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CONGRUENCE-FREE INVERSE SEMIGROUPS WITH ZERO

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A semigroup is said to be congruence-free if it has only two congruences, the identity congruence and the universal congruence. It is almost immediate that a congruence-free semigroup of order greater than two must either be simple or 0–simple. In this paper we describe the semilattices of congruence-free inverse semigroups with zero. Further, congruence-free inverse semigroups with zero are characterized in terms of partial isomorphisms of their semilattices. A general discussion of congruence-free inverse semigroups, with and without zero, is given by Munn (to appear).

Let $S = S^0$ be a semigroup with zero. For $x \in S$ we define $A_S(x)$ as follows:

$$A_S(x) = \{(a,b) \in S^1 \times S^1 \,|\, axb = 0\}.$$

When there is no possibility of confusion we shall write $A(x) = A_S(x)$.

DEFINITION. *A semigroup $S = S^0$ is said to be disjunctive if $A(x) = A(y)$ implies that $x = y$.*

Thus a semigroup $S = S^0$ is disjunctive if the principal congruence $\mathscr{P}_{\{0\}}$ (Clifford and Preston (1961, 1967), Chapter 10) determined by the subset $\{0\}$ of S is the identity congruence on S. The starting point of our discussion is the following result due to Schein.

THEOREM 1. (Schein (1966), Corollary 6.2.1) *A 0-simple semigroup $S = S^0$ free (h-simple in Schein's terminology) if and only if S is a disjunctive semigroup.*

THEOREM 2. *Let $S = S^0$ be a regular semigroup and E the set of idempotents of S. Further, suppose that S is disjunctive. Then $\langle E \rangle$, the semigroup generated by the idempotents of S, is disjunctive.*

PROOF. Let $x, y \in \langle E \rangle$ and suppose that $A_{\langle E \rangle}(x) = A_{\langle E \rangle}(y)$. We are required to prove $x = y$. Take $(a, b) \in A_S(x)$. Then $axb = 0$ and so $a^*axbb^* = 0$, where a^* is an inverse of a and b^* is an inverse of b. Hence $(a^*a, bb^*) \in A_{\langle E \rangle}(x)$ and so by assumption $(a^*a, bb^*) \in A_{\langle E \rangle}(y)$. Thus $a^*aybb^* = 0$ and so $aa^*aybb^*b = 0$, that is, $ayb = 0$. Hence $(a, b) \in A_S(y)$ and $A_S(x) \subseteq A_S(y)$. Similarly, $A_S(y) \subseteq A_S(x)$ and so $A_S(x) = A_S(y)$. Since S is disjunctive $x = y$ and our proof is complete.

A non-trivial group with zero ajoined provides a counterexample to the converse of Theorem 2.

Let us recall that an idempotent semigroup is a semilattice of rectangular bands (Clifford and Preston (1961), page 129 Ex. 1).

LEMMA 1. *Let $E = E^0$ be a disjunctive idempotent semigroup. Then E is commutative* i.e., *E is a semilattice.*

PROOF. Let $E = \bigcup_{j \in \Gamma} E_j$ be a decomposition of E into a semilattice Γ of rectangular bands $E_j, j \in \Gamma$. Then Γ has a least element ω and $E_\omega = \{0\}$. Since $E_\alpha E_\mu E_\beta \subseteq E_{\alpha\mu\beta}$ for $\alpha, \beta \in \Gamma$ and $E_\omega = \{0\}$ it follows that, for $a, b \in E$ and $x, y \in E_\mu$, $axb = 0$ if and only if $ayb = 0$. But E is disjunctive and so $x = y$. Hence $|E_\mu| = 1$ for all $\mu \in \Gamma$ and so E is commutative.

Evidently a semilattice $E = E^0$ is disjunctive if and only if for all $e, f \in E$, $e \neq f$, there exists $g \in E$ such that $eg = 0$ and $fg \neq 0$ OR $eg \neq 0$ and $fg = 0$. It is thus an easy matter to decide whether a semilattice is disjunctive or not.

Following Munn (1970) we say that a semigroup is fundamental if and only if the only congruence contained in Green's equivalence $\mathscr{H}$ is the identity congruence. It follows from Lallement (1966) that a regular semigroup S is fundamental if and only if the maximal idempotent-separating congruence on S is the identity congruence.

THEOREM 3. *Let $S = S^0$ be a 0–simple regular semigroup and E the set of idempotents of S. Further, suppose that S is fundamental and that $\langle E \rangle$, the semigroup generated by the idempotents of S, is disjunctive. Then S is a congruence-free semigroup.*

PROOF. Let σ be a non-identical congruence on S. Then there exist idempotents $e, f \in S$, $e \neq f$, such that $(e, f) \in \sigma$. Otherwise, σ is idempotent-separating and so the identity congruence by hypothesis, a contradiction. Since $\langle E \rangle$ is disjunctive $A_{\langle E \rangle}(e) \neq A_{\langle E \rangle}(f)$ and so we may assume without loss of generality that there exists $(a, b) \in A_{\langle E \rangle}(e)$ such that $(a, b) \notin A_{\langle E \rangle}(f)$. By compatibility we have $(aeb, afb) \in \sigma$ i.e., $(0, afb) \in \sigma$ with $afb \neq 0$. However, the congruence class containing 0 is a two sided ideal. We conclude that $0\sigma = S$, since S is 0–simple. Thus $\sigma = S \times S$ and so S is congruence-free.

We now recall some results of Munn (1970). Let $\mathscr{I}_X$ denote the symmetric inverse semigroup on a set X. We denote the domain, $X\alpha^{-1}$, and range, $X\alpha$, of an element α of $\mathscr{I}_X$ by $\Delta(\alpha)$ and $\nabla(\alpha)$ respectively. Let E be a semilattice and let T_E be the subset of $\mathscr{I}_E$ consisting of all α in $\mathscr{I}_E$ such that $\Delta(\alpha)$ and $\nabla(\alpha)$ are principal ideals of E and α is an isomorphism of $\Delta(\alpha)$ upon $\nabla(\alpha)$. T_E is an inverse subsemigroup of $\mathscr{I}_E$.

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DEFINITION. *A semilattice* $E = E^0$ *is* 0*–uniform if and only if* $Ee \cong Ef$ *for all* $e, f \in E \setminus \{0\}$.

DEFINITION. *A semilattice* $E = E^0$ *is* 0*–subuniform if and only if for all* $e, f \in E \setminus \{0\}$ *there exists* $g \in E$ *such that* $g \leqq f$ *and* $Ee \cong Eg$.

DEFINITION. *Let* $E = E^0$ *be a semilattice. An inverse subsemigroup* S *of* T_E *is called* 0*–subtransitive* [0*–transitive*] *if and only if the following two conditions are satisfied.*

(i) S *contains the zero of* T_E, *and*

(ii) *to each pair of non-zero elements* $e, f \in E$ *there corresponds* $\gamma \in S$ *such that* $Ee = \Delta(\gamma)$ *and* $\nabla(\gamma) \subseteq Ef$ [$\nabla(\gamma) = Ef$].

THEOREM 4. (Munn (1970), Theorems 3.1 and 3.2) (i) Let S *be an* 0*-simple* [0*-simple*] *inverse semigroup with semilattice* E. *Then* E *is* 0*-subuniform.* [0*-uniform*]. *Furthermore, if* S *is fundamental, then it is isomorphic to a* 0*-subtransitive* [0*-transitive*] *inverse subsemigroup of* T_E.

(ii) *Let* $E = E^0$ *be a* 0*–subuniform* [0*–uniform*] *semilattice and let* S *be a* 0*–subtransitive* [0*–transitive*] *inverse subsemigroup of* T_E. *Then* S *is a fundamental* 0*–simple* [0*–bisimple*] *inverse semigroup with semilattice isomorphic to* E.

The next theorem characterizes all congruence-free inverse semigroups with zero.

THEOREM 5. (i) *Let* S *be a congruence-free inverse semigroup* [*congruence free* 0*–bisimple inverse semigroup*] *with zero whose semilattice is* E. *Then* E *is disjunctive and* 0*–subuniform* [0*–uniform*]. *Furthermore,* S *is isomorphic to a* 0*–subtransitive* [0*–transitive*] *inverse subsemigroup of* T_E.

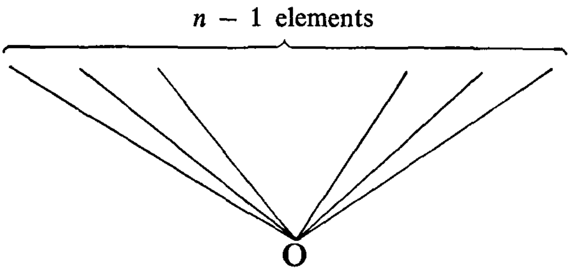
(ii) *Let* $E = E^0$ *be a disjunctive and* 0*–subuniform* [0*–uniform*] *semilattice and let* S *be a* 0*–subtransitive* [0*–transitive*] *inverse subsemigroup of* T_E. *Then* S *is a congruence-free inverse semigroup* [*congruence free* 0*–bisimple inverse semigroup*] *with semilattice isomorphic to* E.

PROOF. (i) Suppose S is a congruence-free inverse semigroup with zero. Clearly S is 0–simple and fundamental and our result follows from theorems 1, 2 and 4 (i). The alternative reading follows similarly.

(ii) Let $E = E^0$ be a disjunctive 0–subuniform semilattice and let S be a 0–subtransitive inverse semigroup of T_E. It follows from theorems 3 and 4 (ii) that S is a congruence free inverse semigroup. The alternative reading follows similarly.

To conclude we exhibit the two simplest types of disjunctive 0–uniform semilattices.

A semilattice $E = E^0$ is called an *M-semilattice* [M for matrix, see below] if $ef = 0$ for $e, f \in E$, $e \neq f$. An M-semilattice with n elements can be represented diagrammatically as follows:



It can be easily verified that a congruence-free inverse semigroup whose idempotents form an M-semilattice is isomorphic to a semigroup of matrix units (Clifford and Preston (1961), page 83, Ex. 7); conversely, every semigroup of matrix units is a congruence-free inverse semigroup whose idempotents form an M-semilattice.

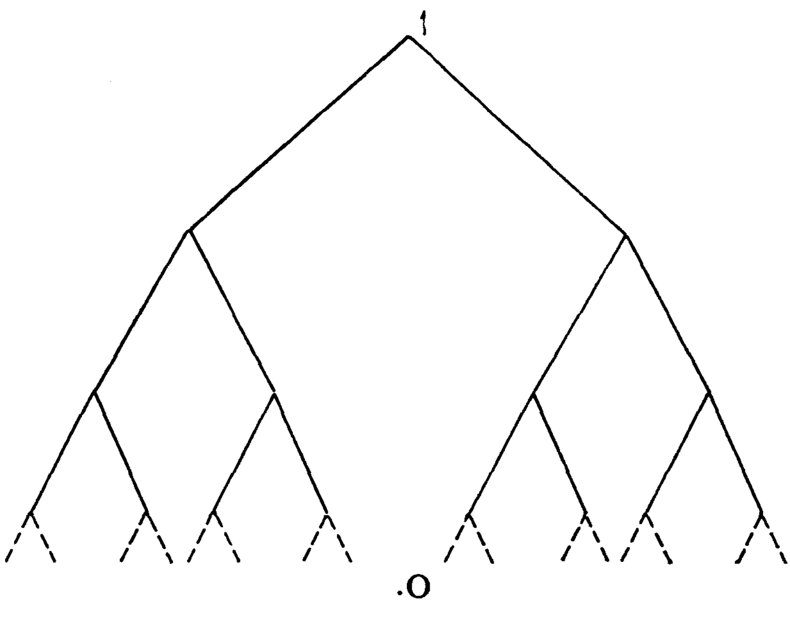
A semilattice E with unit, 1, and zero, 0, is called a *hanging tree* if it satisfies the following conditions;

(i) E is uniform,

(ii) for each $e \in E\backslash\{0\}$ there exists a unique finite subset $\{e_1, e_2, \cdots, e_k\}$ of E such that $1 = e_1 > e_2 > \cdots > e_{k-1} > e_k = e$, where e_i covers e_{i+1} for $i = 1, 2, \cdots, k-1$, and,

(iii) there exist $e, f \in E\backslash\{0\}$ such that $ef = 0$.

The cardinality of the set of elements covered by the identity of a hanging tree E is called the *degree* of E. Condition (iii) implies that the degree of a hanging tree E is always strictly greater than 1. It is easy to show that a hanging tree E is determined up to isomorphism by its degree. A hanging tree of degree 2 can be represented diagrammatically as follows:



We now provide examples of congruence-free inverse semigroups whose idempotents form hanging trees. Let X be a set of cardinality α. Let $\mathscr{F}_X^1$ be the free semigroup with identity on X and put $P_\alpha = \mathscr{F}_X^1 \times \mathscr{F}_X^1 \cup \{0\}$. Define multiplication on P_α as follows:

$$0 \cdot (f,g) = (f,g) \cdot 0 = 0$$

$$(f,g) \cdot (f',g') = \begin{cases} (f, hg') \text{ if } g = hf' \\ (kf, g') \text{ if } g' = kg \\ 0 \quad \text{otherwise} \end{cases}$$

Then, if $\alpha > 1$, P_α is a congruence-free inverse semigroup whose idempotents form a hanging tree of degree α. The semigroups, P_α, are the polycyclic monoids of Nivat and Perrot (1970).

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