spec}(K)$. As in [9] and [10], we consider ℓ-adic cohomology $H^i(\bar{X})$ with $\mathbb{Q}_\ell$-coefficents. Recall briefly that one defines the first coniveau level

$$N^1H^i(\bar{X}) = \{\alpha \in H^i(\bar{X}), \exists \text{ divisor } D \subset X \text{ s.t. } 0 = \alpha|_{X\setminus D} \in H^i(\overline{X \setminus D})\}.$$

As $H^i(\bar{X})$ is a finite dimensional $\mathbb{Q}_\ell$-vector space, one has by localization

$$\exists D \subset X \text{ s.t. } N^1H^i(\bar{X}) = \mathrm{Im}\big(H^i_{\bar{D}}(\bar{X}) \to H^i(\bar{X})\big),$$

where $D \subset X$ is a divisor. One says that $H^i(\bar{X})$ is *supported in codimension* ≥ 1 if $N^1H^i(\bar{X}) = H^i(\bar{X})$. The purpose of this note is twofold. We show the following theorem.

Theorem 1.1. *Let X be a smooth, projective, absolutely irreducible variety defined over a local field K with finite residue field k. Assume that ℓ-adic cohomology $H^i(\bar{X})$ is supported in codimension ≥ 1 for all $i \geq 1$. Let $\mathcal{X}$ be a model of X over the ring of integers R of K. Then there is a projective surjective morphism $\sigma : \mathcal{Y} \to \mathcal{X}$ of R-schemes such that*

$$|\mathcal{Y}(k)| \equiv 1 \bmod |k|.$$

In particular, any model $\mathcal{X}/R$ of X/K has a k-rational point.

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This generalizes [10, Theorem 1.1] where the theorem is proven under the assumption that K has characteristic 0. On the other hand, assuming that $\mathcal{X}$ is regular, we showed in [9, Theorem 1.1] that the number of k-rational points $|\mathcal{X}(k)|$ is congruent to 1 modulo $|k|$. It was in fact the way to show that k-rational points exist on $\mathcal{X}$, as surely $|k|$, being a p-power, where p is the characteristic of k, is > 1. We show that if we drop the regularity assumption, there are models which, according to Theorem 1.1, have a rational point, but do not satisfy the congruence.

Theorem 1.2. *Let $X_0 = \mathbb{P}^2$ over $K_0 := \mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Then there is a finite field extension $K \supset K_0$, which can be chosen to be unramified, and there is a normal model $\mathcal{X}/R$ of $X := X_0 \otimes_{K_0} K$, such that $|\mathcal{X}(k)|$ is not congruent to 1 modulo $|k|$.*

The ℓ-adic proof of Theorem 1.1 closely follows the one of unequal characteristic in [10, Theorem 1.1], and, in addition to Deligne's integrality theorem [7, Corollaire 5.5.3] and [9, Appendix] and purity [11], relies strongly on de Jong's alteration theorem as expressed in [6]. However, we have to replace the trace argument we used there by a more careful analysis of the Leray spectral sequence stemming from de Jong's construction. The construction of the examples in Theorem 1.2 uses Artin's contraction theorem as expressed in [1] and is somewhat inspired by Kollár's construction exposed in [4, Section 3.3].

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Let K be a local field with finite residue field k. Let $R \subset K$ be its valuation ring. Let $\mathcal{X} \to \operatorname{Spec} R$ be a model of a projective variety $X \to \operatorname{Spec} K$. We do not assume here that X is absolutely irreducible, nor do we assume that X/K is smooth. Then by [6, Corollary 5.15], there is a diagram

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{\pi} & \mathcal{Y} & \xrightarrow{\sigma} & \mathcal{X} \\ & \searrow & & \searrow & \downarrow \\ & & & & \operatorname{Spec} R \end{array} \tag{2.1}$$

and a finite group G acting on $\mathcal{Z}$ over $\mathcal{Y}$ with the properties

(i) $\mathcal{Z} \to \operatorname{Spec} R$ and $\mathcal{Y} \to \operatorname{Spec} R$ are flat,
(ii) σ is projective, surjective, $K(\mathcal{X}) \subset K(\mathcal{Y})$ is a purely inseparable field extension,
(iii) $\mathcal{Y}$ is the quotient of $\mathcal{Z}$ by G,
(iv) $\mathcal{Z}$ is regular.

We want to show that this $\mathcal{Y}$ is a model satisfying the congruence $|\mathcal{Y}(k)| \equiv 1$ modulo $|k|$ of Theorem 1.1. Let us set

$$Y = \mathcal{Y} \otimes K, \ Z = \mathcal{Z} \otimes K.$$

The only difference from [10, (2.1)] is that $K(\mathcal{X}) \subset K(\mathcal{Y})$ may be a purely inseparable extension rather than an isomorphism. Thus, the argument there breaks down, as one does not have traces as in [10, (2.3), (2.4)]. We do not have [10, (2.5)] a priori, and we cannot conclude [10, Claim 2.1].

Let us overtake the notations of *loc. cit.*: we endow all schemes considered (which are R-schemes) with the upper subscript ^{u} to indicate the base change $\otimes_R R^u$ or $\otimes_K K^u$, where $K^u \supset K$ is the maximal unramified extension and $R^u \supset R$

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is the normalization of R in K^u. Likewise, we write $\overline{?}$ to indicate the base change $\otimes_R \bar{R}$, $\otimes_K \bar{K}$, $\otimes_k \bar{k}$, where $\bar{K} \supset K$, $\bar{k} \supset k$ are the algebraic closures and $\bar{R} \supset R$ is the normalization of R in $\bar{K}$. We consider as in [9, (2.1)] the F-equivariant exact sequence ([8, 3.6(6)])

$$\dots \to H^i_{\bar{B}}(\mathcal{Y}^u) \xrightarrow{\iota} H^i(\bar{B}) = H^i(\mathcal{Y}^u) \xrightarrow{sp^u} H^i(Y^u) \to \dots, \tag{2.2}$$

where $F \in \mathrm{Gal}(\bar{k}/k)$ is the geometric Frobenius, and $B = \mathcal{Y} \otimes k$. We have [10, Claim 2.2] unchanged:

Claim 2.1. The eigenvalues of the geometric Frobenius $F \in \mathrm{Gal}(\bar{k}/k)$ acting on $H^i_{\bar{B}}(\mathcal{Y}^u)$, thus a fortiori on $\iota(H^i_{\bar{B}}(\mathcal{Y}^u)) \subset H^i(\bar{B})$, lie in $q \cdot \bar{\mathbb{Z}}$ for all $i \geq 1$.

Proof. For sake of completeness, we reproduce the proof of [9, Theorem 2.2], which is itself derived from [10, Claim 2.2]. By (iii), one has $H^i_{\bar{B}}(\mathcal{Y}^u) = H^i_{\bar{C}}(\mathcal{Z}^u)^G \subset H^i_{\bar{C}}(\mathcal{Z}^u)$, where $C = \pi^{-1}(B)$. By (iv), $\mathcal{Z}$ is regular. Thus $\mathcal{Z}^u$, being the base change of $\mathcal{Z}$ by the unramified extension $R^u \supset R$, is regular as well. So it is enough to show that the eigenvalues of F acting on $H^i_{\bar{C}}(\mathcal{Z}^u)$ lie in $q \cdot \bar{\mathbb{Z}}$ for all $i \geq 1$, where now the scheme $\mathcal{Z}^u$ is regular and C has codimension ≥ 1. Let $C^0 \subset C$ be the smooth locus of C, let $C^1 \subset C \setminus C^0$ be the smooth locus of C^0, etc. Then $\bar{C}^i$ is smooth. Using localization

$$\dots \to H^i_{\bar{C}^1}(\mathcal{Z}^u) \to H^i_{\bar{C}}(\mathcal{Z}^u) \to H^i_{\bar{C}^0}(\mathcal{Z}^u \setminus \bar{C}^1) \to \dots$$

and purity $H^{i-2}(\bar{C}^0)(-1) \cong H^i_{\bar{C}^0}(\mathcal{Z}^u \setminus \bar{C}^1)$ ([11, Theorem 2.1.1]), etc., one reduces the problem to integrality of the eigenvalues of F acting on $H^j(\bar{D})$ for any smooth variety D defined over k and any $j \geq 1$. One then applies Deligne's integrality theorem [7, Lemme 5.5.3 iii)] and duality on D or directly [9, Appendix, Corollary 0.4]. □

So the problem is to show that the eigenvalues of F acting on $\mathrm{Im}(sp^u) \subset H^i(Y^u)$ lie in $q \cdot \bar{\mathbb{Z}}$ as well. One has the following claim.

Claim 2.2. The eigenvalues of the geometric Frobenius $F \in \mathrm{Gal}(\bar{k}/k)$ acting on $H^i(Y^u)$, and therefore on $\mathrm{Im}(sp^u) \subset H^i(Y^u)$, lie in $q \cdot \bar{\mathbb{Z}}$ for all $i \geq 1$.

Proof. Let us decompose the morphism σ as

$$\sigma : Y \xrightarrow{\tau} X_1 \xrightarrow{\epsilon} X, \tag{2.3}$$

where X_1 is the normalization of X in $K(Y)$. Thus in particular, τ is birational, and ϵ is finite and purely inseparable. Let us denote by $U \subset X$ a non-empty open set such that $\tau|_{\epsilon^{-1}(U)} : \tau^{-1}\epsilon^{-1}(U) \to \epsilon^{-1}(U)$ is an isomorphism, and let us set $D := X \setminus U$. We define

$$\mathcal{C} := \mathrm{cone}(\mathbb{Q}_\ell \to R\tau_* \mathbb{Q}_\ell)[-1] \tag{2.4}$$

as an object in the bounded derived category of $\mathbb{Q}_\ell$-constructible sheaves on X_1. Since $\tau_* \mathbb{Q}_\ell = \mathbb{Q}_\ell$, the cohomology sheaves of $\mathcal{C}$ are in degree ≥ 1, and have support in $D_1 := D \times_X X_1$. We conclude

$$H^i_{D_1^u}(X_1^u, \mathcal{C}) = H^i(X_1^u, \mathcal{C}) \ \forall i \geq 0. \tag{2.5}$$

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One has the commutative diagram of exact sequences

(2.6)
$$\begin{array}{ccc}
H^{i+1}_{D_1^u}(X_1^u) & & \\
\uparrow & & \\
H^i_{D_1^u}(X_1^u, \mathcal{C}) & \xrightarrow{=(2.5)} & H^i(X_1^u, \mathcal{C}) \\
\uparrow & & \uparrow \\
H^i_{E^u}(Y^u) & \longrightarrow & H^i(Y^u) \\
\uparrow & & \uparrow \\
H^i_{D_1^u}(X_1^u) & \longrightarrow & H^i(X_1^u)
\end{array}$$

where $E = \sigma^{-1}(D)$. So to show the claim, via the right vertical exact sequence, it is enough to show that the eigenvalues of F acting on $H^i(X_1^u)$ and on $H^i(X_1^u, \mathcal{C})$ lie in $q \cdot \bar{\mathbb{Z}}$. This is true on $H^i(X_1^u)$ by [9, Theorem 1.5 and Appendix]. For $H^i(X_1^u, \mathcal{C})$, via the left vertical exact sequence, it is enough to show that the eigenvalues of F acting on $H^i_{E^u}(Y^u)$ and on $H^{i+1}_{D^u}(X_1^u)$ lie in $q \cdot \bar{\mathbb{Z}}$. Writing $H^i_{E^u}(Y^u) = H^i_{L^u}(Z^u)^G$ where $L = D \times_X Z$, one is reduced to showing that the eigenvalues of F acting on $H^j_{V^u}(W^u)$ lie in $q \cdot \bar{\mathbb{Z}}$ for W a regular K-scheme and $V \subset W$ a closed K-subscheme of codimension $c \geq 1$. If V is regular, one applies purity $H^{j-2c}(V^u)(-c) \cong H^j_{V^u}(W^u)$ again, and one is reduced to showing that the eigenvalues of F acting on $H^i(V^u)$ lie in $\bar{\mathbb{Z}}$ for all $i \geq 0$. One applies [9, Appendix, Corollary 0.3]. If V is not regular, one writes the F-equivariant exact sequence $\ldots \to H^i_{(V^1)^u}(W^u) \to H^i_{V^u}(W^u) \to H^i_{(V^0)^u}((W^0)^u) \to \ldots$, where $V^0 \subset V$ is the regular locus, $W^0 = W \setminus V^1$, $V^1 = V \setminus V^0$ and one argues inductively as in the proof of Claim 2.1. $\square$

We now conclude the proof of Theorem 1.1: all the eigenvalues of F acting on $H^i(\bar{B})$ lie in $q \cdot \bar{\mathbb{Z}}$ for $i \geq 1$; thus the Grothendieck-Lefschetz trace formula applied to $H^*(\bar{B})$, together with the absolute connectedness of B, which follows from the absolute irreducibility of Y, imply the congruence. This finishes the proof of Theorem 1.1. To summarize: $\mathcal{Z}$ of course has a complicated cohomology as the covering $\mathcal{Z} \to \mathcal{Y}$ might be non-trivial, while $\mathcal{Y}$ is cohomologically the same as $\mathcal{X}$ and is nearly regular as a quotient of $\mathcal{Z}$.

3. Construction of examples

This section is devoted to the proof of Theorem 1.2.

Let us first recall that if E is a smooth genus 1 curve over a finite field $\mathbb{F}_q$, it is always an elliptic curve, which means that it always carries a $\mathbb{F}_q$-rational point. Furthermore one has

Claim 3.1. Given an elliptic curve $E/\mathbb{F}_q$, there is a finite field extension $\mathbb{F}_{q^n} \supset \mathbb{F}_q$ such that $|E(\mathbb{F}_{q^n})|$ is not congruent to 1 modulo q^n.

Proof. By the trace formula, $|E(\mathbb{F}_{q^n})|$ being congruent to 1 modulo q^n for all $n \geq 1$ is equivalent to saying that the eigenvalues of F^n acting on $H^i(\bar{E})$ lie in $q^n \cdot \bar{\mathbb{Z}}$ for all $n \geq 1$ and $i \geq 1$. By purity (which in dimension 1 is Weil's theorem), this is equivalent to saying that the eigenvalues of F^n acting on $H^1(\bar{E})$ lie in $q^n \cdot \bar{\mathbb{Z}}$ for

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all $n \geq 1$. On the other hand, by duality, if λ is an eigenvalue, then $\frac{q^n}{\lambda}$ is also an eigenvalue. It is then impossible that both λ and $\frac{q^n}{\lambda}$ be q^n-divisible as algebraic integers. □

We now construct the following scheme. Let us set $\mathcal{P}_0 := \mathbb{P}^2$ over $R_0 := \mathbb{Z}_p$ or over $\mathbb{F}_p[[t]]$. Choose an elliptic curve $E_0 \subset \mathcal{P}_0 \otimes \mathbb{F}_p = \mathbb{P}^2_{\mathbb{F}_p}$ defined over $\mathbb{F}_p$. Let $k \supset \mathbb{F}_p$ be a finite field extension such that $|E_0(k)|$ is not k-divisible (Claim 3.1). Set $E := E_0 \otimes_{\mathbb{F}_p} k$, $\mathcal{P} := \mathcal{P}_0 \otimes_{R_0} R$, with $R = W(k)$ or $\mathbb{F}_q[[t]]$, and $K = \mathrm{Frac}(R)$. Choose a smooth projective curve $\mathcal{C} \subset \mathcal{P}$ over R, of degree ≥ 4, such that $C := \mathcal{C} \otimes k$ is transversal to E. Define $\Sigma = E \cap C \subset E$ to be the 0-dimensional intersection subscheme. It has degree ≥ 12, thus in particular > 9. Let $b : \mathcal{Y} \to \mathcal{P}$ be the blow up of $\Sigma \subset \mathcal{P}$. We denote by P_Σ the exceptional locus (which is a trivial $\mathbb{P}^2$ bundle over Σ), by Y the strict transform of $\mathbb{P}^2_k$, and we still denote by $E \subset Y$ the strict transform of the elliptic curve. So one has the following diagram:

$$\begin{array}{ccccc} P_\Sigma & \longrightarrow & Y \cup P_\Sigma & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma & \longrightarrow & \mathbb{P}^2_k & \longrightarrow & \mathcal{P} \end{array}$$

Then the conormal bundle $N^\vee_{E/\mathcal{Y}}$ of E in $\mathcal{Y}$ is an extension of the conormal bundle $N^\vee_{E/Y}$ of E in Y by the restriction to E of the conormal bundle $N^\vee_{Y/\mathcal{Y}}$ of Y in $\mathcal{Y}$, both ample line bundles on E by the condition on the degree of Σ.

Let $I \subset \mathcal{O}_\mathcal{Y}$ be the ideal sheaf of E. For a coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$, we denote by $I^n/I^{n+1} \cdot \mathcal{F}$ the image of $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}$ in $\mathcal{F}$, where $n \in \mathbb{N}$.

Claim 3.2. For every coherent sheaf $\mathcal{F}$ on $\mathcal{Y}$, one has $H^1(E, I^n/I^{n+1} \cdot \mathcal{F}) = 0$ for all $n \in \mathbb{N}$ large enough.

Proof. Since by definition one has a surjection $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F} \to I^n/I^{n+1} \cdot \mathcal{F}$, it is enough to show $H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}) = 0$ for n large enough. As I^n/I^{n+1} is locally free, $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}$ is an extension of $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}_0$ by $I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{T}$, where $\mathcal{T} \subset \mathcal{F}$ is the maximal torsion subsheaf and $\mathcal{F}_0 = \mathcal{F}/\mathcal{T}$ is locally free. Because $H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{T}) = 0$, we may assume that $\mathcal{F}$ is locally free. As I^n/I^{n+1} is a locally free filtered sheaf, with associated graded a sum of ample line bundles of strictly increasing degree as n grows, we have $H^1(E, \mathrm{gr}(I^n/I^{n+1}) \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}) = 0$ for n large enough, and thus $H^1(E, I^n/I^{n+1} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{F}) = 0$ as well. □

Artin's contraction criterion [1, Theorem 6.2] applied to $E \to \mathrm{Spec}(k)$, together with Artin's existence theorem [1, Theorem 3.1], show the existence of a contraction

$$a_1 : \mathcal{Y} \to \mathcal{X}_1, \tag{3.1}$$

where $\mathcal{X}_1$ is an algebraic space over R, $a_1|_{\mathcal{Y} \setminus E}$ is an isomorphism and $a_1(E) = \mathrm{Spec}(k)$. Let $\mathcal{X} \xrightarrow{\nu} \mathcal{X}_1$ be the normalization of $\mathcal{X}_1$ in $K(\mathcal{Y}) = K(\mathcal{P})$. This is a normal algebraic space over R. One has a diagram

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{a} & \mathcal{X} & \xrightarrow{\nu} & \mathcal{X}_1 \\ \downarrow b & & & & \\ \mathcal{P} & & & & \end{array} \qquad a_1 = \nu \circ a \tag{3.2}$$

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Claim 3.3. $|\mathcal{X}(k)|$ is not congruent to 1 modulo $|k|$.

Proof. Recall $a_1(E)$ is a rational point of $\mathcal{X}_1$. By [9, Theorem 1.1] (or by a simple computation in this case), $|\mathcal{Y}(k)|$ is congruent to 1 modulo $|k|$. By Claim 3.1 and the choice of E, $|\mathcal{X}_1(k)|$ is not congruent to 1 modulo $|k|$. On the other hand, as the fibers of a_1 are absolutely irreducible, ν has to be a homeomorphism. Thus $|\mathcal{X}(k)| = |\mathcal{X}_1(k)|$. This finishes the proof. □

In order to finish the proof of Theorem 1.2, it remains to show

Claim 3.4. $\mathcal{X} \to \mathrm{Spec}(R)$ is a model of $X = \mathbb{P}^2/K$.

Proof. We have to show that $\mathcal{X} \to \mathrm{Spec}(R)$ is a flat projective morphism. Since $\mathcal{X}$ is integral and $\mathrm{Spec}(R)$ is regular of dimension 1, then [12, IV Proposition 14.3.8] allows us to conclude that $\mathcal{X}/R$ is flat. Thus we just have to show that $\mathcal{X}/R$ is projective. To this aim, we want a line bundle to descend from $\mathcal{Y}$ to an ample line bundle on $\mathcal{X}$. Recall $P_\Sigma = b^{-1}(\Sigma)$. Let us define the line bundle $\mathcal{M} := b^*\mathcal{O}_{\mathcal{P}}(\mathcal{C})(-P_\Sigma)$ on $\mathcal{Y}$. By definition, one has

$$\mathcal{M}|_E \cong \mathcal{O}_E. \tag{3.3}$$

Claim 3.5. The line bundle $\mathcal{M}$ descends to $\mathcal{X}$, that is, there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ with $a^*\mathcal{L} = \mathcal{M}$.

Proof of Claim 3.5. The proper morphism of algebraic spaces $a : \mathcal{Y} \to \mathcal{X}$, with $a_*\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$, has the property that $a^{-1}a(E) = E$ set-theoretically, that $a|_{\mathcal{Y}\setminus E} : \mathcal{Y} \setminus E \to \mathcal{X} \setminus a(E)$ is an isomorphism, and that $H^1(E, I^n/I^{n+1}) = 0$ for $n \geq 1$. So Keel's theorem [13, Lemma 1.10] asserts that some positive power $\mathcal{M}^{\otimes r}$ descends to $\mathcal{X}$ if the following condition is fulfilled:

$$\forall m > 0, \exists r(m) > 0 \text{ s.t } \mathcal{M}^{\otimes r(m)}|_{E_m} \text{ descends to } a(E_m), \quad \text{where } E_m := \mathrm{Spec}(\mathcal{O}_\mathcal{Y}/I^{m+1}). \tag{3.4}$$

So we just have to check that (3.4) is fulfilled with $r = 1$ in our situation. The scheme $a(E_m)$ has Krull dimension 0. Thus by Hilbert's Theorem 90 (see, e.g. [14, Corollary 11.6]) one has

$$\mathrm{Pic}(a(E_m)) = 0. \tag{3.5}$$

We conclude that to check (3.4) is equivalent to checking that $\mathcal{M}^{\otimes r(m)}|_{E_m} \cong \mathcal{O}_{E_m}$ for some positive power $r(m)$. In fact one has

$$\mathcal{M}|_{E_m} \cong \mathcal{O}_{E_m} \ \forall m \geq 1. \tag{3.6}$$

For $m = 1$, this is (3.3). We argue by induction and assume that for $m > 1$, we have a trivializing section $s_m : \mathcal{O}_{E_m} \xrightarrow{\cong} \mathcal{M}|_{E_m}$. We want to show that it lifts to a trivializing section $s_{m+1} : \mathcal{O}_{E_{m+1}} \xrightarrow{\cong} \mathcal{M}|_{E_{m+1}}$.

One has an exact sequence

$$0 \to I^{m+1}/I^{m+2} \to \mathcal{M}|_{E_{m+1}} \to \mathcal{M}|_{E_m} \to 0. \tag{3.7}$$

Since $H^1(E, I^{m+1}/I^{m+2}) = 0$, as $m \geq 0$, the trivializing section of $s_m : \mathcal{O}_{E_m} \xrightarrow{\cong} \mathcal{M}|_{E_m}$ lifts to a section $s_{m+1} : \mathcal{O}_{E_{m+1}} \to \mathcal{M}|_{E_{m+1}}$, and likewise, its inverse t_m :

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$\mathcal{M}|_{E_m} \xrightarrow{\cong} \mathcal{O}_{E_m}$ lifts to $t_{m+1} : \mathcal{M}|_{E_{m+1}} \to \mathcal{O}_{E_{m+1}}$. The composite $t_{m+1} \circ s_{m+1} : \mathcal{O}_{E_{m+1}} \to \mathcal{O}_{E_{m+1}}$ lifts the identity of $\mathcal{O}_{E_m}$. Therefore it is invertible. This shows that s_{m+1} trivializes. The proof of Keel's theorem (see (2) after [13, (1.10.1)]) then shows that one can take $r = 1$. □

In order the finish the proof of Claim 3.4, it remains to see that $\mathcal{L}$ on $\mathcal{X}$ is ample. We first show the following claim.

Claim 3.6. $\mathcal{L}|_{\mathcal{X}\otimes k}$ is an ample line bundle on $\mathcal{X} \otimes k$.

Proof. We first show that $\mathcal{M}|_{\mathcal{Y}\otimes k}$ is nef and big. In fact, we prove a more precise property: for any irreducible curve Γ on $\mathcal{Y} \otimes k$, one has $\mathcal{M}|_{\mathcal{Y}\otimes k} \cdot \Gamma \geq 0$, and the equality holds if and only if $\Gamma = E$. By construction, $\mathcal{Y} \otimes k = P_\Sigma \cup Y$ and each component over $\bar{k}$ of P_Σ is isomorphic to $\mathbb{P}^2_{\bar{k}}$. Since the restriction of $\mathcal{M}$ on every component of P_Σ is isomorphic to $\mathcal{O}(1)$, we can assume $\Gamma \subset Y$. The embedding $E \subset Y$ is a section of the line bundle $b|_Y^*\mathcal{O}(3)(-E_\Sigma)$, where $E_\Sigma = P_\Sigma \cap Y$. There is also a large enough n, such that $H = b|_Y^*\mathcal{O}(n)(-E_\Sigma)$ is ample. So $\mathcal{M}|_Y = b|_Y^*\mathcal{O}(C)(-E_\Sigma) \equiv_{\mathbb{Q}} e_0E + e_1H$, where $0 < e_0, e_1 < 1$ and $e_0 + e_1 = 1$. From this, we easily see that $\mathcal{M}|_Y \cdot \Gamma > 0$, when $\Gamma \subset Y$ and $\Gamma \neq E$. The above argument also shows the bigness of $\mathcal{M}|_{\mathcal{Y}\otimes k}$: on P_Σ, it is ample; and on Y, it is a convex combination of an effective divisor and of an ample divisor.

Since $a^*(\mathcal{L}) = \mathcal{M}$, the nefness and bigness of $\mathcal{M}|_{\mathcal{Y}\otimes k}$ imply that the same properties hold for $\mathcal{L}|_{\mathcal{X}\otimes k}$. So $\mathcal{L}|_{\mathcal{X}\otimes k}$ is semiample by [13, Corollary 0.3]. Furthermore, the more precise property we proved above for $\mathcal{M}|_{\mathcal{Y}\otimes k}$ implies that the intersection of $\mathcal{L}|_{\mathcal{X}\otimes k}$ with any curve on $\mathcal{X} \otimes k$ is positive; thus we conclude $\mathcal{L}|_{\mathcal{X}\otimes k}$ is ample. □

So by the Serre vanishing theorem, for sufficiently large m, $H^1(\mathcal{X} \otimes k, \mathcal{L}|_{\mathcal{X}\otimes k}^{\otimes m}) = 0$. Base change implies $H^1(\mathcal{X}, \mathcal{L}^{\otimes m}) \otimes k = 0$ ([12, III Theorem 7.7.5]); thus by Nakayama's lemma, one has

$$H^1(\mathcal{X}, \mathcal{L}^{\otimes m}) = 0 \text{ for } m \text{ large enough.} \tag{3.8}$$

As $\mathcal{L}$ is invertible, multiplication $\mathcal{L}^{\otimes m} \xrightarrow{\pi} \mathcal{L}^{\otimes m}$ by the uniformizer π is injective, with quotient $\mathcal{L}|_{\mathcal{X}\otimes k}^{\otimes m}$. Thus (3.8) implies surjectivity

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes m}) \to H^0(\mathcal{X} \otimes k, \mathcal{L}|_{\mathcal{X}\otimes k}^{\otimes m})$$

for m large enough. Thus $H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$ is a free R-module, and the linear system $H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$ maps without base points $\mathcal{X}$ to $\mathbb{P}^N_R$, with $N+1 = \mathrm{rank}_R H^0(\mathcal{X}, \mathcal{L}^{\otimes m})$. As it embeds $\mathcal{X} \otimes k$, it embeds $\mathcal{X}$ as well. This finishes the proof. □

4. Remarks

Remark 4.1. In Theorem 1.1, if X/K has dimension 1, which means concretely if $X/K = \mathbb{P}^1/K$, then any normal model $\mathcal{X}/R$ satisfies the congruence $|\mathcal{X}(k)| \equiv 1$ modulo $|k|$. Thus the examples of Theorem 1.2 have the smallest possible dimension.

Proof. Indeed, using (2.1), the only thing to check is that $H^1(\bar{A})$, which is equal to $H^1(\mathcal{X}^u)$, injects via σ^* into $H^1(\bar{B}) = H^1(\mathcal{Y}^u)$. Here $A := \mathcal{X} \otimes_R k$. Let us denote

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by $\mathcal{X}'$ the normalization of $\mathcal{X}$ in $K(\mathcal{Y})$, with factorization

$$\mathcal{Y} \xrightarrow{\sigma'} \mathcal{X}' \xrightarrow{\nu} \mathcal{X}, \qquad \sigma = \nu \circ \sigma' \tag{4.1}$$

and set $A' := A \times_{\mathcal{X}} \mathcal{X}'$. Then σ' induces an isomorphism $K(\mathcal{X}') \xrightarrow{\cong} K(\mathcal{Y})$. Furthermore, $\mathcal{X}' \xrightarrow{\nu} \mathcal{X}$ and and $A' \xrightarrow{\nu|_A} A$ are homeomorphisms. Thus $H^1(\mathcal{X}^u) = H^1(\bar{A}) \xrightarrow{\nu^*} H^1((\mathcal{X}')^u) = H^1(\bar{A}')$ is an isomorphism. On the other hand, since $\sigma'_*\mathbb{Q}_\ell = \mathbb{Q}_\ell$, the Leray spectral sequence for σ' applied to $H^1(\mathcal{Y}^u)$ yields an inclusion $H^1((\mathcal{X}')^u) = H^1(\bar{A}') \xrightarrow{\text{inj}} H^1(\mathcal{Y}^u) = H^1(\bar{B})$. This finishes the proof. □

Remark 4.2. We generalize Remark 4.1 to the higher dimensional case in the following form. Let X be a smooth projective variety defined over K and let $\mathcal{X}/R$ be a model over R. Let us use the notation of (2.1). We set $A = \mathcal{X} \otimes_R k,\ B = \mathcal{Y} \otimes_R k$. If the assumptions of Theorem 1.1 are fulfilled, that is, if ℓ-adic cohomology $H^i(\bar{X})$ is supported in codimension ≥ 1 for all $i \geq 1$, and if in addition

$$\sigma^* : H^i(\mathcal{X}^u) = H^i(\bar{A}) \to H^i(\mathcal{Y}^u) = H^i(\bar{B}) \tag{4.2}$$

is injective for all $i \geq 0$, then one has

$$|\mathcal{X}(k)| \equiv 1 \text{ modulo } |k|. \tag{4.3}$$

Indeed, the exact sequence (2.2) together with Claim 2.1 and Claim 2.2 show that under the assumptions of Theorem 1.1 one has

$$\text{eigenvalues of } F \text{ acting on } H^i(\bar{B}) \in q \cdot \bar{\mathbb{Z}}\ \forall i \geq 1. \tag{4.4}$$

As σ^* in (4.2) is equivariant (which of course we already used in the proof of Theorem 1.1), we conclude

$$\text{eigenvalues of } F \text{ acting on } H^i(\bar{A}) \in q \cdot \bar{\mathbb{Z}}\ \forall i \geq 1. \tag{4.5}$$

Since $H^i(\mathcal{Y}^u) = H^i(\mathcal{Z}^u)^G \subset H^i(\mathcal{Z}^u)$, injectivity of σ^* in (4.2) is equivalent to injectivity of

$$\tau^* \circ \sigma^* : H^i(\mathcal{X}^u) \to H^i(\mathcal{Z}^u). \tag{4.6}$$

One may ask the following question:

Question 4.3. Let $\mathcal{X}$ be an integral R-scheme. What are the types of singularities of $\mathcal{X}$ which force the following: for any alteration $\pi : \mathcal{Y} \to \mathcal{X}$ in the sense of de Jong, that is, π is proper, dominant with $K(\mathcal{X}) \subset K(\mathcal{Y})$ finite, and with $\mathcal{Y}$ regular, one has that the induced map $\pi^* : H^i_c(\mathcal{X}) \to H^i_c(\mathcal{Y})$ on compactly supported ℓ-adic cohomology is injective?

P. Berthelot ([3]) observes that if π is generically étale, that is, if $K(\mathcal{X}) \subset K(\mathcal{Y})$ is separable and $\mathcal{X}$ is regular, then purity as in [11] implies immediate injectivity of π^*. Of course, from the viewpoint of point counting, since regularity of $\mathcal{X}$ is the assumption under which the main result of [9] was shown, this does not bring any new information. However, this, together with Theorem 1.2 of this note, suggests singling out a good definition of mild singularities for $\mathcal{X}$ which would force injectivity of π^*. There is the extra problem of separability of $K(\mathcal{X}) \subset K(\mathcal{Y})$. It would be nice not to have it as an assumption. Theorem 1.1 perhaps suggests that this is not the main point.

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Remark 4.4. We can lower the level of difficulty of Question 4.3 by considering varieties A defined over finite field k, or even a perfect field. In this situation, a notion of Witt-rational singularities was introduced in [4], which echoes the notion of rational singularities in characteristic zero and which relies on the slope theorem [2, Theorem 1.1] in Berthelot's rigid cohomology. Working ℓ-adically, the corresponding notion may be: let A be a variety defined over a finite field k. Then A has *ℓ-adic rational singularities* if for any alteration $\pi : B \to A$, the induced map $\pi^* : H^i_c(\bar{A}) \to H^i_c(\bar{B})$ is injective on the maximal subspace $H^i_c(\bar{A})^{<1}$ of $H^i_c(\bar{A})$, which is invariant under the geometric Frobenius F, and on which F acts with eigenvalues not in $q \cdot \bar{\mathbb{Z}}$. Such a definition will force the point counting to work as on smooth A. For example, [4, Theorem 1.1] would work similarly, with "Witt-rational singularities" replaced by *ℓ-adic rational singularities*. But somehow, this is of restricted interest: the beauty of rational singularities in chararcteristic 0 is that due to their definition via coherent cohomology, one can understand geometrically well what they are. A definition directly via étale cohomology somehow does not give such an immediate geometric picture.

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